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Nash Equilibria as Limits of Equilibria of Nearby Finite Games

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Abstract

We study finite-player normal-form games with compact metric action spaces and bounded measurable payoffs. Our main theorem shows that every Nash equilibrium of such a game can be recovered as the limit, in the product weak topology, of Nash equilibria of finite games obtained by discretizing the action spaces and perturbing payoffs by a uniformly vanishing amount. The proof samples from the target equilibrium, uses concentration inequalities to control weak convergence and incentive constraints on a growing finite set, and then applies a payoff perturbation to convert the resulting approximate equilibrium into an exact one. We also provide an example of a continuous game with a Nash equilibrium that cannot be approximated through Nash equilibria of finite games without perturbing payoffs.

1 Introduction

Nash equilibrium is the central solution concept in noncooperative game theory. In finite games, its existence was established early on by Nash (1950). In infinite games, however, equilibrium existence is not guaranteed, especially when payoffs are discontinuous. A large literature explores sufficient conditions for the existence and closure of equilibrium objects in discontinuous games by investigating when limits of equilibria, or approximate equilibria of finite approximations, remain equilibria of the

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underlying game; see, among many others, Reny (1999), Carmona (2011), Balder (2011), Bagh (2010), and Dilmé (2026).

This paper asks a complementary question. Suppose that an infinite game already has a Nash equilibrium. Can that equilibrium be represented as the limit of equilibria of finite games? The answer is yes, provided one allows the approximating finite games to have payoffs that differ from the original ones by a uniformly vanishing amount on the chosen finite action sets. In that precise sense, every Nash equilibrium of the infinite game is the limit of Nash equilibria of nearby finite games.

The distinction between *nearby finite games* and *raw finite restrictions* is important. The theorem below does *not* assert that every equilibrium of an infinite game $\langle S, u\rangle$ is the limit of Nash equilibria of the unperturbed restricted games $\langle S_n, u|_{S_n}\rangle$. What it proves is that for every Nash equilibrium σ, there exist finite action sets S_n, perturbed payoffs u_n, and strategy profiles σ_n with

$$\sup_{s\in S_n}\|u_n(s) - u(s)\|_\infty \to 0$$

such that each σ_n is an exact Nash equilibrium of $\langle S_n, u_n\rangle$ and $\sigma_n \Rightarrow \sigma$ (where $\sigma_n \Rightarrow \sigma$ indicates weak convergence). Equivalently, one may state the conclusion without explicit payoff perturbations: there exist finite action sets S_n and strategy profiles σ_n such that $\sigma_n \Rightarrow \sigma$ and each σ_n is an ε_n-Nash equilibrium of the unperturbed restriction $\langle S_n, u|_{S_n}\rangle$ for some sequence $\varepsilon_n \downarrow 0$. In particular, for every $\varepsilon > 0$, there are finite restrictions with an ε-Nash equilibrium arbitrarily close to σ. Our result therefore demonstrates that equilibrium behavior in the infinite model is not an artifact of assuming an infinite action space, as it can always be approximated through finite discretizations. In Section 4, we show that if one insists on not perturbing payoffs, even continuous games may have Nash equilibria that cannot be approximated through discretizations.

The proof proceeds in two main steps. First, for each player, we combine a deterministic dense grid of actions with a large random sample drawn from the target equilibrium. This deterministic grid guarantees that the finite action sets become dense in the original action space. This random sample generates an empirical mixed strategy that approximates the target distribution. Because the equilibrium probability measure is concentrated on pure best replies to the opponents' equilibrium strategies, the sampled actions are almost surely best replies in the limit game. By applying concentration inequalities, we then show that with positive probability three properties hold simultaneously: empirical weak convergence to the target equilibrium, near-optimality of the sampled actions against the empirical opponents, and near-unprofitability of deviations to the deterministic grid. Hence, there is a realization where all are satisfied.

Second, once such a realization is fixed, the resulting empirical profile is already an $O(1/n)$-equilibrium of the unperturbed finite restriction. A targeted payoff perturbation of order $O(1/n)$ on the equilibrium support then turns this approximate equilibrium into an exact one.

Conceptually, the paper is closest to the finite-approximation and refinement literature. Fudenberg and Levine (1986) characterize equilibria that arise as limits of ε-equilibria of smaller games. Simon and Stinchcombe (1995), Carbonell-Nicolau and McLean (2013), and Bajoori et al. (2013) study refinement ideas for infinite games that are also built from finite approximations. The present theorem works at the level of Nash equilibrium itself and establishes a general finite-approximation representation under minimal measurability and compactness assumptions. Methodologically, the argument is closer to concentration-based large-game results such as those of Kalai (2004), Carmona and Podczeck (2012), and Cartwright and Wooders (2009): equilibrium behavior is sampled, concentration inequalities are used to obtain a good deterministic realization, and that realization is then interpreted as an equilibrium object in a suitably nearby finite model.

The rest of the paper is organized as follows. Section 2 introduces the model and states the theorem. Section 3 provides the proof. Section 4 gives the example showing that payoff perturbations are genuinely needed. Section 5 discusses the contribution and its relation to the literature.

2 Model and main result

Let I be a finite set of players and write $N := |I|$. A normal-form game is a pair

$$G = \langle S, u \rangle, \qquad S = \prod_{i \in I} S_i,$$

where each S_i is a compact metric action space and $u : S \to \mathbb{R}^I$ is bounded and measurable. For each player i, let

$$\Sigma_i := \Delta(S_i), \qquad \Sigma := \prod_{i \in I} \Sigma_i,$$

where $\Delta(S_i)$ denotes the set of Borel probability measures on S_i. Because u is bounded and measurable, expected payoffs are well defined for all mixed-strategy profiles. We use the same notation u for the multilinear extension of payoffs to Σ, so $u_i(\sigma)$ denotes the expected payoff of player i under $\sigma \in \Sigma$.

A profile $\sigma \in \Sigma$ is a Nash equilibrium of G if

$$u_i(\sigma) \geq u_i(\tau_i, \sigma_{-i}) \qquad \text{for every } i \in I \text{ and every } \tau_i \in \Sigma_i.$$

Whenever $S_{i,n} \subseteq S_i$, we identify $\Delta(S_{i,n})$ with the subset of Σ_i consisting of measures supported on $S_{i,n}$. A sequence of finite games

$$G_n = \langle S_n, u_n \rangle, \qquad S_n = \prod_{i \in I} S_{i,n},$$

converges to G if, for every player i, the set $S_{i,n} \subseteq S_i$ is finite and

$$d_H(S_{i,n}, S_i) \to 0,$$

and if, in addition,

$$\sup_{s \in S_n} \|u_n(s) - u(s)\|_\infty \to 0.$$

Here d_H denotes Hausdorff distance on S_i. We endow Σ with the product weak topology, so that $\sigma_n \Rightarrow \sigma$ means equivalently that $\sigma_{i,n} \Rightarrow \sigma_i$ weakly for every player i.

We can now state the main result.

Theorem 1. *Let $G = \langle S, u \rangle$ be a game of the preceding form, and let $\sigma \in \Sigma$ be a Nash equilibrium of G. Then there exist finite games $G_n = \langle S_n, u_n \rangle$ converging to G and Nash equilibria $\sigma_n \in \prod_{i \in I} \Delta(S_{i,n})$ such that*

$$\sigma_{i,n} \Rightarrow \sigma_i \qquad \text{for every } i \in I.$$

Equivalently, $\sigma_n \Rightarrow \sigma$ in the product weak topology.

Remark 2. The theorem yields exact equilibria of finite games whose payoff functions converge uniformly to the original payoff function on the chosen finite action sets. It does not claim that the same conclusion holds for the unperturbed restricted games $\langle S_n, u|_{S_n} \rangle$.

3 Proof of Theorem 1

Proof. Fix a Nash equilibrium $\sigma \in \Sigma$.

Step 1: Sample from the target equilibrium. For each player i, define

$$V_i := u_i(\sigma), \qquad B_i := \Big\{ s_i \in S_i : u_i(s_i, \sigma_{-i}) = V_i \Big\}.$$

By Fubini–Tonelli, the map $s_i \mapsto u_i(s_i, \sigma_{-i})$ is Borel measurable, so B_i is measurable. Since σ is a Nash equilibrium,

$$u_i(s_i, \sigma_{-i}) \le V_i \qquad \text{for every } s_i \in S_i,$$

and also

$$\int_{S_i} u_i(s_i, \sigma_{-i})\, d\sigma_i(s_i) = u_i(\sigma) = V_i.$$

Therefore the nonnegative measurable function $s_i \mapsto V_i - u_i(s_i, \sigma_{-i})$ has integral zero under σ_i, and hence

$$\sigma_i(B_i) = 1.$$

So σ_i is concentrated on pure best replies to σ_{-i}.

For each player i, choose a dense sequence $(a_{i,r})_{r\geq 1}$ in S_i and a sequence $(f_{i,r})_{r\geq 1}$ that is dense in the unit ball of $C(S_i)$ under the supremum norm. Since S_i is compact and metric, $C(S_i)$ is separable and such a sequence exists. Define

$$\rho_i(\mu, \nu) := \sum_{r=1}^{\infty} 2^{-r} \left| \int f_{i,r}\, d\mu - \int f_{i,r}\, d\nu \right|.$$

Then ρ_i metrizes the weak topology on Σ_i.

Let

$$M := \max\left\{1, \max_{i\in I} \|u_i\|_\infty\right\}, \qquad \gamma := \min\left\{\frac{1}{2}, \frac{1}{2M^2 N}\right\}.$$

For each $n \geq 1$, choose an integer m_n so large that

$$2N(2n + m_n)\exp(-\gamma m_n/n^2) < 1.$$

This is possible because $(2n+m)e^{-\gamma m/n^2} \to 0$ as $m \to \infty$.

For each n, work on an auxiliary probability space on which, independently for every player i and every $k = 1, \ldots, m_n$, we draw

$$X_{i,k}^{(n)} \sim \sigma_i.$$

Because $\sigma_i(B_i) = 1$, we have $X_{i,k}^{(n)} \in B_i$ almost surely.

Define the empirical strategy

$$\hat{\sigma}_{i,n} := \frac{1}{m_n} \sum_{k=1}^{m_n} \delta_{X_{i,k}^{(n)}}, \qquad \hat{\sigma}_n := (\hat{\sigma}_{i,n})_{i\in I},$$

and the finite action set

$$S_{i,n} := \{a_{i,1}, \ldots, a_{i,n}\} \cup \{X_{i,1}^{(n)}, \ldots, X_{i,m_n}^{(n)}\}.$$

The deterministic points guarantee density of $S_{i,n}$ in S_i, while the sampled points generate the candidate equilibrium. All probabilities and expectations below refer to this auxiliary randomization.

Step 2: Three high-probability events. If $N = 1$, then there are no opponent samples. In that case part (ii) below is replaced by the deterministic inequalities

$u_1(a_{1,r}) \le V_1$ for all r, and part (iii) holds almost surely because $X^{(n)}_{1,k} \in B_1$ almost surely; Step 3 and Step 4 then go through verbatim. Thus, for the concentration estimates below, we may assume that $N \ge 2$.

We use the bounded-differences inequality (also called McDiarmid's or Hoeffding–Azuma inequality).

Lemma 3 (McDiarmid, 1989, Lemma 1.2). *Let $Y_1, \dots, Y_m$ be independent random variables and let $F = F(Y_1, \dots, Y_m)$ be measurable. Suppose that for each $r \in \{1, \dots, m\}$ there exists $c_r \ge 0$ such that, whenever $Y_j = Y'_j$ for all $j \ne r$,*

$$|F(Y_1, \dots, Y_m) - F(Y'_1, \dots, Y'_m)| \le c_r.$$

Then, for every $t > 0$,

$$\Pr\Big(|F - \mathbb{E}F| > t\Big) \le 2\exp\left(-\frac{2t^2}{\sum_{r=1}^m c_r^2}\right).$$

For each n, define the following events.

(i) *Empirical approximation.* For $i \in I$ and $1 \le r \le n$, let

$$A_1(i,r,n) := \left\{\left|\int f_{i,r}\, d\hat{\sigma}_{i,n} - \int f_{i,r}\, d\sigma_i\right| \le \frac{1}{n}\right\}.$$

Since $\|f_{i,r}\|_\infty \le 1$, changing one sample changes $\int f_{i,r}\, d\hat{\sigma}_{i,n}$ by at most $2/m_n$. Lemma 3 therefore yields

$$\Pr\Big(A_1(i,r,n)^c\Big) \le 2\exp\left(-\frac{m_n}{2n^2}\right) \le 2\exp\Big(-\gamma m_n/n^2\Big).$$

(ii) *Grid actions are almost unprofitable.* For $i \in I$ and $1 \le r \le n$, let

$$A_2(i,r,n) := \left\{u_i(a_{i,r}, \hat{\sigma}_{-i,n}) \le V_i + \frac{1}{n}\right\}.$$

The random variable $u_i(a_{i,r}, \hat{\sigma}_{-i,n})$ depends only on the opponents' samples. More explicitly,

$$u_i(a_{i,r}, \hat{\sigma}_{-i,n}) = \frac{1}{m_n^{N-1}} \sum_{k_{-i} \in \{1,\dots,m_n\}^{N-1}} u_i\Big(a_{i,r}, X^{(n)}_{-i,k_{-i}}\Big).$$

There are $(N-1)m_n$ opponent samples in total. If one of them is changed, exactly m_n^{N-2} terms in the sum change, and each changed term moves by at most $2M$. Hence changing one opponent sample changes $u_i(a_{i,r}, \hat{\sigma}_{-i,n})$ by at most $2M/m_n$. Moreover,

$$\mathbb{E}\Big[u_i(a_{i,r}, \hat{\sigma}_{-i,n})\Big] = u_i(a_{i,r}, \sigma_{-i}) \le V_i.$$

Therefore

$$
\begin{aligned}
\Pr\Big(A_2(i,r,n)^c\Big) &= \Pr\left(u_i(a_{i,r},\hat{\sigma}_{-i,n}) > V_i + \frac{1}{n}\right) \\
&\le \Pr\left(u_i(a_{i,r},\hat{\sigma}_{-i,n}) - \mathbb{E}\Big[u_i(a_{i,r},\hat{\sigma}_{-i,n})\Big] > \frac{1}{n}\right) \\
&\le \Pr\left(\Big|u_i(a_{i,r},\hat{\sigma}_{-i,n}) - \mathbb{E}\Big[u_i(a_{i,r},\hat{\sigma}_{-i,n})\Big]\Big| > \frac{1}{n}\right) \\
&\le 2\exp\left(-\frac{m_n}{2M^2(N-1)n^2}\right) \\
&\le 2\exp\left(-\frac{m_n}{2M^2Nn^2}\right) \le 2\exp\Big(-\gamma m_n/n^2\Big)\,.
\end{aligned}
$$

(iii) *Sampled actions remain almost best replies.* For $i \in I$ and $1 \le k \le m_n$, let

$$
A_3(i,k,n) := \left\{\Big|u_i\Big(X_{i,k}^{(n)},\hat{\sigma}_{-i,n}\Big) - V_i\Big| \le \frac{1}{n}\right\}.
$$

Since $\sigma_i(B_i) = 1$, we have $u_i(x,\sigma_{-i}) = V_i$ for σ_i-almost every x. Fix such an x. Conditional on $X_{i,k}^{(n)} = x$, the random variable $u_i(x,\hat{\sigma}_{-i,n})$ depends only on the opponents' samples and satisfies

$$
\mathbb{E}\Big[u_i(x,\hat{\sigma}_{-i,n}) \mid X_{i,k}^{(n)} = x\Big] = u_i(x,\sigma_{-i}) = V_i.
$$

As in part (ii), changing one opponent sample changes $u_i(x,\hat{\sigma}_{-i,n})$ by at most $2M/m_n$. There are $(N-1)m_n$ such samples, so for σ_i-almost every x,

$$
\begin{aligned}
\Pr\Big(A_3(i,k,n)^c \mid X_{i,k}^{(n)} = x\Big) &\le 2\exp\left(-\frac{m_n}{2M^2(N-1)n^2}\right) \le 2\exp\left(-\frac{m_n}{2M^2Nn^2}\right) \\
&\le 2\exp\Big(-\gamma m_n/n^2\Big)\,.
\end{aligned}
$$

Integrating over $X_{i,k}^{(n)}$ gives

$$
\Pr\Big(A_3(i,k,n)^c\Big) \le 2\exp\Big(-\gamma m_n/n^2\Big)\,.
$$

By the union bound,

$$
\begin{aligned}
&\Pr\left(\bigcap_{i\in I}\bigcap_{r=1}^{n} A_1(i,r,n) \,\cap\, \bigcap_{i\in I}\bigcap_{r=1}^{n} A_2(i,r,n) \,\cap\, \bigcap_{i\in I}\bigcap_{k=1}^{m_n} A_3(i,k,n)\right) \\
&\quad \ge 1 - 2N(2n + m_n)e^{-\gamma m_n/n^2} > 0.
\end{aligned}
$$

So for each n there exists at least one realization for which all these events hold simultaneously. Fix such a realization and set

$$
\sigma_n := (\sigma_{i,n})_{i\in I} := \hat{\sigma}_n.
$$

Step 3: Convergence and approximate incentive compatibility. First, each $S_{i,n} \subseteq S_i$ is finite and contains $\{a_{i,1}, \dots, a_{i,n}\}$. Since $(a_{i,r})_{r \geq 1}$ is dense in S_i, it follows that

$$d_H(S_{i,n}, S_i) \to 0 \qquad \text{for every } i \in I.$$

Second, event $A_1(i, r, n)$ implies

$$\rho_i(\sigma_{i,n}, \sigma_i) \leq \sum_{r=1}^{n} 2^{-r} \frac{1}{n} + \sum_{r=n+1}^{\infty} 2^{-r} \cdot 2 \leq \frac{1}{n} + 2^{-n+1}.$$

Hence $\rho_i(\sigma_{i,n}, \sigma_i) \to 0$ for every i, so $\sigma_{i,n} \Rightarrow \sigma_i$ for every player and therefore $\sigma_n \Rightarrow \sigma$ in the product weak topology.

Third, the empirical profile is already an approximate equilibrium of the unperturbed restricted game $\langle S_n, u \rangle$. Fix a player i. If $s_i \in \operatorname{supp}(\sigma_{i,n})$, then $s_i = X_{i,k}^{(n)}$ for some $k \leq m_n$, so event $A_3(i, k, n)$ yields

$$u_i(s_i, \sigma_{-i,n}) \geq V_i - \frac{1}{n}.$$

If $s_i \in S_{i,n}$ is arbitrary, then either $s_i \in \operatorname{supp}(\sigma_{i,n})$, in which case event $A_3(i, k, n)$ gives

$$u_i(s_i, \sigma_{-i,n}) \leq V_i + \frac{1}{n},$$

or $s_i \notin \operatorname{supp}(\sigma_{i,n})$, in which case s_i must be one of the deterministic grid points $a_{i,r}$ with $r \leq n$, and event $A_2(i, r, n)$ gives the same upper bound. Therefore

$$u_i(\sigma_n) \geq V_i - \frac{1}{n} \qquad \text{and} \qquad u_i(\tau_i, \sigma_{-i,n}) \leq V_i + \frac{1}{n} \quad \forall \tau_i \in \Delta(S_{i,n}).$$

So σ_n is a $2/n$-equilibrium of the finite restriction $\langle S_n, u \rangle$.

Step 4: A vanishing payoff perturbation makes the equilibrium exact. For each player i and each action $s_i \in S_{i,n}$, define

$$b_{i,n}(s_i) := \begin{cases} V_i + \frac{1}{n} - u_i(s_i, \sigma_{-i,n}) & \text{if } s_i \in \operatorname{supp}(\sigma_{i,n}), \\ 0 & \text{if } s_i \notin \operatorname{supp}(\sigma_{i,n}), \end{cases}$$

and let

$$u_{i,n}(s) := u_i(s) + b_{i,n}(s_i), \qquad u_n(s) := (u_{i,n}(s))_{i \in I}, \qquad s \in S_n.$$

If $s_i \in \operatorname{supp}(\sigma_{i,n})$, then $|u_i(s_i, \sigma_{-i,n}) - V_i| \leq 1/n$ by Step 3, so

$$|b_{i,n}(s_i)| \leq \frac{2}{n}.$$

If $s_i \notin \operatorname{supp}(\sigma_{i,n})$, then $b_{i,n}(s_i) = 0$. Consequently,

$$\sup_{s \in S_n} \|u_n(s) - u(s)\|_\infty \leq \frac{2}{n}.$$

It remains to verify that σ_n is a Nash equilibrium of $G_n := \langle S_n, u_n\rangle$. Fix a player i and a pure action $s_i \in S_{i,n}$. If $s_i \in \operatorname{supp}(\sigma_{i,n})$, then by construction

$$u_{i,n}(s_i, \sigma_{-i,n}) = u_i(s_i, \sigma_{-i,n}) + b_{i,n}(s_i) = V_i + \frac{1}{n}.$$

If $s_i \notin \operatorname{supp}(\sigma_{i,n})$, then $b_{i,n}(s_i) = 0$ and Step 3 yields

$$u_{i,n}(s_i, \sigma_{-i,n}) = u_i(s_i, \sigma_{-i,n}) \leq V_i + \frac{1}{n}.$$

Therefore every pure deviation yields at most $V_i + 1/n$, while every action in the support of $\sigma_{i,n}$ yields exactly $V_i + 1/n$. By linearity,

$$u_{i,n}(\tau_i, \sigma_{-i,n}) \leq V_i + \frac{1}{n} \qquad \text{for every } \tau_i \in \Delta(S_{i,n}),$$

and

$$u_{i,n}(\sigma_n) = V_i + \frac{1}{n}.$$

Hence player i has no profitable deviation. Since i was arbitrary, σ_n is a Nash equilibrium of G_n.

We have shown that each G_n is finite, that $G_n \to G$, and that $\sigma_n \Rightarrow \sigma$ with every σ_n a Nash equilibrium of G_n. This proves the theorem. □

4 Example

The payoff perturbation in Theorem 1 is genuinely needed. We give a game with a Nash equilibrium σ that cannot be obtained as a weak limit of Nash equilibria of the raw restrictions $\langle S_n, u|_{S_n}\rangle$, although it can be obtained after arbitrarily small payoff perturbations.

Let

$$a_m := 2^{-(m-1)} \quad (m \in \mathbb{N}), \qquad A := \{a_m : m \in \mathbb{N}\} \cup \{0, -1\} \subset \mathbb{R},$$

and set $S_1 = S_2 = A$, with the subspace topology from $\mathbb{R}$.

Define player 1's payoff by

$$u_1(a_m, a_1) = a_m^2, \qquad u_1(a_m, a_{m+1}) = -2a_m \qquad (m \in \mathbb{N}),$$

and let $u_1(s_1, s_2) = 0$ for all other $(s_1, s_2) \in A \times A$. Define player 2's payoff symmetrically by

$$u_2(s_1, s_2) := u_1(s_2, s_1).$$

The payoff function $u = (u_1, u_2)$ is continuous, because its only nonzero values are a_m^2 and $-2a_m$, both of which tend to 0 as $m \to \infty$.

For each player $i = 1, 2$, let

$$\sigma_i := \sum_{m=1}^{\infty} \frac{1}{2^{m+1}} \delta_{a_m} + \frac{1}{4}\delta_0 + \frac{1}{4}\delta_{-1}.$$

Since $\sum_{m=1}^{\infty} 2^{-(m+1)} = \frac{1}{2}$, this is a probability measure.

We first show that $\sigma := (\sigma_1, \sigma_2)$ is a Nash equilibrium. For every $m \in \mathbb{N}$,

$$u_1(a_m, \sigma_2) = a_m^2\, \sigma_2(a_1) - 2a_m\, \sigma_2(a_{m+1}) = a_m^2 \cdot \frac{1}{4} - 2a_m \cdot \frac{1}{2^{m+2}} = 0,$$

and also

$$u_1(0, \sigma_2) = u_1(-1, \sigma_2) = 0.$$

Thus every pure action of player 1 is a best response to σ_2. By symmetry, every pure action of player 2 is a best response to σ_1, so σ is a Nash equilibrium.

Why the raw restrictions fail. Suppose, toward a contradiction, that there exist finite sets $S_{i,n} \subseteq A$ with

$$d_H(S_{i,n}, A) \to 0 \qquad (i = 1, 2),$$

and Nash equilibria

$$\tau_n \in \Delta(S_{1,n}) \times \Delta(S_{2,n})$$

of the raw restricted games

$$\widetilde{G}_n := \langle S_{1,n} \times S_{2,n},\ u|_{S_{1,n} \times S_{2,n}} \rangle$$

such that $\tau_n \Rightarrow \sigma$.

Since a_1 and -1 are isolated points of A, weak convergence implies

$$\tau_{i,n}(a_1) \to \frac{1}{4}, \qquad \tau_{i,n}(-1) \to \frac{1}{4} \qquad (i = 1, 2).$$

Hence, for all sufficiently large n, both a_1 and -1 belong to $S_{i,n}$ and receive strictly positive probability under $\tau_{i,n}$.

For such n, define

$$m_n := \max\{m \in \mathbb{N} : a_m \in S_{1,n}\}, \qquad k_n := \max\{m \in \mathbb{N} : a_m \in S_{2,n}\}.$$

These maxima are well defined because $a_1 \in S_{i,n}$ for large n.

Because $\tau_{1,n}(-1) > 0$ and τ_n is a Nash equilibrium, the action -1 must be a best response to $\tau_{2,n}$. Its payoff is always 0, so no available action of player 1 can yield a strictly positive payoff against $\tau_{2,n}$.

Now suppose that $m_n \geq k_n$. Then $a_{m_n+1} \notin S_{2,n}$, so the negative payoff of action a_{m_n} is absent from the restricted game. Since $a_1 \in S_{2,n}$ and $\tau_{2,n}(a_1) > 0$, we get

$$u_1(a_{m_n}, \tau_{2,n}) = a_{m_n}^2 \, \tau_{2,n}(a_1) > 0,$$

a contradiction. Hence $m_n < k_n$ for all sufficiently large n. By symmetry, interchanging the two players gives $k_n < m_n$ for all sufficiently large n, again a contradiction.

Therefore, there is no sequence of Nash equilibria of the raw restricted games $\widetilde{G}_n$ converging weakly to σ.

How a small payoff perturbation fixes the problem. For each $n \geq 1$, let

$$S_{1,n} = S_{2,n} := \{a_1, \ldots, a_n, 0, -1\}, \qquad S_n := S_{1,n} \times S_{2,n}.$$

Then $d_H(S_{i,n}, A) \leq a_{n+1} \to 0$.

Define, for each player $i = 1, 2$,

$$\sigma_{i,n} := \sum_{m=1}^{n} \frac{1}{2^{m+1}} \delta_{a_m} + \left(\frac{1}{4} + \frac{1}{2^{n+1}}\right) \delta_0 + \frac{1}{4} \delta_{-1}.$$

This is obtained from σ_i by moving the omitted tail mass $\sum_{m>n} 2^{-(m+1)} = 2^{-(n+1)}$ to the point 0.

We claim that $\sigma_{i,n} \Rightarrow \sigma_i$. Let $f \in C(A)$. Then

$$\int_A f \, d\sigma_{i,n} - \int_A f \, d\sigma_i = \sum_{m=n+1}^{\infty} \frac{1}{2^{m+1}} \big(f(0) - f(a_m)\big).$$

Hence

$$\left| \int_A f \, d\sigma_{i,n} - \int_A f \, d\sigma_i \right| \leq \sup_{m>n} |f(0) - f(a_m)| \sum_{m=n+1}^{\infty} \frac{1}{2^{m+1}} \longrightarrow 0,$$

because $a_m \to 0$ in A. Thus $\sigma_{i,n} \Rightarrow \sigma_i$.

Next define perturbed payoffs $u_n = (u_{1,n}, u_{2,n})$ on S_n by

$$u_{1,n}(s_1, s_2) := \begin{cases} 0 & \text{if } (s_1, s_2) = (a_n, a_1), \\ u_1(s_1, s_2) & \text{otherwise}, \end{cases} \qquad u_{2,n}(s_1, s_2) := u_{1,n}(s_2, s_1).$$

Thus the only modified payoff entries are (a_n, a_1) for player 1 and (a_1, a_n) for player 2. Since

$$u_1(a_n, a_1) = a_n^2,$$

we obtain

$$\sup_{s \in S_n} \|u_n(s) - u(s)\|_\infty = a_n^2 \to 0.$$

We now show that $\sigma_n := (\sigma_{1,n}, \sigma_{2,n})$ is an exact Nash equilibrium of the finite game $G_n := \langle S_n, u_n \rangle$. For every $m < n$, the same calculation as above gives

$$u_{1,n}(a_m, \sigma_{2,n}) = 0.$$

For the action a_n, the positive payoff against a_1 has been reset to 0, and the negative payoff against a_{n+1} is absent because $a_{n+1} \notin S_{2,n}$. Hence

$$u_{1,n}(a_n, \sigma_{2,n}) = 0.$$

Also

$$u_{1,n}(0, \sigma_{2,n}) = u_{1,n}(-1, \sigma_{2,n}) = 0.$$

So every pure action in $S_{1,n}$ yields payoff 0 against $\sigma_{2,n}$. Hence player 1 has no profitable deviation from $\sigma_{1,n}$. By symmetry, the same holds for player 2. Therefore σ_n is a Nash equilibrium of G_n for every n.

In conclusion, $G_n = \langle S_n, u_n \rangle$ is a sequence of finite games such that

$$d_H(S_{i,n}, A) \to 0, \qquad \sup_{s \in S_n} \|u_n(s) - u(s)\|_\infty \to 0, \qquad \sigma_n \Rightarrow \sigma,$$

and each σ_n is an exact Nash equilibrium of G_n.

5 Discussion and related literature

The theorem complements the equilibrium-existence literature for discontinuous games. Results such as Reny (1999), Carmona (2011), Balder (2011), and Dilmé (2026) study conditions under which discontinuous games admit equilibria or equilibrium-related objects that are closed under approximation. Our starting point is different: we assume that a Nash equilibrium of the infinite game is already given and ask whether that equilibrium itself can be represented through finite games.

The papers that are conceptually closest are those that use finite approximations as a way to organize equilibrium or equilibrium refinement in infinite games. Fudenberg and Levine (1986) characterize when equilibria arise as limits of ε-equilibria of games with smaller strategy spaces. Simon and Stinchcombe (1995) develop refinement notions for infinite normal-form games using finite approximations. Carbonell-Nicolau and McLean (2013) obtain approximation theorems for discontinuous games and use them for equilibrium refinement, while Bajoori et al. (2013) study perfect equilibrium in compact-action games. Dilmé (2026) provides conditions for Nash equilibria to coincide with the set of strategy profiles that can be approximated through ε-equilibria along all discretizations of a compact game. Relative to that literature, the present result is narrower in one dimension and broader in another: it stays at the level of Nash equilibrium, but it applies under very weak condition and shows that *every* equilibrium can be recovered as the limit of exact equilibria of nearby finite games.

Methodologically, the proof is closer to the concentration-based literature on large Bayesian games. Kalai (2004) and Carmona and Podczeck (2012) sample equilibrium behavior and use concentration to establish approximate ex-post stability with high probability. Cartwright and Wooders (2009) then connect approximate ex-post stability to purification. The present paper uses the same broad template—sample from equilibrium, use concentration, and fix one good realization—but for a different purpose: not ex-post stability in a large realized game, but finite approximability of a given equilibrium in a fixed finite-player game.

Finally, the convergence notion adopted here is intentionally weak. We require only that the finite action sets become dense and that payoffs converge uniformly on those finite sets. This is exactly what the proof needs. At the same time, the deterministic part of the discretization can be chosen arbitrarily subject to density. Thus, when one works with a stronger approximation concept and can supply a suitable deterministic discretization in advance—for instance, one motivated by the variational convergence viewpoint of Bagh (2010)—the sampling part of the present argument can be layered onto that scheme without changing its core logic.

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