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Simultaneous Bidding in Sealed-Bid Auctions

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Abstract

In this paper, we analyze a model of competing sealed-bid first-price and second-price auctions where bidders have unit demand and can bid on multiple auctions simultaneously. We show that there is no symmetric pure equilibrium with strategies that are increasing in the lowest type, unlike in standard auction games. However, for a two-player game a symmetric mixed-strategy equilibrium exists, and bidders place bids on all available auctions with probability one. This holds true for any mixed equilibrium and for any number of bidders. We then solve the case of two auctions and two bidders. Analyzing the case of binary type space, we are able to identify mixed strategy equilibria and analyze the consequences of discrete bid spaces.

KEYWORDS: Simultaneous bidding, concurrent auctions, sealed-bid auctions, first-price auction, second-price auction

JEL CLASSIFICATION: C72, D44

1 Introduction

In this paper, we study the bidding behavior in a model of competing and simultaneous sealed-bid auctions. We make use of second-price auctions and first-price auctions. Buyers are symmetric, have unit demand, and can participate and bid on all the auctions with no entry costs. The structure of the game is as follows. Several sellers hold simultaneous auctions and sell homogeneous goods (one each) to a group of buyers. Each buyer perceives the goods as perfect substitutes: they are interested in acquiring just one unit. The game has incomplete

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information, where valuations are independent and ex-ante symmetric. After the buyers have decided on their bids, the auctions are solved. This framework raises the following trade-off: bidding on multiple auctions increases the probability of winning at least one object, but concurrently it increases the sum of the expected prices, as the bidder may win more than one object, which is undesired. Yet, winning two goods is better than zero if the prices paid are low.

Recall that agents bid their type in the classic equilibrium of a second-price auction. Here, if bidders find it optimal to bid on multiple auctions, they reduce their bids to offset the higher sum of expected prices and hence would not reveal their type. Solving these games is not trivial: Cai and Dimitriou (2014) show that these games are at least PP-hard[1]. Tipically, we aim to identify symmetric and pure equilibria. The first part of Theorem 1 shows that no such equilibrium exists for a large class of 'regular' strategies. The regularity condition we impose that bidders, on at least an auction, use a bidding function that is increasing from the lowest type to any arbitrary higher type. Theorem 1 also states that in any pure equilibrium of a game with excess demand (the number of auctions is lower than the number of buyers), at least one bidder will place bids in more than one auction. Consequently, the assumption that bidders can participate in only a single auction is not without loss of generality, even when bidders have unit demand.

We believe that symmetric behavior is a plausible property of the equilibrium of a symmetric auction game, even if its existence is not trivial. Theorem 2 shows that a symmetric equilibrium exists in the two-player case if we allow for mixed strategies. This existence result relies on Reny (1999). The author shows that under some technical conditions of the strategy space (compactness and Hausdorff) and payoffs (better-reply security), any discontinuous symmetric game possesses a symmetric mixed strategy equilibrium. By showing that the game satisfies these properties, we prove existence. Furthermore, it turns out that in any of these equilibria all the bidders bid on all the auctions with probability 1. We are not able to prove existence in general. In fact, better-reply security relies on the fact that for each $\varepsilon > 0$ the players can always play strategies that avoid ties and lose no more than ε expected utility. When the game has more than two players, we cannot guarantees this. Therefore, the game

[1] A PP-hard problem can be solved by probabilistic polynomial time. NP problems are included in this class.

may not have enough continuity to get the existence result. In Appendix C we provide an example that shows why ties cannot be easily excluded as in standard auction games.

In Appendix B, we propose a brief analysis of non-trivial, pure asymmetric equilibria with increasing strategies[2]. We provide some examples and find their respective equilibria. As anticipated in the discussion of Theorem 1, in equilibrium some of the players bid on multiple auctions.

Building upon Szentes (2007), we characterize symmetric mixed-strategy equilibria in a game with two auctions and two bidders. We consider two kinds of binary type space. In the first one, the low type has a valuation of 0 for the object. This type does not participate in any auctions. Therefore, the only incomplete information for a bidder with high type is whether the other player wants to participate. Then, we consider the case where the low type has a strictly positive valuation, so that both types want to bid positive amounts and so the high type faces stronger competition. Solving a particular functional equation, we find a closed-form expression of a symmetric mixed strategy equilibrium in both cases. In the case of lowest type has zero valuation for the objects, the high types randomize their bid over two decreasing lines. These lines are cut in half by the 45-degree line, and one line is strictly above the other. Moreover, the distribution of bids is the same for both auctions. On the other hand, when the lowest type has positive valuation for the goods, we find a continuum of equilibria where each type randomizes over a decreasing line, and the support of the low type is strictly below the support of the high type.

We also study the case of discrete bids, showing how richer bid spaces are responsible for higher probabilities of positive bids on all the auctions.

The rest of the paper is organized as follows. In section 2, we present the literature review. In section 3 we describe the model and set the basics of the game. Section 4 discusses equilibrium existence for a general number of auctions and bidders. Here the main two theorems are presented. In section 5, we find symmetric equilibria in the specific case of two auctions and

[2]An increasing strategy can be defined in multiple ways in the concurrent auctions setting. As we try to be as general as possible, we posit a prerequisite condition, stipulating that the strategy must exhibit strict monotonicity from the lowest type to a higher type in at least one auction. This definition accommodates functions with diminishing segments as well. The crucial element is the presence of at least one auction where the function strictly increase from the lowest to a higher type.

two bidders and discuss the consequences of discrete bids. Finally, we present the conclusions at the end of the paper.

2 Literature Review

Many works on competing auctions assume that buyers (who desire to acquire just one unit of the good) can participate in one auction only and allow them to randomize their participation decision (McAfee (1993), Peters and Severinov (1997), Delnoij and De Jaegher (2020)). Peters and Severinov (2006) consider instead simultaneous English auctions in which bidders can bid on multiple auctions. When there are no bidding cost and no fixed ending time for the auctions, the authors find that the strategy that bids on the auction with the lowest standing bid is a Bayesian equilibrium. Anwar, McMillan, and Zheng (2006) perform an empirical analysis using evidence from eBay. They find that bidders bid across multiple auctions; their strategy is coherent with what is suggested by Peters and Severinov (2006). We follow this approach in a framework of sealed-bid auctions. Gerding, Dash, Byde, and Jennings (2008b) have also addressed this problem by categorizing bidders into two groups: local and global. A local bidder is a bidder who can bid on one auction only, while a global bidder can bid on multiple auctions at the same time[3]. In their model, there is only one global bidder, and the mechanism is a second-price auction. They study the behavior of the global bidder and prove that no matter the number of local bidders and available auctions, she wants to place a bid on all the auctions. We can interpret this result in the following way: in a model with only global bidders, there exists no equilibrium in which all want to bid on one auction only. Our work diverts from this path, considering games with global bidders only. At the end of their paper, Gerding et al. (2008b) analyze the game with three global bidders and no local bidders. They approach the problem with numerical simulation. Their algorithm oscillates among two states and hence it does not converge. In Appendix *B*, we solve this specific game analytically, and we find a (non-trivial) equilibrium when the distribution of types is uniform,

[3] The authors provide many reasons why a bidder should be local. For example, she may have bounded rationality, and therefore be unable to compute the optimal strategy when considering multiple bids. The bidder may also have a budget constraint: Gerding, Dash, Byde, and Jennings (2008a) prove in their paper that a bidder may prefer to concentrate her resources on one auction when budget-constrained. Alternatively, the bidder may be unaware of the other auctions (unlikely in the case of online auctions).

the same distribution the authors assume in their example.

Our paper also builds on the work of Szentes (2007), who analyzed the scenario of two auctions and two symmetric bidders. Szentes examined both perfect complements and perfect substitutes goods, assuming that bidders have only one type, which implies complete information. The auction mechanism studied was a first-price auction. However, in a separate paper, Szentes (2005) provided methods to convert first-price auction equilibria into second-price auction equilibria. We extend Szentes' results by introducing an additional bidder type into the model, thus incorporating incomplete information.

3 The Model

We consider an auction game G, in which $K \geqslant 2$ independent sealed-bid auctions selling the same good are held simultaneously. In this game, N bidders can participate in any number of auctions. The mechanism can either be a first-price auction, $G = G^{FPA}$, or a second-price auction, $G = G^{SPA}$. When not specified, G can be either of them. Bidders are assumed ex-ante symmetric and with unit demand. In the game's first stage, Nature specifies a type for each bidder from the set $\Theta \subseteq [0, 1]$ according to some distribution F (assumed to be atomless when types are continuous). Types are independent across players. Once they know their type $\theta \in \Theta$, each bidder selects a bid for each auction from the bid space $\mathcal{B}$. Since there are K simultaneous sealed-bid auctions, the action space is $\mathcal{A} = \mathcal{B}^K$. We assume that $0 \in \mathcal{B}$ and consider a bid of 0 on auction j equivalent to the decision of not participating in that auction.

Since goods are homogeneous and bidders have unit demand, the object of interest is the probability of winning at least one good. We denote it with Q. Q is a function of the player's vector of bids $b \in \mathcal{A}$ and of the other players' strategies. A pure strategy in this game is a function $\beta = (\beta^1, \ldots, \beta^K)$ where, for all k, $\beta^k : \Theta \longrightarrow \mathcal{B}$ is a measurable function. Each β^k assigns to each type θ a bid $\beta^k(\theta)$ on auction k. Therefore, for each $\theta \in \Theta$, $\beta(\theta)$ gives the vector of bids of the player.

Preferences are assumed to be linear. Hence, the interim expected payoff of bidder i when her type is θ_i, she bids $\beta_i(\theta) = (\beta_i^1(\theta_i), \dots, \beta_i^K(\theta_i))$ and each $j \neq i$ uses strategy β_j is

$$u_i(\theta_i, \beta_i, \beta_{-i}) = \theta_i Q_i(\beta_i(\theta_i), \beta_{-i}) - \sum_{k=1}^{K} E[P_k | \beta_i^k(\theta), \beta_{-i}^k],$$

where $E[P_k | \beta_i^k(\theta), \beta_{-i}^k]$ denotes the expected price of auction k paid by i given i's bid and strategies β_{-i}^k. Clearly, the expected price depends on the format FPA or SPA. We compute the expected payoff of a mixed strategy in the obvious way.

It is important to note that while the expected gain $\theta_i Q_i$ is influenced by the strategies used in all the auctions, each expected price is independent of the bidding strategies on the other auctions.

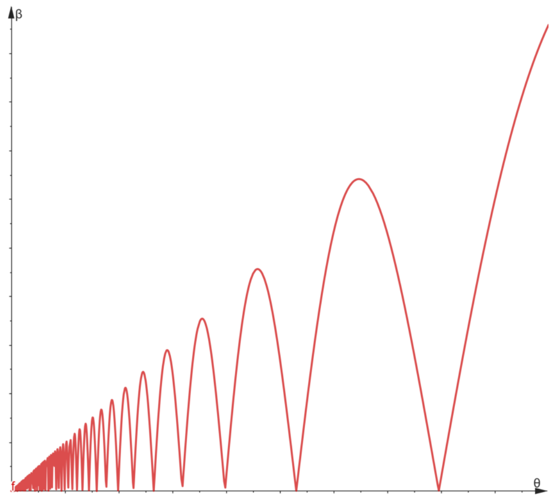
4 Equilibrium Existence

In this section, we let $\Theta = \mathcal{B} = [0, 1]$. In symmetric auction games we usually search for symmetric equilibria. We investigate the nature of pure strategy equilibria first. A symmetric pure strategy equilibrium in this game is a strategy β^* such that $\forall i \in \{1, \dots, N\}$, $\forall \theta_i \in \Theta$, $\beta^*(\theta_i)$ maximizes $u_i(\theta_i, \beta_i, \beta_{-i}^*)$. We know that, in any symmetric equilibrium, $\beta^k(\theta) \leqslant \theta$ for each θ and auction format. Now, for each auction k, define

$$\theta^k := \inf\{\theta \in [0, 1] : \exists \tilde{\theta} < \theta \text{ s.t } \beta^k(\tilde{\theta}) \geqslant \beta^k(\theta)\}.$$

Note that when $\theta^k > 0$, the bidding function β^k is strictly increasing in the interval $[0, \theta^k]$. In fact, consider $\theta < \theta' < \theta^k$. Then, by definition of θ^k we have that $\beta^k(\theta) < \beta^k(\theta')$. Moreover, observe β^k is allowed to have decreasing parts in the interval $[\theta^k, 1]$. An example of a bidding function β^k with $\theta^k = 0$ and $\beta^k(\theta) \leqslant \theta$ is when it exhibits infinite oscillations (decreasing branch followed by an increasing branch) as θ gets close to 0. Figure 4 depicts an example of a bidding function of this kind. Clearly, θ^k is a function of β^k and of β in general. When the set where we take the infimum is empty, let $\theta^k = +\infty$. In the next Theorem, we show that the existence of an auction k with $\theta^k > 0$ is enough to exclude the possibility of a symmetric pure equilibrium. Moreover, we show that whenever there are more buyers than sellers (so

Figure 1: Bidding function k with $\theta^k = 0$



that there is excess of demand), there cannot be an equilibrium where the bidders split and bid on a single auction each. Therefore, if they coordinate and enter one auction only, they have an incentive to deviate and bid across multiple auctions.

Theorem 1. *Consider strategy β and suppose there exists $k \in \{1, \ldots, K\}$ such that $\theta^k > 0$. Then, β cannot be a symmetric pure strategy equilibrium. Moreover, whenever $N > K$ there are no equilibria in pure strategies where all the bidders bid on one auction only.*

Before the proof, we want to report Gerding et al. (2008b) result about participation on multiple auctions, adapting it to the current setting.

Theorem (Gerding et al. (2008b)) let $G = G^{SPA}$. Suppose that $\forall i \in \{1, \ldots, N-1\}$ there exists $k \in \{1, \ldots, K\}$ such that for all $\theta_i \in [0, 1]$, $\beta_i^k(\theta_i) = \theta_i$ and for $j \neq k$, $\beta_i^j(\theta_i) = 0$. Then, β_N is optimal for player N only if $\beta_N^k(\theta_N) > 0$ for all $\theta_N > 0$, and for all $k \in \{1, \ldots, K\}$.

We are now ready to prove Theorem 1.

Proof. First, observe $\forall \theta' \in (0, 1]$ we cannot have more than one auction k where β^k is strictly increasing in $[0, \theta']$. If there are two or more, these auctions have perfectly correlated allocations, and in this case, a profitable deviation is to bid on only one of these auctions. In fact, not that if k and m are two auctions such that β^k and β^m are strictly increasing in $[0, \theta']$, then for $\theta_i \in (0, \theta']$,

$$Q_i(\beta_i^1(\theta_i), \ldots, \beta_i^k(\theta_i), \ldots, \beta_i^m(\theta_i), \ldots, \beta^K(\theta_i), \beta_{-i}) = Q_i(\beta_i^1(\theta_i), \ldots, \beta_i^k(\theta_i), \ldots, 0, \ldots, \beta^K(\theta_i), \beta_{-i}).$$

Therefore, bidding 0 on auction m leaves the expected allocation unchanged and reduces the expected price of auction m to 0.

Hence, suppose there exists a unique $k \in \{1, \ldots, K\}$ such that $\theta^k > 0$. Take $\theta' \leqslant \theta^k$. Now, consider auction $m \neq k$. Let

$$\theta^* := \inf\{\theta \in [0, \theta'] : \exists \tilde{\theta} < \theta \text{ s.t } \beta^m(\tilde{\theta}) > \beta^m(\theta)\}.$$

The set on which we take the infimum, can either be empty or non-empty. Assume it is empty first. Then, $\forall \theta \in [0, \theta']$, and $\forall \hat{\theta} \in (\theta, \theta']$, $\beta^m(\hat{\theta}) \geqslant \beta^m(\theta)$. Hence, β^m is weakly increasing in $[0, \theta']$. First, we exclude $\beta^m(\theta) = 0$ for all $\theta \in [0, \delta)$ for any $\delta < \theta'$. If so, take $\theta < \delta$, and observe that $\beta^k(\theta) = 0$ and $\beta^m(\theta) > 0$ is a profitable deviation (when $G = G^{FPA}$, we consider small $\beta^m(\theta)$). Therefore, $\beta^m(\theta) > 0$ for all $\theta \in (0, \delta]$ for some $\delta > 0$. But then, since β^m is weakly increasing and positive for positive types, winning auction m implies winning auction k. Therefore, $\beta^m(\theta) = 0$ is a profitable deviation for these types, a contradiction.

Now, assume $\{\theta \in [0, \theta'] : \exists \tilde{\theta} < \theta \text{ s.t } \beta^m(\tilde{\theta}) > \beta^m(\theta)\}$ is non-empty. We have two different cases, $\theta^* = 0$ and $\theta^* > 0$. Suppose first $\theta^m = 0$. We first claim that there exists $\theta \in [0, \theta']$ such that $\beta^m(\theta) \geqslant \beta^m(\tilde{\theta})$ for all $\tilde{\theta} < \theta$. Suppose not. Then, for all $\theta \in (0, \theta']$ there exits $\underline{\theta} < \theta$ such that $\beta^m(\theta) < \beta^m(\underline{\theta})$. Now, construct a sequence of types in the following way.

1. Let $\theta_1 \in (0, \theta']$. Then take $\theta_2 < \theta_1$ be such that $\beta^m(\theta_2) > \beta^m(\theta_1)$.

2. Let $\theta_n \in (0, \theta_{n-1})$ be such that $\beta^m(\theta_n) > \beta^m(\theta_{n-1})$.

3. Let $\theta_n \to 0$. Observe this is possible by assumption.

Now, take n^* such that $\theta_{n*} < \beta^m(\theta_1)$. Such n^* exists since $\theta_n \to 0$. Observe that by construction we have $\beta^m(\theta_{n*}) > \beta^m(\theta_1) > \theta_{n*}$, a contradiction. Therefore, there exits $\theta \in (0, \theta']$ such that for all $\tilde{\theta} < \theta$, $\beta^m(\theta) \geqslant \beta^m(\tilde{\theta})$. But then, type θ wins auction m only if she wins auction k. Hence, θ has a profitable deviation by bidding on k.

Hence, let $\theta^* > 0$. Observe that when this is the case, then β^m is weakly increasing in $[0, \theta^*]$ (it cannot be strictly increasing by assumption). By the previous argument, these types win auction m only if they win auction k, so they have a profitable deviation.

The second result for $G = G^{SPA}$ stems from Gerding et al. (2008b) in the following way. By contradiction, suppose that in equilibrium all the bidders $i \in \{1, \dots, N-1\}$ bid on one auction only. Then, player N has the incentive to bid on all the auctions. Hence, in a pure equilibrium, at least one bidder bids on multiple auctions.

For $G = G^{FPA}$ consider the following. Suppose each bidder place her bid on a single auction. Clearly, there exists $i \in \{1, \dots, N\}$ such that $Q_i < 1$, as $N > K$. Moreover, in a FPA, we have that in equilibrium $\beta_j^k(\theta_j) \leqslant \theta_j$ for all bidders j and auction k. Now, suppose i bids on auction m, and consider a deviation that bids the same amount on auction m and $\varepsilon > 0$ on auction k. Then, since the auctions are independent, the new probability of winning at least one object is $Q_i' = Q_i^m + Q_i^k - Q_i^m Q_i^k$ (we suppressed the arguments for readability). Note that $Q_i^k \geqslant F^n(\varepsilon)$, where n is the number of bidders on auction k. Moreover, note that the expected price on auction k is $Q_i^k \varepsilon$. Hence, the new expected utility increases by the amount

$$\theta_i Q_i^k (1 - Q_i^m) - Q_i^k \varepsilon = Q_i^k (\theta_i (1 - Q_i^m) - \varepsilon). \tag{1}$$

Since $Q_i^k > 0$ for $\varepsilon > 0$ and $Q_i = Q_i^m < 1$ by assumption, $\exists \varepsilon > 0$ such that (1) is strictly positive. Therefore, i can deviate and this is a contradiction. ∎

This Theorem tells us that we cannot have any symmetric pure strategy equilibrium unless for each auction k we have $\theta^k = 0$. This excludes any equilibrium for which β^k is increasing in an interval $[0, \theta']$ for any θ' and any auction k. An example of a strategy β with $\theta^k = 0$ for each k, is a function such that for all k, β^k has infinite oscillation as θ gets close to 0 (Figure 4). Our conjecture is, in case an equilibrium like this exists, that oscillations work as a coordination device. Clearly, $\beta^k \neq \beta^m$ (on sets with positive measure) for each pair of auctions k and m, otherwise these auctions are perfectly correlated in terms of allocation and bidders prefer to bid on a single auction. Hence, each auction will have a different strategy. Now, to have an intuition of our conjecture, suppose there are only two auctions. We think that every type will bid 'high' on an auction and bid 'low' on the other one. High and low are calibrated to minimize the probability of winning both auctions under the constraint of monotonic allocation (a higher type has higher probability of winning at least one object, i.e., a higher Q). This attempt of coordination could generate oscillations. Another example of $\theta^k = 0$ is a function β^k that is weakly increasing from 0 to some θ and exhibits constant values in intervals that become progressively smaller as θ approaches zero. As shown in the Appendix for the proof of Theorem 2, we cannot exclude the possibility of ties in equilibrium for a general number of bidders. In fact, we believe that ties allow the players to hedge against the risk of winning too many objects.

Another natural candidate for the equilibrium is the case in which the bidders can coordinate their entry to reduce competition. For example, if there are $N = 4$ bidders and $K = 2$ auctions, naive intuition may expect the participation of two bidders in the first auction and the two other bidders in the second auction. Our second statement says that whenever $N > K$, this kind of coordination fails. In fact, when agents split and participate on one auction only, independent types imply that the allocation on one auction is independent from the allocation on another auction. As Gerding et al. (2008b) suggests, since $Q < 1$, bidders always "demand" for more probability of winning. As they can also control their expected price through their bid they always have incentive to bid on multiple auctions. This may not be achieved if auctions are highly correlated. An intuition of this result is also provided by the next example.

Example 1. Let $G = G^{SPA}$. Suppose there are $N = 2n + 1$ bidders, where $n \in \mathbb{N}$ and $K = 2$. They independently draw their type from the uniform distribution over $[0, 1]$. There are $2n$ local bidders and one global bidder. n local bidders bid on auction 1, and the other n bid on auction 2. Local bidders play either $\beta(\theta) = (\theta, 0)$ or $\beta(\theta) = (0, \theta)$ depending on the auction in which they participate. Suppose the global bidder places a bid of θ (her true type) on the first auction. Her interim utility is then

$$u(\theta, (\theta, 0), \beta_{-i}) = \theta^{n+1} - \frac{n}{n+1}\theta^{n+1}.$$

Now, consider placing a bid $b \in \mathbb{R}_+$ on auction 2 as well. Then

$$u(\theta, (\theta, b), \beta_{-i}) = \theta(\theta^n + b^n - \theta^n b^n) - \frac{n}{n+1}\theta^{n+1} - \frac{n}{n+1}b^{n+1}.$$

Therefore, the expected gain is

$$\begin{aligned}
&u(\theta, (\theta, b), \beta_{-i}) - u(\theta, (\theta, 0), \beta_{-i}) \\
&= \theta(\theta^n + b^n - \theta^n b^n) - \frac{n}{n+1}\theta^{n+1} - \frac{n}{n+1}b^{n+1} - \left(\theta^{n+1} - \frac{n}{n+1}\theta^{n+1}\right) \\
&= \theta b^n - \theta^{n+1} b^n - \frac{n}{n+1}b^{n+1} \\
&= b^n \left(\theta - \theta^{n+1} - \frac{n}{n+1}b\right).
\end{aligned}$$

Observe that the sign of the last expression depends on

$$\theta(1 - \theta^n) - \frac{n}{n+1}b \gtreqless 0. \tag{2}$$

Choose $b \in \left(0, \frac{n+1}{n}\theta(1-\theta^n)\right)$. Expression (2) becomes strictly positive, and then the agent gains from bidding b.

Theorem 1 highlights two fundamental incentives within the game. First, when auctions are highly correlated, bidders tend to bid on fewer auctions. Conversely, when the auctions are independent and each bidder initially bids on a single auction, they wish to deviate by bidding on all of them. Therefore, any equilibrium lies between these two extremes: the auctions will be neither entirely independent nor perfectly correlated.

It is not trivial to obtain symmetric behavior in this game. The following result states symmetric behavior is possible in equilibrium with mixed strategies when $N = 2$. Proving the existence of such solutions can be problematic. Auctions present discontinuity in the payoffs. Therefore, we cannot apply classical results in fixed point theory. Reny (1999) provided Nash equilibrium existence results for a large class of discontinuous games. His main Theorem gives sufficient conditions for the existence of pure strategy equilibria that generalizes the mixed strategy equilibrium existence in the previous literature (e.g., Nash (1950), Glicksberg (1952)). Moreover, he provides additional conditions which are sufficient for the existence of symmetric equilibria. Proving that G possesses all the sufficient conditions requires many technical steps. Thus, we leave the proof of the following Theorem in the Appendix.

Theorem 2. *Let $N = 2$. The game G possesses a symmetric equilibrium in mixed strategies.*

In this section we have seen that pure, regular strategies and symmetric behavior cannot be achieved at the same time. Therefore in the next section we analyze the game G focusing on symmetric strategies. As proved in the previous theorem, we can obtain such an equilibrium when $N = 2$ allowing the bidders to use mixed strategies. The reason why we cannot extend the proof to any $N \geqslant 2$ is related to the fact that we cannot grant that the bidders do not strictly prefer ties in equilibrium. This blocks us from generating enough continuity in the game to use Reny (1999) results. We discuss this in Appendix C. The analysis of (asymmetric) pure strategy equilibria is left in the Appendix B, where we find the equilibria in several different examples and show their properties.

5 Symmetric Equilibria

5.1 Discrete bids

We start the section with a simple example of a game with discrete bids, two players, two auctions and two types. This game allows us to show the role of discrete bids on the equilibrium. Let $G = G^{SPA}$, $\Theta = \{0, 1\}$, $N = 2$ and $\mathcal{B} = \{0, 1/2, 1\}$, and set $Pr(\theta = 0) = 1/2$. We denote with $\sigma(\theta)$ the probability distribution over bids when the player's type is θ. For example, $\sigma(1) = [0.5(x, y), 0.5(y, x)]$ means that type $\theta = 1$ plays (x, y) (i.e., bids x on auction 1 and y on auction 2) and (y, x) with probability of 0.5 each.

Proposition 1. *The game G^{SPA} has only two symmetric mixed equilibria σ_1, σ_2, where*

$$\sigma_1(\theta) = \begin{cases} [1(0,0)] & \text{if } \theta = 0 \\ [{}^1\!/\!_6(1,0), {}^1\!/\!_6(0,1), {}^2\!/\!_3(1/2,1/2)] & \text{if } \theta = 1. \end{cases}$$

and

$$\sigma_2(\theta) = \begin{cases} [1(0,0)] & \textit{if } \theta = 0 \\ [{}^1\!/_2(1,1/2), {}^1\!/_2(1/2,1)] & \textit{if } \theta = 1. \end{cases}$$

In the first equilibrium, bidders have a positive probability of bidding on one auction only (recall 0 bids are equivalent to non-participation). It is natural to ask whether this kind of equilibrium exists because of the low cardinality of the action space. In the next Proposition we show that this is indeed the case. The probability of bidding on one auction decreases as the action space becomes richer. Therefore, consider the sequence of games G_n similar to the previous one, where the type space is $\Theta = \{0,1\}$ and the bid space is $\mathcal{B}_n = \{0 = x_0, x_1, \ldots, x_n, x_{n+1} = 1\}$ such that the points in the set are equidistant. We get the following proposition.

Proposition 2. *Consider the sequence of games G_n^{SPA} and let p_n be the probability that a player bids on one auction only in a symmetric mixed equilibrium of G_n^{SPA}. Then, $p_n \to 0$ as $n \to \infty$.*

Proof. We drop the subscript n for readability. Suppose both bidders play the action $(1,0)$ with probability p by both players in a symmetric equilibrium. Next, consider player i and the deviation $(1,x_1)$. We check the net gain of player i of moving mass p from $(1,0)$ to $(1,x_1)$. This deviation benefits i in the event in which the other bidder plays $(1,0)$ and she loses the tie. The benefit is 1, and the probability that this event happens is $\frac{1}{4}p$ (0.5 for $\theta = 1$ of the other player and 0.5 for losing the tie). Therefore the benefit is at least $\frac{1}{4}p$. Next, observe that playing $(1,x_1)$ over $(1,0)$ can be detrimental to i in the case in which she is already winning auction 1. Yet, if the other player plays $(1,0)$, i does not pay any additional price. Hence, the maximum loss is $\frac{1}{2}(1-p)x_1$. Thus, a necessary condition for the optimality of $(1,0)$ is

$$\frac{1}{4}p \leqslant x_1 \frac{1}{2}(1-p) \Leftrightarrow p \leqslant \frac{x_1}{{}^1\!/_2 + x_1}.$$

Then, as $x_1 \to 0$, we must have that $p \to 0$. ■

Therefore, a richer type space induces the players to bid more frequently on all the auctions. In what follows, we prove that this fact holds in the limit too, where the bid space is the continuum $[0,1]$. Moreover, it holds for any number of auctions and bidders.

Consider again $G \in \{G^{FPA}, G^{SPA}\}$, $\Theta = \mathcal{B} = [0,1]$.

Proposition 3. *In any symmetric mixed strategy equilibrium of G, the players bid on all the available auctions with probability 1.*

This Proposition is true for all $N \geqslant 2$. Yet, as previously discussed, we cannot prove the existence of a symmetric mixed strategy equilibrium when the number of players is more than 2. The intuition of the previous Proposition is pretty straightforward: by symmetry, if one bidder puts positive probability on the strategy that bids on, say, auction 1 only, then everyone assigns the same probability to the same action. Hence, there is a strictly positive probability that another auction, for example, auction 2, will be left with no participants. Therefore, the strategy that bids the same amount on auction 1 and a small amount on auction 2 makes the player strictly better off, as the amount on the second auction can be arbitrarily small[4]. We start with the following Lemma, which is the core of the proof.

Lemma 1. *Suppose $N = 2$, $K = 2$, and types are uniformly distributed over $[0,1]$. Let $b > 0$. Then, in any symmetric mixed strategy equilibrium players do not put positive probability on the strategies $(b, 0)$ and $(0, b)$.*

Proof. Suppose otherwise, that is, both players put $q_1 > 0$ on $(\beta^1(\theta), 0)$, or $q_2 > 0$ on $(0, \beta^2(\theta))$, or both. Without loss of generality, we assume both. Consider bidder $i \in \{1,2\}$. We claim $\exists \varepsilon > 0$ such that $(\beta^1(\theta_i), \varepsilon)$ (where θ_i is i's type) is a profitable deviation against $(\beta^1(\theta_i), 0)$. To do so, we compare the interim payoff provided by $(\beta^1(\theta_i), \varepsilon)$ and $(\beta^1(\theta_i), 0)$ in all the relevant events. We use interim payoffs as all the statements in the following steps hold for all $\theta > 0$.

With probability $q_1 > 0$ bidder $j \neq i$ plays $(\beta^1(\theta_j), 0)$. In equilibrium, $\beta^1(\theta') > \beta^1(\theta'')$ for $\theta' > \theta''$ (this is trivially true when the player decides to bid on a single auction). In this event, the action $(\beta^1(\theta_i), \varepsilon)$ makes i win at least one object. Moreover, the expected prices of $(\beta^1(\theta_i), 0)$ and $(\beta^1(\theta_i), \varepsilon)$ are the same if we condition on j playing $(\beta^1(\theta_j), 0)$ in G^{SPA}, and are arbitrarily close in G^{FPA}. Therefore, $(\beta^1(\theta_i), \varepsilon)$ increases payoff by $\theta_i(1-\theta_i)$ with probability q_1.

[4]The intuition is similar to the result in war of attrition in continuous time where at most one bidder can concede with positive probability. In fact, if both do, one of the player can wait an ε at time 0 and get a strictly higher payoff. See Abreu and Gul (2000), Hendricks, Weiss, and Wilson (1988).

Consider next j playing $(0, \beta^2(\theta_j))$. This event happens with a probability of $q_2 > 0$. Here, both $(\beta^1(\theta_i), 0)$ and $(\beta^1(\theta_i), \varepsilon)$ grant i the object. Yet, the second bid increases the expected price of auction 2 by a maximum of ε.

Finally, with probability q_3 bidder j is playing some bid (β', β''). Then, $(\theta_i, 0)$ and (θ_i, ε) provide the same probability of winning and expected price on auction 1. Winning auction 2 may be correlated with the event of winning auction 1. We consider the worst-case scenario, in which bidding on auction 2 does not increase the probability of winning at least one object. Therefore, $(\beta^1(\theta_i), \varepsilon)$ increases the expected price on auction 2 by no more than ε. In the worst-case scenario the difference in the payoff is at least

$$q_1\theta_i(1-\theta_i) + q_2(-\varepsilon) + q_3(-\varepsilon).$$

Hence, $\exists \varepsilon > 0$ such that $(\beta^1(\theta_i), \varepsilon)$ is strictly better than $(\beta^1(\theta_i), 0)$ (note that if $\varepsilon > 0$ is a profitable deviation for θ_i, then it is a profitable deviation for all $\tilde{\theta}_i > \theta_i$). ■

Now, we are able to prove the Proposition.

Proof. Whenever $q_1 > 0$, there is a strictly positive probability that the other player is giving up auction 2. Hence, the bidder can just put a small amount in that auction and win the object for free. We can extend this result to any number of bidders. In fact, suppose there are $n+1 \in \mathbb{N}$ bidders. Consider again $q_1 > 0$ and $q_2 > 0$. There is still q_1^n probability of increasing the payoff by $\theta_i(1-\theta_i)^n$. Since in the other cases the expected costs can be controlled by the player via $\varepsilon > 0$, $(\beta^1(\theta_i), \varepsilon)$ is still a profitable deviation against $(\beta^1(\theta_i), 0)$. Again, the same holds for $q_2 > 0$. Finally, we can allow for any number of auctions. By the same reasoning as the last part, strategies of the kind $(0, \ldots, \beta^k(\theta_i), \ldots, 0)$ are dominated. When bidders put a positive probability on a strategy that bids on multiple auctions but not all of them, we can still apply the same logic. In fact, suppose the equilibrium strategy puts probability $q > 0$ on $(\beta^1(\theta_i), \beta^2(\theta_i), \ldots, 0, \ldots, \beta^K(\theta_i))$. Hence, there is a probability q^n that one auction has no bidders. Therefore, there exists $\varepsilon > 0$ such that $(\beta^1(\theta_i), \beta^2(\theta_i), \ldots, \varepsilon, \ldots, \beta^K(\theta_i))$ dominates the strategy in the support.

Note that the uniform distribution does not play a role. Hence, we can substitute it with any atomless distribution. ■

5.2 Continuous bids

We have described the property that holds in any symmetric mixed equilibrium, and so we now describe and analyze a particular case. We find symmetric mixed strategy equilibria of a game G in which $K = 2$ and $N = 2$. Theorem 2 proved that this game always have a symmetric mixed strategy equilibrium for this number of bidders, and the Theorem could be easily extended to the discrete types case. This problem was previously considered by Szentes (2007) with type space $\Theta = \{1\}$. Therefore, he assumes complete information in his model. We extend his results. First, we consider the case $\Theta = \{0, 1\}$ and $\mathcal{B} = [0, 1]$. We assume $G = G^{FPA}$ to find a closed form solution, as this ensures an easier payoff structure. Then, we transform the equilibrium into an equilibrium of the second-price auction game G^{SPA} through a modified version of a technique provided in Szentes (2005). We leave the details in the Appendix. Finally, we modify the type space to $\Theta = \{a, 1\}$, $a > 0$. In this case, the competition for the high type is tighter as the low type is interested in the object. On both cases, $Pr(\theta = 1) = 1/2$.

Since the game has perfect recall, we can describe strategies in behavioral form, as in Proposition 1. Therefore, a symmetric equilibrium is a set of strategies such that each type randomizes over the square $[0, 1] \times [0, 1]$ and no profitable deviations are possible. Szentes (2007) proves that, in any symmetric mixed equilibrium with atomless strategies, agents randomize over two decreasing lines that lie in the space $\mathcal{A} = [0, 1] \times [0, 1]$. The reason the support includes decreasing lines only is intuitive: whenever a bidder increases the bid on one auction, say, auction 1, the marginal value of the object sold in auction 2 will decrease, making it optimal to place a lower bid on this auction. We now consider the type space $\Theta = \{0, 1\}$ as in the previous example and seek an explicit solution to the game. With incomplete information and $\Theta = \{0, 1\}$, randomization can occur along a single decreasing line. However, following Szentes' approach, we aim to find an equilibrium with a two-line support, as this equilibrium resembles the one found in the case of $\Theta = \{a, 1\}$. Before delving into the technical analysis, we provide an intuitive example to illustrate why the support must consist of either one or two decreasing lines.

So, take two points P_2 and P_1 in $[0,1]^2$ such that $P_2 >> P_1$. These corresponds to two different vectors of bids. We assume that P_2 has higher bids on auction 1 and auction 2, as in Figure 2. Now consider alternative bids D_1 and D_2 as in the picture. Now compare the randomization $\frac{1}{2}P_1 + \frac{1}{2}P_2$ against $\frac{1}{2}D_1 + \frac{1}{2}D_2$. Note that both provide the same payoff if we condition on the event in which the other bidder plays outside of the red square. Instead, in case the opponent plays inside the square, P_1 wins no object while P_2 wins both; D_1 and D_2 win exactly one object each. Therefore, in order for P_1 and P_2 to be in the equilibrium support, the players cannot put positive mass inside the square, otherwise $\frac{1}{2}D_1 + \frac{1}{2}D_2$ is a deviation.

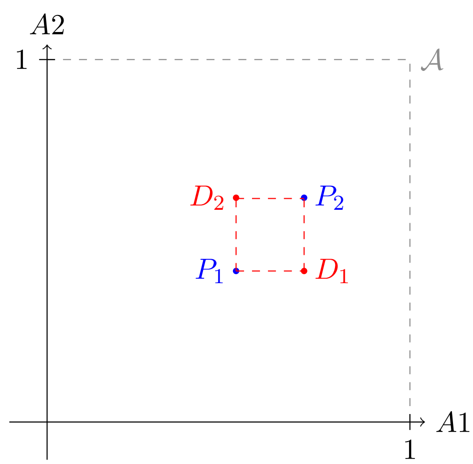


Figure 2: P_1 and P_2 against D_1 and D_2

This exclude bidimensional supports, increasing lines or more than two decreasing lines, as in Figure 3.

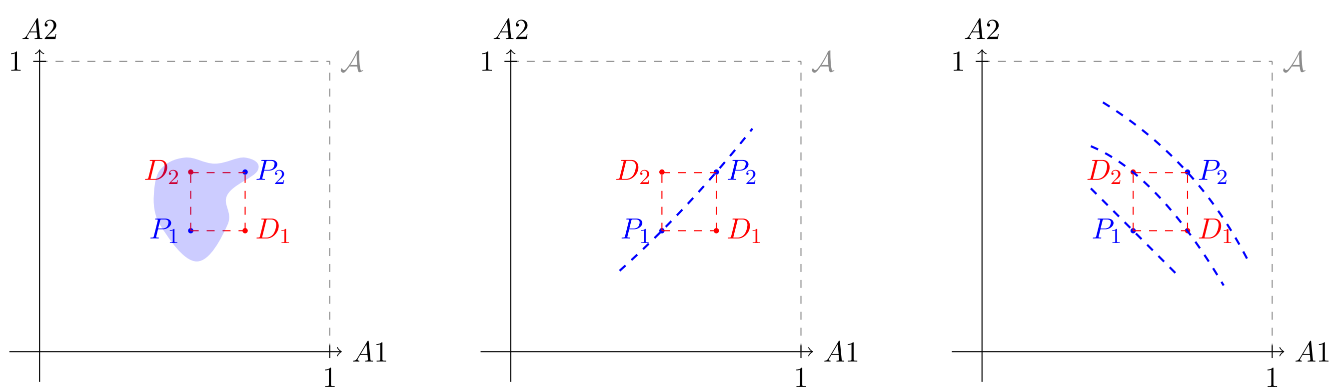


Figure 3: Examples of supports that cannot emerge in equilibrium

Therefore, we are left with the possibility of two or one decreasing line. We proceed with the former. Let g_1 and g_2 be two decreasing functions. The union of their graphs is going to be the support of the equilibrium strategy of the high type as, clearly, $\sigma(0) = [1(0,0)]$. The distribution of this strategy is determined by two other functions that we call G_1 and G_2. Finally, these four functions are related in the following way:

$$G_1(g_1(x)) = \frac{1}{2} - G_1(x)$$
$$G_2(g_2(x)) = \frac{1}{2} - G_2(x),$$

where $G_1(x) = \dfrac{x}{1-x}(1 + G_2(x))$ and $G_2(0) = 0$, $G_2(1/4) = 1/2$. Then, we have the following.

Lemma 2. *The curves g_1 and g_2 are both symmetric with respect to the objects, strictly decreasing and for all $x \in (0, 1/4)$, and $g_1(x) > g_2(x)$ when $G_2(x) > \dfrac{x}{1-2x}$.*

Therefore, the support is symmetric to the auctions, as the curves g_1 and g_2 are cut in half by the 45-degree line. Now, consider the following two functions:

$$F_1(\{(y, g_1(y))|x \in [0, x]\}) = G_1(x)$$
$$F_2(\{(y, g_2(y))|x \in [0, x]\}) = G_2(x).$$

Both F_1 and F_2 are not defined over the entire Borel σ-algebra of $Graph(g_1)$ and $Graph(g_2)$, but they can be extended in the obvious way. Assume also that these extensions take value of zero on the square $[0,1] \times [0,1]$ except on their respective graphs. Call these extensions μ_1 and μ_2. Therefore, for example,

$$\mu_1(\{(x, g_1(x))|x \in [a,b]\}) = F_1(\{(x, g_1(x))|x \in [0,b]\}) - F_1(\{(x, g_1(x))|x \in [0,a]\}).$$

The same goes for μ_2. Observe then that μ_1 and μ_2 are measures over $[0,1] \times [0,1]$[5].

[5] The extension on $[0,1] \times [0,1]$ works in the following way: for all $A \subseteq [0,1] \times [0,1]$, $\mu_i(A) = \mu_i(A \cap Graph(g_i))$, for $i \in \{1, 2\}$.

Proposition 4. *Let* $G_2(x) > \dfrac{x}{1-2x}$. *Then*

1. μ_1 *and* μ_2 *induce the probability measure* $\sigma(1) = \mu_1 + \mu_2$ *over the union of the graphs of* g_1 *and* g_2*;*

2. *The points* $\{(x,y)|x \in [0, 1/4], y \in [g_2(x), g_1(x)]\}$ *provide the same payoff against* σ*, where* $\sigma(0) = [1(0,0)]$*;*

3. σ *is a symmetric mixed strategy equilibrium.*

Proof. 1. Trivially, μ_1 and μ_2 are measures and therefore σ is a measure. Next, observe

$$\sigma(Graph(g_1) \bigcup Graph(g_2)) = \mu_1(Graph(g_1)) + \mu_2(Graph(g_2)) = G_1(1/4) + G_2(1/4) = \frac{1}{2} + \frac{1}{2} = 1.$$

Hence, σ is a probability measure over the graphs of g_1 and g_2.

2. Next, let $x \in [0, 1/4]$ and $y \in [g_2(x), g_1(x)]$ and consider player I with $\theta_I = 1$. The probability of winning at least one object is then

$$\underbrace{\frac{1}{2} \cdot 1}_{\theta_{II}=0} + \underbrace{\frac{1}{2}[\underbrace{(1 - G_1(1/4))}_{II \text{ plays on } g_2} + \underbrace{G_1(x) + G_1(y)}_{II \text{ plays on } g_1}]}_{\theta_{II}=1}$$

and the expected payment is

$$\underbrace{\left(\frac{1}{2} + \frac{1}{2}(G_1(x) + G_2(x))\right)x}_{\text{Win auction 1}} + \underbrace{\left(\frac{1}{2} + \frac{1}{2}(G_1(y) + G_2(y))\right)y}_{\text{Win auction 2}}$$

that can be rewritten as

$$\frac{1}{2}x(G_1(x) + G_2(x)) + \frac{1}{2}y(G_1(y) + G_2(y)) + \frac{1}{2}(x + y).$$

Now, observe that for $x \in [0, 1/4]$

$$G_1(x) = \frac{x}{1-x}(1 + G_2(x)) \Rightarrow G_1(x) = x + x(G_1(x) + G_2(x)).$$

Therefore, the expected payment is

$$\frac{1}{2}(G_1(x) - x) + \frac{1}{2}(G_1(y) - y) + \frac{1}{2}(x + y) = \frac{1}{2}(G_1(x) + G_1(y)).$$

Finally, the expected payoff is

$$\frac{1}{2} + \frac{1}{2}(1 - G_1(1/4)) + \frac{1}{2}(G_1(x) + G_2(y)) - \frac{1}{2}(G_1(x) + G_2(y)) = \frac{1}{2} + \frac{1}{2}(1 - G_1(1/4)),$$

and then it is independent of (x, y) as long as $x \in [0, 1/4]$ and $y \in [g_2(x), g_1(x)]$.

3. We are left to show that σ is an equilibrium, that is, there are no profitable deviations outside the set $\{(x, y) | x \in [0, 1/4], y \in [g_2(x), g_1(x)]\}$ against σ. Observe we only have to check deviations such that $x, y \leqslant \frac{1}{4}$.

   Obviously $\sigma(0) = [1(0, 0)]$ is optimal. For $\theta = 1$, let $x \in [0, 1/4]$ and $y > g_1(x)$. Observe this action provides the same probability of winning as $g_1(x)$ (i.e., 1) but has a strictly higher expected price. Hence, it cannot be optimal. Next, consider $y < g_2(x)$. The probability of winning one object at least is now

$$\frac{1}{2} + \frac{1}{2}(G_1(x) + G_2(x) + G_1(y) + G_2(y))$$

   and together with the expected price we have a payoff of

$$\frac{1}{2} + \frac{1}{2}(G_2(x) + G_2(y)).$$

   Recall that $G_1(1/4) + G_2(1/4) = 1$, and so we can rewrite the previous expression as

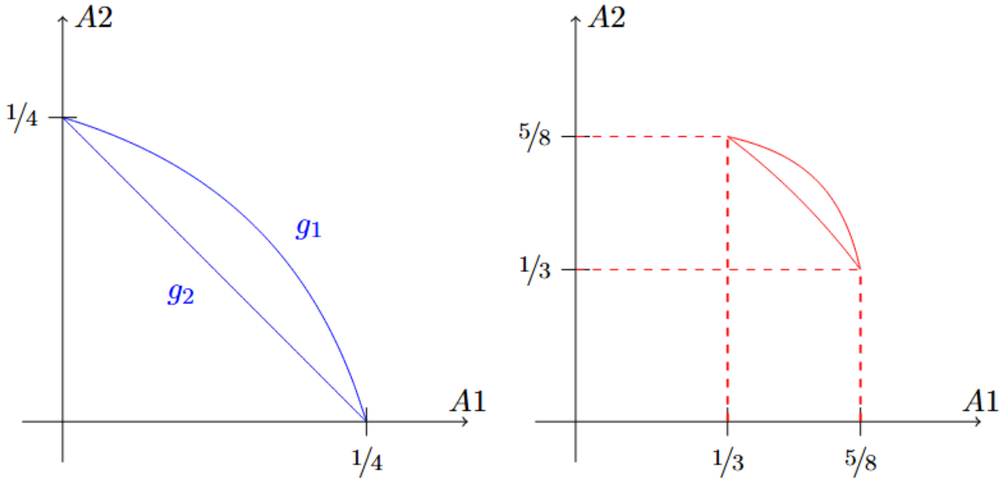
$$\frac{1}{2} + \frac{1}{2}(1 - G_1(1/4)) - \frac{1}{2}(G_2(1/4) - G_2(x) - G_2(y)),$$

   and since $G_2(1/4) - G_2(x) - G_2(y) > 0$, it is not profitable to deviate to $y < g_2(x)$.

∎

Therefore, any G_2 that satisfies the condition in the previous Proposition gives us an equilib-

Figure 4: Mixed equilibria support



(a) FPA Equilibrium Support (b) SPA Equilibrium Support

rium. In the following picture, we show an example of the support of this equilibrium and the corresponding transformation into the support of G^{SPA} equilibrium.

The case just showed presents a particular kind of incomplete information. A player who is interested in acquiring the object ($\theta = 1$) is uncertain about whether the other player wants or does not want the object. Therefore, the information she is missing is about the participation decision of the other agent. It is interesting to ask what kind of symmetric equilibrium emerges when the low type is bounded away from zero. Or, equivalently, when both types are interested in winning the object. It turns out that this time, the support of each type can be just one decreasing line. Therefore, let $\Theta = \{a, 1\}$, where $a > 0$. The auctions are first-price as before. Let g_1 and g_a be defined on $[0, a/2]$ and $[0, b]$ $(b < a/2)$, respectively. The graph of the first function is the support of the equilibrium strategy of $\theta = 1$ while the second one is the support of the equilibrium strategy of $\theta = a$. These functions are implicitly defined by

$$G_1(g_1(x)) = 1 - G_1(x)$$

$$G_a(g_a(x)) = 1 - G_a(x),$$

where $G_a(0) = 0$, $G_a(b) = 1$, G_a strictly increasing over $[0, b]$, and

$$G_1(x) = \begin{cases} \dfrac{x}{a-x} G_a(x) & x \in [0, b] \\ \dfrac{x}{a-x} & x \in (b, a/2]. \end{cases}$$

The distributions on the curves in the equilibrium are

$$\begin{aligned} F_1(\{(y, g_1(y)) | y \in [0, x)\}) =& G_1(x) \\ F_a(\{(y, g_a(y)) | y \in [0, x)\}) =& G_a(x). \end{aligned}$$

We extend these functions to μ_1 and μ_a in the same fashion as before. Observe μ_1 and μ_a are probability measures. This equilibrium looks different to the case $a = 0$. Here every type randomizes over one decreasing line. In fact, $\theta = a$ plays over the graph of g_a while $\theta = 1$ plays over the graph of g_1.

Lemma 3. *The curves g_1 and g_a are both symmetric with respect to the objects, strictly decreasing in their domain and $g_1 > g_a$ in the domain of g_a.*

The proof of this Lemma follows the same lines of Lemma 2, hence it is skipped.

Proposition 5. *The strategy above generates a symmetric mixed strategy equilibrium.*

Proof. We first show that both $\theta = 1$ and $\theta = a$ are indifferent in their support, and then we prove there is no profitable deviation for the players. Then, consider $\theta = 1$. Let $(x, g_1(x))$ be the action she plays. Given that the other player plays (μ_a, μ_1), the payoff is

$$\begin{aligned} &\frac{1}{2}[1 - G_1(x)x - G_1(g_1(x))g_1(x)] + \frac{1}{2}[1 - G_a(x)x - G_a(g_1(x))g_1(x)] \\ =&\frac{1}{2}[2 - (G_1(x) + G_a(x))x - (G_1(g_1(x)) + G_a(g_1(x)))g_1(x))]. \end{aligned}$$

Observe that for all $x \in [0, a/2]$, we have

$$(G_1(x) + G_a(x))x = aG_1(x)$$

$$(G_1(g_1(x)) + G_a(g_1(x)))g_1(x) = (1 - G_1(x))a.$$

Therefore, the payoff reduces to $1 - a/2$. Hence, $\theta = 1$ is indifferent in the graph of g_1.

Next, let $\theta = a$. We show a stronger statement for this type, that is, she is indifferent on the entire set

$$A = \{(x, y) | g_a(x) \leqslant y \leqslant g_1(x)\}$$

Therefore, suppose $\theta = a$ plays $(x, y) \in A$. Observe that her payoff is

$$\begin{aligned}
&\frac{1}{2}a[1 + G_1(x) + G_1(y)] - \frac{1}{2}[(G_a(x) + G_1(x))x + (G_a(y) + G_1(y))y] \\
=&\frac{1}{2}a + \frac{1}{2}(aG_1(x) + aG_1(y)) - \frac{1}{2}(aG_1(x) + aG_1(y)) \\
=&\frac{1}{2}a
\end{aligned}$$

Hence, $\theta = a$ is indifferent on the entire set A, which includes the graph of g_a.

Now, we check optimality. Consider again $\theta = 1$. Clearly, (x, y) where $y > g_1(x)$ is not optimal as in the case $a = 0$. Then, let the agent bid (x, y), $g_a(x) < y < g_1(x)$. The expected payoff is then

$$\begin{aligned}
&\frac{1}{2}[G_1(x) + G_1(y) + 1 - (G_1(x) + G_a(x))x - (G_1(y) + G_a(y))y] \\
=&\frac{1}{2}[G_1(x) + G_1(y) + 1 - aG_1(x) - aG_a(y)] \\
=&\frac{1}{2}[(1 - a)(G_1(x) + G_1(y)) + 1] \\
<&1 - \frac{a}{2},
\end{aligned}$$

where the last inequality follows from $G_1(x) + G_1(y) < 1$. Hence, (x, y) cannot be a profitable deviation. Assume $y < g_a(x)$. The expected profit is then

$$\frac{1}{2}[(1-a)(G_1(x) + G_1(y)) + (G_a(x) + G_a(y))] < 1 - \frac{a}{2}.$$

We conclude that $\theta = 1$ is in equilibrium. We are left to show that $\theta = a$ does not want to deviate to any (x, y) with $y < g_a(x)$. If the low type is playing a bid below g_a, her payoff is

$$\begin{aligned}
&\frac{1}{2}a(G_a(x) + G_a(y) + G_1(x) + G_1(y)) - \frac{1}{2}(G_a(x)x + G_a(y)y + G_1(x)x + G_1(y)y) \\
=&\frac{1}{2}a(G_a(x) + G_a(y) + G_1(x) + G_1(y)) - \frac{1}{2}(aG_1(x) + aG_1(y)) \\
=&\frac{1}{2}a(G_a(x) + G_a(y)) < \frac{1}{2}a,
\end{aligned}$$

where the last inequality follows from $G_a(x)+G_a(y) < 1$. Hence, type $\theta = a$ is in equilibrium. Therefore, the suggested strategies form an equilibrium. ■

As the proof shows, the high type strictly prefers to play in its support than in any other portion of the action space. The low type, instead, is indifferent among all the points between the curves g_a and g_1. Therefore, the low type is characterized by the same payoff condition as in Szentes (2007), where the agent had complete information.

6 Conclusion

In this paper, we have analyzed the bidding behavior of unit-demand buyers who have the opportunity to place bids on multiple sealed-bid auctions. Our analysis reveals several key insights about the strategic behavior of bidders in such environments.

Firstly, we demonstrated that restricting bidders to participate in only one auction is not without loss of generality. In fact, bidders have a strong incentive to place bids on multiple auctions simultaneously. This multi-auction bidding behavior arises due to the strategic trade-offs faced by bidders. While bidding on multiple auctions raises the sum of expected prices, it also increases the likelihood of winning at least one item. However, the correlation among auctions plays a critical role in shaping these incentives. When auctions are highly correlated, the incentive for bidders to participate in all auctions diminishes. Specifically, if one auction is nearly a perfect copy of another, the incentive to bid on both auctions is significantly reduced. Yet, independent auctions, i.e., bidders participate in only one auction, creates incentives to bid on all of them. This leads to the conclusion that in any pure equilibrium, at least one bidder will place bids across multiple auctions.

Secondly, we explored the existence of symmetric equilibria with 'standard' strategies. These strategies include at least one bidding function that is increasing from the lowest type to any arbitrarily higher type. We found that such equilibria are unattainable in our setting. When bidders adopt regular strategies, some auctions become highly correlated, which incentivizes bidders to focus their efforts on a single auction, effectively abandoning the others. To reintroduce symmetry, we must consider mixed strategies. We prove equilibrium existence in the case of two bidders. Moreover, our analysis shows that if we aim to achieve symmetric equilibria through mixed strategies, we should expect all bidders to participate in all auctions with probability one.

Lastly, we provided a detailed characterization of the equilibria in the specific case of two bidders facing two sealed-bid auctions and incomplete information, therefore extending previous literature. As we considered binary type spaces for closed form solutions, future research could extend our model further by investigating equilibria with continuous types, which would offer a better understanding of bidder behavior. Additionally, examining the role of reserve prices in such auctions could provide valuable insights into how sellers can influence bidding strategies and auction outcomes.

A Appendix

B Asymmetric pure equilibria

In this section we find explicit solutions to the case of $K = 2$ and $N = 3$. We start with the case of binary types. We consider discrete and continuous type spaces and show that we get similar equilibria. Assume there are $K = 2$ sealed-bid second-price auctions and $N = 3$ ex-ante symmetric bidders.

Binary types

Let $\Theta = \{0, 1\}$. Types are equally likely, and then $Pr(\theta = 0) = Pr(\theta = 1) = \frac{1}{2}$. The bid space is $\mathcal{B} = [0, 1]$. Finally, label the players with I, II, and III. Ties are broken evenly and randomly. Then, the following set of strategies constitutes an equilibrium of the game G:

1. $\beta_I(\theta) = (\theta, 0)$
2. $\beta_{II}(\theta) = (0, \theta)$
3. $\beta_{III}(\theta) = \begin{cases} (0, 0) & \text{if } \theta = 0 \\ (1/2, 1/2) & \text{if } \theta = 1 \end{cases}$

Observe the following. The third bidder would be in equilibrium with any strategy that has $\beta_{III}(1) = (\varepsilon, \varepsilon)$ where $\varepsilon \in (0, 1)$. He can only win when the other bidders have a type equal to 0. Hence, whenever he wins one or two objects, he gets them for free. One may think that the previous equilibrium is because with a probability of $1/2$ bidders I or II will not bid on the auction, leaving to III the possibility of winning the object for free. This is not entirely true. Consider the following case. Let $\Theta = \{2/5, 1\}$ and $Pr(\theta = 2/5) = 1/2$. Then,

1. $\beta_I(\theta) = (\theta, 0)$
2. $\beta_{II}(\theta) = (0, \theta)$
3. $\beta_{III}(\theta) = \begin{cases} (0, 0) & \text{if } \theta = 2/5 \\ (1/2, 1/2) & \text{if } \theta = 1 \end{cases}$

is an equilibrium of G. Even if the minimum bid from I and II is pretty high, III prefers to bid on both when his type is 1. Moreover, when III has a valuation of $2/5$ for the object, he does not participate in any of the auctions, as winning leaves him indifferent.

It turns out that there are conditions on the type space and type distribution under which an equilibrium of this kind always exists. We fix the upper bound of Θ to 1 as this has no qualitative impact on the following result.

Proposition 6. *Consider the game G where $K = 2$, $N = 3$ and $\mathcal{B} = [0,1]$. Let $\Theta = \{a, 1\}$ and $Pr(\theta = a) = P_a$. Then, for any $\varepsilon \in (a, 1)$, whenever $1 - a \geqslant P_a$, there exists an equilibrium $(\beta_I, \beta_{II}, \beta_{III})$ where*

1. $\beta_I(\theta) = (\theta, 0)$
2. $\beta_{II}(\theta) = (0, \theta)$
3. $\beta_{III}(\theta) = \begin{cases} (0,0) & \text{if } \theta = a \\ (\varepsilon, \varepsilon) & \text{if } \theta = 1 \end{cases}$.

Proof. Consider player III first. Observe that the unique deviation to consider is to play either $(1,0)$ or $(0,1)$, as all the other alternatives are either equivalent/dominated by $(\varepsilon, \varepsilon)$ or dominated by $(1,0)$ and $(0,1)$. Suppose $\theta_{III} = 1$ (optimality for $\theta_{III} = a$ is trivial). Then,

$$u_{III}(1, (1,0), \beta_I, \beta_{II}) = 1\left(P_a + (1 - P_a)\frac{1}{2}\right) - \left(P_a \cdot a + (1 - P_a)\frac{1}{2} \cdot 1\right) = P_a(1 - a).$$

Now compute the expected payoff of $(\varepsilon, \varepsilon)$. First, we find the probability of winning at least one object and then the expected prices. We can write Q_{III} as 1 minus the probability of losing all the objects. Observe that the last event happens with a probability of $(1 - P_a)^2$. Therefore,

$$Q_{III}((\varepsilon, \varepsilon), \beta_I, \beta_{II}) = 1 - (1 - P_a)^2.$$

By symmetry of β_I and β_{II}, the expected prices of auction 1 and 2 are the same given $\beta_{III}(1) = (\varepsilon, \varepsilon)$, and are equal to $P_a \cdot a + (1 - P_a) \cdot 0 = P_a \cdot a$. Therefore, we have

$$u_{III}(1, (\varepsilon, \varepsilon), \beta_I, \beta_{II}) = 1(1 - (1 - P_a)^2) - 2P_a \cdot a.$$

The action $(\varepsilon, \varepsilon)$ is weakly better than $(1, 0)$ or $(0, 1)$ whenever

$$1 - (1 - P_a)^2 - 2P_a \cdot a \geqslant P_a(1 - a) \Leftrightarrow 1 - a \geqslant P_a.$$

Therefore, whenever the last weak inequality is satisfied, player III is in equilibrium with β_{III}.

Now consider player I. There are four possible scenarios:

(i) $\theta_{II} = a, \theta_{III} = a$

(ii) $\theta_{II} = a, \theta_{III} = 1$

(iii) $\theta_{II} = 1, \theta_{III} = a$

(iv) $\theta_{II} = 1, \theta_{III} = 1$

Suppose $\theta_I = a$. In (i), the player cannot do better than playing $(a, 0)$. In (ii), he cannot obtain more than a payoff of 0, and $(a, 0)$ achieves it. Case (iii) is the same as (i). Finally, in case (iv), he cannot gain more than 0 as in (ii). Therefore, $\beta_I(a) = (a, 0)$ is optimal.

Next, suppose $\theta_I = 1$. In case (i), $(1, 0)$ is trivially optimal. In (ii), the minimum price is ε on both auctions. As $1 > \varepsilon$, $(1, 0)$ is optimal. In case (iii) it is strictly better to play on auction 1 and $(1, 0)$ is the optimal bid. In (iv), $(1, 0)$ is again trivially optimal.

Since I and II are symmetric and play symmetric roles in the equilibrium, II's optimality follows from I's optimality. Therefore, $(\beta_I, \beta_{II}, \beta_{III})$ is an equilibrium under $1 - a \geqslant P_a$. ■

The statement tells us that for $a = 0$, any distribution of types sustains such an equilibrium. Observe that if a increases, the expected price of the third player (with $\theta_{III} = 1$) increases as well. To convince the third player to bid on both, the probability of a needs to decrease so that the low type becomes less relevant. When this happens, competition is high, and the event of winning both objects is unlikely. Therefore, $\theta_{III} = 1$ can accept the risk and keep bidding on all the available auctions.

On the other hand, when P_a increases, it becomes easier for θ_{III} to win an object. Then the incentives of bidding on both are lower. A low a reduces the expected prices, and winning all the goods is not too costly. Player III can then bid on both.

Continuum of types

Let $\Theta = [0,1]$. The following is an equilibrium of the game G:

1. $\beta_I(\theta) = (\theta, 0)$
2. $\beta_{II}(\theta) = (0, \theta)$
3. $\beta_{III}(\theta) = \left(\frac{\theta}{1+\theta}, \frac{\theta}{1+\theta}\right)$

Proof. Consider agent III. Her utility is

$$u_{III}(\theta, (b_1, b_2), (\beta_I, \beta_{II})) = \theta(b_1 + b_2 - b_1 b_2) - \frac{b_1^2}{2} - \frac{b_2^2}{2}.$$

This function is concave and hence FOC will be sufficient. The point that satisfies the FOC is

$$\beta_{III}(\theta) = \left(\frac{\theta}{1+\theta}, \frac{\theta}{1+\theta}\right).$$

For what regards players I and II, consider the following. Suppose I bids $\beta_I = (x_1, x_2)$. Then, observe that $x_1 \geqslant x_2$ is not optimal in case I wants to bid on both auctions. In fact, note that I faces $\theta_{III}/(1+\theta_{III})$ on auction 1 and $\theta_{III}/(1+\theta_{III})$ and θ_{II} on auction 2. Hence, when $x_1 \geqslant x_2$, winning auction 2 immediately implies winning auction 1. Winning both is never desired. Hence, $x_2 = 0$ would be an improvement. Therefore, we are going to assume that $x_2 > x_1$. This condition is not optimal too. To see this, suppose we start from $x_2 = x_1$. We study what happens when we go in the direction $x_2 + \varepsilon$ compared to the direction $x_1 + \varepsilon$. Observe that $\forall \varepsilon > 0$, we increase the probability of winning in auction 1 more than in auction 2 as in auction 1 agent I faces only III and in auction 2 she faces II and III (and III bids the same amount on 1 and 2). Moreover, the increase in the expected price in the second auction is at least as high as in the first one, as in 1 player I faces $\theta_{III}/(1+\theta_{III})$ and in the second one, she faces $\theta_{III}/(1+\theta_{III})$ and θ_{II}. Hence, going in the direction of $x_2 + \varepsilon$ is not optimal. Since neither $x_1 \geqslant x_2$ nor $x_2 > x_1$ is optimal, bidding on both auctions is never optimal. Of course, it is better to bid $(\theta_I, 0)$ than $(0, \theta_I)$, as the first auction features a higher probability of winning and a lower expected price. Note that II would apply the same reasoning. Hence the suggested equilibrium is indeed an equilibrium. ■

In this equilibrium, the third bidder bids on both auctions with the same amount. The strategy in each auction is concave with respect to the type. The reason why is the following. Consider for example type $\theta = 1$. This type bids $(1/2, 1/1)$, and therefore the sum of her bids is equal to her type. If we consider type $\theta = 1/2$, then we observe the bid $(1/3, 1/3)$. In this case, the sum of her bids are strictly higher her true valuation. As we approach type 0, we see that the sum of the bids gets closer and closer to 2θ, that is, the bidder is placing a bid of *almost* her type on both auctions. Clearly, this is due to the fact that as the type increases, the probability of winning *both* auctions increases as well if the bidder place her type for both objects. To offset this undesired event, the bidder decreases the sum of the bids with respect to her type.

Four bidders

There are $K = 2$ auctions with $N = 4$ ex-ante symmetric bidders. The type space is $\Theta = \{0, 1\}$ and $P_0 = \frac{1}{2}$. Then,

1. $\beta_I(\theta) = (\theta, 0)$
2. $\beta_{II}(\theta) = (0, \theta)$
3. $\beta_{III}(\theta) = \begin{cases} (0,0) & \text{if } \theta = 0 \\ (3/4, 1/2) & \text{if } \theta = 1 \end{cases}$
4. $\beta_{IV}(\theta) = \begin{cases} (0,0) & \text{if } \theta = 0 \\ (1/2, 3/4) & \text{if } \theta = 1 \end{cases}$

is an equilibrium. This equilibrium is similar to the previous ones, but two bidders are bidding on both auctions this time. Moreover, these bidders do not bid symmetrically in the two auctions. The first one bids more aggressively on the first auction, while the second one does the opposite. Bidding symmetrically for them is not an equilibrium. If they do so, each of them could unilaterally deviate by increasing the bid in the first auction by some small $\varepsilon > 0$

and decrease the amount in the second one by the same $\varepsilon > 0$. This deviation would allow the player, with the same price, to win one object every time he would have tied. Furthermore, they would reduce the probability of winning and paying for both goods. Hence they can be in equilibrium (given I and II behavior) only once they coordinate. Then, not only do bidders coordinate on who should bid where, but also on the amount they bid.

C Equilibrium existence with $N = 2$

A mixed strategy is a distribution over the product of measurable functions from Θ to $\mathcal{B}$. A game, in what follows, is a vector $G = (X_i, u_i)_{i=1}^N$ where X_i is the strategy set of each player i and u_i is her utility function.

The following definitions and results are fundamental for the proof of the existence of a symmetric mixed equilibrium.

Definition 1. *Player i can secure a payoff of $\alpha \in \mathbb{R}$ at $x \in X$ if there exists $\overline{x}_i \in X_i$ such that $\exists U \subseteq X_{-i}$ (open) with $x_{-i} \in U$ such that*

$$\forall x'_{-i} \in U \quad u_i(\overline{x}_i, x'_{-i}) \geqslant \alpha.$$

Therefore, when the game is at $x \in X$, i can secure a payoff α if i has a strategy that grants him that payoff even when the other players deviate slightly from x_{-i}. Next, let $u(x) = (u_1(x), \ldots, u_N(x))$ be the vector payoff function which, for each $x \in X$, gives the utility of all players.

Definition 2. *A game $G = (X_i, u_i)_{i=1}^N$ is better-reply secure if whenever (x^*, u^*) is in the closure of the graph of its vector payoff function and x^* is not an equilibrium, some player i can secure a payoff strictly above u_i^* at x^*.*

Hence, a game is better-reply secure when i can secure a payoff strictly above u_i^* whenever x^* is not an equilibrium. reny99 observes that any continuous game is better-reply secure. Any better reply will provide (at least) one agent with a payoff that is strictly above the payoff of a non-equilibrium and its neighborhood.

Now, let G be a quasi-concave game whenever X_i is convex for each player i, and, for each i, for each $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ is quasi-concave on X_i. The following is reny99's main theorem.

Theorem 3. *(Reny 1999, Theorem 3.1) If $G = (X_i, u_i)_{i=1}^N$ is compact, quasi-concave, and better-reply secure, it possesses a pure strategy Nash equilibrium.*

The proof of the theorem makes the role of better-reply security in the existence of the equilibrium clearer. Better reply security creates a link between continuous and discontinuous games. When the game possesses this property, it is possible to characterize the set of Nash equilibrium in terms of a particular function $\underline{u}_i$ instead of u_i, which is lower-semicontinuous. Then u_i is approachable from below by continuous functions, and this reduces the existence problem to establishing whether there are strategies that are robust against a finite set of deviations. In the final part of the proof, we show that such strategies do indeed exist.

In any case, what we are interested in is not the set of pure strategy equilibria. As previously shown, if we require symmetry in the equilibrium, we need to abandon increasing strategies. The following definitions will guide us through the existence result of a mixed strategy equilibrium first and the existence of a symmetric mixed strategy equilibrium next.

Let M the set of probability measure over (X, B), where B is the Borel σ-algebra over X, where X is equipped with its weak*-topology. With a slight abuse of notation, call $u_i(\mu) = \int_X u_i d\mu$ for $\mu \in M$ for each $i \in \{1, \ldots, N\}$. Then, let $\overline{G} = (M_i, u_i)_{i=1}^N$ be the mixed extension of G.

Theorem 4. *(reny99, Corollary 5.2) Suppose that $G = (X_i, u_i)_{i=1}^N$ is a compact and Hausdorff game. Then G possesses a mixed strategy Nash Equilibrium if its mixed extension $\overline{G}$ is better-reply secure.*

Before showing that our game $\overline{G}$ is better-reply secure, let us see what are the sufficient conditions for a symmetric equilibrium.

Definition 3. *A game $G = (X_i, u_i)_{i=1}^N$ is quasi-symmetric if*

1. $\forall i, j \in \{1, \ldots, N\}$ $X_i = X_j$;
2. $\forall x, y \in X$ $u_1(x, y, \ldots, y) = u_2(y, x, y, \ldots, y) = \cdots = u_N(y, \ldots, y, x)$

Here, $u_i(y, \ldots, y, x, y, \ldots, y)$ denotes the function u_i evaluated at the strategy in which player i chooses x while any other player $j \neq i$ chooses strategy y.
Now, call $v(x) = u_i(x, \ldots, x)$ the diagonal payoff function. Consider the following two definitions.

Definition 4. *Player i can secure a payoff of $\alpha \in \mathbb{R}$ along the diagonal at $(x, \ldots, x) \in X^N$, if there exists $\overline{x} \in X$ such that $u_i(x', \ldots, \overline{x}, \ldots, x') \geqslant \alpha$ for all x' in some open neighborhood of $x \in X$.*

Definition 5. *The game $G = (X_i, u_i)_{i=1}^N$ is diagonally better-reply secure if whenever $(x^*, u^*) \in X \times \mathbb{R}$ is in the closure of the graph of its diagonal payoff function and $(x^*, \ldots, x^*)$ is not an equilibrium, some player i can secure a payoff strictly above u^* along the diagonal at $(x^*, \ldots, x^*)$*

As reny99 points out, diagonal better-reply security is strictly weaker than better-reply security when $N \geqslant 3$. Hence, showing better-reply security implies that the game is diagonally better-reply secure when the game is quasi-symmetric. The next theorem, along the lines of the preceding ones, tells us that diagonal better-reply security is sufficient for the existence of a symmetric mixed strategy equilibrium.

Theorem 5. *(reny99, Corollary 5.3) Suppose that $G = (X_i, u_i)_{i=1}^N$ is a quasi-symmetric, compact, Hausdorff game. Then G possesses a symmetric mixed strategy Nash Equilibrium if its mixed extension, $\overline{G}$, is better reply secure along the diagonal.*

Now that all the definitions and results have been introduced, let us summarize what we need to prove to show the existence of a mixed symmetric equilibrium:

A. X_i has to be compact for each i;

B. X_i has to be Hausdorff for each i;

C. G has to be quasi-symmetric;

D. $\overline{G}$ has to be better-reply secure.

While point B. and C. will be trivial, point A. and D. requires a bit of work.

A. X_i is compact.

In a single auction, each player would have to choose a strategy beforehand, which consists of a measurable function from the interval $[0,1]$ to the interval $[0,1]$. Here, instead, the bidder has to choose K strategies of the kind $x : [0,1] \longrightarrow [0,1]$. For analytical purposes, these functions have to be measurable. Hence, X_i is the product of K spaces, in particular, K copies of the set of measurable functions $x : [0,1] \longrightarrow [0,1]$. We will consider one of these spaces at the time.

Consider the space $L^\infty_\lambda([0,1])$, that is, the space of all (equivalence classes of) measurable functions with domain $[0,1]$ which are λ-essentially bounded, where λ is the Lebesgue measure. Observe that the set of all measurable functions $x : [0,1] \longrightarrow [0,1]$, say $\tilde{X}_i$, is strictly contained in $L^\infty_\lambda([0,1])$. Moreover, if we consider $L^\infty_\lambda([0,1])$ as a normed vector space (equipped with the supremum norm $||\cdot||_\infty$), we have that

$$\tilde{X}_i \subseteq B_\infty := \{f \in L^\infty_\lambda([0,1]) : ||f||_\infty \leqslant 1\},$$

that is, $\tilde{X}_i$ is contained in the unit ball of the space.

Now, let us topologize $L^\infty_\lambda([0,1])$ with its weak*-topology. By the Banach-Alaoglu's Theorem, B_∞ is weakly*-compact. Therefore, to prove that $\tilde{X}_i$ is weakly*-compact, we need to show that it is weakly*-closed. We can use sequences instead of nets to characterize the closedness of $\tilde{X}_i$.

Proposition 7. *$\tilde{X}_i$ is a weakly*-closed subset of B_∞.*

Proof. Observe that $\forall x \in \tilde{X}$, $||x||_\infty \leqslant 1$ λ-a.e.

Now, take a sequence $x_n \in \tilde{X}_i$ such that $x_n \overset{\star}{\rightharpoonup} x$ (that is, x_n weakly* converges to x). By the Riesz Representation Theorem, we have that

$$x_n \overset{\star}{\rightharpoonup} x \Leftrightarrow \int_{[0,1]} x_n g d\lambda \to \int_{[0,1]} x g d\lambda \quad \forall g \in L^1_\lambda([0,1]). \tag{3}$$

Claim 1: $x \leqslant 1$ λ-a.e.

Suppose not, i.e., $\exists A \in B$ such that $\lambda(A) > 0$ and $x(a) > 1 \; \forall a \in A$. Then, consider the function $g = \mathbb{1}_A$, i.e., the index function of A. Since A is a measurable set, g is a measurable function. Moreover,

$$\int_{[0,1]} g d\lambda = \int_{[0,1]} \mathbb{1}_A d\lambda = \int_A 1 d\lambda = \lambda(A) \leqslant \lambda([0,1]) < \infty.$$

Hence, g is λ-integrable and therefore $g \in L^1_\lambda([0,1])$. Moreover, observe

$$\int_{[0,1]} xg d\lambda = \int_{[0,1]} x \mathbb{1}_A d\lambda = \int_A x d\lambda > \int_A 1 d\lambda = \lambda(A),$$

where the inequality sign comes from the monotonicity of the integral. Therefore, by equation (3), we have that

$$\exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N, \qquad \int_{[0,1]} x_n g d\lambda > \lambda(A).$$

But

$$\int_{[0,1]} x_n g d\lambda = \int_{[0,1]} x_n \mathbb{1}_A d\lambda = \int_A x_n d\lambda \leqslant \int_A 1 d\lambda = \lambda(A),$$

a contradiction. Therefore, $x \leqslant 1$ λ-a.e.

Claim 2: $x \geqslant 0$ λ-a.e.

The proof follows the same lines of the previous claim. Suppose $x < 0$ on a set $A \in B$ such that $\lambda(A) > 0$. Consider again $g = \mathbb{1}_A \in L^1_\lambda([0,1])$. Then,

$$\int_{[0,1]} xg d\lambda = \int_{[0,1]} x \mathbb{1}_A d\lambda = \int_A x d\lambda < 0 = \int_A 0 d\lambda.$$

As before, since $\int x_n g d\lambda \to \int xg d\lambda$ in the euclidean topology over $\mathbb{R}$, $\exists N \in \mathbb{N}$ such that $\forall n \geqslant N$

$$\int_{[0,1]} x_n g d\lambda < 0,$$

but again

$$\int_{[0,1]} x_n g d\lambda = \int_{[0,1]} x_n \mathbb{1}_A d\lambda = \int_A x_n d\lambda \geqslant 0 = \int_A 0 d\lambda,$$

a contradiction. Hence, $x \geqslant 0$ λ-a.e.

Therefore, $x \in \tilde{X}_i$. This implies that $\tilde{X}_i$ is weakly*-closed and therefore weakly*-compact. ■

Now, equip X_i with the product topology (recall X_i is the product of K copies of $\tilde{X}_i$). By the Tychonoff Theorem, X_i is compact if and only if every component of the product is compact. Since this is indeed the case, X_i is compact.

B. X_i is Hausdorff.

Observe that the weak*-topology over $\tilde{X}_i$ is metrizable, as B_∞ is metrizable. Hence, $\tilde{X}_i$ is Hausdorff. Since the product of Hausdorff spaces is Hausdorff (ida), X_i is a Hausdorff space.

C. G is quasi-symmetric.

Since $X_i = X_j \ \forall i \neq j$, and the bidders are endowed with the same utility function, the game is trivially quasi-symmetric.

D. $\overline{G}$ is better-reply secure.

Reny proves in his paper that the pay-your-bid auction is a better-reply secure game. We are going to follow the same lines. The only difference is that he only deals with strictly increasing strategies, which is not our case, unfortunately. In any case, we can fix it by allowing players to play strategies that, given the others' strategies, do not permit ties with strictly positive probability. It causes no loss of generality, as the following Lemma states.

Lemma 4. *Let $N = 2$. For $G \in \{G^{FPA}, G^{SPA}\}$, bidders can always use a pure strategy that induces ties with zero probability and lose an arbitrarily small utility.*

Proof. Let $(\beta_i^1(\theta_i), \beta_i^2(\theta_i), \ldots, \beta_i^K(\theta_i))$ be i's bid in the game G when his type is θ_i. Without loss of generality, assume that it induces a tie on auction 1 with strictly positive probability. By assumption, this is true for a set of positive measure $A \subseteq [0,1]$. Therefore, $\theta_i \in A$. Define $\Pr(k \neq 1|\text{tie } 1)$ as the probability of winning any auction $k \in \{2, \ldots, K\}$ given that i ties on auction 1 using the bid $\beta_i^1(\theta_i)$. Then, in the event of the tie, i's utility is

$$u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) =$$

$$(\theta_i - {}^1\!/\!{}_2\beta_i^1(\theta_i))\Pr(k \neq 1|\text{tie } 1) + {}^1\!/\!{}_2(\theta_i - \beta_i^1(\theta_i))(1 - \Pr(k \neq 1|\text{tie } 1)) - \sum_{k=2}^{K} E[P_k|\beta_i^k(\theta_i), \beta_{-i}^k, \text{tie } 1].$$

Consider a small $\varepsilon > 0$ and bid $\beta_i^1(\theta_i) + \varepsilon$ on auction 1 such that it does not induce ties with positive probabilities. Clearly, such ε exists. Then, the strategy $(\beta_i^1(\theta_i) + \varepsilon, \beta_i^2(\theta_i), \ldots, \beta_i^K(\theta_i))$ provides a conditional utility of

$$u_i(\theta_i, (\beta_i^1(\theta_i) + \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) =$$

$$(\theta_i - \beta_i^1(\theta_i) - c^{FPA}\varepsilon)\Pr(k \neq 1|\text{tie } 1) + \theta_i - \beta_i^1(\theta_i) - c^{FPA}\varepsilon)(1 - \Pr(k \neq 1|\text{tie } 1))$$

$$- \sum_{k=2}^{K} E[P_k|\beta_i^k(\theta_i), \beta_{-i}^k, \text{tie } 1],$$

where

$$c^{FPA} = \begin{cases} 1 & \text{if } G = G^{FPA} \\ 0 & \text{if } G = G^{SPA}. \end{cases}$$

Then, we have

$$u_i(\theta_i, (\beta_i^1(\theta_i) + \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) =$$

$$- ({}^1\!/\!{}_2\beta_i^1(\theta_i) + c^{FPA}\varepsilon)\Pr(k \neq 1|\text{tie } 1) + {}^1\!/\!{}_2(\theta_i - \beta_i^1(\theta_i))(1 - \Pr(k \neq 1|\text{tie } 1))$$

$$- c^{FPA}\varepsilon(1 - \Pr(k \neq 1|\text{tie } 1)).$$

If this difference is strictly positive, then i strictly prefers $\beta_i^1(\theta_i) + \varepsilon$ over $\beta_i^1(\theta_i)$. In fact, in the event of tie he can get a strictly higher utility, while in the event of no ties he can have an arbitrarily small loss, controlled through ε. As ε diminishes, the utility gain in the first event does not decreases over a certain constant amount, while the maximum loss in the second event converges to zero. Therefore, assume otherwise that $\forall\varepsilon > 0$, the difference is strictly negative. Therefore, we have

$$-({}^{1}\!/{}_{2}\beta_i^1(\theta_i))\text{Pr}(k \neq 1|\text{tie } 1) + {}^{1}\!/{}_{2}(\theta_i - \beta_i^1(\theta_i))(1 - \text{Pr}(k \neq 1|\text{tie } 1)) \leqslant 0.$$

For the final step, consider the bid $\beta_i^1(\theta_i) - \varepsilon$ on auction 1, for some small ε. Again, an ε that produces no ties exists. The conditional payoff is then

$$u_i(\theta_i, (\beta_i^1(\theta_i) - \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1)) = \theta_i \text{Pr}(k \neq 1|\text{tie } 1) - \sum_{k=2}^{K} E[P_k|\beta_i^k(\theta_i), \beta_{-i}^k, \text{tie } 1].$$

Hence, the difference is

$$u_i(\theta_i, (\beta_i^1(\theta_i) - \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) =$$

$${}^{1}\!/{}_{2}\beta_i^1(\theta_i)\text{Pr}(k \neq 1|\text{tie } 1) - \frac{1}{2}(\theta_i - \beta_i^1(\theta_i))(1 - \text{Pr}(k \neq 1|\text{tie } 1)) \geqslant 0.$$

Since as before on the event of no ties the loss can be made arbitrarily small, we obtain that i can avoid ties (through $\beta_i^1(\theta_i) + \varepsilon$ or $\beta_i^1(\theta_i) - \varepsilon$) and lose an arbitrarily small utility. As this can be applied to any auction $k \in \{1, \ldots, K\}$, where ties happen with positive probability, we get the desired result. ■

Before moving into the proof, we need another technical detail. Recall that each $\tilde{X}_i$ equipped with the relative weak*-topology of the unit ball is metrizable. This implies that X_i with the product topology is metrizable (Theorem 3.36, ida). Hence, since X_i is compact and metrizable, the set of probability measures over X_i is compact and metrizable (Theorem 15.11, ida). This means that the topological properties of our new set of strategies, M_i, can be expressed in terms of sequences without loss of generality. Now, let us show that our game is better-reply secure. The proof follows the same lines as the example in reny99. We include this proof for completeness.

Let $m^* \in M_i$ and suppose it is not an equilibrium and does not imply ties with strictly positive probability. Moreover, suppose that (m^*, u^*) is an element of the closure of the graph of the mixed extensions vector (ex-ante) payoff function. Now, consider a sequence m^n that converges to m^*. By definition, $\lim u(m^n) = u^*$. Since m^* is not an equilibrium, $\exists i \in \{1, \ldots, N\}$ that has a profitable deviation. Observe that, by definition of the supremum, $\forall m_{-i} \in M_{-i}$ $\forall \varepsilon > 0$ i can use a pure strategy x_i^ε that provides a payoff within ε of her supremum and, by the previous argument, does not imply any tie (Lemma 4). If there are no ties, u_i is continuous at $(x_i^\varepsilon, m_{-i}^*)$. Now, since x_i^ε is a profitable deviation, $u_i(x_i^\varepsilon, m_{-i}^*) > u_i(m_i^*, m_{-i}^*) = u_i(m^*)$. By continuity, there exists a neighborhood of m_{-i}^* such that $u_i(x_i^\varepsilon, m'_{-i}) > u_i(m_i^*, m_{-i}^*)$ for each m'_{-i} in the neighborhood. Since there are no ties by assumption, u_i is also continuous at m^*, which implies $u(m^*) = u(\lim m^n) = \lim u(m^n) = u^*$. Hence, $u_i(x_i^\varepsilon, m'_{-i}) > u_i^*$ in the neighborhood, that is, i can secure a payoff strictly above u_i^* at m^*.

Now suppose that ties happen with strictly positive probability at m^*. Then, the function is not continuous at m^* and then $u^* = \lim u(m^n) \neq u(\lim m^n) = u(m^*)$. Moreover, one of the bidders loses with a probability strictly higher than zero for an infinite amount of times along the sequence m^n. This bidder can strictly increase her payoff with x_i^ε for sufficiently small ε as it does not produce ties. Hence, $u_i(x_i^\varepsilon, m_{-i}^n)$ is bounded away from $u_i(m^n)$ for large n. Hence, $u_i(x_i^\varepsilon) > u_i^*$ in the limit and by continuity of u_i at m_{-i}^*, there exists a neighborhood of m_{-i}^* where $u_i(x_i^\varepsilon, m_{-i}^*) > u_i^*$. Then, again, i can secure a payoff strictly higher than u_i^* at m^*.
Therefore, $\overline{G}$ is better-reply secure.

G with more than two players We discuss here why we cannot obtain the same existence result when $N > 2$. It pins down to the proof of Lemma 4. In order to show why the Lemma does not apply with more than two players, consider an N-player game G^{SPA} and suppose that all of the i's opponent play the same strategy, so that the event "tie" does not convey different information based on the identity of the players who tied. Call the probability of tying with $\ell \in \{1, \ldots, N-1\}$ players $Tie(\ell)$. As before, we assume without loss of generality that on auction 1 player i can tie with positive probability using $(\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i))$. The conditional utility is

$$u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) =$$

$$\sum_{\ell=2}^{N}[(\theta_i - {}^1\!/\!_\ell\beta_i^1(\theta_i))\Pr(k \neq 1|\text{tie } 1, \ell) + {}^1\!/\!_\ell(\theta_i - \beta_i^1(\theta_i))(1 - \Pr(k \neq 1|\text{tie } 1, \ell))]Tie(\ell)$$

$$-\sum_{k=2}^{K} E[P_k|\beta_i^k(\theta_i), \beta_{-i}^k, \text{tie } 1].$$

Now, consider the bid $\beta_i^1(\theta_i) + \varepsilon$, where ε is chosen to not induce ties, as before. We have

$$u_i(\theta_i, (\beta_i^1(\theta_i) + \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) =$$

$$\sum_{\ell=2}^{N}[(\theta_i - \beta_i^1(\theta_i))\Pr(k \neq 1|\text{tie } 1, \ell) + (\theta_i - \beta_i^1(\theta_i))(1 - \Pr(k \neq 1|\text{tie } 1, \ell))]Tie(\ell)$$

$$-\sum_{k=2}^{K} E[P_k|\beta_i^k(\theta_i), \beta_{-i}^k, \text{tie } 1].$$

Therefore, the difference is

$$u_i(\theta_i, (\beta_i^1(\theta_i) + \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) =$$

$$\sum_{\ell=2}^{N}[\frac{1-\ell}{\ell}\beta_i^1(\theta_i)\Pr(k \neq 1|\text{tie } 1, \ell) + \frac{\ell-1}{\ell}(\theta_i - \beta_i^1(\theta_i))(1 - \Pr(k \neq 1|\text{tie } 1, \ell))]Tie(\ell) =$$

$$\sum_{\ell=2}^{N}\frac{1-\ell}{\ell}[\beta_i^1(\theta_i) - \theta_i(1 - \Pr(k \neq 1|\text{tie } 1, \ell))]Tie(\ell).$$

As in the previous Lemma, now we consider the bid $\beta_i^1(\theta_i) - \varepsilon$. After some computations, we get

$$u_i(\theta_i, (\beta_i^1(\theta_i) - \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) =$$

$$\sum_{\ell=2}^{N}\frac{1}{\ell}[\beta_i^1(\theta_i) - \theta_i(1 - \Pr(k \neq 1|\text{tie } 1, \ell))]Tie(\ell).$$

Let $\beta_i^1(\theta_i) - \theta_i(1 - \Pr(k \neq 1|\text{tie } 1, \ell)) = a_\ell$. We have

$$u_i(\theta_i, (\beta_i^1(\theta_i)+\varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) = \sum_{\ell=2} \frac{1-\ell}{\ell} a_\ell Tie(\ell),$$

and

$$u_i(\theta_i, (\beta_i^1(\theta_i)-\varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) = \sum_{\ell=2} \frac{1}{\ell} a_\ell Tie(\ell).$$

Therefore, we can have, in principle, a set of strategies that allows for

$$\sum_{\ell=2} \frac{1-\ell}{\ell} a_\ell Tie(\ell) < 0 \quad \text{and} \quad \sum_{\ell=2} \frac{1}{\ell} a_\ell Tie(\ell) < 0,$$

that is, a bid that induces ties with positive probability is strictly better than any bid that reduces this probability to zero. This fact blocks Lemma 4 and so the game cannot have, in principle, enough continuity so that better-reply security goes through.

D Symmetric mixed strategy equilibrium

Lemma 5. *Suppose one player is playing $\sigma(0) = [1(0,0)]$ and $\sigma(1)$ is atomless. Then, the set of best-replies of the other player is an anti-lattice*[6] *when her type is $\theta = 1$.*

Proof. Suppose the other player is playing $\sigma(0) = [1(0,0)]$ and that $\sigma(1)$ is an atomless distribution represented by the cdf F. Take $(x_1, y_1) > (x_2, y_2)$ best responses against σ and consider type $\theta = 1$. Suppose this type plays the strategy $\hat{\sigma} = [0.5(x_1, y_2), 0.5(x_2, y_1)]$. The expected gain for this type is then

$$0.5\left[\frac{1}{2}(F_x(x_1) + F_y(y_2) - F(x_1, y_2)) + \frac{1}{2}\right] + 0.5\left[\frac{1}{2}(F_x(x_2) + F_y(y_1) - F(x_2, y_1)) + \frac{1}{2}\right].$$

The expected gain of playing $\tilde{\sigma} = [0.5(x_1, y_1), 0.5(x_2, y_2)]$ is instead

$$0.5\left[\frac{1}{2}(F_x(x_1) + F_y(y_1) - F(x_1, y_1)) + \frac{1}{2}\right] + 0.5\left[\frac{1}{2}(F_x(x_2) + F_y(y_2) - F(x_2, y_2)) + \frac{1}{2}\right].$$

[6] A set $X \subseteq \mathbb{R}^2$ is an anti-lattice if whenever $a \wedge b, a \vee b \in X$ then $a, b \in X$.

Observe the expected prices of $\hat{\sigma}$ and $\tilde{\sigma}$ are exactly the same and therefore they do not count in the difference. Thus, the difference in terms of expected payoffs in terms of the two strategies is

$$\frac{1}{4}[F(x_1, y_1) + F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1)].$$

Since F is a cdf, this difference is positive and by the assumption of optimality of (x_1, y_1) and (x_2, y_2), we must have that it is exactly zero, i.e., F puts mass of 0 on the square. Then, we have that (x_1, y_2) and (x_2, y_1) are also best replies, and therefore the best-response set is an anti-lattice. ■

The proof, as noted in szentes07, suggests that the support cannot be any two-dimensional set, the union of any number of increasing lines or the union of more than two decreasing lines. In his paper, it is also noted that only one decreasing line cannot constitute the entire support. While we can have a support with one decreasing line here, we analyzed the two-line case.

Proof of Lemma 2

Symmetry with respect of the object is equivalent to proving that $g_i(g_i(x)) = x$, $i = 1, 2$. Then,

$$G_i(g_i(g_i(x))) = \frac{1}{2} - G_i(g_i(x)) = \frac{1}{2} - \left(\frac{1}{2} - G_i(x)\right) = G_i(x).$$

Since both G_1 and G_2 are strictly increasing, it must be that $g_i(g_i(x)) = x$.

In order to prove that the two curves are strictly decreasing, consider $x > y$. We have

$$G_i(g_i(x)) = \frac{1}{2} - G_i(x) < \frac{1}{2} - G_i(y) = G_i(g_i(y)).$$

Since G_i is strictly increasing, $g_1(x) < g_2(y)$. Finally, we prove $g_1 > g_2$. First, observe that

$$G_2(x) > G_1(x) \Leftrightarrow G_2(x) > \frac{x}{1-2x}.$$

Suppose so. Then,

$$G_1(g_1(x)) = \frac{1}{2} - G_1(x) > \frac{1}{2} - G_2(x) = G_2(g_2(x)),$$

and since $G_2 > G_1$, it must be that $g_1(x) > g_2(x)$. ■

E Equilibrium Transformation

szentes05 provides a method to transform FPA equilibria into SPA equilibria in frameworks like the current one. The intuition is simple: we assume that both equilibria (FPA and SPA) have equal expected costs; from this equivalence, we derive the optimal bidding function for the SPA format. We change the strategy description to make the transformation easier. Therefore, a strategy is now a vector (F, p) where F is a probability measure and p is a function $p : Supp(F) \longrightarrow [0,1]^2$. F will describe the randomization process; p transforms the outcomes of the randomization into bids. Observe that this kind of description is without loss of generality. For example, in the previous equilibrium, $F = \sigma$ and $p = id$ (the identity function). In the SPA case, we fix F and find the function q such that (F, q) is an equilibrium strategy of SPA. Of course, when $\theta = 0$, the action $(0,0)$ is still optimal, hence we focus on $\theta = 1$. Once we have found q, we need to check whether it is strictly increasing. If so, (F, q) has the same payoffs of (F, id) in the FPA case, so it is an equilibrium. If a profitable deviation exists in (F, q), then we would be able to recover a profitable deviation in (F, id), which would lead to a contradiction.

Since the equation to compute q is slightly different than in szentes05, we write its derivation explicitly. So, let σ_i be the marginal distribution of the distribution of σ over the bids on auction i, $i = 1, 2$. Then, bidding x on auction i has an expected payoff of $\frac{1}{2}x + \frac{1}{2}\sigma_i(x)x$. If we use (σ, q) on the SPA format, we would get $\frac{1}{2}0 + \frac{1}{2}\int_0^x q_i(y)d\sigma_i(y)$. Therefore, the cost equivalence condition requires

$$\int_0^x q_i(y)d\sigma_i(y) = x(1 + \sigma_i(x)).$$

We assume that σ_i has a density f_i, and so we have

$$\int_0^x q_i(y) d\sigma_i(y) = x(1 + \sigma_i(x))$$

$$\Rightarrow \int_0^x q_i(y) f_i(y) dy - x = x\sigma_i(x)$$

$$\Rightarrow \int_0^x (q_i(y) f_i(y) - 1) dy = x\sigma_i(x)$$

$$\Rightarrow \int_0^x \left(q_i(y) - \frac{1}{f_i(y)} \right) f_i(y) dy = x\sigma_i(x).$$

Denote $z_i(y) = q_i(y) - \frac{1}{f_i(y)}$, so that

$$\int_0^x z_i(y) f_i(y) dy = x\sigma_i(x)$$

$$\Rightarrow \int_0^x z_i(y) d\sigma_i(y) = x\sigma_i(x)$$

$$\Rightarrow z_i(x) = \frac{d(x\sigma_i(x))/dx}{d\sigma_i(x)/dx}.$$

Observe that in our case $\sigma_i(x) = G_1(x) + G_2(x)$ for $i = 1, 2$. Let $G_2(x) = 2x$ (which satisfies all the assumptions imposed on $G_2(x)$). Then, we have

$$z_i(x) = 2x - x^2 \Rightarrow q_i(x) = \frac{4x - 2x^2 + 1}{3},$$

and since q_i is strictly increasing between $[0, 1]$, we get the equivalence of payoffs. Therefore $\sigma(0) = [1(0, 0)]$ together with $(\sigma(1), q)$, where $q = (q_1, q_2)$, constitutes a symmetric mixed strategy equilibrium for the SPA game.

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