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Reputational Bargaining with an Omniscient Type

Silvio Sorbera[1]

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[1] University of Bonn, Email: silvio.sorbera@uni-bonn.de.

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Abstract

This paper investigates the role of second-order beliefs in a reputational bargaining model involving two agents, A and B. Both agents can be either irrational (refusing to concede and sticking to their initial offer) or rational. B can take one of two rational forms: omniscient, who is certain of A's rationality, or ignorant, who is uncertain. In typical reputational bargaining, agents make an offer at the beginning and adhere to it throughout the negotiation. However, we allow B to propose a 'fair' 50-50 split of the surplus, which reveals B's rationality and serves as a potential signal for the omniscient type. Using a hybrid discrete-continuous time framework proposed by Abreu and Pearce (2007), we examine how reputation effects can arise even when one agent (omniscient) is fully aware of the other's true nature and decides whether to reveal or withhold this information. Our analysis reveals multiple equilibria, including scenarios where no fair offers are made, as rational players strategically avoid disclosing their rationality to preserve their advantage. If B's irrational demand exceeds a fair division of the surplus, this scenario is the unique equilibrium. Conversely, when the demand is less than 50%, an equilibrium with a fair offer can occur. Every equilibrium of this type is characterized by a period t in which the fair deal is offered with positive probability exclusively at t.

KEYWORDS: Reputation, reputational bargaining, second-order beliefs

JEL CLASSIFICATION: C7, C78, D82

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1 Introduction

When two agents negotiate over the division of some surplus, they may each attempt to convince the other that they are a tough party to bargain with, refusing to accept anything less than their demands. In other words, they try to build a reputation for being firm. As noted by Milgrom and Roberts (1982), for reputation effects to emerge, it is not necessary for one agent to be uncertain about the nature of their opponent. In fact, it is enough for the opponent to believe that they are building a reputation.

Consider two agents, A (she) and B (he), negotiating over a surplus. Suppose that B is certain, based on information gathered prior to the negotiation, that A is not as tough as she appears. If A is unsure about the information B possesses or what he actually believes, she may still attempt to build a reputation by delaying the agreement. Moreover, would B attempt to convince A that he is omniscient, even if doing so would expose his rationality?

In this paper, we not only study the reputation effects that arise from second-order beliefs, but also examine the incentives for a player, who is certain of the other's rationality, to reveal his information. We analyze a bargaining game in a hybrid discrete-continuous time framework, as proposed in Abreu and Pearce (2007). In this model, two agents, A and B, make initial demands and stick to them until one concedes to the other. Both players can either be irrational (or stubborn)[1] or perfectly rational. An irrational player holds firm to their demand and never concedes, while a rational player can concede at any point. While A is described by just these two types, B has a more complex structure. In fact, if B is rational, his type is further determined by the information he possesses. The first type, called the ignorant type, holds a nondegenerate prior belief about A's true nature, meaning he is uncertain whether A is perfectly rational. The second type, the omniscient type, *knows* that A is rational. Importantly, A does not know whether B is irrational, rational ignorant, or rational omniscient. At the beginning of the game, agents A and B can offer to the other a behavioral contract such that their demands are incompatible. Both can concede to the other at any point in time, yet, at any discrete period, we allow B to propose a fair 50-50 split of the surplus. Such an offer would expose B's rationality, but it could also signal to the rational type of A

[1] We will also refer to this type as the behavioral type.

that B is omniscient and prepared to engage in a war of attrition to prove he has information about her rationality. In our model, we assume that once A believes with certainty that B is omniscient, she immediately concedes to the fair offer. This assumption allows us to avoid the complexities of a war of attrition between two fully rational agents negotiating over the surplus.

To analyze this game, we first examine the continuation game in which B has already revealed his rationality. This helps us understand the consequences of offering a fair deal during the bargaining process. We refer to this continuation game as the one behavioral type model, as only A can be irrational in this scenario. We find that there are two classes of equilibria. In the first, there is a continuum of degenerate equilibria (all outcome-equivalent), where B concedes immediately, regardless of his rational type. We interpret this as B's belief (confirmed in equilibrium) that A's rational type is so stubborn that any informational advantage is irrelevant, leading A to never concede. In the second class, there is a unique nondegenerate equilibrium where the omniscient type of B does not concede. Here, A attempts to convince B that she is a tough negotiator who will not back down, while B tries to convince A that he knows she is only pretending to be tough—that is, that he is omniscient.

Once we have understood the consequences of the fair offer, we return to the original game (referred to as the two behavioral types model). First, we find that an equilibrium without any fair offers is always possible, regardless of the parameters. This equilibrium can be sustained by having the agents play the degenerate equilibrium from the one behavioral type game. In this case, the omniscient type prefers not to reveal his rationality, as doing so would put him at a disadvantage, with A never conceding. Naturally, this discourages the ignorant type from proposing a fair deal as well. Moreover, when B's irrational demand exceeds the fair offer (i.e., more than 50%), this is the unique equilibrium. In this scenario, the omniscient type is unwilling to reveal his information if it means proposing a deal that leaves him with less than the irrational demand. We demonstrate that this outcome holds even when the war of attrition following the fair offer is shorter. On the other hand, when B's irrational offer is less than 50%, an equilibrium with a fair contract becomes possible. We prove that, in equilibrium, there is exactly one period in which this offer can be made with positive probability. During this period, the omniscient type proposes the fair contract with probability 1, while the ignorant type mixes between proposing the fair contract and sticking with the irrational demand. An example of an equilibrium for certain parameter values is provided at the end.

The rest of the paper is organized as follows. Section 2 presents the literature review. Sections 3 and 4 analyze the models with one and two behavioral types, respectively, where the main results of the paper and an equilibrium example are presented. Finally, Section 5 concludes. In the Appendix, we provide a brief summary of the key results from Abreu and Gul (2000), which we use in subsequent sections, a discussion of equilibrium consistency, and a model of the informational structure based on the Appendix of Milgrom and Roberts (1982) to illustrate the derivation of the type structure we employ. At the end of the Appendix, we include the proofs.

2 Literature review

The work on reputation started with Kreps and Wilson (1982) and Milgrom and Roberts (1982). Both papers analyze a model where an incumbent threatens potential entrants to start a price war so that new firms are discouraged from entering the market. Even if a price war is not immediately convenient to the incumbent, it is shown that building a reputation for being aggressive has long-term advantages. These concepts are then developed in the bargaining framework by Abreu and Gul (2000) (AG in the following sections). The authors study a model where two agents can assume at the beginning of the bargaining process one out of many different 'unfair' postures, that is, an infinite path of high demands. This model predicts that the game ends before some fixed time (and therefore the negotiation does not proceed forever) and that some party concedes to the unfair demand of the other. Many papers on reputational bargaining then followed. For example, Abreu and Pearce (2007) builds a similar model where players in each period can also take some action that affects the utility. Moreover, they allow behavioral non-stationary strategies. More recently, in Fanning (2016), the author explains how 'deadline effects' can be caused by reputational effects when the deadline is stochastic.

This work is closely related to two recent contributions. The first one is Wolitzky (2012).

In this study, the author explores the effect of reputational bargaining when the agents *know* that the other is rational but may be committed to a certain behavioral posture with some small probability. The solution concept used in that work is *minmax*. In contrast, our work employs the classic *sequential equilibrium* solution.

The second paper is Zhao (2023). In this study, the author considers a model where the rational agent may have either an optimistic or pessimistic view regarding the other's rationality. An optimistic (rational) agent assigns a high probability to the other's rationality, while a pessimistic agent assigns a small probability to the same event. The game in that paper follows the structure of a classic war of attrition. In our model, we allow the agent who may possess full information to signal their level of information, deviating from the model presented in Zhao (2023).

This paper also contributes to the literature on war of attrition with more than two actions. Two recent examples are von Leeuwen, Offerman, and van de Ven (2020) and Hörner and Sahuguet (2011). The first paper study a war of attrition where the players have the possibility to fight. The players are uncertain about the strength of their opponent and the fight resolves the conflicts. This differs from our model as the additional action starts a new war of attrition. The second paper analyzes a war of attrition with alternate moves, and players make arbitrary payments. Their opponent can either match the payment or concede.

A recent contribution to this literature and the literature on reputational bargaining is Ekmekci and Zhang (2024), where the agents have the opportunity to end the conflict through an external resolution whose outcome depends on on the strength of their claims.

3 The one behavioral type model

We first analyze a game in which player B is commonly known to be rational. Therefore, A knows B is rational, B knows that A knows B is rational and so on. B can only offer the contract $(1/2, 1/2)$, that we will call 'fair' throughout the paper. Consequently, in this game, B can only be either ignorant (lacking knowledge of A's type) or omniscient (knowing that A is rational).

This game features two distinct classes of equilibria. In the first class, there is a continuum of equilibria where B concedes immediately, even if he is omniscient. In this scenario, A is perceived as too stubborn even when rational, and thus, B's knowledge of her rationality does not help him achieve a higher payoff. In the second class, there is a unique equilibrium characterized by a war of attrition between A and B. Here, A attempts to convince B that she is a behavioral type (without knowing if this is possible, as she is uncertain whether B is ignorant), while B tries to convince A that he is omniscient and therefore knows she is pretending to be irrational.

To simplify the analysis and avoid detailing the complexities of a war of attrition that occurs after A is convinced that B is omniscient, we assume that as soon as A is certain of B's omniscience, she accepts the fair deal of $(1/2, 1/2)$.

This simplified game is then used to address the larger game where B can also be irrational and has the choice to either imitate the behavioral type or reveal his rationality by offering $(1/2, 1/2)$, thereby attempting to convince A that he possesses information about her rationality. By using backward induction, we solve this game to understand the consequences of offering (or not offering) a fair contract in the middle of the standard war of attrition and determine whether B will ever choose to reveal his rationality.

3.1 The bargaining game

The bargaining protocol is defined in continuous time, on the interval $[0, +\infty)$. The agents have to split a surplus of 1. If they never reach an agreement, both get a payoff of 0. At $t = 0$, A and B make the offers $(a, 1-a)$ and $(1-b, b)$ respectively, where the first element of the vector is the quantity of the surplus for player A and the second element is the quantity for player B. If the offers are compatible, i.e., $a + b < 1$, the game ends, and an even randomization decides the contract to be implemented. When $a + b = 1$, this share of the split is enforced. If $a + b > 1$, instead, the game continues. Payoffs are then exponentially discounted according to the rate δ, which is symmetric across the agents. The behavioral type θ_b^A is restricted to the offer $(\gamma, 1 - \gamma)$ where $\gamma > 1/2$, which is what we call the 'unfair' offer. On the other hand, since B is known to be rational, we restrict his strategy to either offering a 'fair' contract $(1/2, 1/2)$ or a split which is compatible with A's unfair offer. Without loss of generality, we can restrict the action space of B to $\{(1/2, 1/2), (\gamma, 1-\gamma)\}$. This structure resembles a classic war of attrition, where 'waiting' corresponds to $(1/2, 1/2)$ for player B and $(\gamma, 1-\gamma)$ for player A and 'conceding' corresponds to $(\gamma, 1-\gamma)$ for B and $(1/2, 1/2)$ for A. Whenever a player waits at some t and the other concedes, the game ends and each one gets the split assigned. Therefore, what determines the outcome of the game is the time t at which a player switches from waiting to conceding.

Player A can have type $\theta^A \in \{\theta^A_r, \theta^A_b\}$, where θ^A_r is referred to as the rational type, while θ^A_b is the behavioral type. The latter one can only offer the unfair contract $(\gamma, 1-\gamma)$ and never concedes to worse contract. Player B, on the other hand, can have type $\theta^B \in \{\theta^B_i, \theta^B_o\}$. We refer to the first one as the ignorant type and to the second one as the omniscient type. Throughout the game, the agents have beliefs about the type of the other player. Let $\mu^B_i(\tau)$ be the probability that type θ^B_i assigns to the event $\theta^A = \theta^A_b$, which is a function of the period that the game has reached. Observe $\mu^B_i(0) = z$. In the same fashion, let μ^A_r be the probability that θ^A_r assigns to the event $\theta^B = \theta^B_i$. Therefore, $\mu^A_r(0) = q$. These beliefs are updated through Bayes rule. Then, consider also the following assumption.

Assumption 1. *Whenever $\mu^A_r(\tau) = 0$, θ^A_r accepts the fair contract $(1/2, 1/2)$ at τ.*

This assumption makes sure that whenever we reach a complete information game (i.e., A's rationality becomes common knowledge), the players do not start a new war of attrition. In fact, we aim to capture the types' behavior *before* full rationality is revealed. Observe that θ^A_r may believe with probability 1 that $\theta^B = \theta^B_o$ even when $\theta^B = \theta^B_i$. In this case, we do not have a complete information game, yet we want to ignore all the strategic interactions that happen after θ^A_r is sure of the other's type. Whenever θ^A_r puts probability 1 to $\theta^B = \theta^B_o$, she gives up the behavioral posture and accepts B's fair split. This can be interpreted as a new continuation game in which there is common knowledge of rationality and both players have an expected payoff of $(1/2, 1/2)$ from the bargain they are going to start. Even if $\theta^B = \theta^B_i$, as soon as A quits the irrational behavior, θ^B_i *knows* that A is rational.

Now, we provide the formal definition of strategy. In the spirit of Laraki, Solan, and Vieille (2005), we define strategies in the following way. Let $\Delta(\overline{\mathbb{R}}_+)$ be the set of probability measure over $\overline{\mathbb{R}}+$, the positive extended real numbers. We topologize it with the the topology of weak convergence.

Definition 1. *A strategy is a function $\sigma : \mathbb{R}_+ \longrightarrow \Delta(\overline{\mathbb{R}}_+)$, that satisfies the following properties.*

(i) *Properness: σ_t assigns probability 1 to $[t, +\infty) \cup \{+\infty\}$;*

(ii) *Conditioning requirement: For all $t \in [0, \tau)$ and borel set $\mathcal{B} \subseteq [\tau, +\infty) \cup \{+\infty\}$, we have*

$$\sigma_t(\mathcal{B}) = (1 - \sigma_t([t, \tau)))\sigma_\tau(\mathcal{B}).$$

The first property guarantees that σ_t is a concession strategy of the continuation game t. The second property says that the measure's distribution must be computed through Bayes rule when possible. Therefore, σ_t represent the plan of action that a type of player B wants to play in continuation game t.

We define the utilities of A and B for their rational types. Each utility function depends on the (potentially mixed) strategy of the other player and the time t at which the player decides to concede. We assume that when both concede at the same time, the contract implemented is randomized. Denote with σ_r^A, σ_i^B, and σ_o^B the strategies of type θ_r^A, θ_i^B, and θ_o^B respectively. Then, consider continuation game τ. When player A with type θ_r^A, concedes at $t \geqslant \tau$, has continuation utility[2]

$$u_r^A((\sigma_i^B, \sigma_o^B), t|\tau) = \mu_r^A(\tau) \cdot \left[\int_0^t \gamma e^{-\delta x} d\sigma_{i,\tau}^B(x) + \left(\sigma_{i,\tau}^B(t) \frac{1}{2} (\gamma + 1/2) + \sigma_{i,\tau}^B((t, +\infty]) \frac{1}{2} \right) e^{-\delta t} \right] +$$
$$(1 - \mu_r^A(\tau)) \cdot \left[\int_0^t \gamma e^{-\delta x} d\sigma_{o,\tau}^B(x) + \left(\sigma_{o,\tau}^B(t) \frac{1}{2} (\gamma + 1/2) + \sigma_{o,\tau}^B((t, +\infty]) \frac{1}{2} \right) e^{-\delta t} \right],$$

That is, with probability $\mu_r^A(\tau)$ player B is of type θ_i^B and therefore is playing according to σ_i^B. If θ_i^B stops before t, then θ_r^A receives γ. If θ_i^B stops exactly at t, then a random contract is enforced. Finally, if θ_i^B decided to stop after t, the rational type θ_r^A receives a fair split of $1/2$. With probability $1 - \mu_r^A(\tau)$ player B is omniscient and therefore plays the strategy σ_o^B.

[2]Throughout the paper, we assume

$$\int_0^t f(x)dF(x) = \lim_{\tau \uparrow t} \int_0^\tau f(x)dF(x).$$

That is, the integral does not include the mass point at t. This holds in case F is a CDF or a measure.

Player B with type θ_i^B concedes at time $t \geqslant \tau$ gets utility

$$u_i^B(\sigma_r^A, t|\tau) =$$

$$(1-\mu_i^B(\tau)) \cdot \left[\int_0^t \frac{1}{2} e^{-\delta x} d\sigma_{r,\tau}^A(x) + \left(\sigma_{r,\tau}^A(t)\frac{1}{2}\left(3/2 - \gamma\right) + \sigma_{r,\tau}^A((t,+\infty])(1-\gamma)\right) e^{-\delta t}\right] +$$

$$\mu_i^B(\tau) \cdot (1-\gamma)e^{-\delta t},$$

Observe that with probability $\mu_i^A(\tau)$, the other player is behavioral and therefore will never concede. Hence θ_i^B receives the share $1-\gamma$ at the time he decided to stop.

Finally, when type θ_o^B concedes at time t he gets utility

$$u_o^B(\sigma_r^A, t|\tau) = \int_0^t \frac{1}{2} e^{-\delta x} d\sigma_{r,\tau}^A(x) + \left(\sigma_{r,\tau}^A(t)\frac{1}{2}\left(3/2 - \gamma\right) + \sigma_{r,\tau}^A((t,+\infty])(1-\gamma)\right) e^{-\delta t}.$$

In what follows, we use the solution concept of sequential equilibrium. For completeness, we list the properties of a sequential equilibrium of this game.

Definition 2. *A sequential equilibrium of the bargaining game is a vector of strategies* $\left(\sigma_i^B, \sigma_o^B, \sigma_r^A\right)$ *and beliefs* (μ_i^B, μ_r^A) *such that*

1. σ_i^B *maximizes* θ_i^B*'s expected utility in any continuation game* $t \in [0, +\infty)$ *given beliefs* μ_i^B *and* σ_r^A*;*
2. σ_o^B *maximizes* θ_o^B*'s expected utility in any continuation game* $t \in [0, +\infty)$ *given* σ_r^A*;*
3. σ_r^A *maximizes* θ_r^A*'s expected utility in any continuation game* $t \in [0, +\infty)$ *given beliefs* μ_r^A *and* σ_i^B *and* σ_o^B*;*
4. $\left((\sigma_i^B, \sigma_o^B, \sigma_r^A), (\mu_i^B, \mu_r^A)\right)$ *is a consistent assessment.*

The discussion and definition of consistent assessment is in the Appendix. The next Proposition is key for finding the equilibria of the game. It provides the relation between the incentives of the ignorant and the omniscient types, showing how the informational advantage of θ_o^B over θ_i^B manifests. Then, the next Corollary states an important property of the equilibrium.

Proposition 1. *Suppose θ_r^A plays the strategy σ_r^A. Consider continuation game τ and let $t \in [\tau, \infty)$ and $\hat{t} > t$. Then*

(i) *θ_i^B weakly prefers concession at $\hat{t} \Rightarrow \theta_o^B$ strictly prefers concession at $\hat{t}$.*

(ii) *θ_o^B weakly prefers concession at $t \Rightarrow \theta_i^B$ strictly prefers concession at t.*

Corollary 1. In any sequential equilibrium, θ_o^B plays a pure strategy in every continuation game.

Proof. Suppose otherwise, i.e., θ_o^B plays a mixed strategy in the continuation game $t \geqslant 0$. Then, by the indifference of θ_o^B and Proposition 1, the type θ_i^B strictly prefers to concede in the support of θ_o^B's strategy. Then, whenever θ_r^A observes waiting at any time of the support, he updates his beliefs and assigns probability 1 to the event $\theta^B = \theta_o^B$ and concedes immediately. Then, at the minimum time in the support, θ_i^B can wait, and in case $\theta^A = \theta_r^A$, he gets his best payoff $\frac{1}{2}$; in case $\theta^A = \theta_b^A$, he immediately observes waiting and can concede. Therefore θ_i^B has a profitable deviation, a contradiction. Therefore θ_o^B plays a pure strategy in every continuation game $t \geqslant 0$. ■

3.2 Degenerate equilibria

The game exhibits a continuum of degenerate equilibria, wherein player B readily concedes. In the subsequent analysis, we aim to characterize this collection. Initially, we establish a specific degenerate equilibrium and subsequently demonstrate the process of generating additional equilibria from it. Finally, we establish that the equilibria we characterize are the only degenerate ones, ruling out the existence of any others.

The equilibrium candidate is the following:

- θ_i^B and θ_o^B concede with probability 1 at any $t \geqslant 0$;
- θ_r^A chooses the strategy σ_r^A such that $\sigma_{r,\tau}^A([a,b]) = 0$ for all $[a,b] \subseteq [\tau, +\infty)$[3].

Consequently, every type of player B always concedes. Conversely, the behavior of θ_r^A resembles that of the irrational type θ_b^A. Our task is to identify consistent beliefs that can

[3]Therefore, the probability measure $\sigma_{r,0}^A$ assigns probability 1 to $\{+\infty\}$

support these strategies as sequentially rational choices. Consider θ_r^A first. Observe $\sigma_{r,\tau}^A(0) = 0$ is optimal given B's strategy as long as $\mu_r^A(\tau) > 0$. Therefore, suppose we reached time $\tau > 0$. Let θ_r^A's beliefs μ_r^A be such that $\mu_r^A(\tau) > 0$. That is, upon reaching time τ, the type θ_r^A does not assign probability 1 to the omniscient type. Then, θ_r^A knows that B concedes with probability 1 at τ. Hence, at τ, θ_r^A is indifferent among all distributions F_r^A such that

$$\sigma_{r,\tau}^A(\tau) = 0. \tag{1}$$

This condition is necessary and sufficient for the sequential rationality of F_r^A. Therefore, the strategy proposed is sequentially rational together with μ_r^A. We show in the appendix that these strategies can be sustained by consistent beliefs.

Now consider player B. By Proposition 1, it is sufficient to show that θ_o^B weakly prefers to concede at every $t \geqslant 0$. In fact, if this is the case, then θ_i^B strictly prefers to adopt the same strategy. Suppose we are in continuation game τ. Since θ_r^A never concedes in this continuation game ($\sigma_{r,\tau}([\tau, +\infty)) = 0$), θ_o^B is indifferent among all those strategies σ_o^B such that $\sigma_{o,\tau}^B(\tau) = 1$. In fact, for any deviation, θ_r^A would still not assign probability 1 to the event $\theta^B = \theta_o^B$, and would still consider the equilibrium strategies where B immediately concedes. Since θ_o^B optimally concedes, so does θ_i^B by Proposition 1. Therefore, the candidate equilibrium is a degenerate sequential equilibrium.

Note, however, that there is a continuum of degenerate equilibria. In fact, θ_r^A is indifferent among all the strategies $\overline{\sigma}_r^A$ such that $\overline{\sigma}_{r,\tau}^A(\tau) = 0$. Hence, any strategy $\overline{\sigma}_r^A$ with no jumps and $\overline{\sigma}_{r,0}^A(0) = 0$ is still optimal for θ_r^A. Therefore, any strategy $\overline{\sigma}_r^A$ that makes θ_o^B weakly better off by conceding immediately (in any continuation game) can still be part of a degenerate equilibrium. In fact, in this case, by Proposition 1, θ_i^B is strictly better off by conceding immediately.

Now we are going to characterize the entire set of degenerate equilibria. Consider the following condition

$$\forall \tau \geqslant 0,\ \forall t' > \tau,$$

$$\int_{\tau}^{t'} \frac{1}{2} e^{-\delta x} d\sigma_{r,\tau}^{A}(x) + \left[\sigma_{r,\tau}^{A}(t')\,(3/2-\gamma) + \sigma_{r,\tau}^{A}([t',+\infty])(1-\gamma)\right] e^{-\delta t'}$$

$$\leqslant (1-\gamma)e^{-\delta\tau}. \tag{2}$$

This condition states that for every continuation game $t \geqslant 0$, the type θ_o^B weakly prefers to concede immediately than to wait any time after t. Note that the payoff is conditioned on the event that we have reached continuation game t. Observe that when this condition is satisfied, θ_o^B concedes immediately with probability 1 in any continuation game. Define the set:

$$DE := \{\sigma_r^A | \forall \tau \geqslant 0\ \sigma_{r,\tau}^A(\tau) = 0 \wedge (2)\}.$$

Let $\hat{\sigma}_i^B$ and $\hat{\sigma}_o^B$ be such that $\hat{\sigma}_{i,\tau}^B(\tau) = \hat{\sigma}_{o,\tau}^B(\tau) = 1$ for all $\tau \geqslant 0$. Then, we get the following result.

Proposition 2. *The set of all strategies sustainable as degenerate equilibria of the bargaining game is*

$$\{(\sigma_i^B, \sigma_o^B, \sigma_r^A) | \sigma_i^B = \hat{\sigma}_i^B \wedge \sigma_o^B = \hat{\sigma}_o^B \wedge \sigma_r^A \in DE\}.$$

3.3 Nondegenerate equilibrium

In a nondegenerate equilibrium, no player concedes with a probability of 1 at time $t = 0$. We will demonstrate the existence of a unique nondegenerate equilibrium. In this equilibrium, players randomize over a finite support interval $[0, T^0]$. However, as stated in Corollary 1, the omniscient type cannot play a mixed strategy equilibrium. Hence, the ignorant type engages in mixing. According to Proposition 1, the omniscient type does not concede within the interval $[0, T^0]$, thereby exhibiting behavior akin to an irrational type. This allows us to analyze the game in a manner similar to the standard AG model, where each player has only one behavioral contract available. Unlike the standard model, the strategy of the omniscient type is endogenous in our context. Since player B is limited to being either rational or irrational in the standard version, the strategy of the irrational player is exogenously determined. In contrast, Proposition 1 informs us that θ_o^B adopts the same strategy, but this choice arises from a strategic decision. We define only strategies σ_0, and derive the equilibrium with their equivalent cdf. Conditional distribution are derived through Bayes rule until the last period of possible concession.

First, define $F^A(t) = (1 - \mu_i^B(t)) \cdot F_r^A(t)$ and $F^B(t) = \mu_r^A(t) F_i^B(t)$, where F_r^A and F_i^B are CDFs representing $\sigma_{r,0}^A$ and $\sigma_{i,0}^B$ respectively. The conditional measures $\sigma_{r,\tau}^A$ and $\sigma_{i,\tau}^B$ can be represented by $F_r^A/(1 - F_r^A(\tau))$ and $F_i^B/(1 - F_i^B(\tau))$. Clearly, $\sigma_{o,\tau}^B([\tau, +\infty)) = 0$ for each τ. Then, let

$$\lambda^A = \frac{(1-\gamma)\delta}{1/2 - (1-\gamma)} \quad \text{and} \quad \lambda^B = \frac{1/2\delta}{1/2 - (1-\gamma)}.$$

Define $T^A = -\log(z)/\lambda^A$ and $T^B = -\log(1-q)/\lambda^B$. As in AG, we have $T^0 = \min\{T^A, T^B\}$. Then, we have that

$$F^A(t) = 1 - c^A e^{-\lambda^A t} \quad \text{and} \quad F^B(t) = 1 - c^B e^{-\lambda^B t},$$

where $c^i = e^{-\lambda^i (T^i - T^0)}$. Observe that if $T^i = T^0$, then player i never concedes at $t = 0$. By Proposition 1 in AG[4], $(F^B/q, F_o^B, F^A/(1-z))$, where $F_o^B(t) = 0$ for all $t \in [0, T^0]$, constitutes the unique nondegenerate equilibrium. Conditional distributions for $\tau \in [0, T^0]$ are computed through Bayes' rule.

Now that we have characterized the entire set of equilibria in this game (later referred to as the game after the signal, or GAS), we are ready to apply these findings to the larger game where B can also be an irrational type. This allows us to study the incentives for θ_i^B and θ_o^B to reveal their rationality and initiate a new war of attrition. As we will see, whenever A and B are expected to play a degenerate equilibrium, θ_o^B lacks the incentive to offer the fair contract. In this case, he understands that doing so would weaken his position, as A would anticipate an immediate concession (or an early one in case B deviates). Furthermore, when B's behavioral demand is high, he is disincentivized to reveal his rationality, as he stands to gain a better deal by continuing to pretend to be irrational.

[4] A summary of Abreu and Gul (2000) Proposition 1 can be found in the appendix.

4 The two behavioral types model

In this section, we enlarge B's type space to include a behavioral type θ_b^B, having then two behavioral types, one for each player. We therefore distinguish between γ_A and γ_B, the behavioral demand of players A and B respectively. The type θ_b^B has all the features of θ_b^A. Therefore, throughout the game he offers a split of the surplus corresponding to $(1-\gamma_B, \gamma_B)$, where $\gamma_A + \gamma_B > 1$, and does not accept anything less. Whenever player A offers $(x, 1-x)$ with $1 - x \geqslant \gamma_B$, θ_b^B accepts immediately. A's types remain unchanged, while B's new type space is $\Theta_1^B := \{\theta_b^B, \theta_o^B, \theta_i^B\}$, where θ_o^B and θ_i^B are as described in the previous sections. We let q_b, q_o, q_i be the prior probabilities of B being irrational, omniscient and ignorant, respectively. A is irrational with probability $z \in (0,1)$ as before. Now, define $\mathcal{Q} := \{(q_o, q_i, z) \in (0,1)^3 | q_o + q_i \in (0,1)\}$. This space of initial beliefs will be helpful when we characterize the equilibrium set. Clearly, $q_b = 1 - q_o - q_i$.

In this new version of the model, the ignorant type θ_i^B decides whether to imitate the behavioral type θ_b^B or try to signal the information he possesses by pretending to know that his opponent is rational. Type θ_o^B has the same dilemma, with the difference that he actually knows about A's rationality. Hence, contrary to the classic war of attrition we want to give one of the players, B, the possibility of using more than two actions (wait and concede). Therefore, suppose that B has offered $(1-\gamma_B, \gamma_B)$ at the beginning of the bargaining procedure. In this continuation game, B can either reject A's offer (waiting), accept A's offer (concede) or offer the fair contract $(1/2, 1/2)$. The last action reveals B's rationality, putting him in a difficult position. Yet, at the same time, it signals A that B is ready to start a new war of attrition with his rationality exposed, which can be taken by A as a sign of strength (omniscient type).

In order to model these choices, we follow Abreu and Pearce (2007) and Abreu, Pearce, and Stacchetti (2015). They formulate a new hybrid model with both discrete and continuous time features. As they explain, this allows for easier calculations in the war of attrition, avoiding openness problems when a new offer is made and there is no "first" time to accept it. Therefore, we allow the players to concede at any period, while B can change his offers only at integer times. For every $t \in \mathbb{N}$, we split the date into three subdates, $(t, -1)$, $(t, 0)$ and $(t, +1)$. At $(t, -1)$, A has her last opportunity to accept the pending offer of her opponent. B can also accept $1 - \gamma_A$ at this dates. At $(t, 0)$, B can offer the fair contract $(1/2, 1/2)$ in case he has not done it before. Finally, at $(t, +1)$ A and B can accept the standing offer of their opponent. There is no discounting among subdates of the same period. A and B can concede at any $t \in \mathbb{R}_{++} \backslash \mathbb{N}$ as well. Date $t = 0$ is split into $(0, 0)$ and $(0, +1)$. At $(0, 0)$ the players choose an initial contract. For simplicity and without loss of generality A is restricted to $(\gamma_A, 1-\gamma_A)$ while B can offer $(1-\gamma_B, \gamma_B)$ and $(1/2, 1/2)$ only. At $(0, +1)$ both can accept their opponent's contract. If we extend any non-integer number to two dimensions, writing $(t, +1)$ for $t \in \mathbb{R}_{++} \backslash \mathbb{N}$, we can put a complete order on our periods. We put the lexicographic order, where $(t, k) \geqslant (t', k')$ in one of these two cases: $t > t'$ OR $t = t'$ and $k \geqslant k'$. Hence, the end of a period and the beginning of a new period are separated. This implies that a player can condition his or her choice of the action at the beginning of a period on the events happened at a previous period.

We denote the continuation game at $(n, 0)$, $n \in \mathbb{N}_0$ after the fair contract has been offered as the *game after the signal* (GAS).

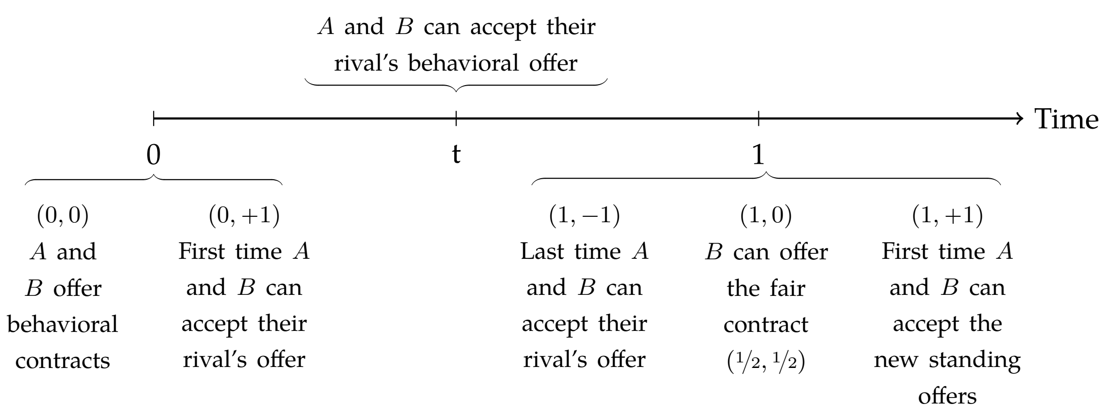

Figure 1: Timeline representation

In Figure 1, we illustrate a game scenario unfolding over the time interval from $(0, 0)$ to $(1, +1)$. In this depiction, players A and B commence by proposing their behavioral contracts. Subsequently, at $(0, +1)$, they are presented with their initial opportunity to accept the behavioral contracts proposed by their respective opponents. Throughout any point in time $t \in (0, 1)$, they retain the option to accept these standing offers. This opportunity persists until period $(0, -1)$, marking the final chance to accept before B can alter his offer. Following this, B can present the fair contract at $(1, 0)$. Should B choose to do so, A can accept this new offer at $(1, +1)$. At this date, B also has the option to accept $(\gamma_A, 1-\gamma_A)$. If B has not proposed the fair contract by $(1, 0)$, the players can start their concession again from $(1, +1)$.

We turn to the description of the mixed strategies. First, note that a pure strategy for player B in this game can be one of three things:

1. The offer of the behavioral contract at $(0, 0)$ and a concession time (t, k);
2. The offer of the behavioral contract at $(0, 0)$, a time $n \in \mathbb{N}$ for the fair offer contract and a concession time $(t, k) > (n, 0)$;
3. The offer of the fair contract at $(0, 0)$ and a concession time (t, k)[5].

In the first case, B decides to not offer the fair contract, while in the second and third case he offers $(1/2, 1/2)$ before conceding to A's behavioral demand. This is a heuristic description of available pure strategies. In fact, the players have to specify an action for any possible continuation game [6]. For this reason, we turn directly to the description of mixed strategies as done in the previous section. In order to introduce the modeling of mixed strategies, we make use of the following notation. Let $n \in \mathbb{N}$ and $t \in (n-1, n)$. Then, concession in the interval $[t, n]$ means concession from t up until $(n, -1)$. On the other hand, when $t \in (n, n+1)$, concession in $[n, t]$ means concession from $(n, +1)$ to t. Finally, concession in $[n-1, n]$ is from $(n-1, +1)$ to $(n, -1)$.

We require A and B to play strategies σ^A and σ^B described in Definition 1 after B offers the fair contract. Therefore, suppose B changes offer from $(1-\gamma_B, \gamma_B)$ to $(1/2, 1/2)$ at $(\tau^*, 0)$, $\tau^* \in \mathbb{N}$, henceforth revealing his rationality. Then, A and B play strategies σ^A and σ^B that satisfy Definition 1 with the difference that their domain and codomain are now $[\tau^*, +\infty)$ and $\Delta([\tau^*, +\infty))$. Condition (i) is left unchanged while (ii) becomes:

[5] Player B can decide to never concede. In this case, $t = +\infty$ and the choice of k would have no meaning.

[6] Note, however, that as long as the fair contract is not offered, there are no unexpected events for θ_r^A. In fact, any continuation game with standing offer $(1-\gamma_B, \gamma_B)$ has positive probability of being reached, since B can be behavioral. This is different from the model with one behavioral type as B is commonly known as rational. For example, in its degenerate equilibrium, B waiting at $t = 0$ is an unexpected event, and strategies that describe the actions after this event must be specified.

- *Conditioning requirement: For all $t \in [\tau^*, \tau)$ and Borel set $\mathcal{B} \subseteq [\tau, +\infty) \cup \{+\infty\}$, we have*

$$\sigma_t(\mathcal{B}) = (1 - \sigma_t([t, \tau)))\sigma_\tau(\mathcal{B}).$$

Therefore, for each $n \in \mathbb{N}_0$, A and B need to specify strategies $\sigma^A[n]$, $\sigma^B[n]$ that satisfy the new version of Definition 1.

Player B at the beginning of the game chooses whether to offer $(1/2, 1/2)$ at some point or concede to A's demand first. Suppose B concedes first. Then, he selects (t_0, k) such that $t_0 \in \mathbb{R}_+$ and $k \in \{-1, +1\}$ such that $k = +1$ for $t_0 \in \mathbb{R}_+ \backslash \mathbb{N}$. If B offers $(1/2, 1/2)$ first, instead, he chooses $t_1 \in \mathbb{N}_0$ and offers the fair contract at $(t_1, 0)$. Hence, we define a function $Y^B : \mathbb{N}_0 \to [0, 1]$ that assigns to each $n \in \mathbb{N}_0$ the probability that B offers $(1/2, 1/2)$ at $(n, 0)$. We denote with $Y_i^B(n)$ and $Y_o^B(n)$ the probabilities assigned by θ_i^B and θ_o^B respectively. Finally, we let $X^B = (X_n^B)_{n \in \mathbb{N}_0}$ be a sequence of measures, such that $X_n^B : \mathbb{B}([n, n+1]) \to [0, 1]$ and

$$\sum_{n=0}^{+\infty} X_n^B([n, n+1]) + \sum_{n=0}^{+\infty} Y^B(n) \leqslant 1 - q_b. \tag{3}$$

Each X_n^B describes the concession of B to A's demand in the interval $[n, n+1]$, hence from $(n, +1)$ to $(n+1, -1)$ [7]. The sigma-algebra $\mathbb{B}$ imposed on each interval is the Borel sigma-algebra derived from the relative Euclidean topology. We write X_i^B and X_o^B to distinguish the strategies used by θ_i^B and θ_o^B respectively. X^A is similarly defined.

Summarizing, B's strategy includes:

- A sequence of measures $X^B = (X_n^B)_{n \in \mathbb{N}_0}$;
- A function $Y^B : \mathbb{N}_0 \to [0, 1]$;
- A sequence $(\sigma^B[n])_{n \in \mathbb{N}_0}$ such that $\forall n \in \mathbb{N}_0$ the modified version of Definition 1 is satisfied,

and (3) holds. A's strategy, on the other hand, includes

[7] The probability of concession in a set $S_n \cup S_m$ (both measurable sets) where $S_n \subseteq [n, n+1]$, $S_m \subseteq [m, m+1]$ with $n \neq m$ can be calculated by $X_n^B(S_n) + X_m^B(S_m)$.

- A sequence of measures $X^A = (X_n^A)_{n \in \mathbb{N}_0}$;
- A sequence $(\sigma^A[n])_{n \in \mathbb{N}_0}$ such that $\forall n \in \mathbb{N}_0$ the modified version of Definition 1 is satisfied,

and

$$\sum_{n=0}^{+\infty} X_n^A([n, n+1]) \leqslant 1 - z.$$

Beliefs are expressed as follows: the belief of B's ignorant type θ_i^B at period (t, k) is denoted by $\mu_i^B((t, k); \theta_b^A)$ and represents the probability that θ_i^B assigns to the event $\theta^A = \theta_b^A$. Beliefs also depend on the history up to period (t, k), but we omit this from the notation for simplicity. When necessary, we specify the history preceding (t, k). The same applies to θ_r^A's beliefs. Clearly, in the game after the signal beliefs coincide with the beliefs of the one behavioral type model. Therefore, for each n we have

$$\sigma^A[n] = (1 - \mu_i^B((n, 0); \theta_b^A))\sigma_r^A[n],$$

and

$$\sigma^B[n] = \mu_r^A((n, 0); \theta_i^B)\sigma_i^B[n] + \mu_r^A((n, 0); \theta_o^B)\sigma_o^B[n],$$

where $\sigma_i^B[n]$, $\sigma_o^B[n]$, $\sigma_r^A[n]$ are the strategies used by types θ_i^B, θ_o^B and θ_r^A respectively and beliefs are computed considering the fair offer happening at $(n, 0)$.

Before we give the definition of equilibrium, we provide a brief description of the utilities. Utilities are written at time 0, the beginning of the bargaining game. Consider θ_i^B and suppose he chooses to offer the fair contract at t_1, before conceding to A's contract, and concedes at t_2 in the game after the signal. His utility is then

$$\begin{aligned} U_i^B((X^A, (\sigma^A[n])_{n \in \mathbb{N}_0}), t_1, t_2) = & \sum_{n=0}^{t_1 - 1} \int_n^{n+1} \gamma_B e^{-\delta t} dX_n^A(t) \\ & + \left(1 - \sum_{n=0}^{t_1 - 1} X_n^A([n, n+1])\right) u_i^B(\sigma_r^A[t_1], t_2 | t_1) e^{-\delta t_1}. \end{aligned}$$

The term in the first line refers to the event in which A concedes to $(1 - \gamma_B, \gamma_B)$ before t_1.

In the second term, $1 - \sum_{n=0}^{t_1-1} X_n^A([n, n+1])$ is the probability that A does not concede before the behavioral demand at t_1. Then, this probability is multiplied by $u_i^B(\sigma_r^A[t_1], t_2|t_1)e^{-\delta t_1}$, the expected utility of θ_i^B in the GAS. The omniscient type knows that A is rational, therefore if he follows the previous example of strategies, he gets

$$U_o^B((X^A, (\sigma^A[n])_{n\in\mathbb{N}_0}), t_1, t_2) = \frac{1}{1-z} \sum_{n=0}^{t_1-1} \int_n^{n+1} \gamma_B e^{-\delta t} dX_n^A(t)$$
$$+ \left(1 - \frac{1}{1-z} \sum_{n=0}^{t_1-1} X_n^A([n, n+1])\right) u_o^B(\sigma_r^A[t_1], t_2|t_1)e^{-\delta t_1}.$$

Finally, suppose θ_r^A chooses $(t_0, +1)$ such that $t_0 \in (n^*, n^*+1)$ for some $n^* \in \mathbb{N}_0$ and concession $(t_2^n)_{n=0}^{n*}$. Each t_2^n represents θ_r^A's concession in case B offers $(1/2, 1/2)$ at n. Let $\Sigma^B = (X^B, Y^B, (\sigma^B[n])_{n\in\mathbb{N}_0})$. Her utility is then,

$$U_r^A(\Sigma^B, (t_0, +1), (t_2^n)_{n=0}^{n*}) = \sum_{n=0}^{n*-1} \int_n^{n+1} \gamma_A e^{-\delta t} dX_n^B(t) + \int_{n*}^{t_0} \gamma_A e^{-\delta t} dX_{n*}^B(t)$$
$$+ \sum_{n=0}^{n*} Y^B(n) u_r^A((\sigma_i^B[n], \sigma_o^B[n]), t_2^n|n)$$
$$+ \left(1 - \sum_{n=0}^{n*-1} X_n^B([n, n+1]) - X_{n*}^B([n^*, t_0]) - \sum_{n=0}^{n*} Y^B(n)\right)(1-\gamma_B)e^{-\delta t_0}$$

The first line captures the events in which B concedes to A's demand before offering the fair contract. The second line includes the probability that B offers $(1/2, 1/2)$ before A concedes. In the third line we have the probability that B neither offers the fair contract nor concedes to A before t_0. In this event θ_r^A's profit is $1-\gamma_B$.

Remark. We can express the utility of θ_i^B in relation to the utility of θ_o^B. Consider the previous case as an example, that is, θ_i^B is offering the fair contract at t_1 and conceding in the GAS at t_2. Observe that at t_1, beliefs are not anymore z and $1-z$. Since A has not conceded in case the players arrive to time t_1, θ_i^B believes A is irrational with probability $z/\left(1 - \sum_{n=0}^{t_1-1} X_n^A([n, n+1])\right)$.

For ease of notation, let $\sum_{n=0}^{t_1-1} X_n^A([n, n+1]) = X$. We have

$$u_i^B(\cdot) = \left(1 - \frac{z}{1-X}\right) u_o^B(\cdot) + \frac{z}{1-X} e^{-\delta t_2 + \delta t_1}(1-\gamma_A).$$

We discounted by $e^{+\delta t_1}$ the term $e^{-\delta t_2}(1-\gamma_A)$ because the GAS is shifted from $t = 0$ to $t = t_1$. Hence,

$$u_i^B(\cdot)e^{-\delta t_1} = \left(1 - \frac{z}{1-X}\right) u_o^B(\cdot)e^{-\delta t_1} + \frac{z}{1-X} e^{-\delta t_2}(1-\gamma_A).$$

Finally,

$$\begin{aligned}
U_i^B((X^A, (\sigma^A[n])_{n\in\mathbb{N}_0}), t_1, t_2) &= \sum_{n=0}^{t_1-1} \int_n^{n+1} \gamma_B e^{-\delta t} dX_n^A(t) + (1-X)\, u_i^B(\cdot)e^{-\delta t_1} \\
&= (1-z)\left(\frac{1}{1-z}\sum_{n=0}^{t_1-1} \int_n^{n+1} \gamma_B e^{-\delta t} dX_n^A(t)\right) \\
&+ (1-X)\left(\left(1 - \frac{z}{1-X}\right) u_o^B(\cdot)e^{-\delta t_1} + \frac{z}{1-X} e^{-\delta t_2}(1-\gamma_A)\right) \\
&= (1-z)\left(\frac{1}{1-z}\sum_{n=0}^{t_1-1} \int_n^{n+1} \gamma_B e^{-\delta t} dX_n^A(t)\right) \\
&+ (1-z)\left(1 - \frac{X}{1-z}\right) u_o^B(\cdot)e^{-\delta t_1} + z e^{-\delta t_2}(1-\gamma_A) \\
&= (1-z)\left(\frac{1}{1-z}\sum_{n=0}^{t_1-1} \int_n^{n+1} \gamma_B e^{-\delta t} dX_n^A(t) + \left(1 - \frac{X}{1-z}\right) u_o^B(\cdot)e^{-\delta t_1}\right) \\
&+ z e^{-\delta t_2}(1-\gamma_A) \\
&= (1-z) U_o^B((X^A, (\sigma^A[n])_{n\in\mathbb{N}_0}), t_1, t_2) + z e^{-\delta t_2}(1-\gamma_A).
\end{aligned}$$

Therefore, with probability $1-z$, θ_i^B has θ_o^B's payoff, with probability z he will get $1-\gamma_A$ at time $t = t_2$. This resembles the GAS payoff of the previous section.

We need a last step before the equilibrium definition. We now solve a game where the last possible chance of sending the signal has not been taken, and therefore the players continue a standard reputational bargaining with behavioral offers.

Game with No Signal Suppose $\tau^* \in \mathbb{N}_0$ is the last period where B can offer $(1/2, 1/2)$, and assume that instead he offers again $(1-\gamma_B, \gamma_B)$. We are left with an AG game where the first period is shifted from 0 to τ^*. Therefore, the solution is unique and can be recovered from the AG results. We also know from the previous section that θ_o^B mimics θ_b^B and therefore θ_i^B is the only type of B that concedes. Hence, let $T_N^A = -\log(\mu_i^B((\tau^*, 0; \theta_b^A)))/\lambda_N^A + \tau^*$, $T_N^B = -\log(\mu_r^A((\tau^*, 0); \theta_o^B, \theta_b^B))/\lambda_N^B + \tau^*$, where

$$\lambda_N^A = \frac{(1-\gamma_A)\delta}{\gamma_A + \gamma_B - 1} \quad \text{and} \quad \lambda_N^B = \frac{(1-\gamma_B)\delta}{\gamma_A + \gamma_B - 1}.$$

Thus, B's concession distribution from τ^*, $(X_n^B)_{n \geqslant \tau^*}$ can be represented by the cdf $F_N^B(t) = 1 - c^B e^{-\lambda_N^B(t-\tau^*)}$. A's concession is instead distributed according to $F_N^A(t) = 1 - c^A e^{-\lambda_N^A(t-\tau^*)}$, where $c^i = e^{-\lambda_N^i(T_N^i - T_N^0)}$ and $T_N^0 = \min\{T_N^A, T_N^B\}$. We refer this continuation game as *game with no signal* (GNS).

We are ready to provide the equilibrium definition. Observe that we exploit Definition 2 to specify the sequential rationality imposed on the equilibrium at $t \in \mathbb{N}_0$ after $(1/2, 1/2)$ has been offered.

Definition 3. *Let $\mathbf{q} \in \mathcal{Q}$. A sequential equilibrium given $\mathbf{q}$ of the full bargaining game is a vector of strategies $(\Sigma_i^B, \Sigma_o^B, \Sigma_r^A)$, where $\Sigma_s^B = (X_s^B, Y_s^B, (\sigma_s^B[n])_{n\in\mathbb{N}_0})$, $s \in \{i, o\}$, and $\Sigma_r^A = (X_r^A, (\sigma_r^A[n])_{n\in\mathbb{N}_0})$, beliefs (μ_i^B, μ_r^A), such that*

1. *Σ_i^B maximizes θ_i^B's expected utility in any continuation game (t, k) given beliefs μ_i^B and Σ_r^A;*
2. *Σ_o^B maximizes θ_o^B's expected utility in any continuation game (t, k) given Σ_r^A;*
3. *Σ_r^A maximizes θ_r^A's expected utility in any continuation game (t, k) given beliefs μ_r^A and Σ_i^B and Σ_o^B;*
4. *For each $n \in \mathbb{N}_0$, $(\sigma_i^B[n], \sigma_o^B[n], \sigma_r^A[n])$ is GAS degenerate or nondegenerate equilibrium;*
5. *A and B play the unique equilibrium in GNS;*
6. *$((\Sigma_i^B, \Sigma_o^B, \Sigma_r^A), (\mu_i^B, \mu_r^A))$ is a consistent assessment.*

We also make another assumption on the equilibrium behavior of B. This assumption states that in case any B's type decides to offer $(1/2, 1/2)$ at $(n, 0)$, implying $\mu_r^A((n,0);\theta_i^B) = 1$, then we force this type to concede at $(n, -1)$. These two actions are equivalent. In fact, when $\mu_r^A((n,0);\theta_i^B) = 1$, B optimally concedes at $(n, +1)$. Since there is no discounting between $(n, -1)$ and $(n, +1)$, we get the equivalence.

Assumption 2. *In any equilibrium, there exists no* $n \in \mathbb{N}_0$ *such that* $(1/2, 1/2)$ *offered at* $(n, 0) \Rightarrow \mu_r^A((n,0);\theta_i^B) = 1$.

Assumption 3. *Suppose* $\exists n \in \mathbb{N}_0$ *such that* $X_n^B([n, n+1]) > 0$ *or* $Y^B(n) > 0$*. Then, there is strictly positive probability that continuation game* $(n, 0)$ *is reached.*

In the next Proposition, we show that B can offer the fair contract only for a limited amount periods in any sequential equilibrium. For this, we use the following notation. Let $T_S^0(n)$ be the time at which the game is certain to end before it, with a probability of 1, provided that A and B play the strategy profile $(\sigma^A[n], \sigma^B[n])$ in the GAS. In the next results, we define $T_N^0(n)$ in the same way for the GNS.

Proposition 3. *In any sequential equilibrium, both holds:*

1. $supp(Y_i^B) = supp(Y_o^B)$;
2. $|supp(Y^B)| < +\infty$.

This Proposition tells us that either there is no event in which $(1/2, 1/2)$ is offered, or there exists a last time τ' where the fair contract can be offered. Therefore for all $\tau > \tau'$, $Y^B(\tau) = 0$.

4.1 Equilibrium with no signal

We now find an equilibrium in the game where the fair contract is not offered. Consequently, the opportunity for B to signal their rationality is never utilized. This outcome must be optimal in equilibrium, therefore we explore one way to achieve this through the use of degenerate equilibria. As anticipated, when A and B play a degenerate equilibrium, θ_o^B does not offer the fair contract, and thus θ_i^B refrains from it as well. We show that in this case, the equilibrium is unique and possesses the properties of the AG solution. The only difference is that θ_o^B

does not concede to A 's irrational demands due to its maximization problem, making θ_o^B's behavior endogenous, unlike θ_b^B. In the next section, we demonstrate that an equilibrium with no signal is the unique possible equilibrium when B's behavioral demand is high.

Lemma 1. *Let $n \in \mathbb{N}_0$. In equilibrium, if $(\sigma^A[n], \sigma^B[n])$ represents the degenerate equilibrium of GAS, then $Y^B(n) = 0$ whenever $\mu_r^A((n,0);\theta_o^B) < 1$.*

Proof. Clear since θ_o^B obtains a higher payoff by waiting instead of offering $(1/2, 1/2)$ at n. Hence, $Y_o^B(n) = 0$. From Proposition 3, $Y_i^B(n) = 0$ and so $Y^B(n) = 0$. ■

Therefore, if we let A and B play the degenerate equilibrium for each $n \in \mathbb{N}_0$, then $\sum_{n=0}^{+\infty} Y^B(n) = 0$. We show that this can be part of an equilibrium[8]. First, consider the following result.

Lemma 2. *Suppose $Y^B(n) = 0$ for each $n \in \mathbb{N}_0$, and take $\hat{t}, t \in \mathbb{R}_+$ such that $\hat{t} > t$ Then,*

(i) *θ_i^B weakly prefers concession at $\hat{t} \Rightarrow \theta_o^B$ strictly prefers concession at $\hat{t}$.*

(ii) *θ_o^B weakly prefers concession at $t \Rightarrow \theta_i^B$ strictly prefers concession at t.*

Proof. Apply the same steps of the proof of Proposition 1, substituting γ with γ_A and $\frac{1}{2}$ with γ_B and $1 - \gamma_B$. ■

This implies that, under the assumption that $Y^B(n) = 0$ for each $n \in \mathbb{N}_0$, in equilibrium, θ_o^B never concedes as long as θ_r^A has not conceded first. To see this, suppose θ_r^A does not concede at time t with probability 1, but assume, for the sake of contradiction, that θ_o^B concedes at t. According to the previous Lemma, θ_i^B strictly prefers to concede no later than t with probability 1.

Now, let $\tau^* = \sup\{\bigcup_{n=0}^{+\infty} \mathrm{supp}(X_{n,o}^B)\}$. Note that θ_r^A concedes with probability 1 no later than τ^*. For any $\varepsilon > 0$, there exists some $t' \in [\tau^- \varepsilon, \tau^*]$ such that t' is in the support of θ_o^B's

[8]Note however that in case θ_r^A updates her beliefs to $\mu_r^A((n,0);\theta_o^B) = 1$ after the fair contract is offered at $(n,0)$, then by Assumption 1 θ_r^A concedes immediately. Note, however, that if $\mu_r^A((n,0);\theta_o^B) = 1$ in equilibrium and this is optimal for θ_o^B, then it is optimal for θ_i^B too, and so $Y_i^B(n) > 0$, implying $\mu_r^A((n,0);\theta_o^B) < 1$, a contradiction. Therefore, in equilibrium, $\mu_r^A((n,0);\theta_o^B) < 1$.

concession strategy. However, if θ_o^B does not concede by τ^*, then θ_r^A will accept B's behavioral demand.

Conceding at t' gives θ_o^B a payoff of $(1-\gamma_A)e^{-\delta t'}$, whereas waiting to concede after τ^* yields a payoff of at least $\gamma_B e^{-\delta\tau^*}$. Since by assumption $\gamma_B > 1-\gamma_A$, there exists an $\varepsilon > 0$ such that waiting is strictly better than conceding at t'. Thus, t' cannot be in the support of θ_o^B's equilibrium strategy, leading to a contradiction. Therefore, θ_o^B does not concede as long as θ_r^A has not done so.

As in the previous section, we have that θ_o^B assumes the posture of the behavioral types, who never concede in the war of attrition. Therefore, we can recover the equilibrium from AG. The types θ_i^B and θ_r^A randomize over some support $[0, T^0]$, with rates

$$\lambda^A = \frac{(1-\gamma_A)\delta}{\gamma_A+\gamma_B-1} \quad \text{and} \quad \lambda^B = \frac{(1-\gamma_B)\delta}{\gamma_A+\gamma_B-1}.$$

The final period T^0 is computed as $\min\{T^A, T^B\}$, where

$$T^A = -\frac{\log(z)}{\lambda^A} \quad \text{and} \quad T^B = -\frac{\log(q_o+q_b)}{\lambda^B}.$$

Clearly, if $T^k > T^j$, for $k \neq j$, player k concedes with positive probability at $(0,+1)$ to compensate for his or her reputation reaching 1 later than the opponent. In this equilibrium, it is necessary that after the unexpected event of the fair contract offer at $(n,0)$, θ_r^A does not update beliefs with $\mu_r^A((n,0);\theta_o^B) = 1$. In fact, in this case A would immediately concede and for some parameters θ_o^B prefers this deviation. If, for example, θ_r^A assumes that θ_o^B and θ_i^B made the mistake with the same probability, beliefs are not degenerate and θ_o^B does not want to deviate. Therefore, in the game with two behavioral types, there always exists an equilibrium where the fair contract is not used. The intuition is that the fair contract is perceived by A and B as a signal for weakness, and therefore it is not used. In fact, in case it is offered, both B's types prefer to concede immediately, as A would be too stubborn in that continuation game. This behavior and beliefs resemble second-order optimism in Friedenberg (2019). In this paper, the author shows that under the assumptions of rationality and common strong belief of rationality, two agents who bargain over a surplus can delay their agreement because any

Pareto improved and earlier offer from one agent may make the other player too optimistic, letting her believe she can obtain even more from a longer negotiation.

4.2 Signaling equilibrium

Now we turn to the analysis of equilibria that feature signaling, that is, equilibria where player B offer the fair contract with positive probability. As anticipated, these equilibria not always exist, and their existence depend on the parameter γ_B.

For the next proposition, let $F_S^A(\cdot;\tau^*)$ be the cdf representing A's concession strategy after B has offered the fair contract at $(\tau^*,0)$. Then, we denote with $Y_s^B(\tau^*|\tau^*)$ the probability that type θ_s^B, $s \in \{i,o\}$, offers the fair contract at $(\tau^*,0)$ conditioning on the event the game arrives at $(\tau^*,0)$. We get the following result.

Lemma 3. *Let $\tau^* = \max supp(Y^B)$. Then, for $\tau^* > 0$, $Y_i^B(\tau^*|\tau^*) \in (0,1)$ and $Y_o^B(\tau^*|\tau^*) \in (0,1]$.*

Hence, if τ^* is the last period where the fair contract is offered with positive probability, at $\tau^*,0)$, θ_i^B mixes between the offer $(1/2, 1/2)$ and the offer $(1-\gamma_B,\gamma_B)$. This implies he has to be indifferent between the two.

Theorem 1. *In any equilibrium we have $\sum_{n=0}^{+\infty} Y^B(n) = 0$ for all $\mathbf{q} \in \mathcal{Q}$ whenever $\gamma_B > 1/2$.*

Therefore, in any equilibrium where $\gamma_B > 1/2$, the omniscient player B opts to conceal their information. This choice arises because if B reveals information by offering the fair contract, player A might still suspect that B is bluffing. At this point, the best outcome he can achieve is the split $(1/2, 1/2)$. Conversely, by continuing to offer the behavioral contract, θ_o^B can secure γ_B. Consequently, the omniscient type θ_o^B cannot prevent the ignorant type θ_i^B from also offering the fair contract with positive probability, making it difficult for A to distinguish between the two.

As established in Lemma 2, in equilibrium, θ_o^B does not concede until θ_r^A has done so. Thus, the omniscient type behaves like a behavioral type, leading to a unique equilibrium, as described in the preceding section. The AG model applies, with the distinction that the probability of B behaving like an irrational type is $q_o + q_b$.

Having resolved the case where $\gamma_B > 1/2$, we now focus on the scenario where $\gamma_B < 1/2$ for the remainder of the paper. Here, the dynamics differ significantly. A fair contract offer not only signals the potential possession of information but also allows B to secure a better deal if A concedes. However, this advantage comes at a cost: B's reputation for being omniscient grows more slowly than the reputation for being behavioral (or omniscient) in the war of attrition before the fair contract was offered.

As stated in the previous proof, concession distributions in the GAS and GNS can be represented by CDFs. Therefore, let τ_1^* be the last period for the fair contract offer. For player $m \in A, B$, denote the concession distribution in the GAS by $F_S^m(\cdot;\tau_1^*)$ and the concession distribution in the GNS by $F_N^m(\cdot;\tau_1^*)$.

Lemma 4. *Let $\tau_1^*, \tau_0^* \in \mathbb{N}$ be such that τ_1^* is the last period and τ_0^* is the second to last period in which B offers $(1/2, 1/2)$ with positive probability. Then, A does not concede with positive probability to either contract at $(\tau_1^*, +1)$. Moreover, in the event B does not offer $(1/2, 1/2)$ at τ_0^*, A and B concede over (τ_0^*, τ_1^*) with rates λ_N^A and λ_N^B, respectively. If τ_0^* cannot be defined, set $\tau_0^* = 0$.*

Lemma 5. *Let $\tau_0^* \in \mathbb{N}$ be the second to last period in which B offers $(1/2, 1/2)$ with positive probability. Then, A does not concede with positive probability to either contract at $(\tau_0^*, +1)$.*

We proceed by establishing a theorem crucial for grasping the dynamics of signaling equilibria. This theorem states that B can only manifest his rationality through signaling in a single period $t \in \mathbb{N}_0$. Consequently, if B foregoes the opportunity to propose the fair contract at t, his offer remains unchanged throughout the game. To gain insight into why this holds true in any signaling equilibrium, consider the following scenario. Suppose B is randomizing his fair contract offer between τ_0^* and τ_1^*. Then θ_o^B is indifferent, and the same holds for θ_i^B. Recall that in equilibrium, θ_i^B can optimally choose θ_o^B's posture, so he can concede in the last period $T_S^0(\cdot)$ in the GAS. Hence, by our previous remark, we know that θ_i^B's utility is the utility of θ_o^B with probability $1 - z$, while with probability z he gets the worst possible outcome, that is $1 - \gamma_A$ on the very last period $T_S^0(\cdot)$. Therefore, since with probability $1 - z$ he is indifferent between τ_0^* and τ_1^* (by θ_o^B's indifference), we know that θ_i^B must be indifferent even in the event that A is irrational, which happens with probability z. Since this worst case scenario depends only on $T_S^0(\cdot)$, indifference necessitates $T_S^0(\tau_0^*) = T_S^0(\tau_1^*)$. Hence, even if B

postpones offering the fair contract until τ_1^*, the GAS still concludes at $T_S^0(\tau_0^*)$. From Lemma 4 and Lemma 5, we know that A does not concede to the fair contract with positive probability at $(\tau_0^*, +1)$ and $(\tau_1^*, +1)$, implying that $T_S^0(\cdot)$ depends on A's reputation and concession rate at both τ_0^* and τ_1^*. However, to maintain uniform deadlines, it must be that if B refrains from proposing the fair contract at τ_0^*, A's reputation grows at a rate of λ_S^A from τ_0^* to τ_1^*. This guarantees that even in the absence of the fair contract at τ_0^*, A's reputation progresses as it actually happened, ensuring that when the players reach τ_1^*, the absence of $(1/2, 1/2)$ at τ_0^* is inconsequential due to A's reputation evolving at rate λ_S^A, thereby maintaining identical deadlines $T_S^0(\tau_0^*)$ and $T_S^0(\tau_1^*)$. Nonetheless, as hinted by Lemma 4, if B refrains from offering $(1/2, 1/2)$ at τ_0^*, A's reputation progresses at a rate of λ_N^A. Given that $\lambda_N^A \neq \lambda_S^A$, randomization between τ_0^* and τ_1^* is untenable.

Theorem 2. *In any signaling sequential equilibrium,* $|supp(Y^B)| = 1$.

Now, we know that in any separating sequential equilibrium, the fair contract can be offered, with positive probability, on a single period τ^* only. By Lemma 4 we also know that θ_r^A does not concede at $(\tau^*, +1)$, no matter the contract offered by B at $(\tau^*, 0)$. Therefore, in the GAS at $(\tau^*, 0)$ we have

$$T_S^0(\tau^*) = T_S^A(\tau^*) = -\frac{\mu_i^B((\tau^*, 0); \theta_b^A)}{\lambda_S^A} + \tau^*$$

and

$$T_N^0(\tau^*) = T_N^A(\tau^*) = -\frac{\mu_i^B((\tau^*, 0); \theta_b^A)}{\lambda_N^A} + \tau^*.$$

Since

$$\lambda_N^A = \frac{(1-\gamma_A)\delta}{\gamma_A + \gamma_B - 1} > \frac{(1-\gamma_A)\delta}{\gamma_A + 1/2 - 1} = \lambda_S^A,$$

we get

$$T_S^0(\tau^*) > T_N^0(\tau^*) \quad \forall \tau^*.$$

From this, the next Corollary follows.

Corollary 2. In any separating sequential equilibrium, $Y_o^B(\tau^*) = 1$.

Proof. Recall $Y_i^B(\tau^*|\tau^*) \in (0,1)$ by Lemma 3, and so $Y_i^B(\tau^*) \in (0,1)$. That is, θ_i^B offers the fair contract at $(\tau^*, 0)$ with a probability strictly less than 1. Moreover, from Theorem 2, $Y_i^B(n) = 0$ $\forall n \neq \tau^*$. Hence, θ_i^B is indifferent between offering $(1/2, 1/2)$ at $(\tau^*, 0)$ and conceding at $T_S^0(\tau^*)$ or concession to A's behavioral demand at $T_N^0(\tau^*)$. That is, in equilibrium,

$$U_i^B(\Sigma^A, (\tau^*, T_S^0(\tau^*))) = U_i^B(\Sigma^A, T_N^0(\tau^*)).$$

Therefore,

$$(1-z)U_o^B(\Sigma^A, (\tau^*, T_S^0(\tau^*))) + z(1-\gamma_A)e^{-\delta T_S^0(\tau^*)}$$

$$=$$

$$(1-z)U_o^B(\Sigma^A, T_N^0(\tau^*)) + z(1-\gamma_A)e^{-\delta T_N^0(\tau^*)}$$

Since $T_S^0(\tau^*) > T_N^0(\tau^*)$, $z(1-\gamma_A)e^{-\delta T_N^0(\tau^*)} > z(1-\gamma_A)e^{-\delta T_S^0(\tau^*)}$, and so

$$U_o^B(\Sigma^A, (\tau^*, T_S^0(\tau^*))) > U_o^B(\Sigma^A, T_N^0(\tau^*)).$$

Since $T_N^0(\tau^*)$ is θ_o^B's optimal action in the GNS, θ_o^B strictly prefers to offer the fair contract at τ^*, and so $Y_o^B(\tau^*|\tau^*) = 1$. Now, since τ_o^* of Lemma 4 is equal to 0, we have that B concedes at rate λ_N^B from 0 to τ^*. Yet, since concession from θ_i^B is necessary in equilibrium (Proposition 1 can be applied to the GNS as well), we have that θ_i^B is indifferent in any concession in the interval $[0, \tau^*]$. Therefore, θ_o^B strictly prefers to wait until $(\tau^*, 0)$ at least. Hence,

$$\sum_{n=0}^{\tau^*-1} X_{n,o}^B([n, n+1]) = 0,$$

and so $Y_o^B(\tau^*) = 1$. ■

We summarize now the properties of any separating equilibrium we have found. For $\gamma_B > 1/2$, there exists a unique equilibrium in which the fair contract is never offered. For $\gamma_B < 1/2$, instead, we get the following.

1. $\exists! \tau^* \in \mathbb{N}_0$ such that $Y^B(\tau^*) > 0$. For all $n \neq \tau^*$, $Y^B(n) = 0$;

2. $Y_i^B(\tau^*|\tau^*) \in (0,1)$ and $Y_o^B(\tau^*) = 1$;

3. A and B concede with rates λ_N^A and λ_N^B respectively in the interval $[0, \tau^*]$;

4. At $(\tau^*, +1)$ player A does not concede with positive probability, no matter B's standing offer.

Note that these condition are necessary for any separating equilibrium, therefore we need to make sure that they do not create a contradiction. In particular, we need to check the payoff indifference conditions for the rational types. Hence, we show how to make sure that all four condition holds in equilibrium so that no rational type has profitable deviations.

1. For the first condition, we need that both θ_i^B and θ_o^B are not willing to offer the fair contract at $n < \tau^*$. This is trivial when $\tau^* = 0$, so suppose $\tau^* > 0$ and take $n \in \mathbb{N}_0$ such that $n < \tau^*$. From Definition 3.4, we have that $(\sigma_i^B[n], \sigma_o^B[n], \sigma_r^A[n])$ is a GAS degenerate or nondegenerate equilibrium. Hence, we can assume that for all $n < \tau^*$, $(\sigma_i^B[n], \sigma_o^B[n], \sigma_r^A[n])$ is a GAS degenerate equilibrium. In this case, any offer of the fair contract before τ^* implies a payoff of $(1-\gamma_A)e^{-\delta n}$ for θ_i^B and θ_o^B. Clearly, θ_i^B is indifferent between the deviation and the equilibrium strategy, and therefore θ_o^B strictly prefers to not offer it at $(n, 0)$. Hence, $Y^B(n) = 0$ can be part of the equilibrium. For what regards its consistency, it is enough that any deviation at n is sustained by A's beliefs that put equal probability of mistake by θ_i^B and θ_o^B (clearly, these are not the unique beliefs that can sustain it).

2. By the previous point 4., we have that A does not concede at $(\tau^*, +1)$, no matter the contract. By Lemma 1, we have that $(\sigma_i^B[\tau^*], \sigma_o^B[\tau^*], \sigma_r^A[\tau^*])$ is the GAS nondegenerate equilibrium. Hence, θ_i^B's payoff in case of fair contract offer is $(1-\gamma_A)e^{-\delta\tau^*}$. In case the fair contract is not offered, A and B play the unique GNS equilibrium, and since A does not concede at $(\tau^*, +1)$, we get that θ_i^B's expected payoff of offering $(1-\gamma_B, \gamma_B)$ is $(1-\gamma_A)e^{-\delta\tau^*}$. Therefore, θ_i^B is indifferent and we can have $Y_i^B(\tau^*|\tau^*) \in (0,1)$. In the proof of Corollary 2 we have shown that θ_i^B's indifference at $(\tau^*, 0)$ implies θ_o^B strictly prefers to offer the fair contract at $(\tau^*, 0)$. Therefore, we have $Y_i^B(\tau^*|\tau^*) = 1$, and since θ_o^B does not concede in $[0, \tau^*]$ (proof of Corollary 2) we get $Y_o^B(\tau^*) = 1$.

3. Clear from Lemma 4.

4. Clear from Lemma 4.

We need one last condition for the signaling equilibrium. We have that $(\tau^*, -1)$ is in θ_r^A concession support, and we know that at $(\tau^*, 0)$ θ_r^A's jump, depending on the contract offer. This may cause a jump in θ_r^A's payoff. Moreover, the contract offered from B may change, and this is another source of jump in θ_r^A's expected utility. Therefore, in order for θ_r^A to be indifferent between concession at $(\tau^*, -1)$ and any period after τ^*, we need to calibrate B's concessions at $(\tau^*, +1)$ and the probability of fair contract offer $Y^B(\tau^*)$. So, for ease of notation let $\overline{U}_r^A$ the equilibrium utility of θ_r^A of concession after $(\tau^*, +1)$, and $\underline{U}_r^A$ the equilibrium utility of concession at $(\tau^*, -1)$. We have

$$\begin{aligned}
\overline{U}_r^A &= \sum_{n=0}^{\tau^*-1} \int_n^{n+1} \gamma_A e^{-\delta t} dX_n^B(t) \\
&+ Y^B(\tau^*)[F_S^B(\tau^*;\tau^*)\gamma_A + (1 - F_S^B(\tau^*;\tau^*))1/2]e^{-\delta\tau^*} \\
&+ \left(1 - \sum_{n=0}^{\tau^*-1} X_n^B([n, n+1]) - Y^B(\tau^*)\right)[F_N^B(\tau^*;\tau^*)\gamma_A + (1 - F_N^B(\tau^*;\tau^*))(1-\gamma_B)]e^{-\delta\tau^*},
\end{aligned}$$

and

$$\underline{U}_r^A = \sum_{n=0}^{\tau^*-1} \int_n^{n+1} \gamma_A e^{-\delta t} dX_n^B(t) + \left(1 - \sum_{n=0}^{\tau^*-1} X_n^B([n, n+1])\right)(1-\gamma_B)e^{-\delta\tau^*}.$$

Hence, $\overline{U}_r^A = \underline{U}_r^A$ implies

$$\begin{aligned}
&Y^B(\tau^*)[F_S^B(\tau^*;\tau^*)\gamma_A + (1 - F_S^B(\tau^*;\tau^*))1/2]e^{-\delta\tau^*} \\
&+ \left(1 - \sum_{n=0}^{\tau^*-1} X_n^B([n, n+1]) - Y^B(\tau^*)\right)[F_N^B(\tau^*;\tau^*)\gamma_A + (1 - F_N^B(\tau^*;\tau^*))(1-\gamma_B)]e^{-\delta\tau^*} \\
&= \\
&\left(1 - \sum_{n=0}^{\tau^*-1} X_n^B([n, n+1])\right)(1-\gamma_B)e^{-\delta\tau^*}.
\end{aligned}$$

This equation can be rewritten as

$$\frac{Y^B(\tau^*)}{1-\sum_{n=0}^{\tau^*-1} X_n^B([n,n+1])}[F_S^B(\tau^*;\tau^*)\gamma_A + (1-F_S^B(\tau^*;\tau^*))1/2]$$

$$+\left(1-\frac{Y^B(\tau^*)}{1-\sum_{n=0}^{\tau^*-1} X_n^B([n,n+1])}\right)[F_N^B(\tau^*;\tau^*)\gamma_A + (1-F_N^B(\tau^*;\tau^*))(1-\gamma_B)]$$

$$=$$

$$(1-\gamma_B),$$

and so,

$$Y^B(\tau^*|\tau^*)F_S^B(\tau^*;\tau^*)\gamma_A + (1-F_S^B(\tau^*;\tau^*))1/2]$$

$$+\left(1-Y^B(\tau^*|\tau^*)\right)[F_N^B(\tau^*;\tau^*)\gamma_A + (1-F_N^B(\tau^*;\tau^*))(1-\gamma_B)]$$

$$=$$

$$(1-\gamma_B).$$

Note that this is the indifference condition of θ_r^A at continuation game $(\tau^*,-1)$ for immediate concession and any concession afterwards. Therefore, $Y^B(\tau^*|\tau^*)$, $F_S^B(\tau^*;\tau^*)$ and $F_N^B(\tau^*;\tau^*)$ must satisfy this equation. Note that $F_S^B(\tau^*;\tau^*)$ and $F_N^B(\tau^*;\tau^*)$ depends on θ_r^A's beliefs, and these beliefs depend on $Y^B(\tau^*;\tau^*)$. Therefore, in order to conclude, we give an example of a separating equilibrium, showing the relation among these variables. The probabilities of concession at $(0,0)$ are going to be fundamental as the allow for beliefs manipulation. Moreover, recall that in equilibrium $T_S^0(\tau) = T_S^A(\tau^*)$ and $T_N^0(\tau^*) = T_N^A(\tau^*)$, which add two additional constraint on the equilibrium, that is, $T_S^A(\tau^*) \leqslant T_S^B(\tau^*)$ and $T_N^A(\tau^*) \leqslant T_N^B(\tau^*)$. These constraint are dependent on beliefs as well and so they are dependent also on the probabilities of concession at $(0,0)$. The following Proposition and remark are useful for the computation of $Y^B(\tau^*|\tau^*)$ and θ_r^A's indifference condition at $(\tau^*,-1)$.

Proposition 4. *We have*

$$Y^B(\tau^*|\tau^*) = \mu_r^A((\tau^*,-1);\theta_o^B) + \mu_r^A((\tau^*,-1);\theta_i^B)Y_i^B(\tau^*|\tau^*).$$

Remark. Let $x = Y_i^B(\tau^*|\tau^*)$, $\mu_r^A((\tau^*,-1);\theta_i^B) = Q_i$ and $\mu_r^A((\tau^*,-1);\theta_o^B) = Q_o$ for ease of notation. Then, when $(1/2, 1/2)$ is offered, θ_r^A's belief is:

$$\mu_r^A((\tau^*,0);\theta_o^B) = \frac{Q_o}{Q_o + Q_i x},$$

while if $(1-\gamma_B, \gamma_B)$ is offered, her belief is:

$$\mu_r^A((\tau^*,0);\theta_b^B) = \frac{Q_b}{Q_b + Q_i(1-x)}.$$

Moreover, note

$$\begin{aligned} F_S^B(\tau^*;\tau^*) &= 1 - e^{-\lambda_S^B(T_S^B - T_S^A)} \\ &= 1 - e^{-\lambda_S^B\left(-\log\left(\frac{Q_o}{Q_o+Q_i x}\right)\frac{1}{\lambda_S^B} + \log(\mu_i^B((\tau^*,0);\theta_b^A))\frac{1}{\lambda_S^A}\right)} \\ &= 1 - \frac{Q_o}{Q_o+Q_i x}\mu_i^B((\tau^*,0);\theta_b^A)^{-\frac{1/2}{1-\gamma_A}}. \end{aligned}$$

In the same fashion, we can prove

$$F_N^B(\tau^*|\tau^*) = 1 - \frac{Q_b}{Q_b + Q_b(1-x)}\mu_i^B((\tau^*,0);\theta_b^A)^{-\frac{1-\gamma_B}{1-\gamma_A}}.$$

Let $Z = \mu_i^B((\tau^*,0);\theta_b^A)$. θ_r^A's indifference condition is

$$\begin{aligned} &(Q_o + Q_i x)\left[\left(1 - \frac{Q_o}{Q_o+Q_i x}Z^{-\frac{1/2}{1-\gamma_A}}\right)\gamma_A + \frac{Q_o}{Q_o+Q_i x}Z^{-\frac{1/2}{1-\gamma_A}}\frac{1}{2}\right] \\ &+ (\overbrace{1-Q_i}^{Q_b+Q_i} - Q_i x)\left[\left(1 - \frac{Q_b}{Q_b+Q_i(1-x)}Z^{-\frac{1-\gamma_B}{1-\gamma_A}}\right)\gamma_A + \frac{Q_b}{Q_b+Q_i(1-x)}Z^{-\frac{1-\gamma_B}{1-\gamma_A}}(1-\gamma_B)\right] \\ &= \end{aligned}$$

$(1-\gamma_B)$

The LHS of this equation can be rewritten as:

$$\left(Q_o + Q_i x - Q_o Z^{-\frac{1/2}{1-\gamma_A}}\right)\gamma_A + Q_o Z^{-\frac{1/2}{1-\gamma_A}}\frac{1}{2} + \left(Q_b + Q_i(1-x) - Q_b Z^{-\frac{1-\gamma_B}{1-\gamma_A}}\right)\gamma_A + Q_b Z^{-\frac{1-\gamma_B}{1-\gamma_A}}(1-\gamma_B)$$

$$=$$

$$\left(Q_o - Q_o Z^{-\frac{1/2}{1-\gamma_A}}\right)\gamma_A + Q_o Z^{-\frac{1/2}{1-\gamma_A}}\frac{1}{2} + \left(Q_b + Q_i - Q_b Z^{-\frac{1-\gamma_B}{1-\gamma_A}}\right)\gamma_A + Q_b Z^{-\frac{1-\gamma_B}{1-\gamma_A}}(1-\gamma_B)$$

Hence, the indifference condition is independent from x, i.e., $Y_i^B(\tau^*|\tau^*)$, as Q_o and Q_i are beliefs computed at $(\tau^*, -1)$. The only role that $Y_i^B(\tau^*|\tau^*)$ plays is in making sure that the constraints $T_S^A(\tau^*) \leqslant T_S^B(\tau^*)$ and $T_N^A(\tau^*) \leqslant T_N^B(\tau^*)$ are satisfied. This generates a continuum of separating equilibria.

Consistency

We have not specified how consistency is obtained so far. Clearly, consistency in the GAS is taken from the results in the section of the model with one behavioral type. Instead, the consistency before the fair contract is not offered is obtained through Bayes' rule as long as there are no surprise events. Note that the only surprise event in which the war of attrition does not end is an unexpected offer of the fair contract, for some $n \neq \tau^*$. For simplicity, consider $n < \tau^*$. In this case, as previously described, we can just assume that A believes that the mistake was made with equal probability by θ_i^B and θ_o^B. Here, as the probability of mistake goes to zero, beliefs are

$$\mu_r^A((n,0);\theta_i^B) = \frac{\mu_r^A((n,-1);\theta_i^B)}{\mu_r^A((n,-1);\theta_i^B) + \mu_r^A((n,-1);\theta_o^B)}$$

and

$$\mu_r^A((n,0);\theta_o^B) = \frac{\mu_r^A((n,-1);\theta_o^B)}{\mu_r^A((n,-1);\theta_i^B) + \mu_r^A((n,-1);\theta_o^B)}.$$

Both are strictly less than 1. Then, we can easily sustain the equilibrium by letting A and B to play the nondegenerate GAS equilibrium.

Example 1. Consider the following set of parameters.

Parameter	**Value**
γ_A	0.8
γ_B	0.4
δ	1
q_o	$0.2e^{-3}$
q_b	$0.1e^{-3}$
z	0.2

Table 1: Example table with parameters and values

Clearly, we have $q_i = 1 - q_o - q_b$. Note that $\gamma_A + \gamma_B > 1$, so that behavioral demands are incompatible. We let $\tau^* = 1$. That is, in equilibrium, B offers the fair contract with positive probability only at period 1. The concessions rate are the following:

$$\lambda_S^A = \frac{(1-\gamma_A)\delta}{\gamma_A + 1/2 - 1} = \frac{2}{3} \qquad \lambda_S^B = \frac{1/2\delta}{\gamma_A + 1/2 - 1} = \frac{5}{3}$$

$$\lambda_N^A = \frac{(1-\gamma_A)\delta}{\gamma_A + \gamma_B - 1} = 1 \qquad \lambda_N^B = \frac{(1-\gamma_B)\delta}{\gamma_A + \gamma_B - 1} = 3$$

By Lemma 4, we know that A and B concede with rates λ_N^A and λ_N^B respectively. Therefore, we have

$$Q_o + Q_b = \mu_r^A((\tau^*, -1); \theta_b^B, \theta_o^B) = (0.3e^{-3})e^{\lambda_N^B} = 0.3.$$

Note that this implies $Q_b = \mu_r^A((1,-1);\theta_b^B) = 0.1$ and $Q_o = \mu_r^A((1,-1);\theta_o^B) = 0.2$. Now, let Z^* be the value that satisfies the previous indifference condition. We can derive $F^A(0;0)$, i.e., the probability of concession from A at time $(0,0)$ that makes her reputation jump from z to $Z^* e^{-\lambda_N^A}$. Note that given the parameters given and $Q_o = 0.2$, $Q_b = 0.1$ we have $Z^* \approx 0.7$, which gives $Z^* e^{-\lambda_N^A} \approx 0.26$. Hence, we compute θ_r^A's probability of concession at $(0,+1)$ that allows this shift in θ_i^B's beliefs. Denote y the probability of θ_r^A's concession at $(0,+1)$.

$$Z^* e^{-1} = \frac{z}{z + (1-z)(1-y)} \Rightarrow y = \frac{1}{4}(5 - e/Z^*) \approx 0.28.$$

Therefore, for a reputation jump from $z = 0.2$ to $Z^* e^{-1} \approx 0.26$ type θ_r^A needs to concede with approximate probability of 0.28. Concession at $(0,+1)$ from A implies no concession from B at the same date. Finally, we need to check that the constraints $T_S^A(\tau^*) \leqslant T_S^B(\tau^*)$ and $T_N^A(\tau^*) \leqslant T_N^B(\tau^*)$ are satisfied. Therefore,

$$T_S^A(1) = -\frac{\log(Z^*)}{2/3} + 1 \approx 1.52$$

$$T_S^B(1) = -\log\left(\frac{Q_o}{Q_o + Q_i x}\right)\frac{3}{5} + 1 = -\log\left(\frac{2}{2+7x}\right)\frac{3}{5} + 1$$

$$T_N^A(1) = -\frac{-\log(Z^*)}{1} + 1 \approx 1.34$$

$$T_N^B(1) = -\log\left(\frac{Q_b}{Q_b + Q_i(1-x)}\right)\frac{1}{3} + 1 = -\log\left(\frac{1}{8-7x}\right)\frac{1}{3} + 1.$$

From these, we get

$$x = Y_i^B(1|1) \in \left(\frac{2(Z^*)^{-5/2} - 2}{7}, \frac{8-(Z^*)^{-3}}{7}\right) \approx (0.06, 0.73).$$

Figure 2 describes the unfolding of the equilibrium graphically.

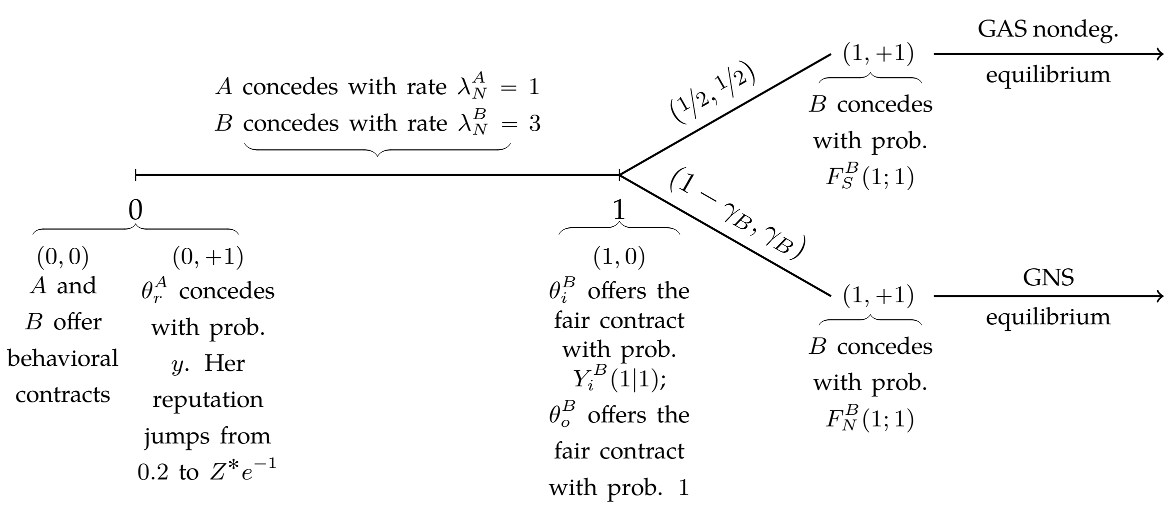

Figure 2: Example of a Signaling Equilibrium

5 Conclusion

In conclusion, this paper explores the role of reputation in bargaining scenarios where an agent may be aware of the other's rationality. Hence, we examined how reputation effects can arise not only from direct uncertainty about an opponent's type, but also from second-order beliefs, where one agent attempts to appear tough despite the opponent's awareness of their true nature. The other main ingredient of the model is the possibility of the informed type to signal his knowledge by a fair split 50-50 of the surplus. By analyzing a bargaining game with both discrete and continuous time dynamics, we focused on the strategic choices player B, and his respective types—rational, ignorant, or omniscient.

In the one behavioral type model, our results highlight two main classes of equilibria. In one, B concedes immediately regardless of his type, due to the belief that A is so stubborn that any informational advantage is useless. In the other, the omniscient type of B does not concede, leading to a situation where both agents engage in signaling games to convince the other of their toughness or knowledge.

We then linked this analysis to the original two-behavioral-type model, demonstrating that an equilibrium without fair offers is always possible, especially when B's irrational demand exceeds the fair offer. However, when B's irrational offer is less than 50%, an equilibrium with a fair contract becomes viable, with a specific period in which this contract can be proposed.

Appendix

AG Proposition 1 summary

We present here a summary of AG results of Proposition 1. In their initial model, there are two players A and B, and each of them is one of two possible types: rational and behavioral. Player i is irrational with probability z_i, and demand a fixed amount γ_i, where $\gamma_A + \gamma_B > 1$. Each player is allowed to concede to the other's demand, or wait. Therefore, since each player has only one rational type, strategies can be described by the cdf F_i.

Players discount payoffs exponentially at the rate of δ_i. Utility functions u_i are written in the same way of θ_i^B's utility in the model with one behavioral type. Then, this result follows:

1. Let $\tau^i = \inf\{t \geqslant 0 | F_i(t) = \lim_{t' \to \infty} F_i(t')\}$. Then $\tau^A = \tau^B$;
2. If $\lim_{t' \to t^-} F_i(t') \neq F_i(t)$, then $\lim_{t' \to t^-} F_j(t') = F_j(t)$, for $j \neq i$;
3. If F_i is continuous at t, then u_i is continuous at $($concession at$)$ t;
4. There is no $t_1, t_2 \in \mathbb{R}_+$ such that $0 \leqslant t_1 < t_2 \leqslant \tau_i$ such that F_A and F_B are constant over (t_1, t_2);
5. F_i is strictly increasing over $(0, \tau^i)$;
6. F_i is continuous for $t > 0$.

From these properties, AG proves that both players concede at constant rate that makes the opponent indifferent, for every $t > 0$, between immediate concession and waiting. In order to calculate this rate, denote it first by λ_i for player i. Then, in order to make j indifferent, the cost and the benefit of waiting must be the same. The cost of not conceding at t, instead of some $t + dt$, is $\delta_j(1 - \gamma_i)dt$, that is, the lost of the interest of i's offer, $1 - \gamma_i$. The benefit of waiting until $t + dt$ is instead the probability that i concedes in the interval $(t, t + dt)$ times the gain j gets from i's acceptance. Therefore, the benefit is $\lambda_i(\gamma_j - (1 - \gamma_i))dt$. Hence, the rate λ_i of concession that makes j indifferent is

$$\lambda_i = \frac{\delta_j(1-\gamma_i)}{\gamma_i+\gamma_j-1}.$$

Hence, we have that $F_i(t) = 1-(1-F_i(0))e^{-\lambda_i t}$. From the previous results, we have that $\tau^A = \tau^B$. For this to happen, we need that both players reach reputation[9] 1 at the same time, at some period T^0. Since in general $\lambda_i \neq \lambda_j$, one of the player concedes with positive probability at time $t = 0$ in order to boost her reputation and make sure she reaches reputation 1 at the same time of her opponent. Note that i's reputation reaches 1 at T^i if

$$T^i = -\frac{\log(z_i)}{\lambda_i}.$$

Therefore, let $T^0 = \min\{T^A, T^B\}$. In case $T^0 = T^j$, then i has to concede at time zero with strictly positive probability $F_i(0)$. We have

$$F_i(0) = 1 - e^{-\lambda_i(T^i - T^0)}.$$

Consistency in the model with one behavioral type

First, we define the definition of convergence of sequence of strategies.

Definition 4. *A sequence of strategies $(\sigma^k)_{k\in\mathbb{N}}$ is said to converge to the strategy σ if and only if*

$$\forall t \in \mathbb{R}, \quad \sigma_t^k \xrightarrow{w} \sigma_t.$$

Definition 5. *Let $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ be the vectors collecting the strategies and beliefs of all the players. We say that $(\boldsymbol{\sigma}, \boldsymbol{\mu})$ is a consistent assessment if and only if there exists a sequence of completely mixed strategies $(\boldsymbol{\sigma}^k)_{k\in\mathbb{N}}$ converging to $\boldsymbol{\sigma}$ and a sequence of beliefs $(\boldsymbol{\mu}^k)_{k\in\mathbb{N}}$ converging to $\boldsymbol{\mu}$ in Euclidean space with the property that for each k, $\boldsymbol{\mu}^k$ is derived from $\boldsymbol{\sigma}^k$ using Bayes' rule.*

When we write completely mixed, we mean that σ_τ assigns positive probability to each $(a, b) \subseteq [\tau, +\infty)$. Therefore, a sequence $(\sigma^k)_{k\in\mathbb{N}}$ converges to some strategy σ if and only if

[9] The other player must believe with probability 1 that she is irrational.

every continuation game measure of the sequence weakly converges to the limit continuation game measure.

Recall that weakly convergence is necessary and sufficient for convergence in distribution. Hence, let F^k and F be the cdf associated with σ_t^k and σ_t respectively. We have that

$$\sigma_t^k \xrightarrow{w} \sigma_t \Leftrightarrow F_k(x) \to F(x),\ x \text{ continuity point of } F.$$

Therefore, even though the definition are described using measures $\sigma(t)$, we prove the statements using their respective cdf $F(\cdot|t)$, i.e., conditional distributions. Moreover, we use $F(\cdot)$ to denote $F(\cdot|0)$. Observe there is no ambiguity as we can always derive a cdf from a measure and vice versa. Conditional utilities are derived using continuation game beliefs $\mu(\cdot|\tau)$ and conditional distribution $F(\cdot|\tau)$.

In section 3.2, we proposed the following candidate equilibrium:

- $F_i^B(t|\tau) = F_o^B(t|\tau) = \begin{cases} 1 & \text{if } t \geqslant \tau \\ 0 & \text{otherwise} \end{cases}$
- $F_r^A(t) = 0$ for all $t \geqslant 0$.

We already showed sequential rationality for all the types. Now we prove consistency of F_i^B and F_o^B (as F_r^A can be trivially be proven to be consistent since μ_i^B is always updated through Bayes' rule).

We now prove consistency. That is, there exists a sequence of completely mixed strategy such that it converges to the equilibrium assessment and beliefs are always derived from Bayes' rule in the sequence. We start with F_r^A and beliefs μ_i^B. Observe that in equilibrium, we must have $\mu_i^B(t) = z$ for each t, as the rational type never concedes. Consider a sequence of completely mixed strategies $\sigma_{r,k}^A$, that assigns probability $1/k$ over the interval $[0, +\infty)$ and $1 - 1/k$ to $\{+\infty\}$. When θ_i^B observe waiting at time t, he updates beliefs $\mu_{i,k}^B(t) = \dfrac{(1-z)(1-F_{r,k}^A(t))}{(1-z)(1-F_{r,k}^A(t)) + z}$, where $F_{r,k}^A$ is the cdf that describes how $\sigma_{r,k}^A$ distributes the mass $1/k$ over $[0, +\infty)$. Since $F_{r,k}^A(t) \to 0$ for each $t \geqslant 0$, we have that $\mu_{i,k}^B \to \mu_i^B$. Hence, F_r^A and μ_i^B satisfy the consistency requirement.

Now consider F_i^B, F_o^B and μ_r^A. In order to sustain the equilibrium, we set $\mu_r^A(t) = q$ for each $t \geqslant 0$. Observe that these beliefs allow F_r^A to satisfy the sequential rationality requirement. Consider the sequence of completely mixed strategies $\sigma_{i,k}^B$, $\sigma_{o,k}^B$ such that $\sigma_{i,k}^B = \sigma_{o,k}^B$ for all k, and both have cdf $F_k(t) = 1 - e^{-kt}$. Then, the conditional distribution upon reaching continuation game τ is

$$\begin{aligned} F_k(t|\tau) &= \frac{F(t) - F(\tau)}{1 - F(\tau)} \\ &= \frac{(1 - e^{-kt}) - (1 - e^{-k\tau})}{1 - (1 - e^{-k\tau})} \\ &= 1 - e^{-k(t-\tau)}. \end{aligned}$$

Then, for each continuity point $t > \tau$, we have that $F_k(t|\tau) \to 1 = F_i^B(t|\tau) = F_o^B(t|\tau)$, as $k \to +\infty$. Therefore, this strategy converges to the candidate equilibrium strategy. Moreover, since θ_i^B and θ_o^B use the same strategy for each k, θ_r^A have constant beliefs in every continuation game τ, that is, $\mu_r^A(\tau) = Pr(\theta^B = \theta_i^B|\tau) = q$ for each $\tau \geqslant 0$. Hence, $((\sigma_{i,k}^B, \sigma_{o,k}^B), \mu_{r,k}^A)) \to ((\sigma_i^B, \sigma_o^B), \mu_r^A)$ where $\mu_{r,k}^A(\tau) = q$ for each k and τ and (σ_i^B, σ_o^B) is the candidate strategy for B. Thus, $F_i^B(\cdot|\tau)$, F_o^B and μ_r^A satisfy the consistency requirement for each τ. Therefore, $\big((F_i^B(\cdot|\tau), F_o^B(\cdot|\tau), F_r^A), (\mu_i^B, \mu_r^A)\big)_{\tau \geqslant 0}$ is a consistent assessment and therefore it is a sequential equilibrium.

The information structure

We propose an information structure that could generate the type space introduced in the model with one behavioral type. From this, we can easily construct a larger information structure capable of generating the model with two behavioral types. We do this in the spirit of Milgrom and Roberts (1982), who propose in their appendix an information structure that could induce reputation effects even when one of the agents knows the other is rational. As they emphasize, the key factor is that the agent attempting to build a reputation is unaware that the other knows about her rationality.

We start by constructing an Aumann model of incomplete information and then we derive the corresponding Harsanyi type space. [10] There are two players, A and B, who bargain over some surplus. There is a set of states of nature $\mathcal{S} = \{s_1, s_2\}$. In s_1, A is irrational (or behavioral), while in s_2 is rational. B is known to be rational by both players and this constitutes common knowledge. We then construct a set of states of the world $\Omega := \{a, b, c\}$. The function that associates each state of the world to the states of nature is $\mathfrak{s} : \Omega \longrightarrow \mathcal{S}$, such that

$$\mathfrak{s}(a) = s_1$$

$$\mathfrak{s}(b) = \mathfrak{s}(c) = s_2.$$

Hence, in the first state of the world, A is irrational, while in the other two states, he is rational. The players' information sets are the following:

$$\mathcal{F}_A := \{\{a\}, \{b, c\}\}$$

$$\mathcal{F}_B := \{\{a, b\}, \{c\}\}.$$

Observe that A can only distinguish the set of states in which he is rational or not, while B can either have complete information (in $\omega = c$) or be completely ignorant (state $\omega \in \{a, b\}$).

In this framework, A's rationality cannot be common knowledge in any of the states. In fact, in state $\omega = b$ player B cannot distinguish the rational type from the behavioral type. In state $\omega = c$, B knows that player A is rational but the latter does not possess this information. As noted in Appendix B of Milgrom and Roberts (1982), reputation effects can emerge even when both players are rational and know that the other is rational. What is key, is the absence of its common knowledge. In this model, common knowledge of rationality fails because player A never knows whether B possesses information about the true state of nature. This missing link will generate reputation strategies from the rational player A. From this information structure, we can derive the usual Harsanyi types. Consider player A. Associate to the first partition element, $\{a\}$, the type θ_b^A, which corresponds to his behavioral type. Then, associate with his second element, $\{b, c\}$, the type θ_r^A, the rational type. Apply the same process to player B. We obtain θ_i^B, the ignorant type, corresponding to the partition

[10] For a reference, see Maschler, Solan, and Zamir (2013), Chapter 9.

element $\{a, b\}$, and θ_o^B, the omniscient type, for $\{c\}$. Note that Harsanyi players' types are not independent. In fact, when $\theta^B = \theta_o^B$, B assigns probability 1 to the event (hence, knows) $\theta^A = \theta_r^A$. Therefore, with this type structure, we get the following joint mass distribution of types:

- $p(\theta_b^A, \theta_i^B) = Pr(\omega = a) = p_a$;
- $p(\theta_b^A, \theta_o^B) = 0$;
- $p(\theta_r^A, \theta_i^B) = Pr(\omega = b) = p_b$;
- $p(\theta_r^A, \theta_o^B) = Pr(\omega = c) = p_c$.

Therefore, for example, when B is of type θ_i^B, he has the following beliefs:

$$z := Pr(\theta_b^A|\theta_i^B) = \frac{p_a}{p_a + p_b} \quad Pr(\theta_r^A|\theta_i^B) = \frac{p_b}{p_a + p_b} = 1 - z.$$

Note we defined with z the probability that A is irrational when B is ignorant. We also define $q := Pr(\theta_i^B|\theta_r^A)$.

Proofs

Proof of Proposition 1

Proof. We only consider the case of indifference between concession at t and t'. The case of strict preference easily follows.

(i) By assumption,

$$(1 - \mu_i^B(\tau)) \cdot \left[\int_0^t \frac{1}{2} e^{-\delta x} d\sigma_{r,\tau}^A(x) + \left(\sigma_{r,\tau}^A(t)\frac{1}{2}\left(3/2 - \gamma\right) + \sigma_{r,\tau}^A((t, +\infty])(1 - \gamma)\right) e^{-\delta t}\right] +$$

$$\mu_i^B(\tau) \cdot (1 - \gamma)e^{-\delta t}$$

$$=$$

$$(1-\mu_i^B(\tau))\cdot\left[\int_0^{\hat{t}}\frac{1}{2}e^{-\delta x}d\sigma^A_{r,\tau}(x)+\left(\sigma^A_{r,\tau}(\hat{t})\frac{1}{2}\left(3/2-\gamma\right)+\sigma^A_{r,\tau}((\hat{t},+\infty])(1-\gamma)\right)e^{-\delta\hat{t}}\right]+$$

$$\mu_i^B(\tau)\cdot(1-\gamma)e^{-\delta\hat{t}}.$$

Observe that $\mu_i^B(\tau)(1-\gamma)e^{-\delta t}>\mu_i^B(\tau)(1-\gamma)e^{-\delta\hat{t}}$, hence

$$\int_0^{\hat{t}}\frac{1}{2}e^{-\delta x}d\sigma^A_{r,\tau}(x)+\left(\sigma^A_{r,\tau}(\hat{t})\frac{1}{2}\left(3/2-\gamma\right)+\sigma^A_{r,\tau}((\hat{t},+\infty])(1-\gamma)\right)e^{-\delta\hat{t}} \tag{4}$$

$$>$$

$$\int_0^{t}\frac{1}{2}e^{-\delta x}d\sigma^A_{r,\tau}(x)+\left(\sigma^A_{r,\tau}(t)\frac{1}{2}\left(3/2-\gamma\right)+\sigma^A_{r,\tau}((t,+\infty])(1-\gamma)\right)e^{-\delta t}, \tag{5}$$

where (4) and (5) are θ_o^B's payoffs when he concedes at t' and t respectively. Therefore, θ_o^B prefers to concede at t'.

(ii) Following the same lines, we assume

$$\int_0^{\hat{t}}\frac{1}{2}e^{-\delta x}d\sigma^A_{r,\tau}(x)+\left(\sigma^A_{r,\tau}(\hat{t})\frac{1}{2}\left(3/2-\gamma\right)+\sigma^A_{r,\tau}((\hat{t},+\infty])(1-\gamma)\right)e^{-\delta\hat{t}}$$

$$=$$

$$\int_0^{t}\frac{1}{2}e^{-\delta x}d\sigma^A_{r,\tau}(x)+\left(\sigma^A_{r,\tau}(t)\frac{1}{2}\left(3/2-\gamma\right)+\sigma^A_{r,\tau}((t,+\infty])(1-\gamma)\right)e^{-\delta t}.$$

Then,

$$(1-\mu_i^B(\tau))\cdot\left[\int_0^{t}\frac{1}{2}e^{-\delta x}d\sigma^A_{r,\tau}(x)+\left(\sigma^A_{r,\tau}(t)\frac{1}{2}\left(3/2-\gamma\right)+\sigma^A_{r,\tau}((t,+\infty])(1-\gamma)\right)e^{-\delta t}\right]+$$

$$\mu_i^B(\tau)\cdot(1-\gamma)e^{-\delta t}$$

$$>$$

$$(1-\mu_i^B(\tau))\cdot\left[\int_0^{\hat{t}}\frac{1}{2}e^{-\delta x}d\sigma^A_{r,\tau}(x)+\left(\sigma^A_{r,\tau}(\hat{t})\frac{1}{2}\left(3/2-\gamma\right)+\sigma^A_{r,\tau}((\hat{t},+\infty])(1-\gamma)\right)e^{-\delta\hat{t}}\right]+$$

$$\mu_i^B(\tau)\cdot(1-\gamma)e^{-\delta\hat{t}}.$$

Therefore θ_i^B strictly prefers to concede at t.

■

Proof of Proposition 2

Proof. First, we show that whenever $F_r^A \notin DE$, then no vector (F_i^B, F_o^B, F_r^A) can be sustained as a degenerate sequential equilibrium. Consider the first condition of the set DE, and suppose otherwise, i.e., $F_r^A(0) > 0$. Then, B (each type) has an incentive to wait an $\varepsilon > 0$ at $t = 0$, and hence $F_i^B(0) = F_o^B(0) = 0$. Since we assume the equilibrium is degenerate, it must be that $F_r^A(0) = 1$. Yet, if θ_r^A wait an $\varepsilon > 0$, then θ_i^B assigns probability 1 to the event $\theta^A = \theta_b^A$ and concedes immediately. Hence, θ_r^A strictly prefers to concede at $t = \varepsilon$ than at $t = 0$, a contradiction. Hence $F_r^A(0) = 0$. Now consider the second condition, that is, F_r^A does not admits a jump at some t. On the contrary, suppose there exists $t > 0$ such that $\Delta(F_r^A(t)) > 0$. Therefore θ_r^A and so A concedes with positive probability at time t. But then, each type of B does not concede in the interval $[t - \varepsilon, t]$ for some $\varepsilon > 0$. Hence, F_i^B and F_o^B are constant over the interval $[t - \varepsilon, t]$. In this case, θ_r^A either prefers to concede at $t - \varepsilon$ or strictly after t. This implies that F_r^A is constant too on the interval $[t - \varepsilon, t + \sigma]$ for some $\sigma > 0$. But this is a contradiction, as $\Delta(F_r^A(t)) > 0$. Thus, in any degenerate equilibrium, F_r^A does not admit jumps.

Consider then DE final condition (2). Suppose it is violated for some $t \geqslant 0$, some $t' > t$ such that $t' = t + \varepsilon$. Observe that in any equilibrium, the set of optimal ε is such that there exists ε' where $t + \varepsilon' = t_{r,max}^A$, where $t_{r,max}^A = \inf\{t | F_r^A(t) = \lim_{\tau \to +\infty} F_r^A(\tau)\}$. In fact, suppose otherwise, i.e., in a sequential equilibrium the type θ_o^B wants to concede at $t + \varepsilon < t_{r,max}^A$ after they reached continuation game t, and they strictly prefer this choice to concession at $t_{r,max}^A$. By Proposition 1, θ_i^B concedes no later than $t + \varepsilon$. Therefore, θ_r^A knows that B concedes before $t + \varepsilon$ with probability 1. Then there exists $\sigma > 0$ such that θ_r^A waits in the interval $[t + \varepsilon - \sigma, t + \varepsilon]$. But if this is the case, θ_o^B either concedes before $t + \varepsilon - \sigma$ or strictly after $t + \varepsilon$, a contradiction. We can conclude that when the players reach continuation game t, θ_o^B's strategy support includes $t_{r,max}^A$. Let t^* be the infimum of the set of t such that θ_o^B can optimally concede at $t_{r,max}^A$ when players reach continuation game t. Suppose first $t^* > 0$.

At t^*, either θ_o^B is indifferent between concession at t^* and concession at $t_{r,max}^A$ or he strictly prefers to concede at $t_{r,max}^A$. Suppose he is indifferent. Then, by Proposition 1 type θ_i^B strictly prefers immediate concession. By continuity, there exists $\sigma > 0$ such that θ_i^B strictly prefers to concede at $\tau \in [t^* + \sigma, t_{r,max}^A)$ than at $t_{r,max}^A$. Again, by continuity and definition of t^*, θ_o^B strictly prefers to concession at $t_{r,max}^A$ than concession at τ in the continuation game starting at $t^* + \sigma$. Hence, at τ, B is playing a separating strategy. Then θ_i^B can profitably deviate imitating θ_o^B at τ, a contradiction.

Now suppose $t^* = 0$. θ_o^B cannot be indifferent between concession at t^* and $t_{r,max}^A$ by the same argument. Yet, if θ_o^B strictly prefers to concede at $t_{r,max}^A$, then $F_o^B(0) = 0$, and then they are not playing a degenerate sequential equilibrium. Hence, in a degenerate equilibrium, condition (2) is satisfied.

Since in any degenerate equilibrium $F_r^A \in DE$, we have that θ_o^B weakly prefers to concede at every t. Therefore, by Proposition 1, θ_i^B strictly concedes in every continuation game, hence $F_i^B = \hat{F}_i^B$. Now we show that when θ_o^B is indifferent, we cannot have an equilibrium in which he does not concede immediately with probability 1. Recall by Corollary 1 that θ_o^B cannot play a mixed strategy in any sequential equilibrium. Therefore, suppose that θ_o^B plays $F_o^B(t) = 0$ for some continuation game $t > 0$. Then, since by the previous argument $F_i^B(t) = \hat{F}_i^B(t) = 1$, B is playing a separating strategy. But then θ_i^B has the incentive to imitate θ_o^B, so that in the event $\theta^A = \theta_r^A$ he gets the best contract. Moreover, in the event $\theta^A = \theta_b^A$ he observes waiting and can then concede. Hence, θ_i^B has a profitable deviation. Therefore, in any degenerate equilibrium $F_o^B = \hat{F}_o^B$. ■

Proof of Proposition 3

Proof. First, we claim that $supp(Y_i^B) = supp(Y_o^B)$. By contradiction, we have two cases:

1. $\exists n_i \in supp(Y_i^B)$ such that $Y_o^B(n_i) = 0$;
2. $\exists n_o \in supp(Y_o^B)$ such that $Y_i^B(n_o) = 0$.

The first case can be easily excluded. Since $Y_o^B(n_i) = 0$, we have

$$\langle (1/2, 1/2) \text{ offered at } (n_i, 0) \rangle \Rightarrow \mu_r^A((n_i, 0); \theta_i^B) = 1,$$

which contradicts Assumption 2. Therefore, consider case 2., and check continuation game $(n_o, -1)$. First, we claim that $\mu_r^A((n_o, -1); \theta_i^B) > 0$, that is, θ_i^B does not concede with probability 1 before $(n_o, -1)$. In order to prove it, note that by Assumption 3, since $Y^B(n_o) > 0$, it must be the case that $\bigcup_{n=n_o}^{+\infty} X_n^A([n, n+1]) > 0$. Therefore, in case θ_i^B concedes with probability 1 no later than some $t < n_o$, we have that $\mu_i^B((t,k); \theta_r^A) > 0$. But then, if θ_i^B deviates by waiting at (t,k), θ_r^A has beliefs $\mu_r^A((t,k); \theta_i^B) = 0$. Now, observe that $Y^B(n) = 0$ for $n > n_o$ since the fair contract is accepted at $(n_o, +1)$ by Assumption 1. Hence, if $(1/2, 1/2)$ is not offered at $(n_o, 0)$, θ_r^A accepts $(1 - \gamma_B, \gamma_B)$ at $(n_o, +1)$ (since no rational types in the following war of attrition believes she is irrational with positive probability). Hence, we must have $1/2 \geqslant \gamma_B$ since θ_o^B offers the fair contract with positive probability. But then, θ_r^A is better off by accepting $(1-\gamma_B, \gamma_B)$ at (t,k), as she is certain, in that continuation game, that she cannot get more than $1 - \gamma_B$. This is a contradiction, and so θ_i^B does not concede before $(n_o, -1)$ with probability 1.

Since $Y_o^B(n_o) > 0$, we can assume that θ_o^B has not offered $(1/2, 1/2)$ yet. Then,

$$\mu_r^A((n_o, -1); \theta_o^B) \in (0, 1).$$

When $(1/2, 1/2)$ is offered at $(n_o, 0)$, we have $\mu_r^A((n_o, 0), \theta_o^B) = 1$. By Assumption 1 θ_r^A immediately concedes. Hence, the action that offers $(1/2, 1/2)$ at $(n_o, 0)$ and concedes to $(\gamma_A, 1 - \gamma_A)$ at $n_o + \varepsilon$ is not a profitable deviation for θ_i^B for any $\varepsilon > 0$, since $Y_i^B(n_o) = 0$. This implies that θ_i^B can obtain at least the same payoff through his strategy. Note that in the event $\theta^A = \theta_b^A$, the former action is strictly dominant for some $\varepsilon > 0$ to any other strategy that does not offer $(1/2, 1/2)$. But then, θ_i^B's action dominates the other in the event $\theta^A = \theta_r^A$. Yet, any strategy s that provides θ_i^B a payoff of $\tilde{u}$ in the event A is rational, can be replicated by θ_o^B. Hence, θ_o^B can obtain $\tilde{u}$ with probability 1. Since $Y_o^B(n_o) > 0$, s cannot provide a strictly higher payoff $\tilde{u}$ to θ_o^B than offering $(1/2, 1/2)$. But this is a contradiction, since $\tilde{u}$ is strictly higher than the utility of the fair contract offer at $(n_o, 0)$ in the event $\theta^A = \theta_r^A$. Therefore, we can conclude $supp(Y_i^B) = supp(Y_o^B)$.

Now, suppose $|supp(Y^B)| = +\infty$. Take $t' \in supp(Y_i^B)$. Then $t' \in supp(Y_o^B)$. There exists a sequence $(t_n)_{n \in \mathbb{N}}$ such that $t_n > t'$ and $t_n \in supp(Y^B)$ for each $n \in \mathbb{N}$, and moreover $t_n \to +\infty$.

Since θ_o^B is indifferent between t' and any t_n, and θ_o^B does not concede in GAS, we have

$$U_o^B(\Sigma^A, t', T_S^0(t')) = U_o^B(\Sigma^A, t_n, T_S^0(t_n)). \tag{6}$$

Now, since θ_i^B is indifferent, and waiting until $T_S^0(\cdot)$ is optimal in any GAS, we have

$$U_i^B(\Sigma^A, t', T_S^0(t')) = U_i^B(\Sigma^A, t_n, T_S^0(t_n)),$$

which implies

$$(1-z)U_o^B(\Sigma^A, t', T_S^0(t')) + ze^{-\delta T_S^B(t')}(1-\gamma_A) = (1-z)U_o^B(\Sigma^A, t_n, T_S^0(t_n)) + ze^{-\delta T_S^B(t_n)}(1-\gamma_A).$$

By equation (6), we get $ze^{-\delta T_S^0(t')}(1-\gamma_A) = ze^{-\delta T_S^0(t_n)}(1-\gamma_A)$, and so $T_S^0(t') = T_S^0(t_n)$. Yet, since $t_n \leqslant T_S^0(t_n)$ for each n, and $t_n \to +\infty$, $T_S^0(t_n) \to +\infty$. Therefore, $\exists n' \in \mathbb{N}$ such that $T_S^0(t') < T_S^0(t_{n'})$. Hence, indifference of θ_o^B implies θ_i^B strictly prefers $(t', 0)$ over $(t_{n'}, 0)$, a contradiction since $n' \in supp(Y_i^B)$ by definition. Therefore, $|supp(Y^B)| < +\infty$. ■

Proof of Lemma 3

Proof. First, suppose $Y_i^B(\tau^*|\tau^*) = 0$. Then $Y_i^B(\tau^*) = 0$. By Proposition 3, $Y_o^B(\tau^*) = 0$, and so $\tau^* \notin supp(Y^B)$, a contradiction. Therefore, $Y_i^B(\tau^*|\tau^*) > 0$.

Now assume $Y_i^B(\tau^*|\tau^*) = 1$. We compare this action with a deviation in which θ_i^B offers the behavioral contract at $(\tau^*, 0)$ and concedes to $\tau^* + \varepsilon$ for some arbitrary $\varepsilon > 0$ in case A does not accept the contract at $(\tau^*, +1)$. In case $Y_o^B(\tau^*|\tau^*) = 1$, then the behavioral offer at $(\tau^*, 0)$ implies beliefs $\mu_r^A((\tau^*, 0); \theta_b^B) = 1$, while in case $Y_o^B(\tau^*|\tau^*) < 1$, beliefs are $\mu_r^A((\tau^*, 0); \theta_o^B, \theta_b^B) = 1$. Since $Y_o^B(n) = 0$ for all $n > \tau^*$, τ_r^A accepts the behavioral contract at $(\tau^*, +1)$ since θ_o^B does not concede in that continuation game. Therefore, θ_i^B can concede at any $\tau^* + \varepsilon$, $\varepsilon > 0$ in case A does not concede at $(\tau^*, +1)$, since in this event A is behavioral. Since ε is arbitrary, we obtain that θ_i^B's payoff from this deviation is

$$\mu_i^B((\tau^*, -1); \theta_b^A)(1-\gamma_A)e^{-\delta\tau^*} + (1 - \mu_i^B((\tau^*, -1); \theta_b^A))\gamma_B e^{-\delta\tau^*}.$$

The payoff of the ignorant type in case of no deviation is

$$\frac{1}{2}F_S^A(\tau^*;\tau^*)e^{-\delta\tau^*} + (1-\gamma_A)(1-F_S^A(\tau^*;\tau^*))e^{-\delta\tau^*}.$$

In equilibrium, we must have

$$\frac{1}{2}F_S^A(\tau^*;\tau^*) + (1-\gamma_A)(1-F_S^A(\tau^*;\tau^*)) \geqslant \mu_i^B((\tau^*,0);\theta_b^A)(1-\gamma_A) + (1-\mu_i^B((\tau^*,0);\theta_b^A))\gamma_B.$$

Clearly, $F_S^A(\tau^*;\tau^*) < 1-\mu_i^B((\tau^*,0);\theta_b^A)$. Therefore, if $\gamma_B > 1/2$, θ_i^B has a profitable deviation. Hence, assume $\gamma_B < 1/2$. Then, at $(\tau^*,-1)$, θ_r^A is certain to receive the fair offer contract on the next period $(\tau^*,0)$ in case B is rational. Otherwise, she receives again the behavioral offer. Since $1-\gamma_B > 1/2$ by assumption and θ_b^B, θ_o^B do not accept her behavioral offer $(\gamma_A, 1-\gamma_A)$ at $(\tau^*,+1)$, θ_r^A is better off by accepting $(1-\gamma_B,\gamma_B)$ at $(\tau^*,-1)$ than waiting $(\tau^*,0)$. But then, $(\tau^*,0)$ cannot be reached with positive probability, a contradiction to Assumption 3. Therefore, $Y_i^B(\tau^*|\tau^*) < 1$.

The proof of $Y_o^B(\tau^*|\tau^*) > 0$ is clear by the same argument that proves $Y_i^B(\tau^*|\tau^*) > 0$. ■

Proof of Theorem 1

Proof. Let τ^* be the last period at which B offers the fair contract. We can analyze the game as such B can offer the fair contract at $\tau' = 0$ only. In fact, the two continuation games are equivalent, except for the discount factor and beliefs. In the second continuation game A and B are splitting a surplus of 1 instead of $e^{-\delta\tau^*}$. Hence, we consider the second continuation game only as it is strategically equivalent to the first one.

We proceed by contradiction, therefore, suppose that one of B's rational type send the signal with positive probability. Then, we can only have an equilibrium where θ_i^B mixes and θ_o^B offers $(1/2, 1/2)$ with positive probability at $\tau^* = 0$, by Lemma 3.

Since $(0,0)$ is the last period for the fair contract offer, the concession distributions following either $(1/2, 1/2)$ or $(1-\gamma_B,\gamma_B)$ can be represented by CDFs. Therefore, call F_S^m the concession cdf for player $m \in \{A,B\}$ in the GAS, and denote with F_N^m the concession cdf for player $m \in \{A,B\}$ in the GNS. From AG Proposition 1, we know these two functions are exponential distributions and hence differentiable. Denote their densities with f_S^m and f_N^m.

First, suppose $T_S^A \leqslant T_S^B$. Then, θ_i^B's payoff in the continuation equilibrium after he offered $(1/2, 1/2)$ is $1-\gamma_A$. In the continuation equilibrium after $(1-\gamma_B, \gamma_B)$ his expected utility is $F_N^A(0)\gamma_B + (1-F_N^A(0))(1-\gamma_A)$. Hence, in order to make θ_i^B indifferent between $(1/2, 1/2)$ and $(1-\gamma_B, \gamma_B)$, we need $F_N^A(0) = 0$, and therefore $T_N^A \leqslant T_N^B$. Now, consider the type θ_o^B. His expected payoff after the offer $(1/2, 1/2)$ is $\int_0^{T_S^A} 1/2 e^{-\delta t}\frac{f_S^A(t)}{1-z}dt$. The offer $(1-\gamma_B, \gamma_B)$ provides, instead, $\int_0^{T_N^A} \gamma_B e^{-\delta t}\frac{f_N^A(t)}{1-z}dt$. Since in the candidate equilibrium θ_o^B offers $(1/2, 1/2)$ with positive probability, we must have

$$\int_0^{T_S^A} 1/2 e^{-\delta t}\frac{f_S^A(t)}{1-z}dt \geqslant \int_0^{T_N^A} \gamma_B e^{-\delta t}\frac{f_N^A(t)}{1-z}dt.$$

$$\Leftrightarrow$$

$$(1-\gamma_A)(1-e^{-(\delta+\lambda_S^A)T_S^A}) \geqslant (1-\gamma_A)(1-e^{-(\delta+\lambda_N^A)T_N^A})$$

and so, we require

$$\frac{\delta+\lambda_N^A}{\delta+\lambda_S^A} \leqslant \frac{T_S^A}{T_N^A} = \frac{\lambda_N^A}{\lambda_S^A}.$$

This inequality is satisfied if and only if $\lambda_S^A \leqslant \lambda_N^A$, but this is true if and only if $\gamma_B \leqslant 1/2$, a contradiction. Hence, we consider the case $T_S^A > T_S^B$.

Type θ_i^B's payoff of offering $(1/2, 1/2)$ is $F_S^A(0)1/2 + (1-F_S^A(0))(1-\gamma_A)$, with $F_S^A(0) > 0$. As stated above, the payoff from $(1-\gamma_B, \gamma_B)$ is $F_N^A(0)\gamma_B + (1-F_N^A(0))(1-\gamma_A)$, so $F_N(0) > 0$ which implies $T_N^B < T_N^A$. Therefore, $T_S^0 = T_S^B$ and $T_N^0 = T_N^B$. Assume first that $T_S^0 < T_N^0$, i.e., $T_S^B < T_N^B$.

Recall that waiting until the end of the continuation game is always optimal for θ_i^B in a nondegenerate equilibrium, independently from the contract offered at $\tau = 0$. Therefore, θ_i^B indifference can be rewritten as

$$(1-z)\left[\frac{F_S^A(0)}{1-z}\frac{1}{2} + \int_0^{T_S^B} \frac{1}{2}e^{-\delta t}\frac{f_S^A(t)}{1-z}dt\right] + ze^{-\delta T_S^B}(1-\gamma^A)$$

$$=$$

$$(1-z)\left[\frac{F_N^A(0)}{1-z}\gamma_B + \int_0^{T_N^B}\gamma_B e^{-\delta t}\frac{f_N^A(t)}{1-z}dt\right] + ze^{-\delta T_N^B}(1-\gamma^A).$$

Since $T_S^B < T_N^B$, we have $ze^{-\delta T_S^B}(1-\gamma_A) > ze^{-\delta T_N^B}(1-\gamma_A)$, therefore

$$\frac{F_S^A(0)}{1-z}\frac{1}{2} + \int_0^{T_S^B}\frac{1}{2}e^{-\delta t}\frac{f_S^A(t)}{1-z}dt < \frac{F_N^A(0)}{1-z}\gamma_B + \int_0^{T_N^B}\gamma_B e^{-\delta t}\frac{f_N^A(t)}{1-z}dt,$$

and so θ_o^B does not offer $(1/2, 1/2)$ in equilibrium, a contradiction.

Next, assume $T_S^B \geqslant T_N^B$. Type θ_i^B's payoff indifference in equilibrium is

$$F_S^A(0)\frac{1}{2} + (1-F_S^A(0))(1-\gamma_A) = F_N^A(0)\gamma_B + (1-F_N^A(0))(1-\gamma_A).$$

Since $\gamma_B > 1/2$, we need $F_S^A(0) > F_N^A(0)$, hence $c_S^A < c_N^A$, which implies

$$\begin{aligned} e^{-\lambda_S^A(T_S^A - T_S^B)} < e^{-\lambda_N^A(T_N^A - T_N^B)} &\Rightarrow \lambda_S^A(T_S^A - T_S^B) > \lambda_N^A(T_N^A - T_N^B) \\ &\Rightarrow -\log(z) - \lambda_S^A T_S^B > -\log(z) - \lambda_N^A T_N^B \\ &\Rightarrow \lambda_S^A T_S^B < \lambda_N^A T_N^B. \end{aligned}$$

Since $\gamma_B > 1/2$, $\lambda_S^A > \lambda_N^A$, and so $T_S^B < T_N^B$, contradiction.

Therefore, in any equilibrium, for each vector of parameters $\mathbf{q}$, we have $Y^B(n) = 0$ for each n. ∎

Proof of Lemma 4

Proof. Assume first that A concedes with a positive probability at $(\tau_1^*, +1)$ following the offer $(1/2, 1/2)$. It's important to note that since τ_1^* marks the final opportunity for the fair contract to be offered, A concedes with a non-zero probability even if B doesn't propose the fair deal (otherwise θ_i^B would strictly prefer to offer the fair contract over the behavioral). Consequently, there exists a time $t < \tau_1^*$ at which the rational player B strictly prefers proposing $(1/2, 1/2)$ over conceding at any time within the interval $(t, \tau_1^*]$. This observation implies that A refrains from conceding during this interval as well[11]. Let t' be the last time before

[11] See Abreu and Gul (2000), Proposition 1

B prefers to wait, i.e., $t' := \sup\{\bigcup_{n=\tau_0^*}^{\tau_1^*-1} supp(X_n^B)\}$. Given that A's strategy includes conceding to both $(1/2, 1/2)$ and $(1-\gamma_B, \gamma_B)$, her expected payoff at the continuation game t' is $[1/2Y^B(\tau_1^*|\tau_1^*) + (1-\gamma_B)(1-Y^B(\tau_1^*|\tau_1^*))]e^{-\delta\tau_1^*}$. However, conceding at t' yields $(1-\gamma_B)e^{-\delta t'}$, which is evidently strictly higher, leading to a contradiction.

Now, let's assume that A doesn't concede with a positive probability at τ_1^* for $(1/2, 1/2)$. Consequently, A also refrains from conceding for $(1-\gamma_B, \gamma_B)$. Suppose there exists $t' < \tau_1^*$ such that A and B do not concede in $(t', \tau_1^*]$. Since A does not concede at $(\tau_1^*, +1)$, θ_i^B's payoff in the continuation game t' is $(1-\gamma_A)e^{-\delta\tau_1^*}$. Yet, concession at t' provides $(1-\gamma_A)e^{-\delta t'}$, a contradiction. Hence, there exists $t \in (\tau_0^*, \tau_1^*)$ such that A and B concede with positive density in $[t, \tau_1^*]$. Observe that if there exists an interval $(t_1, t_2) \subseteq [\tau_0^*, t)$ with no concessions, then θ_r^A and θ_i^B strictly prefer concession at t_1 over any $\tau > t_2$, again leading to a contradiction. Therefore, A and B concede with everywhere positive probability in $[\tau_0^*, \tau_1^*]$. In fact, consider the following. Define the function $X_A^* : [\tau_0^*, \tau_1^*] \to [0, 1]$ where:

$$X_A^0 = \sum_{n=0}^{\tau_0^*-1} X_n^A([n, n+1]),$$

for $m \in [\tau_0^*, \tau_1^*]$, $m \in \mathbb{N}$,

$$X_A^*(t) = X_A^0 + \sum_{n=\tau_0^*}^{m-1} X_n^A([n, n+1]) + X_{m+1}^A(\{m\}),$$

and for $t \in (m, m+1)$,

$$X_A^*(t) = X_A^0 + \sum_{n=\tau_0^*}^{m-1} X_n^A([n, n+1]) + X_{m+1}^A([m, t])$$

Observe that X_A^* is weakly increasing in $[\tau_0^*, \tau_1^*]$. This function represents the cumulative distribution of concession of player A in the interval $[\tau_0^*, \tau_1^*]$, with no distinction between $(t, +1)$ and $(t, -1)$ for $t \in \mathbb{N}$. In fact, since $Y^B(t) = 0$, we can treat the two subdates as the same date. We define X_B^* in the same manner. Type θ_i^B's utility of concession at $t \in [\tau_0^*, \tau_1^*]$

can be written as

$$U_i^B((X^A,(\sigma^A[n])_{n\in\mathbb{N}_0}),t) = \sum_{n=0}^{\tau_0^*-1}\int_n^{n+1}\gamma_B e^{-\delta z}dX_n^A(z) + \int_{\tau_0^*}^{t}\gamma_B e^{-\delta z}dX_A^*(z)$$

$$+ (1-X_A^*(t))(1-\gamma_A)e^{-\delta t}.$$

We now describe the properties of X_A^* and X_B^*. We follow AG Proposition 1, in particular the points $(b)-(f)$.

(i) *If X_A^* jumps at $t\in(\tau_0^*,\tau_1^*]$, X_B^* does not jump at t.*
In fact, B can just wait an instant after t. If $t=\tau_1^*$, B can wait until $(\tau_1^*,+1)$.

(ii) *If X_A^* is continuous at $t\in(\tau_0^*,\tau_1^*)$, then U_i^B is continuous at t. If X_B^* is continuous at $t\in(\tau_0^*,\tau_1^*)$ then U_r^A is continuous at t.*
These properties stem directly from definitions.

(iii) *There is no interval (t',t'') with $\tau_0\leqslant t'<t''\leqslant\tau_1^*$ such that X_A^* and X_B^* are both constant in (t',t'').*
First we claim that A does not concede at $(\tau_1^*,-1)$ with positive probability, i.e., $\lim_{\tau\to\tau_1^*}X_A^*(\tau)=X_A^*(\tau_1^*)$. If A concedes at $(\tau_1^*,-1)$ with positive probability, $\exists\varepsilon>0$ such that B does not concede in $[\tau_1^*-\varepsilon,\tau_1^*]$ and prefers instead concession at $(\tau_1^*,+1)$. If this is the case, A's concession at $\tau_1^*-\varepsilon$ provides $(1-\gamma_B)e^{-\delta(\tau_1^*-\varepsilon)}$ (in the continuation game $\tau_1^*-\varepsilon$), while concession at $(\tau_1^*,-1)$ gives A $(1-\gamma_B)e^{-\delta\tau_1^*}$ (in the same continuation game) since B does not concede in $[\tau_1^*-\varepsilon,\tau_1^*]$, a contradiction.
Now assume there is a time interval (t',t''), as described in the statement. Let t^* be the supremum of t'' for which $\exists t\in[\tau_0^*,\tau_1^*)$ such that (t',t^*) has the property stated above. We first argue that $t^*<\tau_1^*$. Assume otherwise, i.e., $t^*=\tau_1^*$. Then by assumption A does not concede in $(t',\tau_1^*]$ (since A does not concede at $(\tau_1^*,-1)$ too). Moreover, A does not concede with positive probability to the fair contract at $(\tau_1^*,+1)$. Therefore, at continuation game $t\in(t',\tau_1^*)$, θ_i^B's utility of immediate concession is $(1-\gamma_A)e^{-\delta t}$, while the offer of the fair contract at $(\tau_1^*,0)$ provides him with $(1-\gamma_A)e^{-\delta\tau_1^*}$ (in continuation game t). Hence concession at t is a profitable deviation, a contradiction. Therefore $t^*<\tau_1^*$.
The remaining part of the proof follows AG Proposition 1 closely, and we include it for

completeness.

Fix $t \in (t', t^*)$. Observe that for both players (in particular for types θ_i^B and θ_r^A) $\exists \varepsilon_t > 0$ such that concession at t is strictly better than any concession in $(t^* - \varepsilon_t, t^*)$. Furthermore, by (i) and (ii) there exists one type between θ_i^B and θ_r^A for which their utility is continuous at t^*. Hence, since $t^* < \tau_1^*$, $\exists \eta > 0$ such that concession at t^* is still strictly better than concession in $(t^*, t^* + \eta)$ for this type. But then X_K^* is constant in $(t^*, t^* + \eta)$, where K is the player whose type has continuous utility at t^*. Yet, if X_K^* is constant in $(t^*, t^* + \eta)$, by optimality X_j^* is constant in $(t^*, t^* + \eta)$ too for $j \neq k$, a contradiction to the definition of t^*.

(iv) *For $t' < t'' < \tau_1^*$, $X_K^*(t') < X_K^*(t'')$, $K \in \{A, B\}$.*

If X_K^* is constant in (t', t''), by optimality X_J^* is constant in (t', t''), but this contradicts (iii).

(v) *X_K^* is continuous in (τ_0^*, τ_1^*).*

A jump at $t \in (\tau_0^*, \tau_1^*)$ implies the opponent waits in some $(t - \varepsilon, t)$, a contradiction to (iv).

Since X_A^* and X_B^* are strictly increasing, A and B randomize over the entire interval (τ_0^*, τ_1^*). Clearly, θ_o^B strictly prefers to wait and hence does not concede in (τ_0^*, τ_1^*). Therefore, U_r^A and U_i^B are constant through (τ_0^*, τ_1^*) and so these utilities are differentiable. From θ_i^B's utility, we get

$$\gamma_B e^{-\delta t} x_A^*(t) - \delta(1 - X_A^*(t))(1 - \gamma_A)e^{-\delta t} - x_A^*(t)(1 - \gamma_A)e^{-\delta t} = 0,$$

where x_A^* is the derivative of X_A^*. Hence, for all $t \in (\tau_0^*, \tau_1^*)$ we get

$$\frac{x_A^*(t)}{1 - X_A^*(t)} = \frac{(1 - \gamma_A)\delta}{\gamma_A + \gamma_B - 1} = \lambda_N^A.$$

Note that $x_A^*(t)/(1 - X_A^*(t))$ represents A's rate of concession at t. Then, we get that A concedes with a constant rate of λ_N^A in the interval $[\tau_0^*, \tau_1^*]$. Through the same calculations, we conclude that B must concede with a constant rate of λ_N^B in the same interval. Note that in order to find the concession rates in $[\tau_0^*, \tau_1^*]$ we never used the fact that a fair contract could be offered at $(\tau_0^*, 0)$ with positive probability. Therefore, in case that is not true, we can have $\tau_0^* = 0$. ■

Proof of Lemma 5

Proof. Suppose first that A concedes with positive probability x_0 to $(1/2, 1/2)$ at τ_0^*. Then, there exists $t < \tau_0^*$ such that B does not concede in the interval (t, τ_0^*). This implies that neither A concedes in the same interval. Now, if B does not concede in the continuation game τ_0^* after offering again $(1-\gamma_B, \gamma_B)$, then A concedes to $(1-\gamma_B, \gamma_B)$ no later than t. In fact, concession at t provides a payoff of $(1-\gamma_B)e^{-\delta t}$, while any other action that moves at τ_0^* or after gives $[1/2Y^B(\tau_0^*|\tau_0^*) + (1-\gamma_B)(1-Y^B(\tau_0^*|\tau_0^*))]e^{-\delta\tau_0^*}$, which is strictly lower than the previous one. Therefore, when $x_0 > 0$, it must be that B concedes with positive probability at τ_0^* after offering again $(1-\gamma_B, \gamma_B)$. Now, since θ_i^B and θ_o^B randomize the offer of the fair contract between τ_0^* and τ_1^*, we have that it is optimal for θ_i^B to offer $(1-\gamma_B, \gamma_B)$ at τ_0^*. Moreover, he is conceding with positive probability to $(\gamma_A, 1-\gamma_A)$ in that event. Hence, θ_i^B's indifference condition implies

$$(1-\gamma_A)e^{-\delta\tau_0^*} = [1/2x_0 + (1-x_0)(1-\gamma_A)]e^{-\delta\tau_0^*}.$$

Yet, this equation cannot be true for $x_0 > 0$. Therefore, we have to assume $x_0 = 0$. Suppose again that A is conceding to $(1-\gamma_B, \gamma_B)$ at τ_0^* with positive probability. Then, θ_i^B's payoff from offering $(1-\gamma_B, \gamma_B)$ is strictly greater than the payoff from the offer of the fair contract, and so θ_i^B does not offer $(1/2, 1/2)$ at τ_0^*, a contradiction.

Therefore, A cannot concede with positive probability to either contract at τ_0^*. ■

Proof of Theorem 2

Proof. By contradiction, assume $|supp(Y^B)| \geqslant 2$. From Proposition 3 we know $|supp(Y^B)| < +\infty$. Therefore, $\exists \tau_0^*, \tau_1^* \in \mathbb{N}_0$ such that $\tau_0^* < \tau_1^*$ and

$$\sum_{n=\tau_0^*+1}^{\tau_1^*-1} Y^B(n) + \sum_{n=\tau_1^*+1}^{\infty} Y^B(n) = 0,$$

so that τ_0^* and τ_1^* are as described in Lemma 4 and Lemma 5. From Proposition 3 we know that $supp(Y_i^B) = supp(Y_o^B)$, so that both θ_o^B and θ_i^B randomize between τ_0^* and τ_1^* (and possibly, between not offering the fair contract at all). Now, since θ_o^B randomizes, we need

$$U_o^B(\Sigma^A, \tau_0^*, T_S^0(\tau_0^*)) = U_o^B(\Sigma^A, \tau_1^*, T_S^0(\tau_1^*))$$

in equilibrium. Recall that θ_o^B waits until the end of GAS $T_S^0(\cdot)$ in equilibrium. Since $T_S^0(\cdot)$ is in θ_i^B's support as well, and θ_i^B randomizes between τ_0^* and τ_1^*, we need

$$U_i^B(\Sigma^A, \tau_0^*, T_S^0(\tau_0^*)) = U_i^B(\Sigma^A, \tau_1^*, T_S^0(\tau_1^*)).$$

Therefore, we have

$$(1-z)U_o^B(\Sigma^A, \tau_0^*, T_S^0(\tau_0^*)) + z(1-\gamma_A)e^{-\delta T_S^0(\tau_0^*)} = (1-z)U_o^B(\Sigma^A, \tau_1^*, T_S^0(\tau_1^*)) + z(1-\gamma_A)e^{-\delta T_S^0(\tau_1^*)}.$$

By θ_o^B's indifference, we get

$$T_S^0(\tau_0^*) = T_S^0(\tau_1^*).$$

From Lemma 4 and Lemma 5 we know that A does not concede with positive probability at $(\tau_0^*, +1)$ and $(\tau_1^*, +1)$ to the fair contract. Therefore, $T_S^0(\tau_0^*)$ and $T_S^0(\tau_1^*)$ depend on λ_S^A, $\mu_i^B((\tau_0^*, 0); \theta_b^A)$ and $\mu_i^B((\tau_1^*, 0); \theta_b^A)$. Since the GAS which starts at $n \in \mathbb{N}$ has the same concession rates of a game started at 0, with the difference that the share of the pie shrinks (with no impact on the concession probability), we have that

$$T_S^0(\tau_0^*) = -\frac{\log(\mu_i^B((\tau_0^*, 0); \theta_b^A))}{\lambda_S^A} + \tau_0^*$$

and

$$T_S^0(\tau_1^*) = -\frac{\log(\mu_i^B((\tau_1^*, 0); \theta_b^A))}{\lambda_S^A} + \tau_1^*.$$

Hence, the GAS at τ_0^* is equivalent to a game started at 0 shifted by τ_0^* periods. The same holds for τ_1^*. We have

$$-\frac{\log(\mu_i^B((\tau_0^*, 0); \theta_b^A))}{\lambda_S^A} + \tau_0^* = -\frac{\log(\mu_i^B((\tau_1^*, 0); \theta_b^A))}{\lambda_S^A} + \tau_1^*,$$

which implies

$$\mu_i^B((\tau_0^*, 0); \theta_b^A)e^{-\tau_0^* \lambda_S^A} = \mu_i^B((\tau_1^*, 0); \theta_b^A)e^{-\tau_1^* \lambda_S^A},$$

and so

$$\mu_i^B((\tau_1^*,0);\theta_b^A) = \mu_i^B((\tau_0^*,0);\theta_b^A)e^{\lambda_S^A(\tau_1^*-\tau_0^*)}.$$

Yet, by Lemma 4, we know that A concedes at rate λ_N^A in the time interval $[\tau_0^*,\tau_1^*]$. Therefore, A's reputation grows at rate λ_N^A. Hence, in equilibrium, we get

$$\mu_i^B((\tau_1^*,0);\theta_b^A) = \mu_i^B((\tau_0^*,0);\theta_b^A)e^{\lambda_N^A(\tau_1^*-\tau_0^*)}.$$

Since $\gamma_B < 1/2$, $\lambda_N^A \neq \lambda_S^A$, and so A's reputation grows at a rate that, given θ_o^B's indifference, does not make θ_i^B indifferent between τ_0^* and τ_1^*. This is a contradiction since $\tau_0^*,\tau_1^* \in supp(Y_i^B)$. ■

Proof of Proposition 4

Proof. First, note

$$Y^B(\tau^*) = q_o + q_i Y_i^B(\tau^*),$$

as $Y_o^B(\tau^*) = 1$. Therefore,

$$Y^B(\tau^*|\tau^*) = \frac{Y^B(\tau^*)}{1-X} = \frac{q_o}{1-X} + \frac{q_i Y_i^B(\tau^*)}{1-X},$$

where $X = \sum_{n=0}^{\tau^*-1} X_n^B([n,n+1])$. Observe that $q_o/(1-X) = \mu_r^A((\tau^*,-1);\theta_o^B)$. Therefore, we are left to prove that $\mu_r^A((\tau^*,-1);\theta_i^B)Y_i^B(\tau^*|\tau^*) = q_i Y_i^B(\tau^*)/(1-X)$. Note that

$$\mu_r^A((\tau^*,-1);\theta_i^B) = 1 - (\mu_r^A((\tau^*,-1);\theta_b^B) + \mu_r^A((\tau^*,-1);\theta_o^B)) = 1 - \frac{q_b+q_o}{1-X}.$$

Hence,

$$\begin{aligned}
\mu_r^A((\tau^*,-1);\theta_i^B)Y_i^B(\tau^*|\tau^*) &= \mu_r^A((\tau^*,-1);\theta_i^B)\frac{Y_i^B(\tau^*)}{1-\frac{1}{q_i}X} \\
&= q_i\mu_r^A((\tau^*,-1);\theta_i^B)\frac{Y_i^B(\tau^*)}{q_i - X} \\
&= q_i\left(1-\frac{q_b+q_o}{1-X}\right)\frac{Y_i^B(\tau^*)}{1-(q_b+q_o)-X}
\end{aligned}$$

$$
= q_i \left(\frac{1 - X - (q_b + q_o)}{1 - X} \right) \frac{Y_i^B(\tau^*)}{1 - (q_b + q_o) - X}
$$

$$
= q_i \frac{Y_i^B(\tau^*)}{1 - X}.
$$

■

References

Abreu, D., & Gul, F. (2000). Bargaining and reputation. *Econometrica*, *68*, 85-117.

Abreu, D., & Pearce, D. (2007). Bargaining, reputation, and equilibrium selection in repeated games with contracts. *Econometrica*, *75*, 653-710.

Abreu, D., Pearce, D., & Stacchetti, E. (2015). One sided uncertainty and delay in reputational bargaining. *Theoretical Economics*, *10*, 719-773.

Ekmekci, M., & Zhang, H. (2024). Reputational bargaining with external resolution opportunities. *Review of Economic Studies*, 1-30.

Fanning, J. (2016). Reputational bargaining and deadlines. *Econometrica*, *84*, 1131–1179.

Friedenberg, A. (2019). Bargaining under strategic uncertainty: the role of second-order optimism. *Econometrica*, *87*, 1835-1865.

Hörner, J., & Sahuguet, N. (2011). A war of attrition with endogenous effort levels. *Economic Theory*, *47*, 1-27.

Kreps, D. M., & Wilson, R. (1982). Reputation and imperfect information. *Journal of Economic Theory*, *27*, 280-312.

Laraki, R., Solan, E., & Vieille, N. (2005, February). Continuous-time games of timing. *Journal of Economic Theory*, *120*(2), 206–238. Retrieved 2023-07-14, from https://www.sciencedirect.com/science/article/pii/S0022053104000493 doi: 10.1016/j.jet.2004.02.001

Maschler, M., Solan, E., & Zamir, S. (2013). *Game Theory*. Cambridge University Press.

Milgrom, P., & Roberts, J. (1982). Predation, reputation, and entry deterrence. *Journal of Economic Theory*, *27*, 280-312.

von Leeuwen, B., Offerman, T., & van de Ven, J. (2020). Fight or flight: endogenous timing in conflicts. *The Review of Economics and Statistics*, *104*, 217-231.

Wolitzky, A. (2012). Reputational bargaining with minimal knowledge of rationality. *Econometrica*, *80*, 2047-2087.

Zhao, Z. (2023). Bargaining with heterogeneous beliefs. *https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4318267*.