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Oligopoly, Complementarities, and Transformed Potentials

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Abstract

We develop a potential games approach to multiproduct-firm pricing games. We introduce the concept of transformed potential and characterize classes of demand systems that give rise to pricing games admitting such a potential. The resulting demand systems may contain nests (of closer substitutes) or baskets (of products that are purchased jointly), or combinations thereof. These demand systems allow for flexible substitution patterns, and can feature product complementarities arising from joint purchases and substitution away from the outside option. Combining the potential games approach with a competition-in-utility approach, we derive powerful results on existence and uniqueness of a pure-strategy Nash equilibrium.

Keywords: Multiproduct firms, potential game, oligopoly pricing, complementary goods, joint purchases, nests, equilibrium existence, equilibrium uniqueness

Journal of Economics Literature Classification: L13, D43

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1 Introduction

We use the theory of potential games (Monderer and Shapley, 1996b) to study multiproduct-firm pricing games in which products can be substitutes or complements. The key feature of a potential game is that deviation incentives can be summarized by a *potential function* common to all players. We characterize the set of demand systems such that the associated multiproduct-firm pricing game admits a *transformed potential*, a novel concept based on applying monotone transformations to payoff functions. These demand systems can be derived from a multi-stage discrete/continuous choice process. They include not only all demand systems satisfying the IIA (Independence of Irrelevant Alternatives) property but also demand systems with much richer, price-dependent patterns of substitutability and complementarity. Complementarities may arise from: (i) substitution away from/towards the outside option; (ii) substitution away from/towards different “nests”; and (iii) joint purchases of products in the same “basket”. We show that, under minimal assumptions, the potential function associated with these pricing games has a global maximizer, thereby proving the existence of a pure-strategy Nash equilibrium. Under stronger assumptions, we leverage the strict concavity of the potential function to establish the uniqueness of a pure-strategy Nash equilibrium, as well as the uniqueness of a correlated equilibrium.

Multiproduct firms selling horizontally differentiated goods are ubiquitous and many markets are dominated by a small number of firms wielding market power. This is reflected in the empirical industrial organization literature, where multiproduct-firm oligopoly features prominently (e.g., Berry, Levinsohn, and Pakes, 1995; Nevo, 2001; Miller and Weinberg, 2017). The products sold by these firms may be substitutes or complements, often depending on the level of prices (see, e.g., Rey and Tirole, 2019, for examples of such price-dependent patterns of substitutability and complementarity). Complementarities arise naturally when products are purchased and consumed jointly—think of chips and soda (Ershov, Laliberté, Marcoux, and Orr, 2025). Moreover, products that are otherwise substitutes or independent in demand can often become complements if they are sold in the same market place and some consumers have a preference for one-stop shopping (Thomassen, Smith, Seiler, and Schiraldi, 2017). Whether products are substitutes or complements is of critical importance in merger control, as it determines whether the internalization of competitive externalities post merger leads to higher or lower prices (see, e.g., Asker and Nocke, 2021). Despite this, the literature has not yet provided a flexible framework to study multiproduct-firm oligopoly with price-dependent patterns of substitutability and complementarity.

In this paper, we adopt a potential games approach to study multiproduct-firm pricing in oligopoly. A normal-form game is said to admit an *exact potential* if there exists a function, called the potential function, such that whenever a player changes her action, the variation in her payoff is equal to the variation in the potential function (Monderer and Shapley, 1996b). Under the weaker concept of an *ordinal potential*, all that is required is that the variation in the deviating player's payoff has the same sign as the variation in the ordinal potential function. In such games, equilibrium existence can be established without solving a multidimensional fixed point problem: an action profile that globally maximizes the potential is a pure-strategy Nash equilibrium.

We begin by studying multiproduct-firm pricing games with constant marginal costs in which the underlying demand system can take any form, provided it satisfies the IIA property. The IIA class includes the multinomial logit and CES demand systems as special cases. Unlike in those, however, products may be substitutes or complements, depending on the level of prices. We provide a multi-stage discrete/continuous choice micro-foundation in which the source of potential complementarities stems from consumers being drawn away from the outside option as prices decrease. We show that any multiproduct-firm pricing game with IIA demand admits an ordinal potential.[1] Based on this insight, we prove existence of a pure-strategy Nash equilibrium by showing that the potential function has a global maximizer.

Next, we seek broader classes of multiproduct-firm pricing games that admit a potential. To this end, we introduce the novel concept of a *transformed potential*: a normal-form game is said to admit a transformed potential if there exists a strictly monotone transformation function such that the game that results from applying this transformation to all players' payoffs admits an exact potential.[2,3,4] We completely characterize the set of demand systems admitting a transformed potential in that the associated pricing game has a transformed potential, regardless of the ownership structure of products (i.e., which product is offered by which firm) and the vector of marginal costs. Solving (systems of) ordinary and partial differential equations, we show that the only classes of demand systems having this property are of the "generalized linear" or IIA forms.[5] In the latter case, the corresponding

[1]Thus, such games are both aggregative games (as shown in Nocke and Schutz, 2018a) and ordinal potential games. Connections between (variants of) aggregative games and (variants of) potential games have been explored in earlier work by Dubey, Haimanko, and Zapechelnyuk (2006) and Jensen (2010).

[2]The advantage of studying transformed potentials rather than ordinal potentials is that, for a given transformation function, Monderer and Shapley (1996b)'s if-and-only-if cross-partial derivatives test for exact potentials can be applied. By contrast, no such test is known for the weaker concept of ordinal potential (see Ewerhart, 2020).

[3]In a related spirit, Zenou and Zhou (2024) transform the system of variational inequalities that characterize Nash equilibria in a class of network games. This allows them to obtain an integrable system and thereby to construct a best-reply potential.

[4]Classic examples of games admitting an ordinal potential that is not exact—such as the homogeneous-goods Cournot model with symmetric firms (Kukushkin, 1994; Monderer and Shapley, 1996b) and thus the lottery contest with symmetric players—admit a transformed potential, with the logarithm as the transformation function.

[5]The "generalized linear" demand system is linear in the prices of other products but potentially nonlinear

transformation function is of the log type, whereas it is of the linear type in the former case.

To explore whether a potential games approach can be used to study oligopoly models with richer patterns of substitutability and complementarity, we then relax the requirement that the pricing game admits a transformed potential *regardless of the ownership structure*. For any *given* ownership structure, we continue to find that the only admissible transformations are of the linear and log types. The classes of demand systems we identify, however, are much broader. The demand systems corresponding to linear transformation functions continue to be of the generalized linear type, albeit in a slightly richer form, which we completely characterize. For the case of two firms, we provide a complete solution to the system of partial differential equations that characterizes the class of demand systems corresponding to log transformation functions. For the case of three or more firms, we provide rich classes of demand systems that admit a log-potential. All of these demand systems permit arbitrary patterns of substitutability or complementarity within a firm's product portfolio.

Next, we study two important classes of these demand systems in more detail. The first class involves partitioning the set of firms into a set of nests. For example, nests may correspond to shopping malls and firms to multiproduct stores within these malls. By virtue of the nest structure, the IIA property need no longer hold, so that products within the same nest may be closer substitutes to each other than to products in other nests. The nest structure also introduces an additional source of potential complementarity: as the price of a product decreases, the demand for other products within the same nest may increase because consumers are drawn away from other nests. In the context of competing shopping malls, this is akin to complementarities arising from one-stop shopping (Stahl, 1982; Bliss, 1988; Chen and Rey, 2012, 2019).

In the second class, consumers make discrete choices among baskets of products offered by one or more firms and purchase from all of the firms within the chosen basket. For example, a basket may consist of one or more product categories, such as beer and chips, with each firm specializing in one category. This class permits an additional source of potential complementarity, stemming from joint purchases: as the price of one product decreases, the attractiveness of all the baskets containing that product increases, which may boost the demand of those products that are purchased predominantly in these baskets. We show that the strength of this effect is measured by a sufficient statistic, which is called the *lift* in the literature on statistical learning, and in particular on market basket analysis (e.g., Hastie, Tibshirani, and Friedman, 2009).[6]

All of the demand systems that admit a log-potential for a given ownership structure

in own price.

[6] Market basket analysis is a data mining technique used by large retailers to understand consumer purchasing patterns and identify which items are frequently bought together.

have the property that the associated indirect utility function is weakly separable in the firm partition (see Goldman and Uzawa, 1964, for the concept of weak separability). Hence, firms can be thought of as competing in utility space (Armstrong and Vickers, 2001): firms' behavior and impact on other firms can be summarized by a uni-dimensional sufficient statistic, which corresponds to the mean utility delivered to consumers. For the cases of demand systems with a nest or basket structure, we show that the ordinal potential of the associated game of competition in utility has a global maximizer under minimal assumptions. It follows that any such game has a pure-strategy Nash equilibrium.

The potential games approach is useful not only to establish equilibrium existence, but also for equilibrium uniqueness. We derive conditions under which the exact potential of a strategically equivalent version of the game of competition in utility space is strictly concave. This gives rise to powerful equilibrium uniqueness theorems for wide classes of demand systems with a basket or a nest structure (with the IIA class being a special case). Specifically, we prove the uniqueness of a pure-strategy Nash equilibrium, no matter what the transformation of payoffs, and the uniqueness of a correlated equilibrium for the pricing game with logged payoffs. The concept of correlated equilibrium is very appealing for applied work, if the modeler is imperfectly informed about firms' information structure.

We also provide several applications and extensions. We first discuss how demand structures with baskets and nests can be combined. An example of such a demand structure may feature nests that correspond to shopping malls and baskets that correspond to combinations of stores within the same mall. Next, we show how the potential games approach can be extended to study multiproduct-firm pricing games where firms have private information about the characteristics of their product portfolio (including marginal costs, qualities, and sets of products). To the best of our knowledge, this is the first treatment of multiproduct-firm oligopoly under incomplete information. Finally, we perform comparative statics, studying the effects of firm-level profit and utility shifters, focusing on the case of basket demand.

Related literature. Our paper is motivated by, and contributes to, the literature on multiproduct-firm pricing games with horizontally differentiated products.[7] As a multiproduct firm's profit function typically fails to be quasi-concave in own price (see Spady, 1984; Hanson and Martin, 1996), Caplin and Nalebuff (1991)'s existence result for single-product-firm pricing games does not extend.[8] As a result, equilibrium existence (and uniqueness) had, until recently, been shown only in special cases of demand systems satisfying some variants of the IIA property: multinomial logit demand (Spady, 1984; Konovalov and Sándor,

[7]The focus on horizontally differentiated products is shared by the empirical industrial organization literature (e.g., Berry, Levinsohn, and Pakes, 1995; Nevo, 2001; Miller and Weinberg, 2017).

[8]Similarly, as payoff functions are typically not (log-)supermodular in own price (see, e.g., Whinston, 2007, footnote 8), standard existence results based on the Tarski theorem do not apply.

2010), CES demand (Konovalov and Sándor, 2010), and nested multinomial logit demand where each firm owns a nest of products (Gallego and Wang, 2014). In recent work, Nocke and Schutz (2018a, 2023, 2025) adopt an aggregative games approach to unify and extend those results to the larger class of demand systems that can be derived from (multi-stage) discrete/continuous choice. Garrido (2024) uses a multidimensional version of this approach to establish equilibrium existence (but not uniqueness) under nested CES or nested logit demand without any restriction on the relationship between ownership and nest structures. All of these papers confine attention to substitutes. By contrast, adopting a potential games approach, the present paper allows for price-dependent patterns of substitutability and complementarity.[9] Additionally, in our micro-foundation of basket demand systems, we go beyond standard discrete/continuous choice by allowing for joint purchases of products.

Our paper also contributes to the literature on potential games, pioneered by Slade (1994) and Monderer and Shapley (1996b). Potential games have been shown to have desirable properties. For example, the Nash equilibrium that maximizes the potential function satisfies the finite improvement property (Monderer and Shapley, 1996b), the fictitious play property (Monderer and Shapley, 1996a), local asymptotic stability (Slade, 1994), and is robust to incomplete information (Ui, 2001). We contribute to this literature by introducing the concept of transformed potential and applying it to oligopoly settings.[10] Closer to our work, Slade (1994) proposes a class of inverse demand systems for differentiated products such that the induced single-product firm quantity-setting game admits an exact potential. We add to this by completely characterizing the set of demand systems such that the induced multiproduct-firm pricing game admits a transformed potential.[11]

Building on the seminal paper of Gentzkow (2007), there is a growing literature in empirical industrial organization focusing on the estimation of consumer demand in the presence of complementarities. Recent contributions include Thomassen, Smith, Seiler, and Schiraldi (2017), Iaria and Wang (2020), Sovinsky, Jacobi, Allocca, and Sun (2024), Wang (2024), and Ershov, Laliberté, Marcoux, and Orr (2025). Unlike the existing literature on multiproduct-firm oligopoly, our approach can accommodate such complementarities.

[9] Quint (2014) and Zhang (2024) study oligopoly settings in which final products are combinations of components supplied by monopolists, with each component being used in the production of one and only one final product. In such settings, there is perfect complementarity between the components used for a given final product and substitutability between the components used for different final products.

[10] Our work is also related to the literature that asks "what does an oligopoly maximize?" (Spence, 1976; Bergstrom and Varian, 1985); see Armstrong and Vickers (2018) for a recent application of these ideas.

[11] In unpublished work, Quint (2006) notes that the single-product-firm pricing game with logged payoffs, multinomial logit demand, and costless production admits an exact potential. We show that this property holds for a considerably larger class of demand systems with multiproduct firms and costly production.

Road map. The remainder of the paper is organized as follows. In Section 2, we study multiproduct-firm oligopoly with IIA demand. In Section 3, we (completely) characterize classes of demand systems admitting a transformed potential. In Section 4, we provide micro-foundations and equilibrium existence results for demand systems with a nest or basket structure. Next, in Section 5, we present equilibrium uniqueness results. We consider various applications and extensions in Section 6. Finally, we conclude in Section 7.

2 Multiproduct-Firm Pricing with IIA Demand

In this section, we illustrate the power of the potential games approach by studying multiproduct-firm pricing games where demand satisfies the IIA property.

The Model. Consider an industry with a finite set of differentiated products $\mathcal{I}$. The representative consumer's quasi-linear indirect utility is given by:

$$y + V(p) = y + \Psi\left(\sum_{j \in \mathcal{I}} h_j(p_j)\right),$$

where y denotes income, p_j the price of product j, and Ψ and h_j are differentiable functions of a single variable. Roy's identity yields the demand for product i:

$$D_i(p) = -h_i'(p_i)\Psi'\left(\sum_{j \in \mathcal{I}} h_j(p_j)\right). \tag{1}$$

Well-known special cases of this class of demand systems include multinomial logit demand (with $h_i(p_i) = \exp[(a_i - p_i)/\lambda]$ and $\Psi(H) = \log(1 + H)$) and CES demand ($h_i(p_i) = a_i p_i^{1-\sigma}$ and $\Psi = \log$). Another special case, which is commonly used in the empirical industrial organization literature (see Ciliberto, Murry, and Tamer, 2021; Betancourt, Hortacsu, Öry, and Williams, 2022; Miller, Osborne, Sheu, and Sileo, 2025), is nested logit with two nests, one for the inside goods and one for the outside option ($h_i(p_i) = \exp[(a_i - p_i)/\lambda]$ and $\Psi(H) = \log(1 + H^\alpha)$, where $\alpha \in (0, 1)$ parametrizes the substitutability between the inside goods and the outside option).

Nocke and Schutz (2018a) show that the demand system of equation (1) can be derived from multi-stage discrete/continuous choice if and only if Assumption C below holds.[12] The choice process is sequential. At the first stage, the consumer observes all prices, the value of the outside option, and a taste shock to the inside goods, and decides whether to take up the outside option. The taste shock to the inside goods can be interpreted as a shopping cost or a search cost. Next, conditional on declining the outside option, the consumer observes

[12] We describe a more general version of this choice process in Section 4.1.

a vector of product-specific taste shocks, and chooses the product that delivers the highest indirect utility. Finally, the consumer decides how much of that product to consume. Under this micro-foundation, $\log h_j$ corresponds to the mean utility delivered by good j, whereas the function Ψ reflects the distribution of the difference between the value of the outside option and the stage-1 shock to the inside goods.[13]

Assumption C. *The following conditions hold:*

(i) *Each h_i is $\mathcal{C}^1$, strictly positive, strictly decreasing, and log-convex.*

(ii) *Ψ is $\mathcal{C}^1$ on $\mathbb{R}_{++}$ and $H \mapsto H\Psi'(H)$ is strictly positive and non-decreasing.*[14]

This demand system has the IIA property, as

$$D_i(p)/D_j(p) = h_i'(p_i)/h_j'(p_j)$$

is independent of the price of any third product k. Note that products are (local) complements if Ψ' is locally increasing and local substitutes if Ψ' is locally decreasing. Such price-dependent patterns of complementarity/substitutability are at the core of Rey and Tirole (2019). It is immediate that a demand system that features complementarities cannot be derived from a *single-stage* discrete/continuous choice model, in which the consumer observes all taste shocks and all prices before choosing a product.[15] Complementarities can arise, however, under *multi-stage* discrete/continuous choice, where a consumer decides whether to take up the outside option *before* observing the taste shocks to the individual products.[16] The intuition is that a reduction in the price of good j reduces the probability that a consumer takes the outside option, thereby potentially increasing the *ex ante* choice probability for good $k \neq j$.

On the supply side, the set of firms, $\mathcal{F}$, is a partition of the set of products, $\mathcal{I}$. We assume that there are at least two firms. Firms produce under constant returns to scale; the vector of constant unit costs for all products is denoted $c = (c_j)_{j\in\mathcal{I}} \in \mathbb{R}^{\mathcal{I}}_{++}$.

Setting $h_j(\infty) \equiv \lim_{p_j\to\infty} h_j(p_j)$, and adopting the convention that the empty sum is equal to zero, the profit of firm f is given by:

$$\Pi^f(p) = \sum_{\substack{k\in f:\\ p_k<\infty}} (p_k - c_k)(-h_k'(p_k))\Psi'\left(\sum_{j\in\mathcal{I}} h_j(p_j)\right), \ \forall p \in (0,\infty]^{\mathcal{I}}.$$

[13]Throughout the paper, assumptions that ensure that Consumer demand has sound micro-foundations will be labeled with a C, regularity conditions for equilibrium Existence with an E, and conditions for equilibrium Uniqueness with a U.

[14]Strictly speaking, Nocke and Schutz (2018a) require Ψ' to be non-negative rather than strictly positive.

[15]By contrast, complementarities arising from joint purchases—as in the case of the basket demand systems analyzed in Section 4.1—are consistent with a single-stage choice process.

[16]Such a setup is consistent with an important strand of the literature on consumer search, where consumers need to inspect products to learn their match values (e.g., Wolinsky, 1986; Choi, Dai, and Kim, 2018).

We allow firms to set infinite prices on some of their products; a product priced at infinity receives zero demand and therefore generates no profit.[17]

Firms compete by setting prices simultaneously. For every firm $f \in \mathcal{F}$, define

$$\mathcal{P}^f \equiv \left\{ p^f \in (0,\infty]^f : \quad \sum_{\substack{k\in f:\\ p_k<\infty}} (p_k - c_k)(-h_k'(p_k)) > 0 \right\}.$$

As price vectors outside $\mathcal{P}^f$ are strictly dominated for firm f, we redefine the action set of firm f as $\mathcal{P}^f$ in the following.

Equilibrium Existence: A Potential Games Approach. For every $p \in \prod_{g\in\mathcal{F}} \mathcal{P}^g$, define

$$O(p) \equiv \Psi'\left(\sum_{j\in\mathcal{I}} h_j(p_j)\right) \prod_{g\in\mathcal{F}} \sum_{\substack{k\in g:\\ p_k<\infty}} (p_k - c_k)\left(-h_k'(p_k)\right). \tag{2}$$

Observe that, for every $f \in \mathcal{F}$, $O(p)$ can be rewritten as

$$O(p) = \Pi^f(p) \times \prod_{g\neq f} \sum_{\substack{k\in g:\\ p_k<\infty}} (p_k - c_k)\left(-h_k'(p_k)\right),$$

where the product on the right-hand side is strictly positive and independent of p^f. It follows that, for every $p = (p^f, p^{-f}) \in \prod_{g\in\mathcal{F}} \mathcal{P}^g$ and $p^{f\prime} \in \mathcal{P}^f$,

$$\Pi^f(p^{f\prime}, p^{-f}) - \Pi^f(p^f, p^{-f}) > 0 \Longleftrightarrow O(p^{f\prime}, p^{-f}) - O(p^f, p^{-f}) > 0.$$

The function O is therefore an ordinal potential for the pricing game defined above.

As shown in Monderer and Shapley (1996b), the ordinal potential can be used to obtain a simple proof of equilibrium existence: if p^* solves the maximization problem

$$\max_{p\in\prod_{g\in\mathcal{F}} \mathcal{P}^g} O(p),$$

then for every $f \in \mathcal{F}$ and $p^f \in \mathcal{P}^f$, $\Pi^f(p^{*f}, p^{*-f}) \geq \Pi^f(p^f, p^{*-f})$, and so p^* is a Nash equilibrium. Equilibrium existence can thus be established by showing the existence of a global maximizer of the ordinal potential.

Applying this insight to our multiproduct-firm pricing game, we obtain equilibrium existence under minimal restrictions:

[17] The compactification of action sets permitted by infinite prices will be useful to establish existence of equilibrium. See Section II.3 in Nocke and Schutz (2018b) for a detailed discussion of infinite prices.

Proposition 1. *Consider a multiproduct-firm pricing game with IIA demand, satisfying Assumption C. The potential function O of equation* (2) *admits a global maximizer. Therefore, the pricing game has a pure-strategy Nash equilibrium.*

Proof. The result follows by combining Theorem 2 in Section 4.2 and Proposition E.1 in Appendix E.1. □

A substantial economic contribution relative to Nocke and Schutz (2018a), and to the existing literature more generally, consists in showing equilibrium existence when products can be complements. Importantly, the equilibrium existence result continues to hold even in the presence of arbitrary price caps and floors. Such price caps (and floors) may arise because of regulation or, as advocated by Rey and Tirole (2019), due to cooperative agreements. Suppose that for all $i \in \mathcal{I}$, there exists a price cap $\overline{p}_i \leq \infty$ and a price floor $\underline{p}_i > 0$ such that p_i has to satisfy $\underline{p}_i \leq p_i \leq \overline{p}_i$. As this type of regulation breaks the convex-valuedness of best responses, standard approaches to equilibrium existence based on the Kakutani fixed point theorem or aggregative games techniques do not apply.[18] By contrast, the potential games approach still delivers equilibrium existence. More generally, an equilibrium exists provided action sets are closed, as would be the case if firms faced uniform-pricing constraints for a subset of their products.

A more technical contribution of Proposition 1 consists in deriving equilibrium existence under weaker regularity assumptions: for instance, in the baseline model of Nocke and Schutz (2018a), it is assumed that Ψ is equal to the logarithm, and each h_i is $\mathcal{C}^3$ with non-decreasing curvature. Without such regularity and monotonicity assumptions, it is easy to construct examples of multiproduct-firm pricing games with IIA demand where best responses are neither convex-valued nor monotone. Despite such classic conditions failing to hold, Proposition 1 implies that those pricing games have a pure-strategy Nash equilibrium.

Finally, note that the potential games approach provides a new method for computing equilibria. Instead of solving a multidimensional fixed point problem (as with the best-response approach) or a nested fixed point problem (as with the aggregative games approach), it involves finding the global maximizer of the ordinal potential function O.

3 Transformed Potentials

While the pricing game analyzed above only has an ordinal potential, the game with payoff functions $\log \Pi^f$ for every f has an *exact* potential: $E \equiv \log O$, where O is the ordinal potential defined in equation (2). That is, the demand system of Section 2 has the following

[18] Recall from Spady (1984) and Hanson and Martin (1996) that multinomial logit profit functions can fail to be quasi-concave, which—with price caps or floors—can result in the failure of uni-modality.

property: there exists a transformation function G (here, $G \equiv \log$) such that—regardless of the vector of marginal costs c and of the firm partition $\mathcal{F}$—the normal form game with payoff function $G \circ \Pi^f$ for every firm f has an exact potential. In such a case, we say that D admits a *transformed potential* or, more specifically, a *G-potential*. In Section 3.1, we fully characterize the set of demand systems that admit a transformed potential and provide admissible transformation functions. Next, in Section 3.2, we relax the requirement that the demand system admits a transformed potential no matter what the firm partition.

3.1 Partition-Independent Characterization

Let the demand system D be a continuous mapping from $\mathbb{R}^{\mathcal{I}}_{++}$ to $\mathbb{R}^{\mathcal{I}}_{+}$. Let $\mathcal{Q} \equiv \{p \in \mathbb{R}^{\mathcal{I}}_{++} : D(p) \in \mathbb{R}^{\mathcal{I}}_{++}\}$ be the set of price vectors at which the demand for all products is strictly positive.[19] By continuity, $\mathcal{Q}$ is open. We impose the following technical restrictions on D. The set $\mathcal{Q}$ is non-empty and convex. Moreover, D is $\mathcal{C}^2$ on $\mathcal{Q}$ and satisfies Slutsky symmetry and strict monotonicity: for all $p \in \mathcal{Q}$ and all $i, j \in \mathcal{I}$, $\partial_i D_j(p) = \partial_j D_i(p)$ and $\partial_i D_i(p) < 0$.[20] It also satisfies non-zero substitution almost everywhere: for all $i, j \in \mathcal{I}$ and almost every $p \in \mathcal{Q}$, $\partial_j D_i(p) \neq 0$. We also assume that for every product $i \in \mathcal{I}$, there exists a price vector $p \in \mathcal{Q}$ such that $\partial_i[p_i D_i(p)] < 0$; that is, the revenue on product i is not everywhere increasing in the price of that product. Slutsky symmetry and the convexity of $\mathcal{Q}$ imply the existence of a function V such that $\partial_i V(p) = -D_i(p)$ for every $p \in \mathcal{Q}$ and $i \in \mathcal{I}$. We assume that the level sets of V are connected surfaces, in the sense that any two points on the same level set can be connected by a continuously differentiable path.[21]

In the following, we confine attention to the game in which firms choose their prices in $\mathcal{Q}$. We restrict attention to transformation functions G that have the following two properties: first, G is defined on an interval of strictly positive reals that include all attainable, strictly positive profit levels; second, G is $\mathcal{C}^2$ with $G' > 0$. We now provide a complete characterization of the classes of demand system admitting a transformed potential:

Theorem 1. *Let D be a demand system. The following assertions are equivalent:*

(a) D admits a transformed potential.

[19] We seek to characterize the demand system D only on the set $\mathcal{Q}$. The reason is that our differential techniques do not allow us to deal with kinks in demand. Such kinks typically occur at price vectors at which the demand for one product vanishes.

[20] Notation: $\partial_i \kappa$ denotes the partial derivative of the function κ with respect to its ith argument; $\partial^2_{ij}\kappa$ denotes the cross-partial derivative with respect to the ith and jth arguments.

[21] This assumption will later allow us to invoke results by Goldman and Uzawa (1964) and Anderson, Erkal, and Piccinin (2020) to integrate systems of partial differential equations. If $\mathcal{Q} = \mathbb{R}^{\mathcal{I}}_{++}$, then the assumption is automatically satisfied if V is convex, i.e., if the demand system D can be derived from quasi-linear utility maximization.

(b) *Either (i) the demand system D takes the IIA form of equation* (1)*; or (ii) the demand system D takes the generalized linear form*

$$D_i(p) = -h_i'(p_i) + \sum_{j \neq i} \alpha_{ij} p_j, \tag{3}$$

with $\alpha_{ij} = \alpha_{ji}$ for every i, j; or both.

If (i) holds (resp., (ii) holds), then the logarithm (resp., the identity function) is an admissible transformation function for demand system D.

Proof. See Appendix A. □

The theorem thus shows that D admits a transformed potential if and only if one (or both) of the following conditions holds. First, D takes the IIA form analyzed in Section 2. In this case, D admits a log-potential. Second, D takes the generalized linear form, which extends the linear demand system of Shubik and Levitan (1980). In this case, D admits an identity-potential.

The potential function for part (b)-(i) of the theorem can be found by taking the logarithm of the ordinal potential function in equation (2):

$$E(p) = \log \Psi' \left(\sum_{j \in \mathcal{I}} h_j(p_j) \right) + \sum_{f \in \mathcal{F}} \log \left(\sum_{j \in f} (p_j - c_j)(-h_j'(p_j)) \right). \tag{4}$$

A potential function for part (b)-(ii) can be obtained by integrating the payoff gradient:

$$E(p) = \sum_{k \in \mathcal{I}} (p_k - c_k)(-h_k'(p_k)) + \frac{1}{2} \sum_{\substack{j,k \in \mathcal{I} \\ j \neq k}} \alpha_{jk} p_j p_k + \frac{1}{2} \sum_{f \in \mathcal{F}} \sum_{\substack{j,k \in f \\ j \neq k}} \alpha_{jk} (p_k - c_k)^2. \tag{5}$$

Hence, the logarithm (resp., the identity function) is an admissible transformation function for IIA demand (resp., generalized linear demand); that is, (b) implies (a).

The proof that (a) implies (b) is significantly more involved. It relies first on applying the cross-partial derivatives test in Theorem 4.5 in Monderer and Shapley (1996b) to obtain a parameterized ordinary differential equation, the integration of which gives the set of admissible transformation functions. A further application of that same theorem yields a system of partial differential equations for the demand system, which we then integrate using results by Goldman and Uzawa (1964).

3.2 Partition-Specific Characterization

Above, we showed that the only two classes of demand systems that give rise to a transformed potential *regardless of the firm partition* are the IIA demand systems and the generalized linear demand systems. For empirical work, both classes have some unattractive properties. First, IIA demand imposes strong restrictions on substitution patterns. Second, a drawback of generalized linear demand is that the demand system is characterized only on a strict subset of the positive orthant, namely the set of prices such that the demand for all products is strictly positive.[22]

In the following, we therefore relax the restriction that the demand system gives rise to a transformed potential for any firm partition. We thus fix the firm partition $\mathcal{F}$ and investigate whether there are richer demand systems that induce a multiproduct-firm pricing game admitting a transformed potential. We say that $(D, \mathcal{F})$ admits a transformed potential if there exists a transformation function G such that, for every marginal cost vector c, the multiproduct-firm pricing game with payoff function $G \circ \Pi^f$ for any $f \in \mathcal{F}$ has an exact potential. We continue to focus on demand systems and transformation functions satisfying the technical conditions introduced in Section 3.1. We further confine attention to demand systems that are $|\mathcal{F}|$ times continuously differentiable.

Proposition 2. *Let D be a demand system and $\mathcal{F}$ a firm partition. The following assertions are equivalent:*

(a) $(D, \mathcal{F})$ admits a transformed potential.

(b) Either (i) The demand system D satisfies the following properties: for every $f, g \in \mathcal{F}$ with $f \neq g$, $i, j \in f$, and $k \in g$, $\partial_k D_i / D_j = 0$ and $\partial^2_{ik} \log D_i / D_k = 0$; or (ii) The demand system D takes the flexible generalized linear form: for any $i \in f \in \mathcal{F}$,

$$D_i(p) = -\partial_i h^f(p^f) + \sum_{\substack{\mathcal{F}' \subseteq \mathcal{F}: \\ f \in \mathcal{F}'}} \sum_{\substack{\iota \in \prod_{g \in \mathcal{F}'} g: \\ \iota(f) = i}} \alpha(\iota) \prod_{\substack{g \in \mathcal{F}': \\ g \neq f}} p_{\iota(g)};$$

or both.

If (i) holds (resp., (ii) holds), then the logarithm (resp., the identity function) is an admissible transformation function for demand system D.

Proof. See Appendix B. □

The class of demand systems in part (b)-(ii) of the proposition is more general than that in part (b)-(ii) of Theorem 1 in the following sense.[23] First, D_i is now allowed to depend

[22] To fix ideas, suppose that the cross-price coefficients α_{ij} are strictly positive. If $-h'_i(p_i)$ is everywhere non-negative, then $p_i D_i(p) \underset{p_i \to \infty}{\longrightarrow} \infty$, provided $p_j > 0$ for some $j \neq i$. If instead $-h'_i(p_i)$ were to become negative for some p_i, then demand would be negative at some price vectors.

[23] Despite its more flexible form, this class of demand systems shares the same drawback as that in part (b)-(ii) of Theorem 1, namely that demand is characterized only on a strict subset of the positive orthant.

non-linearly not only on p_i but also on p^f, the vector of prices of the firm owning product i. Second, for $k \notin f$, the substitution effect $\partial_k D_i$ is not necessarily constant, in that it is allowed to depend on products of prices set by firms that own neither good i nor good k. The associated potential function is:

$$E(p) = \sum_{f \in \mathcal{F}} \sum_{j \in f} (p_j - c_j) \left(-\partial_j h^f(p^f)\right) + \sum_{\mathcal{F}' \subseteq \mathcal{F}} \sum_{\iota \in \prod_{g \in \mathcal{F}'} g} \alpha(\iota) \prod_{g \in \mathcal{F}'} p_{\iota(g)}. \quad (6)$$

In contrast to part (b)-(i) of Theorem 1, part (b)-(i) of the proposition does not fully characterize the resulting demand system, but instead provides a system of partial differential equations that the demand system must solve. Below, we provide rich classes of demand systems satisfying condition (b)-(i) of Proposition 2. The following proposition identifies such a class and establishes uniqueness in the case of two firms:

Proposition 3. *Let $\mathcal{F}$ be a firm partition. Then, $(D, \mathcal{F})$ admits a log-potential if—and, in the case of $|\mathcal{F}| = 2$, only if—the demand system D takes the following form. For every $i \in f \in \mathcal{F}$ and $p \in \mathcal{Q}$,*

$$D_i(p) = -\partial_i h^f(p^f) \Psi' \left(\sum_{g \in \mathcal{F}} h^g(p^g) \right).$$

An associated indirect (sub-)utility function is

$$V(p) = \Psi \left(\sum_{g \in \mathcal{F}} h^g(p^g) \right).$$

Proof. For the "if" part, define

$$E(p) = \log \Psi' \left(\sum_{f \in \mathcal{F}} h^f(p^f) \right) + \sum_{f \in \mathcal{F}} \log \left(\sum_{j \in f} (p_j - c_j) \left(-\partial_j h^f(p^f)\right) \right). \quad (7)$$

This function is an exact potential for the transformed pricing game, as its gradient coincides with the transformed payoff gradient. That V is an indirect utility function is immediate, as $D_i = -\partial_i V$. The proof of the "only if" part (in the case of two firms), which is relegated to Appendix C, relies on Lemma 1 and Theorem 3 in Goldman and Uzawa (1964). □

The substantial difference compared to the class of demand systems of Section 2 is that the functions h^g are completely unrestricted and so, in particular, need not be additively separable in p^g. This allows for arbitrary substitution patterns across products offered by the same firm. In addition, the ratio of demands for goods i and j can depend on the price of a third product k provided that product k is owned by a firm that also owns at least one of the two products i and j. A special case is a nested multinomial logit (or nested CES) demand system where each firm owns one or several nests of products.[24]

The IIA demand systems of Section 2 have the property that, at any given vector of prices, all products are either local substitutes or local complements to one another. The new class of demand systems of Proposition 3 permits more flexibility in this regard: for example, product 1 could be a complement to product 2 and a substitute to product 3, with all three products owned by the same firm, and at the same time a substitute to all products owned by the rival firm. Such demand patterns frequently arise through "one-stop shopping", where products offered by different stores are substitutes, but products offered by the same store can be complements.

The class of demand systems of Proposition 3 satisfies the IIA property across firms (rather than across products) in the following sense: the ratio of demands between products i and j, where i is sold by firm f and j is sold by firm $f' \neq f$, is independent of the prices set by any third firm. This means that a product of firm f is an equally good substitute (or complement) to a product of firm f' as to one of firm f''. The following class of demand systems relaxes this feature by partitioning the set of firms $\mathcal{F}$ into a set of nests $\mathcal{N}$:

Proposition 4. *Let $\mathcal{F}$ be a firm partition and $\mathcal{N}$ a partition of $\mathcal{F}$. Then, $(D, \mathcal{F})$ admits a log-potential if the demand system D takes the following form. For every $i \in f \in n \in \mathcal{N}$ and $p \in \mathcal{Q}$,*

$$D_i(p) = -\partial_i h^f(p^f) \, (\Phi^n)' \left(\sum_{g \in n} h^g(p^g) \right) \Psi' \left[\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} h^g(p^g) \right) \right]. \tag{8}$$

An associated indirect (sub-)utility function is

$$V(p) = \Psi \left[\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} h^g(p^g) \right) \right]. \tag{9}$$

Proof. The result follows immediately by defining the exact potential

$$\begin{aligned} E(p) = \log \Psi' &\left[\sum_{n \in \mathcal{N}} \Phi^n \left(\sum_{g \in n} h^g(p^g) \right) \right] \\ &+ \sum_{n \in \mathcal{N}} \log \, (\Phi^n)' \left(\sum_{g \in n} h^g(p^g) \right) + \sum_{n \in \mathcal{N}} \sum_{g \in n} \log \left(\sum_{j \in g} (p_j - c_j) \, (-\partial_j h^g(p^g)) \right) \end{aligned}$$

and noticing that its gradient does indeed coincide with the transformed payoff gradient. That V is an indirect utility function is immediate, as $D_i = -\partial_i V$. □

[24] Another, more flexible special case is a multi-level nested multinomial logit (or nested CES) demand system as in Garrido (2024), in which each upper-tier nest is entirely owned by a single firm.

The class of demand systems of Proposition 3 arises as a special case when the nest partition is trivial (i.e., there is only one nest or each firm forms its own nest). Otherwise, more flexible patterns of substitutability and complementarity emerge. In Section 4.3, we provide a multi-stage discrete/continuous choice micro-foundation. As discussed there, a nest may be viewed as a shopping mall and a firm as a multiproduct store in that mall.

Another generalization of the demand systems of Proposition 3 involves a "basket" structure, where a basket B contains the products of one or more firms:

Proposition 5. *Let $\mathcal{F}$ be a firm partition and $\mathcal{B} \equiv 2^{\mathcal{F}} \setminus \{\emptyset\}$. Then, $(D, \mathcal{F})$ admits a log-potential if the demand system D takes the following form. For every $i \in f \in \mathcal{F}$ and $p \in \mathcal{Q}$,*

$$D_i(p) = -\partial_i h^f(p^f) \left(\sum_{\substack{B \in \mathcal{B}: \\ f \in B}} a(B) \prod_{\substack{g \in B \\ g \neq f}} h^g(p^g) \right) \Psi' \left[\sum_{B \in \mathcal{B}} a(B) \prod_{g \in B} h^g(p^g) \right]. \tag{10}$$

An associated indirect (sub-)utility function is

$$V(p) = \Psi \left[\sum_{B \in \mathcal{B}} a(B) \prod_{g \in B} h^g(p^g) \right]. \tag{11}$$

Proof. The result follows immediately by defining the exact potential

$$E(p) = \log \Psi' \left[\sum_{B \in \mathcal{B}} a(B) \prod_{g \in B} h^g(p^g) \right] + \sum_{f \in \mathcal{F}} \log \left(\sum_{j \in f} (p_j - c_j) \left(-\partial_j h^f(p^f) \right) \right).$$

and noticing that its gradient does indeed coincide with the transformed payoff gradient. That V is an indirect utility function is immediate, as $D_i = -\partial_i V$. □

The class of demand systems of Proposition 3 arises as a special case when $a(B)$ is equal to 0 whenever $|B| \geq 2$. More flexible patterns of sustitutability and complementarity emerge whenever $a(B) \neq 0$ for some non-singleton baskets. In Section 4.1, we provide a discrete/continuous choice micro-foundation for this new class of demand systems. As we will see there, a given consumer can be thought of as making a discrete choice among baskets, and may therefore end up jointly purchasing from all the firms in the chosen basket. A basket may, for example, contain a beer producer and a chips producer—or only a beer producer or only a chips producer.

4 Multiproduct-Firm Pricing with Baskets and Nests

In this section, we consider in more detail the demand systems with nests of Proposition 4 and those with baskets of Proposition 5. We provide multi-stage discrete/continuous choice micro-foundations for basket demand systems in Section 4.1 and prove equilibrium existence for pricing games with such basket demand in Section 4.2. We repeat those steps for demand systems with nests in Section 4.3.

4.1 Micro-Foundations for Baskets

In this subsection, we provide a micro-foundation for the basket demand systems introduced in Proposition 5. Recall that $\mathcal{B} = 2^{\mathcal{F}} \setminus \{\emptyset\}$ is the set of all baskets, each consisting of all of the products of at least one firm.

The choice process. Consider the following multi-stage discrete/continuous choice process for a continuum of consumers. At stage 1, consumers observe all prices and each receives a taste shock $\varepsilon_0 \in [-\infty, \infty)$ to the outside option and another taste shock $\varepsilon \in [-\infty, \infty)$ to the inside goods. The taste shocks are jointly distributed according to the measure μ, which satisfies the following properties: for every $x \in \mathbb{R}$, the function $(\varepsilon_0, \varepsilon) \in [-\infty, \infty)^2 \mapsto \max(\varepsilon_0, x + \varepsilon)$ is μ-integrable; the function $K: \ x \in \mathbb{R} \mapsto \mu\left(\{(\varepsilon_0, \varepsilon) \in [-\infty, \infty)^2 : \ \varepsilon_0 \leq x + \varepsilon\}\right)$ is continuous and strictly positive. Each consumer then decides whether to take up the outside option, which yields a payoff of ε_0, or move on to the next stage of the choice process.[25]

At stage 2, among those consumers who turned down the outside option, each receives a taste shock ε_B for every basket B, drawn i.i.d. from a standard type-I extreme value distribution, and then chooses one of these baskets. The conditional indirect utility from choosing basket B is

$$\log a(B) + \sum_{f \in \mathcal{B}} v^f(p^f) + \varepsilon_B + \varepsilon,$$

where $a(B) \geq 0$ is a basket-level utility shifter, common to all consumers. We assume that v^f is a $\mathcal{C}^1$ conditional indirect (sub-)utility function; that is, it is strictly decreasing and convex on $\mathbb{R}^f_{++}$.

A stage 3, after having chosen basket B, a consumer decides in which quantities to purchase the products in that basket, according to the conditional indirect utility function $\sum_{f \in \mathcal{B}} v^f(p^f)$. Note that v^f could be micro-founded by either a model of continuous choice (in which the consumer ends up purchasing all of firm f's products), or by a discrete/continuous choice model (in which the consumer purchases a continuous quantity of a single product). The latter micro-foundation seems more appropriate in the case in which firms may specialize in product categories, e.g., either chips or beer, and each consumer ends up selecting a single product within each category.

[25] Our assumptions on μ ensure that consumer surplus is finite and there is always a strictly positive mass of consumers turning down the outside option.

Optimal consumer behavior. Assume that, for every $f \in \mathcal{F}$, there exists a $B \in \mathcal{B}$ such that $a(B) > 0$ and $f \in B$. This ensures that the demand for each firm's products is strictly positive. We proceed by backward induction. At stage 3, Roy's identity implies that, conditional on having chosen basket B, a consumer optimally purchases $-\partial_i v^f(p^f)$ units of every good $i \in f \in B$. At stage 2, the Holman-Marley theorem implies that, conditional on having turned down the outside option, a consumer chooses basket B with probability

$$s_B = \frac{1}{H} a(B) \exp \sum_{f \in B} v^f(p^f),$$

where

$$H \equiv \sum_{B' \in \mathcal{B}} a(B') \exp \sum_{f \in B'} v^f(p^f).$$

Hence, the expected utility from declining the outside option is $\log H + \varepsilon$. At stage 1, a consumer therefore turns down the outside option if (and only if) $\varepsilon_0 \leq \log H + \varepsilon$. Hence, the mass of consumers forgoing the outside option is $K(\log H)$.

Summing up, the demand for product $i \in f \in B$ equals

$$D_i(p) = K(\log H) \times s^f \times (-\partial_i v^f(p^f)), \tag{12}$$

where $s^f \equiv \sum_{B: f \in B} s_B$ is the conditional probability of choosing a basket in which firm f is present. Consumer surplus is given by $\Psi(H) \equiv \int \max(\varepsilon_0, \log H + \varepsilon) d\mu$. Differentiating under the integral sign, we obtain, $\Psi'(H) = K(\log H)/H$ (see the proof of Proposition IX in Nocke and Schutz, 2018b). Let $h^f \equiv \exp v^f$. Replacing $K(\log H)$ by $H\Psi'(H)$ and v^f by $\log h^f$ in equation (12), we obtain the demand system of equation (10). The functions h^f and Ψ satisfy the following adapted version of Assumption C.

Assumption C′. *The following conditions hold:*

(i) *Each h^f is $\mathcal{C}^1$, strictly positive, strictly decreasing, and log-convex.*

(ii) *Ψ is $\mathcal{C}^1$ on $\mathbb{R}_{++}$ and $H \mapsto H\Psi'(H)$ is strictly positive and non-decreasing.*

Conversely, suppose that the demand system of equation (10) satisfies Assumption C′. Then, the argument in the proof of Proposition IX in Nocke and Schutz (2018b) implies the existence of a measure μ satisfying the assumptions made above such that the multi-stage discrete/continuous choice process with measure μ, basket-level utility shifters $a(B)$, and conditional indirect utility functions $v^f \equiv \log h^f$ generates demand system (10) and the associated indirect utility function (11). The following proposition summarizes these insights and shows that the demand system is *quasi-linearly integrable* (Nocke and Schutz, 2017):

Proposition 6. *The demand system of Proposition 5 can be derived from multi-stage discrete/continuous choice if and only if Assumption C′ holds. Under the same assumption, the demand system admits a representative consumer with quasi-linear preferences. Regardless of the micro-foundation, consumer surplus is given by equation* (11)*.*

Proof. The proof of the first part is in the text above. The proof of the second part can be found in Appendix D. □

Sources of complementarity. As discussed before, in the micro-foundation of the IIA demand systems of Section 2, the only source of potential complementarity between products arises from drawing consumers towards (or away from) the outside options; as such, products are locally either all substitutes or all complements with each other. While the demand systems of Proposition 3 allow for arbitrary patterns of complementarity or substitutability within firms, they still have the property that the products of different firms are locally either all substitutes or all complements.[26]

The basket demand systems of Proposition 5 feature a new source of complementary between the products sold by different firms: joint purchases. This source of complementarity is at the heart of a growing empirical literature (Gentzkow, 2007; Iaria and Wang, 2020; Sovinsky, Jacobi, Allocca, and Sun, 2024; Ershov, Laliberté, Marcoux, and Orr, 2025).

Suppose for expositional convenience that the demand system is differentiable. Then, the derivative of the demand for product $i \in f$ with respect to the price of product $j \in g \neq f$, $\partial_j D_i$, has the same sign as

$$\rho(H) - \frac{s^{fg}}{s^f s^g}, \tag{13}$$

where $\rho(H) \equiv -H\Psi''(H)/\Psi'(H)$ is the curvature of Ψ, and $s^{fg} \equiv \sum_{B:f,g\in B} s_B$ is the conditional probability of choosing a basket in which both firms f and g are present. In the discrete/continuous choice micro-foundation, $\rho(H) = 1 - K'(\log H)/K(\log H)$; hence, $\rho(H) \leq 1$.

If firms f and g are never in the same basket, so that $s^{fg} = 0$, the only source of potential complementarity works through the outside option. In this case, the products are (local) complements if and only if $\rho(H) < 0$, i.e., if and only if the reversed hazard rate of $K(\log H)$ is greater than 1. The ratio on the right-hand side of equation (13), $\ell^{fg} \equiv s^{fg}/(s^f s^g)$, captures complementarities arising from joint purchases, and is larger, the greater is the extent of overlap of the baskets in which firm f and g are present. In statistical learning, ℓ^{fg} is called the *lift* of f and g and is used for evaluating association rules for market basket analysis (e.g., Hastie, Tibshirani, and Friedman, 2009). An increase in ℓ^{fg} makes it more "likely" that the products of the two firms are complements. Indeed, if the lift of firms f and g is

[26]Suppose that, at price vector p, products i and j sold by firms f and $g \neq f$ are substitutes (resp., complements). Then, for any pair of products k and l sold by two distinct firms, the same is true.

greater than 1, then the products of the two firms are necessarily complements, no matter the sign of ρ. This arises, for example, if firm f's products are predominantly purchased in conjunction with those of firm g, so that s^{fg}/s^f is close to 1, implying that $\ell^{fg} \simeq 1/s^g > 1$. Note that, with basket demand, the products of firm f may be complements to those of firm g but substitutes to those of firm g', as $\rho(H)$ could be smaller than ℓ^{fg} but larger than $\ell^{fg'}$. In Appendix D, we study the comparative statics of the lift, and show that a decrease in firm f's prices tends to make it more "likely" that that firm's products become complements with those of *any other* firm.

4.2 Equilibrium Existence for Baskets

We now turn to proving equilibrium existence for pricing games with basket demand systems. As in the equilibrium existence result of Proposition 1, it is useful to compactify firms' action sets by allowing them to price some (or all) of their products at infinity. This requires extending the domain of h^f and its partial derivatives to price vectors with infinite components:

Assumption E1. *For every $f \in \mathcal{F}$, the function h^f and its partial derivatives have continuous extensions to $(0,\infty]^f$. Moreover, $h^f(p^f) > h^f(\infty)$ whenever p^f has at least one finite component.*

Next, we make an assumption to ensure that each firm can ensure itself positive demand (even) when all rival prices are infinite.[27]

Assumption E2. *At least one of the following conditions holds:*

(i) $a(\{f\}) > 0$ *for all* $f \in \mathcal{F}$*;*

(ii) $h^f(\infty) > 0$ *for all* $f \in \mathcal{F}$*.*

At price vector $p \in (0,\infty]^{\mathcal{I}}$, the profit of firm f is:

$$\Pi^f(p) = \sum_{\substack{j \in f: \\ p_j < \infty}} (p_j - c_j)(-\partial_j h^f(p^f)) \left[\sum_{\substack{B \in \mathcal{B}: \\ f \in B}} a(B) \prod_{\substack{g \in \mathcal{F} \\ g \neq f}} h^g(p^g) \right] \Psi' \left(\sum_{B \in \mathcal{B}} a(B) \prod_{g \in B} h^g(p^g) \right). \quad (14)$$

Infinite prices have the same interpretation as in Section 2: a product priced at infinity generates no sales and therefore no profit.

The next step in proving equilibrium existence involves rewriting the model as a game of competition in utility space. To this end, we first consider each firm f's associated Ramsey

[27] This assumption thus rules out the trivial equilibrium in which all prices are infinite.

problem of choosing p^f to maximize $\Pi^f(p)$ subject to providing consumers with a given utility level u^f.[28] Define $\underline{u}^f \equiv h^f(\infty)$, $\overline{u}^f \equiv \lim_{p^f \to 0} h^f(p^f)$, and $u_0^f \equiv h^f(c^f)$, where c^f is firm f's marginal cost vector. Define also

$$\tilde{\pi}^f(p^f) \equiv \sum_{\substack{j \in f: \\ p_j < \infty}} (p_j - c_j)(-\partial_j h^f(p^f))$$

for every $p^f \in (0, \infty]^f$. So, firm f's Ramsey problem can be written as

$$\max_{p^f \in (0,\infty]^f} \tilde{\pi}^f(p^f) \quad \text{s.t.} \quad h^f(p^f) = u^f, \tag{15}$$

where the utility target u^f takes values in $[\underline{u}^f, \overline{u}^f)$. To see why this is equivalent to maximizing $\Pi^f(p)$ subject to $h^f(p^f) = u^f$, note that the term inside square brackets in equation (14) is independent of p^f and the argument of the function Ψ' depends on p^f only through $h^f(p^f)$, which is held fixed at u^f.

Denote the set of solutions to problem (15) by $P^f(u^f)$ and let

$$\pi^f(u^f) \equiv \sup\left\{\tilde{\pi}^f(p^f) : \ p^f \in (0, \infty]^f \text{ and } h^f(p^f) = u^f\right\}. \tag{16}$$

Clearly, $c^f \in P^f(u_0^f)$ (aggregate surplus is maximized when all products are priced at marginal cost) and $(\infty, \ldots, \infty) \in P^f(\underline{u}^f)$. Hence, $\pi^f(u_0^f) = \pi^f(\underline{u}^f) = 0$. Moreover, $\pi^f(u^f) > 0$ for every $u^f \in (\underline{u}^f, u_0^f)$, as there exists a $\mu > 0$ such that $h^f((c_j + \mu)_{j \in f}) = u^f$ by the intermediate value theorem. Observe also that $\pi^f(u^f) < 0$ whenever $u^f > u_0^f$, implying that firm f will never provide those utility levels in the pricing game. The following assumption ensures that the Ramsey problem is well behaved in that the optimal prices are bounded away from zero.

Assumption E3. *There exists $\varepsilon > 0$ such that for every $u^f \in (\underline{u}^f, u_0^f)$, problem* (15) *has a solution in $[\varepsilon, \infty]^f$.*

As we show in Proposition E.1 in Appendix E, a sufficient condition for Assumptions C′-(i), E1, and E3 to hold jointly is that the conditional indirect utility function $\log h^f$ can be derived from multi-stage discrete/continuous choice. Equivalently, $h^f(p^f) = \Phi^f(\sum_{j \in f} h_j(p_j))$ for some functions h_j satisfying Assumption C-(i) and some function Φ^f whose logarithm satisfies Assumption C-(ii).

[28] The fact that firms can be thought of as competing in utility space holds not only for the basket demand systems considered here, but also for any other demand system satisfying property (b)-(i) of Proposition 2. This follows because that property implies that the indirect utility function $V(p)$ is weakly separable with respect to the firm partition in the following sense: $\partial_k(\partial_i V / \partial_j V) = 0$ for every $i, j \in f$ and $k \in g \neq f$. Theorem 3 in Goldman and Uzawa (1964) then implies that $V(p)$ can be written as $V(p) = \Lambda\left((h^g(p^g)_{g \in \mathcal{F}})\right)$, so that each firm f can indeed be thought of as choosing $u^f = h^f(p^f)$.

In Appendix E, we show that the pricing game is strategically equivalent to one in which each firm f chooses $u^f \in (\underline{u}^f, u_0^f)$, with payoff function

$$\Pi^f(u) = \pi^f(u^f) \left[\sum_{\substack{B \in \mathcal{B}: \\ f \in B}} a(B) \prod_{\substack{g \in B \\ g \neq f}} u^g \right] \Psi' \left(\sum_{B \in \mathcal{B}} a(B) \prod_{g \in B} u^g \right). \tag{17}$$

By the same argument as in Section 2, the function

$$O(u) = \left(\prod_{g \in \mathcal{F}} \pi^g(u^g) \right) \Psi' \left(\sum_{B \in \mathcal{B}} a(B) \prod_{g \in B} u^g \right) \quad \forall u \in \prod_{g \in \mathcal{F}} (\underline{u}^g, u_0^g) \tag{18}$$

is an ordinal potential for that game. We obtain:

Theorem 2. *Consider a multiproduct-firm pricing game with basket demand, satisfying Assumptions C′ and E1–E3. The ordinal potential function O of equation* (18) *admits a global maximizer. Therefore, the pricing game has a pure-strategy Nash equilibrium.*

Proof. See Appendix E. □

Note that the equilibrium identified here is non-trivial in that each firm sets at least one finite price and therefore makes a strictly positive profit. Note also that Theorem 2 subsumes Proposition 1 as a special case; the reason is that the assumptions of the theorem are satisfied whenever the conditional indirect utility functions can be derived from discrete/continuous choice (see Proposition E.1 in Appendix E), which necessarily holds for IIA demand.

4.3 Nests: Micro-Foundations and Equilibrium Existence

We now turn to providing a multi-stage discrete/continuous choice micro-foundation for the class of nested demand systems of Proposition 4. Here, we give only a brief summary; details are relegated to Appendix D. Each consumer first observes all prices, receives taste shocks to the outside option and the inside goods, and decides whether to take up the outside option. Conditional on turning it down, the consumer observes a vector of nest-level taste shocks, drawn i.i.d. from a type-I extreme-value distribution with dispersion parameter $\lambda > 0$, and chooses one of the nests. Having picked a nest, the consumer observes taste shocks to the nest-level outside option and to the inside goods in the nest, and decides whether to take up the outside option. Conditional on declining that outside option, the consumer observes a vector of i.i.d. standard type-I extreme-value taste shocks to the firms in the nest, and chooses one of the firms. Finally, the consumer decides in which quantities to purchase the products of the chosen firm f according to the conditional indirect utility function v^f.

We find that the class of demand systems generated by this choice process is that of Proposition 4 under an adapted version of Assumption C.

Assumption C″. *The following conditions hold:*

(i) *Each* h^f *is* $\mathcal{C}^1$*, strictly positive, strictly decreasing, and log-convex.*

(ii) Ψ *is* $\mathcal{C}^1$ *on* $\mathbb{R}_{++}$ *and* $H \mapsto H\Psi'(H)$ *is strictly positive and non-decreasing.*

(iii) *Each* Φ^n *is* $\mathcal{C}^1$ *and strictly positive on* $\mathbb{R}_{++}$*, with an elasticity that is strictly positive, non-decreasing, and bounded.*

We obtain the analog of Proposition 6 for nested demand systems:[29]

Proposition 7. *The demand system of Proposition 4 can be derived from multi-stage discrete/continuous choice if and only if Assumption C″ holds. Under the same assumption, the demand system admits a representative consumer with quasi-linear preferences. Regardless of the micro-foundation, consumer surplus is given by equation* (9)*.*

Proof. See Appendix D. □

Using the above micro-foundation, the demand for product $i \in f \in n$ can be written as

$$D_i(p) = K(\lambda \log H) \times s^n \times K^n(\log H^n) \times s^{f|n} \times (-\partial_i v^f(p^f)),$$

where $s^{f|n}$ is the probability of choosing firm f conditional on being in nest n and on not having taken the nest-level outside option; s^n is the conditional choice probability of nest n; K^n is the c.d.f. of the difference between the taste shock for the nest-level outside option and the taste shock for the inside goods in nest n; K is the c.d.f. of the difference between the taste shock for the stage-1 outside option and the taste shock for the inside goods; and $\lambda \log H$ and $\log H^n$ are the inclusive values of the set of nests and the set of firms in nest n, respectively. Taking the logarithm and differentiating with respect to the price of good $j \in g \in n$, with $g \neq f$, we obtain that $\partial_j \log D_i$ has the same sign as

$$-K^n(\log H^n) s^n \frac{K'(\lambda \log H)}{K(\lambda \log H)} - \frac{1}{\lambda} K^n(\log H^n)(1 - s^n) - \frac{K^{n\prime}(\log H^n)}{K^n(\log H^n)} + 1.$$

The first and fourth terms were already present under IIA demand. The former captures the source of complementarity working through the stage-1 outside option, whereas the latter reflects the substitutability of different products within the same nest conditional on purchasing from one of the firms in that nest. The second and third terms summarize new sources of complementarity: the second term captures the complementarity that arises

[29] By considering a broader class of multi-stage discrete/continuous choice models, this result extends Proposition IX in Nocke and Schutz (2018b) in two ways: first, it allows indirect utility within a firm to be completely unrestricted (rather than additively separable); second, it only requires the elasticity of Φ^n to be bounded (rather than bounded above by 1).

from inducing consumers to switch to different nests, whereas the third term represents the complementarity working through the nest-level outside option. Note that products can be complements even in the absence of stage-1 and nest-level outside options. To see this, suppose indeed that $K^{n\prime} = K' = 0$ and $K^n = 1$. Then, $\partial_j D_i$ has the same sign as

$$-\frac{1 - s^n}{\lambda} + 1.$$

Hence, products i and j are complements if and only if $\lambda < 1$ and s^n is sufficiently small. The intuition is that, when λ is small, there is little horizontal differentiation between the nests, which implies that a decrease in the attractiveness of nest n results in many consumers switching to other nests.[30]

Next, we turn to equilibrium existence for price competition under nested demand systems. We continue to impose Assumption E1 so that the demand system can be extended to price vectors with infinite components. The profit of firm $f \in n$ is then given by

$$\Pi^f(p) = \sum_{\substack{j \in f: \\ p_j < \infty}} (p_j - c_j)(-\partial_j h^f(p^f)) \, (\Phi^n)' \left(\sum_{g \in n} h^g(p^g) \right) \Psi' \left[\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} h^g(p^g) \right) \right].$$

Under Assumption E3, the pricing game is strategically equivalent to one of competition in utility space. Using the notation of Section 4.2, this amounts to each firm $f \in n$ choosing $u^f \in (\underline{u}^f, u_0^f)$ to maximize

$$\Pi^f(u) = \pi^f(u^f) \, (\Phi^n)' \left(\sum_{g \in n} u^g \right) \Psi' \left[\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} u^g \right) \right].$$

An ordinal potential is given by

$$O(u) = \left(\prod_{f \in \mathcal{F}} \pi^f(u^f) \right) \left[\prod_{n \in \mathcal{N}} (\Phi^n)' \left(\sum_{g \in n} u^g \right) \right] \Psi' \left[\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} u^g \right) \right]. \tag{19}$$

We are now in the position to state our existence result.

Theorem 3. *Consider a multiproduct-firm pricing game with nested demand, satisfying Assumptions C″, E1, and E3. The ordinal potential function O of equation* (19) *admits a global maximizer. Therefore, the pricing game has a pure-strategy Nash equilibrium.*

Proof. See Appendix E. □

[30] The case where λ is strictly smaller than 1 corresponds to the nesting parameter being strictly greater than 1 in the standard formulation of the nested logit model. While such a nesting parameter cannot be rationalized by a *single-stage* discrete choice model (McFadden, 1978), it is fully compatible with multi-stage discrete(/continuous) choice.

5 Equilibrium Uniqueness

In this section, we provide conditions under which a multiproduct-firm pricing game with baskets or nests has a unique pure-strategy Nash equilibrium. Importantly, we show that under an additional, mild technical condition, this equilibrium is also the unique correlated equilibrium of the pricing game with transformed (i.e., logged) payoffs. Correlated equilibrium is the appropriate concept in applied work, when the modeler does not observe all signals on which players may base their behavior.[31]

In the following, we focus on a multiproduct-firm pricing game with a basket demand system satisfying the assumptions of Section 4.2, relegating the treatment of nested demand systems to Appendix F.2. Consider the strategically equivalent version of the game of competition in utility space, in which each firm f chooses $t^f \equiv -\log u^f$ to maximize log-profit. This game has an exact potential,

$$E(t) \equiv \sum_{f \in \mathcal{F}} \log \pi^f(\mathrm{e}^{-t^f}) + \log \Psi' \left(\sum_{B \in \mathcal{B}} a(B) \prod_{f \in B} \mathrm{e}^{-t^f} \right).$$

We will show that, under some mild technical conditions, each of the terms on the right-hand side is (strictly) concave and smooth. This, in turn, implies that the pricing game has a unique pure-strategy Nash equilibrium for the following reason. Let the strategy profile $t = (t^f)_{f \in \mathcal{F}}$ be a Nash equilibrium. Then, the first-order condition of each firm's (log-)profit maximization must hold at t. It follows that the first-order conditions for the maximization of the potential function also hold at t. As E is strictly concave, this implies that t is equal to the unique global maximizer of the exact potential function. This establishes the uniqueness of a pure-strategy Nash equilibrium.

The following assumption ensures that $\log \pi^f(\mathrm{e}^{-t^f})$ is smooth and strictly concave on an interval that includes all of firm f's strategies that are not strictly dominated:

Assumption U1. *For every f, h^f takes the form $h^f(p^f) = \Phi^f(\sum_{j \in f} h_j(p_j))$. The function Φ^f is $\mathcal{C}^2$ and strictly positive, with a non-decreasing and non-negative curvature and a non-decreasing and strictly positive elasticity. Moreover, either (i) there exists $\sigma^f > 1$ such that for every $j \in f$, $h_j(p_j) = a_j(p_j + \beta_j)^{1-\sigma^f}$ for some $a_j > 0$ and $\beta_j \geq 0$; or (ii) for every $j \in f$, $h_j(p_j) = \exp \frac{a_j - p_j}{\lambda_j}$ for some $a_j \in \mathbb{R}$ and $\lambda_j > 0$.*

The next assumption guarantees that $\log \Psi' \left(\sum_B a(B) \prod_{f \in B} \mathrm{e}^{-t^f} \right)$ is smooth and concave:

[31] See, for example, Bergemann and Morris (2013) for a version of this argument in games of incomplete information when the modeler does not know players' information structure. A related issue is whether a given Nash equilibrium is robust to the introduction of incomplete information. For finite games admitting a unique correlated equilibrium, Kajii and Morris (1997) show that the answer is positive.

Assumption U2. Ψ *is* $\mathcal{C}^2$ *with non-negative and non-decreasing curvature.*

Finally, the following assumption implies that strategies with a very high t^f are strictly dominated. This allows us to compactify action sets and apply Theorem 2 in Neyman (1997) to establish the uniqueness of a correlated equilibrium in the pricing game with logged payoffs.

Assumption U3. *At least one of these conditions holds: (a)* $\Psi'(H)$ *is bounded in the neighborhood of* $H = 0$*; (b) for every* f, $\underline{u}^f > 0$.

Note that condition (a) of Assumption U3 holds if $\Psi(H) = \tilde{\Psi}(H^0 + H)$ for some function $\tilde{\Psi}$ satisfying the above assumptions and some $H^0 > 0$. Intuitively, this corresponds to the case where consumers have access to an outside option at the second stage of the choice process introduced in Section 4.1.

We are now in a position to state our uniqueness result:

Theorem 4. *Consider a multiproduct-firm pricing game with basket demand satisfying Assumptions C′-(ii), E2, and U1–U2. The exact potential function of the logged pricing game in which each firm* f *chooses* t^f *is strictly concave on a rectangle that includes all the pure strategies that are not strictly dominated. Therefore, the pricing game has a unique pure-strategy Nash equilibrium (no matter what the transformation of payoffs), which corresponds to the unique potential maximizer. If, in addition, Assumption U3 holds, then the potential maximizer also corresponds to the unique correlated equilibrium of the logged pricing game.*

Proof. See Appendix F.1. □

Theorem 4 illustrates that the potential games approach to multiproduct-firm oligopoly is useful not only to establish equilibrium existence but also for equilibrium uniqueness, including in environments where firms may base their behavior on (imperfectly) correlated signals. The version of the theorem proven in the Appendix (Theorem F.1) is more general, as it does not rely on the assumption that the functions h_j are either power functions or exponential functions. Applying that theorem to the special case where $a(B) = 0$ for non-singleton baskets delivers an equilibrium uniqueness result for IIA demand systems. We also refer the reader to the Appendix for a uniqueness result (Theorem F.2) applying to nested demand systems.

6 Extensions and Applications

In this section, we discuss extensions to "hybrid" demand systems that combine baskets and nests and to competition under incomplete information, and briefly report on comparative statics.

6.1 Combining Baskets and Nests

So far, we have focused on demand systems featuring either baskets or nests. However, as we show now, hybrid demand structures can easily be obtained by combining baskets and nests. A multi-stage discrete/continuous choice micro-foundation can be derived by blending the choice processes considered in Sections 4.1 and 4.3 above. Ordinal potentials can be identified by adapting those given earlier.

Nests of baskets. First, partition the set of firms $\mathcal{F}$ into a set of nests $\mathcal{N}$. Second, make available in each nest the set of baskets containing the firms in that nest. The associated indirect utility function takes the form

$$V(p) = \Psi\left[\sum_{m\in\mathcal{N}} \Phi^m\left(\sum_{B\in 2^m\setminus\{\emptyset\}} a(B)\prod_{g\in B} h^g(p^g)\right)\right].$$

In this example, nests may correspond to shopping malls and baskets to combinations of stores within the same mall. Each consumer would first choose whether to go to a shopping mall or not, and if so, to which mall. Once at a mall, a consumer would decide from which combination of stores to purchase.

Baskets of nests. First, partition again the set of firms, $\mathcal{F}$, into a set of nests, $\mathcal{N}$. Second, create a set of baskets, each consisting of one or more nests. The associated indirect utility function takes the form

$$V(p) = \Psi\left[\sum_{B\in 2^{\mathcal{N}}\setminus\{\emptyset\}} a(B)\prod_{m\in B} \Phi^m\left(\sum_{g\in m} h^g(p^g)\right)\right].$$

In this example, a nest may correspond to a product category (e.g., beer or chips) and a basket may consist of different combinations of these categories (e.g., only beer, only chips, and beer and chips). Each consumer would first select a combination of product categories (if any) and then, from each chosen category, a firm.

6.2 Competition under Incomplete Information

To the best of our knowledge, multiproduct-firm oligopoly has to date been studied only in complete-information settings. The existing literature on *single*-product oligopoly under incomplete information has focused on either homogeneous products (e.g., Vives, 1988; Hansen, 1988; Spulber, 1995) or on differentiated products, typically with linear demand or strong symmetry assumptions (e.g., Vives, 1984; Raith, 1996). In this subsection, we show that the potential games approach can also be deployed to study multiproduct-firm oligopoly with incomplete information and arbitrary firm and product heterogeneity.

For conciseness, we focus here on the competition-in-utility framework for basket demand systems we introduced in Section 4.2. Specifically, suppose that firms are expected utility maximizers with a Bernoulli utility function given by the log profit. There exists a finite state space $\Theta = \prod_{f\in\mathcal{F}} \Theta^f$, with $\omega(\theta)$ denoting the probability of state $\theta \in \Theta$. Firm f's type $\theta^f \in \Theta^f$ is private information and may affect that firm's marginal costs, qualities, and set of products. The value of firm f's Ramsey problem (see equation (16)), which does not depend on other firms' types, can thus be written as $\pi^f(u^f, \theta^f)$, with $u^f \in (\underline{u}^f(\theta^f), u_0^f(\theta^f))$.

The expanded game associated with this incomplete-information pricing game is strategically equivalent to one in which each firm f chooses $(u^f(\theta^f))_{\theta^f\in\Theta^f}$ to maximize

$$\widetilde{\Pi}^f(u(\cdot)) \equiv \sum_{\theta\in\Theta} \left[\log \pi^f(u^f(\theta^f), \theta^f) + \log \Psi'\left(\sum_{B\in\mathcal{B}} a(B) \prod_{g\in B} u^g(\theta^g)\right)\right] \omega(\theta),$$

where we have eliminated the log of the term inside square brackets in equation (17), as it does not depend on u^f. An exact potential for this game is given by

$$\widetilde{E}(u(\cdot)) \equiv \sum_{\theta\in\Theta} \left[\sum_{f\in\mathcal{F}} \log \pi^f(u^f(\theta^f), \theta^f) + \log \Psi'\left(\sum_{B\in\mathcal{B}} a(B) \prod_{g\in B} u^g(\theta^g)\right)\right] \omega(\theta). \tag{20}$$

The argument in the proof of Theorem 2 implies that the exponential of the bracketed term in equation (20) has a continuous extension on $\prod_{f\in\mathcal{F}}[\underline{u}^f(\theta^f), u_0^f(\theta^f)]$. It follows that the exponential of the exact potential also has a continuous extension on $\prod_{\theta\in\Theta}\prod_{f\in\mathcal{F}}[\underline{u}^f(\theta^f), u_0^f(\theta^f)]$. By the Weierstrass theorem, there exists a global maximizer. As boundary points result in a value of zero, that global maximizer must be interior. It therefore corresponds to a pure-strategy Bayes-Nash equilibrium of the multiproduct-firm pricing game under incomplete information. Under the same assumptions as in Section 5 and using again the transformation $t^f = -\log u^f$, the exact potential $\widetilde{E}$ is strictly concave, implying that the pure-strategy Bayes-Nash equilibrium is unique and, in fact, corresponds to the unique agent normal-form correlated equilibrium (see Forges, 1993, 2006).

6.3 Comparative Statics

We now briefly discuss the comparative statics of multiproduct-firm pricing games, focusing on demand systems with baskets. To fix ideas, suppose that the demand system is smooth and there exists a unique global maximizer of the log potential

$$E(u) = \sum_{g\in\mathcal{F}} \log \pi^g(u^g) + \log \Psi'\left(\sum_{B\in\mathcal{B}} a(B) \prod_{g\in B} u^g\right)$$

at which the second-order condition holds strictly. Recall from the introduction that the Nash equilibrium corresponding to that global maximizer has attractive properties. In performing comparative statics, we therefore select this equilibrium.

Strategic substitutes vs. complements. We find that the cross-partial derivative of $E(u)$ with respect to u^f and u^g, which is also equal to $\partial^2 \log \Pi^f / \partial u^f \partial u^g$ and $\partial^2 \log \Pi^g / \partial u^f \partial u^g$, has the same sign as

$$\rho(H)[\eta(H) - \ell^{fg}], \tag{21}$$

where ρ is again the curvature of Ψ, $\eta(H) \equiv 1 - H\rho'(H)/\rho(H)$, and ℓ^{fg} denotes again the lift of firms f and g.

It is instructive to consider the special case in which consumers do not have access to an outside option, implying that $\Psi = \log$, and so $\rho(H) = \eta(H) = 1$ for every H. In that case, as we have seen in Section 4.1, firm f's and g's products are complements if $\ell^{fg} > 1$ and substitutes if $\ell^{fg} < 1$. It follows that the firms' actions are strategic complements if their products are substitutes, and strategic substitutes if their products are complements. Going beyond that special case, there is no longer an unequivocal relationship between substitutability/complementarity and strategic substitutability/complementarity. For example, an increase in the lift ℓ^{fg} makes it more (resp., less) "likely" that firms' actions are strategic substitutes if $\rho > 0$ (resp., $\rho < 0$).

Effects of firm-level profit shifters. Consider a change in a parameter θ^f that smoothly affects the value of firm f's Ramsey problem, $\pi^f(u^f, \theta^f)$, assuming that $\partial^2 \pi^f / \partial u^f \partial \theta^f > 0$. The fact that the equilibrium profile of utilities globally maximizes the potential $E(u)$ implies that the Hessian of $E(u)$ is negative definite. It follows from the implicit function theorem that firm f's equilibrium response consists in providing a higher u^f.[32] If firms' actions are local strategic complements, i.e., if expression (21) is positive for every pair of firms, then the monotone comparative statics theorem implies that the equilibrium responses of *all* firms consist in providing higher utility levels. In the case of two firms, the rival firm $g \neq f$ will provide a higher utility level if expression (21) is positive and a lower utility level if the inequality is reversed.

[32] Let $\Delta u \equiv (du^g / d\theta^f)_{g \in \mathcal{F}}$ denote the column vector of equilibrium responses, $\mathcal{H}$ the Hessian of $E(u)$, and $x = (x^g)_{g \in \mathcal{F}}$ the column vector with $x^g = 0$ for $g \neq f$ and $x^f = \partial^2 E / \partial u^f \partial \theta^f$. By the implicit function theorem, $\Delta u = -\mathcal{H}^{-1} x$, so that

$$\frac{du^f}{d\theta^f} \underbrace{\frac{\partial^2 E}{\partial u^f \partial \theta^f}}_{>0} = x^T \Delta u = x^T (-\mathcal{H}^{-1}) x > 0,$$

where the superscript T denotes the transpose operator and the inequality follows by the negative definiteness of $\mathcal{H}$.

Effects of basket-level utility shifters. Consider an increase in $a(B_0)$ for some basket B_0. We find that the cross-partial derivative of $E(u)$ with respect to $a(B_0)$ and u^f has the same sign as

$$\rho(H)[-\mathbf{1}_{f\in B_0} + \eta(H)s^f],$$

where $\mathbf{1}_{f\in B_0}$ is an indicator function taking the value of 1 if and only if firm f is in basket B_0. For simplicity, we focus on the case in which $\rho(H) > 0$ and $\eta(H) \leq 1$ (which holds, for example, if Ψ is the logarithm or a power function). Observe that the impact of the increase in $a(B_0)$ on firm f's incentive to deliver utility is negative if firm f is in basket B_0 and positive otherwise. In the special case of the "grand basket" containing all firms, this implies that all firms have an incentive to deliver less utility. In equilibrium, at least one firm responds by lowering its utility provision and, in the case of strategic complementarity (see expression (21)), all firms will do so.

7 Conclusion

In this paper, we have pioneered the potential games approach to equilibrium existence and uniqueness in multiproduct-firm pricing games. We have started by showing that multi-product pricing games based on any IIA demand system admit an ordinal potential. As the potential function has a global maximizer, these pricing games have a pure-strategy Nash equilibrium. An important feature of this class of demand systems is that, depending on the level of prices, products can be local substitutes or complements. In the discrete/continuous choice micro-foundation, all products are substitutes conditional on not choosing the outside option, but may become complements through substitution away from (or towards) the outside option.

Next, we have introduced the novel concept of a transformed potential. The advantage of this concept is that, in contrast to an ordinal potential, Monderer and Shapley (1996b)'s cross-partial derivatives test is available for transformed potentials. We have provided a complete characterization of the class of demand systems admitting such a transformed potential regardless of which firm produces which products, along with the associated transformation functions. Those demand systems take the generalized linear or IIA forms.

For a given ownership structure of products, we have shown that the only admissible transformation functions continue to be either linear or logarithmic. Under the linear transformation, the demand systems are more flexible versions of generalized linear demand. Under the logarithmic transformation, the demand systems may contain nests (of closer substitutes) or baskets (of products that are jointly purchased), or combinations thereof. These demand systems have sound micro-foundations and permit richer patterns of substitution and complementarity, going well beyond the IIA property. For both classes of demand systems, which encompass IIA demand as a special case, we have combined the potential games approach with a competition-in-utility space approach to derive powerful equilibrium existence and uniqueness theorems. These results rely on showing that the potential function has a global maximizer under minimal conditions, and that it is strictly concave under stronger conditions.

While demand systems featuring nests are well studied, those featuring baskets are not. We show that, with basket demand, whether products are substitutes or complements depends on the interaction of two channels. The first is the channel working through the outside option, which is already present under IIA demand. The second works through joint purchases. The extent to which the products of two different firms are predominantly purchased together or not is measured by their *lift* and determines the strength of complementarity conditional on forgoing the outside option.

Appendix

A Proof of Theorem 1

We begin by noting that the gradient of the potential functions defined in equations (4) and (5) is equal to the transformed payoff gradient. This implies that the logarithm (resp., the identity function) is an admissible transformation function for IIA demand (resp., generalized linear demand); see Monderer and Shapley (1996b). Hence, (b) implies (a).

In the remainder of this subsection, we show that (a) implies (b). The proofs of all the intermediate lemmas stated below can be found in Appendix A. Suppose that the demand system D admits a G-potential. We introduce new notation. For every $i \in \mathcal{I}$, let $\overline{\pi}_i \equiv \sup_{p\in\mathcal{Q}} p_i D_i(p)$ and $\overline{\pi} \equiv \max_{i\in\mathcal{I}} \overline{\pi}_i$. Define

$$\pi_i : \ (p, c_i) \in \{(p, c_i) \in \mathcal{Q} \times \mathbb{R}_{++} : \ p_i > c_i\} \mapsto (p_i - c_i)D_i(p).$$

The range of π_i is the open interval $(0, \overline{\pi}_i)$. For every $\pi \in (0, \overline{\pi}_i)$, let

$$Q_i(\pi) = \{p \in \mathcal{Q} : \ p_i D_i(p) > \pi\}.$$

For every π, $Q_i(\pi)$ is non-empty and open, and the set function $Q_i(\cdot)$ is non-increasing: $Q_i(\pi) \subseteq Q_i(\pi')$ whenever $\pi \geq \pi'$. Moreover, $p \in Q_i(\pi)$ if and only if there exists $c_i < p_i$ such that $\pi_i(p, c_i) = \pi$.

Let $\varphi(\pi) \equiv \pi G'(\pi)$ for every π. Applying Theorem 4.5 in Monderer and Shapley (1996a), we show that φ solves a certain parameterized ordinary differential equation:

Lemma A.1. *For every $i, j \in \mathcal{I}$ with $i \neq j$, there exists a function $\kappa_{ij}(\cdot)$ such that for every $\pi \in (0, \overline{\pi}_i)$ and $p \in Q_i(\pi)$,*

$$\partial_j D_i \left(1 + \pi \frac{\partial_i D_i}{D_i^2}\right) \varphi'(\pi) + \left(\frac{\partial^2_{ij} D_i}{D_i} - \frac{\partial_i D_i \partial_j D_i}{D_i^2}\right) \varphi(\pi) = \kappa_{ij}(p), \tag{22}$$

where the function D_i and its partial derivatives are all evaluated at p.

Exploiting Lemma A.1, we characterize the admissible transformation functions $G(\pi)$ for π sufficiently close to 0:

Lemma A.2. *There exist constants $\hat{\pi} > 0$, A, B, and C such that $B + C\pi > 0$ and*

$$G(\pi) = A + B \log \pi + C\pi \tag{23}$$

for every $\pi \in (0, \hat{\pi})$.

Using again Theorem 4.5 in Monderer and Shapley (1996a) and the above transformation functions, we show that the demand system must satisfy certain partial differential equations:

Lemma A.3. *If $B \neq 0$ in equation (23), then for every $p \in \mathcal{Q}$,*

$$\forall (i, j, k) \in \mathcal{I}^3 \text{ with } k \neq i, j, \quad \partial_k \frac{D_i(p)}{D_j(p)} = 0,$$

$$\forall (i, j) \in \mathcal{I}^2, \quad \partial^2_{ij} \log \frac{D_i(p)}{D_j(p)} = 0.$$

If $C \neq 0$ in equation (23), then for every $p \in \mathcal{Q}$,

$$\forall (i, j, k) \in \mathcal{I}^3 \text{ with } k \neq i, j, \quad \partial^2_{ik} D_j(p) = 0.$$

Integrating the system of partial differential equations from the second part of the previous lemma (which is straightforward) as well as from the first part (which relies on earlier results by Goldman and Uzawa (1964) and Anderson, Erkal, and Piccinin (2020)) yields:

Lemma A.4. *If $B \neq 0$ in equation (23), then the demand system D takes the IIA form of equation (1) on the domain $\mathcal{Q}$.*
If $C \neq 0$ in equation (23), then the demand system D takes the generalized linear form of equation (3) on the domain $\mathcal{Q}$.

A.1 Proof of Lemma A.1

Proof. Let $i, j \in \mathcal{I}$ with $i \neq j$ and $p \in \mathcal{Q}$. For any vector of marginal costs $c = (c_k)_{k \in \mathcal{I}}$ such that $c_k < p_k$ for every k, Theorem 4.5 in Monderer and Shapley (1996b), applied to the pricing game in which all firms are single-product firms and the marginal cost vector is $(c_k)_{k \in \mathcal{I}}$, implies that

$$\frac{\partial^2}{\partial p_i \partial p_j} G(\pi_i(p, c_i)) = \frac{\partial^2}{\partial p_i \partial p_j} G(\pi_j(p, c_j)). \tag{24}$$

As the right-hand side does not depend on c_{-j} while the left-hand side does not depend on c_{-i}, there exists a function $\kappa_{ij}(p)$, which is independent of the marginal cost vector, such that

$$\frac{\partial^2}{\partial p_i \partial p_j} G(\pi_i(p, c_i)) = \kappa_{ij}(p)$$

for every $c_i < p_i$.

Next, let $\pi \in (0, \overline{\pi}_i)$, and $p \in Q_i(\pi)$, and $c_i > 0$ such that $\pi_i(p, c_i) = \pi$. We have:

$$\begin{aligned}
\kappa_{ij}(p) &= \frac{\partial}{\partial p_i} \left[(p_i - c_i) \partial_j D_i G'(\pi_i(p, c_i)) \right] \\
&= (p_i - c_i) \partial_j D_i \left[D_i + (p_i - c_i) \partial_i D_i \right] G''(\pi_i(p, c_i)) + \left[\partial_j D_i + (p_i - c_i) \partial^2_{ij} D_i \right] G'(\pi_i(p, c_i)) \\
&= \pi \frac{\partial_j D_i}{D_i} \left[D_i + \pi \frac{\partial_i D_i}{D_i} \right] G''(\pi) + \left[\partial_j D_i + \pi \frac{\partial^2_{ij} D_i}{D_i} \right] G'(\pi) \\
&= \partial_j D_i \left[1 + \pi \frac{\partial_i D_i}{D_i^2} \right] (\pi G''(\pi) + G'(\pi)) + \left[\frac{\partial^2_{ij} D_i}{D_i} - \frac{\partial_j D_i \partial_i D_i}{D_i^2} \right] \pi G'(\pi) \\
&= \partial_j D_i \left[1 + \pi \frac{\partial_i D_i}{D_i^2} \right] \varphi'(\pi) + \left[\frac{\partial^2_{ij} D_i}{D_i} - \frac{\partial_j D_i \partial_i D_i}{D_i^2} \right] \varphi(\pi).
\end{aligned}$$

□

A.2 Proof of Lemma A.2

To prove Lemma A.2, we split it into a series of technical lemmas. We introduce new notation. Let $\hat{p} \in \mathcal{Q}$ such that, at $p = \hat{p}$, $\partial_i(p_i D_i) < 0$ and $\partial_j D_i \neq 0$ for some $j \neq i$. (Such a price vector exists, as D is $\mathcal{C}^1$, $\partial_j D_i \neq 0$ almost everywhere, and $\partial_i(p_i D_i(p)) < 0$ for some p.) There exists $\hat{c}_i \in (0, \hat{p}_i)$ such that $(\hat{p}_i - \hat{c}_i)\partial_i D_i(\hat{p}) + D_i(\hat{p}) = 0$. Define $\hat{\pi} \equiv (\hat{p}_i - \hat{c}_i) D_i(\hat{p})$, and note that

$$D_i(\hat{p}) + \hat{\pi} \frac{\partial_i D_i(\hat{p})}{D_i(\hat{p})} = 0,$$

i.e., $\hat{\pi} = -D_i(\hat{p})^2 / \partial_i D_i(\hat{p})$.

We begin by solving differential equation (22) on $(0, \hat{\pi})$ for $p = \hat{p}$:

Lemma A.5. *There exist constants $q \in \mathbb{R}$, $s \geq 0$, and $t \in \mathbb{R}$, such that the function φ takes the form*

$$\varphi(\pi) = q \left(\hat{\pi} - \pi \right)^s + t \tag{25}$$

on the interval $(0, \hat{\pi})$.

Proof. Define

$$\alpha \equiv \partial_j D_i, \ \beta \equiv \frac{\partial^2_{ij} D_i}{D_i} - \frac{\partial_i D_i \partial_j D_i}{D_i^2}, \text{ and } \kappa \equiv \kappa_{ij}(\hat{p}),$$

where the function D_i and its derivatives are evaluated at $\hat{p}$. For every $\pi \in (0, \hat{\pi})$, we have that $\hat{p} \in Q_i(\pi)$, so that Lemma A.1 applies to profit level π at price vector $\hat{p}$. Making use of the above notation, equation (22) can be rewritten as:

$$\alpha \left(1 - \frac{\pi}{\hat{\pi}}\right) \varphi'(\pi) + \beta\varphi(\pi) = \kappa.$$

Dividing both sides by $\alpha(1 - \pi/\hat{\pi})$ yields

$$\varphi'(\pi) + \frac{\beta\hat{\pi}}{\alpha(\hat{\pi} - \pi)}\varphi(\pi) = \frac{\kappa\hat{\pi}}{\alpha(\hat{\pi} - \pi)}.$$

This first-order, inhomogeneous, linear differential equation can be solved using standard techniques.

Suppose first that $\beta \neq 0$. The solutions to the corresponding *homogeneous* differential equation take the form

$$\tilde{\varphi}(\pi) = K(\hat{\pi} - \pi)^{\frac{\beta\hat{\pi}}{\alpha}},$$

where K is a constant of integration. A particular solution to the inhomogeneous differential equation is $\tilde{\varphi}(\pi) = \kappa/\beta$. Hence, φ, as a solution to the inhomogeneous differential equation, must take the form

$$\varphi(\pi) = K(\hat{\pi} - \pi)^{\frac{\beta\hat{\pi}}{\alpha}} + \frac{\kappa}{\beta}.$$

If $K = 0$, then $\varphi(\pi) = \kappa/\beta$ for every $\pi \in (0, \hat{\pi})$, and we obtain functional form (25) by setting $q = 0$, $s = 1$, and $t = \kappa/\beta$. If instead $K \neq 0$, then we obtain functional form (25) by setting $q = K$, $s = \beta\hat{\pi}/\alpha$, and $t = \kappa/\beta$. In the latter case, if s where strictly negative, then φ would tend to $\pm\infty$, implying that G would fail to be continuously differentiable at $\hat{\pi}$, a contradiction. Hence, $s \geq 0$.

Suppose instead that $\beta = 0$. Integrating $\varphi' = \kappa\hat{\pi}/(\alpha(\hat{\pi} - \pi))$ yields

$$\varphi(\pi) = -\frac{\kappa\hat{\pi}}{\alpha}\log(\hat{\pi} - \pi) + K.$$

If $\kappa \neq 0$, then we obtain the contradiction that $\varphi(\pi) \underset{\pi \to \hat{\pi}}{\longrightarrow} \pm\infty$. Hence, $\varphi(\pi) = K$ for every $\pi \in (0, \hat{\pi})$, and it is thus as in equation (25) with $t = K$, $q = 0$, and $s = 1$. □

Next, we use the fact that equation (22) must hold for any p to show that φ must be affine in π for $\pi \in (0, \hat{\pi})$:

Lemma A.6. *There exist constants* B *and* C *such that* $\varphi(\pi) = B + C\pi$ *for every* $\pi \in (0, \hat{\pi})$.

Proof. By Lemma A.5, φ must take the form of equation (25) with $s \geq 0$ on the interval $(0, \hat{\pi})$. Assume for a contradiction that $q \neq 0$ and $s \neq 1$. Note that φ is $\mathcal{C}^2$ on $(0, \hat{\pi})$, and satisfies

$$\frac{\varphi''(\pi)}{\varphi'(\pi)} = \frac{1-s}{\hat{\pi}-\pi}.$$

Let $\tilde{\pi} \in (0, \hat{\pi})$. As $\hat{p} \in Q_i(\tilde{\pi})$, $\partial_j D_i(\hat{p}) \neq 0$, and the demand system is $\mathcal{C}^1$, there exist an open and convex set $O \subseteq \mathcal{Q}$ and an $\zeta > 0$ such that $p \in Q_i(\pi)$ and $\partial_j D_i(p) \neq 0$ for every $p \in O$ and $\pi \in (\tilde{\pi} - \zeta, \tilde{\pi} + \zeta)$. By Lemma A.1, equation (22) must hold for every such p and π. We can therefore differentiate that equation with respect to π to obtain

$$\partial_j D_i(p) \left[1 + \pi \frac{\partial_i D_i(p)}{D_i(p)^2}\right] \varphi''(\pi) + \frac{\partial^2_{ij} D_i(p)}{D_i(p)} \varphi'(\pi) = 0$$

for every $p \in O$ and $\pi \in (\tilde{\pi} - \zeta, \tilde{\pi} + \zeta)$. Dividing both sides by $\varphi'(\pi)$ and using the above expression for φ''/φ' yields

$$\partial_j D_i(p) \frac{1-s}{\hat{\pi}} \frac{1 + \pi \frac{\partial_i D_i(p)}{D_i(p)^2}}{1 - \frac{\pi}{\hat{\pi}}} + \frac{\partial^2_{ij} D_i(p)}{D_i(p)} = 0.$$

As the above condition must hold for every $p \in O$ and $\pi \in (\tilde{\pi} - \zeta, \tilde{\pi} + \zeta)$ and $(1-s)\partial_j D_i(p) \neq 0$, it follows that

$$\frac{\partial_i D_i(p)}{D_i(p)^2} = -\frac{1}{\hat{\pi}} \tag{26}$$

$$\text{and } \partial^2_{ij} D_i(p) = -\frac{1-s}{\hat{\pi}} D_i(p) \partial_j D_i(p) \tag{27}$$

for every $p \in O$.

Condition (26) can be rewritten as $\partial_i(1/D_i(p)) = 1/\hat{\pi}$. As it holds for every p in the open and convex set O, there exists a $\mathcal{C}^2$ function ϕ such that

$$\frac{1}{D_i(p)} = \frac{p_i}{\hat{\pi}} + \phi(p_{-i})$$

for every $p \in O$. Differentiating this with respect to p_j yields

$$-\frac{\partial_j D_i(p)}{D_i(p)^2} = \partial_j \phi(p_{-i}),$$

i.e., $\partial_j D_i(p) = -D_i(p)^2 \partial_j \phi(p_{-i})$. Further differentiating with respect to p_i, we obtain:

$$\partial^2_{ij} D_i(p) = -2 D_i(p) \partial_i D_i(p) \partial_j \phi(p_{-i}) = \frac{2}{\hat{\pi}} D_i^3(p) \partial_j \phi(p_{-i}),$$

where we have used equation (26) to obtain the second equality. Hence,

$$\frac{\partial^2_{ij} D_i(p)}{\partial_j D_i(p)} = -\frac{2}{\hat{\pi}} D_i(p)$$

for every $p \in O$. Combining this with condition (27), we obtain that $1 - s = 2$, i.e., $s = -1$, which is a contradiction, as s must be non-negative. □

We are now in a position to prove Lemma A.2. By Lemma A.6, we have that $G'(\pi) = C + B/\pi$ for every $\pi \in (0, \hat{\pi})$. Hence, for some constant of integration A, $G(\pi) = A + B \log \pi + C\pi$. Moreover, as G' must be strictly positive, it must be that $B + C\pi > 0$ for every $\pi \in (0, \hat{\pi})$.

A.3 Proof of Lemma A.3

Proof. Let $p \in \mathcal{Q}$, $i, j \in \mathcal{I}$, and $k, l \in \mathcal{I} \setminus \{i, j\}$, where i may or may not be equal to j, and k may or may not be equal to l. Let $f = \{i, j\}$ and $g = \{k, l\}$, and consider the firm partition $\mathcal{F} \equiv \{f, g, \mathcal{I} \setminus (f \cup g)\}$. For every $i' \in \mathcal{I} \setminus (f \cup g)$, fix some $c_{i'} \in (0, p_{i'})$. Choose $c_j \in (0, p_j)$ such that $\pi_j(p, c_j) < \hat{\pi}$ and, if $i \neq j$, let $c_i = p_i$. Similarly, choose $c_l \in (0, p_l)$ such that $\pi_l(p, c_l) < \hat{\pi}$ and, if $k \neq l$, let $c_k = p_k$. We have thus defined a multiproduct-firm pricing game. Note that, by construction, $\Pi^f(p) \in (0, \hat{\pi})$ and $\Pi^g(p) \in (0, \hat{\pi})$, where $\Pi^h(p) \equiv \sum_{n \in h}(p_n - c_n)D_n(p)$ for $h \in \{f, g\}$. That is, both firms' profits are within the domain to which Lemma A.2 applies. Moreover, the firms' profits remain in that domain for small perturbations of the marginal cost vector.

By Theorem 4.5 in Monderer and Shapley (1996b),[33] we have that

$$\frac{\partial^2}{\partial p_i \partial p_k} G\left[\Pi^f(p)\right] = \frac{\partial^2}{\partial p_i \partial p_k} G\left[\Pi^g(p)\right].$$

If $i \neq j$, then, by Lemma A.2,

$$\begin{aligned}
\partial^2_{ik} G(\Pi^f) &= \partial^2_{ik}\left[B \log \Pi^f + C\Pi^f\right] \\
&= B\partial_k \frac{D_i + (p_j - c_j)\partial_i D_j}{(p_j - c_j)D_j} + C\partial_k\left[D_i + (p_j - c_j)\partial_i D_j\right] \\
&= B\left[\frac{1}{p_j - c_j}\partial_k \frac{D_i}{D_j} + \partial^2_{ik} \log D_j\right] + C\left[\partial_k D_i + (p_j - c_j)\partial^2_{ik} D_j\right].
\end{aligned}$$

If instead $i = j$, then we obtain the same expression using again Lemma A.2:

$$\begin{aligned}
\partial^2_{ik} G(\Pi^f) &= B\partial^2_{ik} \log D_j + C\left[\partial_k D_j + (p_j - c_j)\partial^2_{ik} D_j\right] \\
&= B\left[\frac{1}{p_j - c_j}\partial_k \frac{D_i}{D_j} + \partial^2_{ik} \log D_j\right] + C\left[\partial_k D_i + (p_j - c_j)\partial^2_{ik} D_j\right].
\end{aligned}$$

Similarly, we obtain

$$\partial^2_{ik} G(\Pi^g) = B\left[\frac{1}{p_l - c_l}\partial_i \frac{D_k}{D_l} + \partial^2_{ik} \log D_l\right] + C\left[\partial_i D_k + (p_l - c_l)\partial^2_{ik} D_l\right].$$

Plugging those expressions into the above condition on cross-partial derivatives and using the fact that $\partial_k D_i = \partial_i D_k$ yields:

[33] Although Monderer and Shapley stated their theorem for uni-dimensional action sets, it is straightforward to extend it to multi-dimensional action sets.

$$B\left[\frac{1}{p_j - c_j}\partial_k \frac{D_i}{D_j} + \partial^2_{ik}\log D_j - \frac{1}{p_l - c_l}\partial_i \frac{D_k}{D_l} - \partial^2_{ik}\log D_l\right] + C\left[(p_j - c_j)\partial^2_{ik}D_j - (p_l - c_l)\partial^2_{ik}D_l\right] = 0. \quad (28)$$

As condition (28) must hold on an open set of costs c_j and c_l, we can differentiate it twice with respect to c_j and c_l to obtain $B\partial_k D_i/D_j = 0$ and $B\partial_i D_k/D_l = 0$. Hence $\partial_k D_i/D_j = 0$ and $\partial_i D_k/D_l = 0$ if $B \neq 0$. Moreover, regardless of whether $B \neq 0$, condition (28) reduces to

$$B\partial^2_{ik}\log\frac{D_j}{D_l} + C\left[(p_j - c_j)\partial^2_{ik}D_j - (p_l - c_l)\partial^2_{ik}D_l\right] = 0.$$

As this condition must again hold on an open set of costs c_j and c_l, we can differentiate it once with respect to c_j and c_l to obtain $C\partial^2_{ik}D_j = 0$ and $C\partial^2_{ik}D_l = 0$, which implies that $\partial^2_{ik}D_j = 0$ and $\partial^2_{ik}D_l = 0$ if $C \neq 0$. Moreover, regardless of whether $C \neq 0$, the condition reduces to $B\partial^2_{ik}\log(D_j/D_l) = 0$. Hence, $\partial^2_{ik}\log(D_j/D_l) = 0$ if $B \neq 0$. □

A.4 Proof of Lemma A.4

To prove Lemma A.4, we split it into two technical lemmas. We begin by integrating the system of partial differential equations in the second part of Lemma A.3:

Lemma A.7. *Suppose that $\partial^2_{ik}D_j = 0$ for every $i, j, k \in \mathcal{I}$ with $k \neq i, j$. Then, the demand system D takes the generalized linear form of equation* (3).

Proof. Fix some j in $\mathcal{I}$. As $\partial_k(\partial_j D_j) = 0$ for every $k \neq j$, we have that $\partial_j D_j$ is independent of p_{-j}. Therefore, there exist functions ϕ_j and ψ_j such that $D_j(p) = \phi_j(p_j) + \psi_j(p_{-j})$ for every $p \in \mathcal{Q}$. Moreover, for every $i, k \neq j$, we have that $\partial^2_{ik}\psi_j(p_{-j}) = \partial^2_{ik}D_j = \partial_{ij}D_k = 0$, where we have used Slutsky symmetry to obtain the second equality. It follows that, for every $i \neq j$, $\partial_i\psi_j$ is equal to some constant α_{ji}. Hence, $\partial_i\left(\psi_j(p_{-j}) - \sum_{j\neq i}\alpha_{ji}p_j\right) = 0$, and so $\psi_j(p_{-j}) = \beta_j + \sum_{j\neq i}\alpha_{ji}p_j$ for some constant of integration β_j. Setting $h'_j(p_j) = -\phi_j(p_j) - \beta_j$ for every j, we obtain the generalized linear form of equation (3). The fact that $\alpha_{ij} = \alpha_{ji}$ follows immediately by Slutsky symmetry. □

Next, we turn to the system of partial differential equations in the first part of Lemma A.3:

Lemma A.8. *Suppose that, for every $i, j, k \in \mathcal{I}$ such that $k \neq i, j$, $\partial_k(D_i/D_j) = 0$, and, for every $i, j \in \mathcal{I}$, $\partial^2_{ij}\log(D_i/D_j) = 0$. Then, the demand system D takes the IIA form of equation* (1).

Proof. Suppose first that $|\mathcal{I}| \geq 3$. Then, the result follows from Proposition 1 in Anderson, Erkal, and Piccinin (2020), the proof of which we replicate here. We have that, for every $p \in \mathcal{Q}$ and every $i, j, k \in \mathcal{I}$ such that $k \neq i, j$, $\partial_k(\partial_i V(p)/\partial_j V(p)) = 0$. Thus, using terminology introduced by Goldman and Uzawa (1964), the function $-V$ is strongly separable with respect to the partition $\{\{n\}\}_{n\in\mathcal{I}}$. Moreover, that function is $\mathcal{C}^3$ on $\mathcal{Q}$, its level sets are connected surfaces, and its partial derivatives are strictly positive everywhere on $\mathcal{Q}$. Theorem 1 in Goldman and Uzawa (1964) then implies that $-V$ takes the form[34]

$$-V(p) = -\Psi\left(\sum_{j\in\mathcal{I}} h_j(p_j)\right).$$

Suppose instead that $|\mathcal{I}| = 2$, and write $\mathcal{I} = \{1, 2\}$. As $\partial^2_{12}\log(D_1/D_2) = 0$, there exist functions ϕ_1 and ϕ_2 such that

$$\log\frac{D_1(p)}{D_2(p)} = \phi_1(p_1) - \phi_2(p_2)$$

for every $p \in \mathcal{Q}$. Taking exponentials, this implies that

$$\frac{D_1(p)}{D_2(p)} = \frac{\mathrm{e}^{\phi_1(p_1)}}{\mathrm{e}^{\phi_2(p_2)}}.$$

For $i = 1, 2$, let h_i be an anti-derivative of e^{ϕ_i}, so that

$$\frac{\partial_1 V(p)}{\partial_2 V(p)} = \frac{h_1'(p_1)}{h_2'(p_2)},$$

which means that there exists a function λ such that

$$\frac{\partial_1 V(p)}{h_1'(p_1)} = \lambda(p) = \frac{\partial_2 V(p)}{h_2'(p_2)}.$$

By Lemma 1 in Goldman and Uzawa (1964), there thus exists a function Ψ such that

$$V(p) = \Psi\left(h_1(p_1) + h_2(p_2)\right). \qquad \square$$

B Proof of Proposition 2

We use the following notation throughout this section: for every $p \in \mathcal{Q}$, $c \in \mathbb{R}^{\mathcal{I}}_{++}$, and $f \in \mathcal{F}$,

$$\Pi^f(p, c) \equiv \sum_{j\in f}(p_j - c_j)D_j(p).$$

[34] Although Goldman and Uzawa stated their results for utility functions defined on the entire non-negative orthant, their proofs continue to go through for utility functions defined over a convex subset of that orthant.

B.1 Proof that (b) implies (a)

In this subsection, we show that (b) implies (a) and that the logarithm (resp. the identity function) is an admissible transformation function for the demand system. If (b)-(ii) holds, then this follows immediately from the fact that the payoff gradient is equal to the gradient of the potential function defined in equation (6).

Suppose instead that (b)-(i) holds. To prove that the logarithm is an admissible transformation function for the demand system, all we need to do is show that Monderer and Shapley (1996b)'s necessary and sufficient condition holds for the pricing game with transformed payoffs at every $p \in Q$ and $c \in \prod_{j\in\mathcal{I}}(0, p_j)$. That is, we need to show that, for every $f, g \in \mathcal{F}$ such that $f \neq g$, for every $i \in f$ and $k \in g$,

$$\frac{\partial^2}{\partial p_i \partial p_k} \log \Pi^f(p, c) = \frac{\partial^2}{\partial p_i \partial p_k} \log \Pi^g(p, c). \tag{29}$$

We have that

$$\partial^2_{ik} \log \Pi^f = \partial^2_{ik} \left(\log D_i + \log \left[\sum_{j\in f} (p_j - c_j) \frac{D_j}{D_i} \right] \right) = \partial^2_{ik} \log D_i,$$

where the second equality follows as $\partial_k(D_i/D_j) = 0$ for every $j \in f$. Similarly, $\partial^2_{ik} \log \Pi^g = \partial^2_{ik} \log D_k$, so that condition (29) reduces to $\partial^2_{ik} \log D_i = \partial^2_{ik} \log D_k$, which holds by assumption.

B.2 Proof that (a) implies (b)

We split this part into a series of technical lemmas. We begin by stating the analogue of Lemma A.1:

Lemma B.1. *Let $f, g \in \mathcal{F}$ with $f \neq g$, $i \in f$ and $j \in g$. There exists a function $\kappa(\cdot)$ such that for every $\pi \in (0, \overline{\pi}_i)$ and $p \in Q_i(\pi)$,*

$$\partial_j D_i \left(1 + \pi \frac{\partial_i D_i}{D_i^2}\right) \varphi'(\pi) + \left(\frac{\partial^2_{ij} D_i}{D_i} - \frac{\partial_i D_i \partial_j D_i}{D_i^2}\right) \varphi(\pi) = \kappa(p), \tag{30}$$

where the function D_i and its partial derivatives are all evaluated at p.

Proof. Let $p \in \mathcal{Q}$ and c be a marginal cost vector such that $c_k = p_k$ for every $k \in f \cup g \setminus \{i, j\}$ and $c_k \in (0, p_k)$ for every other product k. Theorem 4.5 in Monderer and Shapley (1996b) implies that $\partial^2_{ij} G^f(\Pi^f(p, c)) = \partial^2_{ij} G^g(\Pi^g(p, c))$. As $p_k = c_k$ for every $k \in f \cup g \setminus \{i, j\}$, this is equivalent to $\partial^2_{ij} G(\pi_i(p, c_i)) = \partial^2_{ij} G(\pi_j(p, c_j))$, i.e., condition (24) in the proof of Lemma A.1 holds. The rest of the proof of that lemma can then be replicated word for word to obtain the result. □

Next, we state the analogue of Lemma A.2:

Lemma B.2. *There exist constants $\hat{\pi} > 0$, A, B, and C such that $B + C\pi > 0$ and $G(\pi) = A + B \log \pi + C\pi$ for every $\pi \in (0, \hat{\pi})$.*

Proof. Given Lemma B.1, the proof of Lemma A.2 can be replicated to obtain the result. □

Next, we state the analogue of Lemma A.3:

Lemma B.3. *Let $f, g \in \mathcal{F}$, $f \neq g$, and $(i, j, k) \in f \times f \times g$. If $B \neq 0$ in Lemma B.2, then $\partial_k(D_i/D_j) = 0$ and $\partial^2_{ik} \log(D_i/D_k) = 0$. If $C \neq 0$ in Lemma B.2, then $\partial^2_{ik} D_j = 0$.*

Proof. Given Lemma B.2, we can proceed as in the proof of Lemma A.3. □

To integrate the system of partial differential equations in the second part of Lemma B.3, we require the following technical lemma:

Lemma B.4. *Let $n \geq 2$ and R be a partition of $\{1, \ldots, n\}$ containing at least two elements. Let $F : X \longrightarrow \mathbb{R}$ be $|R|+1$ times continuously differentiable over its open and convex domain $X \subseteq R^n$. Suppose that, for every $r \in R$, for every $i, j \in r$, $\partial^2_{ij} F = 0$. Then, F takes the form*

$$F(x) = \alpha + \sum_{R' \subseteq R} \sum_{\iota \in \prod_{r \in R'} r} \alpha(\iota) \prod_{r \in R'} x_{\iota(r)}.$$

Proof. We prove the result by induction on $|R|$. If $|R| = 1$, then $\partial^2_{ij} F = 0$ for every $1 \leq i, j \leq n$, and so $\partial_i F$ is equal to some constant α_i for every i. It follows that $F(x) = \alpha + \sum_{i=1}^n \alpha_i x_i$ for some α, establishing the property for $|R| = 1$.

Next, suppose that $|R| > 1$ and that the property holds for $|R| - 1$. Let $r_0 \in R$. We have that $\partial^2_{ij} F = 0$ for every $i, j \in r_0$, implying that, for every $i \in r_0$, $\partial_i F(x)$ is equal to some $\mathcal{C}^{|R|}$ function $\beta_i(x^{-r_0})$ of the subvector $x^{-r_0} = (x_j)_{\substack{1 \leq j \leq n \\ j \notin r_0}}$. Observe that Y, the domain of β_i, is open and convex.[35] Thus, for every $i \in r_0$, we have that $\partial_i \left(F(x) - \sum_{j \in r_0} \beta_j(x^{-r_0}) x_j \right) = 0$, which implies the existence of a $\mathcal{C}^{|R|}$ function $\beta_0(x^{-r_0})$ such that

$$F(x) = \beta_0(x^{-r_0}) + \sum_{j \in r_0} \beta_j(x^{-r_0}) x_j. \tag{31}$$

Let $r \in R \setminus \{r_0\}$. For every $k, \ell \in r$, we have that

$$\partial^2_{k\ell} F = \partial^2_{k\ell} \beta_0(x^{-r_0}) + \sum_{j \in r_0} \partial^2_{k\ell} \beta_j(x^{-r_0}) x_j = 0.$$

[35] Convexity is immediate. To see why Y is open, note that $Y = \bigcup_{z \in \mathbb{R}^{r_0}} \{y : (z, y) \in X\}$ and each of the sets in the union is open.

For every $x^{-r_0} \in Y$, the above condition has to hold on the open set $\{x^{r_0} = (x_j)_{j\in r_0} : (x^{r_0}, x^{-r_0}) \in X\}$. Hence, $\partial^2_{k\ell}\beta_0 = 0$ and $\partial^2_{k\ell}\beta_j = 0$ for every $j \in r$. We can therefore apply the induction hypothesis to each of the β functions, $\mathcal{C}^{|R|}$ over the open and convex domain Y, with partition $R \setminus \{r_0\}$, to obtain that, for every $j \in r_0 \cup \{0\}$, β_j takes the form

$$\beta_j(y) = \alpha_j + \sum_{R' \subseteq R\setminus\{r_0\}} \sum_{\iota \in \prod_{r\in R'} r} \alpha(\iota) \prod_{r\in R'} y_{\iota(r)}.$$

Combining this with equation (31) proves the lemma. □

Armed with Lemma B.4, we integrate the system of partial differential equations in the second part of Lemma B.3:

Lemma B.5. *Assume that, for every $f, g \in \mathcal{F}$ with $f \neq g$, and $(i, j, k) \in f \times f \times g$, $\partial^2_{ik} D_j = 0$. Then, D takes the following form: For any $f \in \mathcal{F}$ and $i \in f$,*

$$D_i(p) = -\partial_i h^f(p^f) + \sum_{\substack{\mathcal{F}'\subseteq\mathcal{F}:\\ f\in\mathcal{F}'}} \sum_{\substack{\iota\in\prod_{g\in\mathcal{F}'} g:\\ \iota(f)=i}} \alpha(\iota) \prod_{\substack{g\in\mathcal{F}':\\ g\neq f}} p_{\iota(g)}.$$

Proof. Let $f \in \mathcal{F}$ and $i \in f$. As $\partial^2_{jk} D_i = 0$ for every $j \in f$ and $k \notin f$, we have that, for every $j \in f$, $\partial_j D_i = \phi^j(p^f)$, for some function ϕ^j of firm f's price vector p^f. As the functions $(\phi^j(p^f))_{j\in f}$ satisfy $\partial_m \phi^j = \partial_{jm} D_i = \partial_j \phi^m$ for every $j, m \in f$, the Poincaré lemma implies the existence of a function $d_i(p^f)$ such that $\partial_j d_i(p^f) = \phi^j(p^f)$ for every $j \in f$. Hence, $\partial_j(D_i(p) - d_i(p^f)) = 0$ for every $j \in f$, which implies that D_i can be written as $D_i(p) = d_i(p^f) + \delta_i(p^{-f})$. Moreover, as $\partial_j d_i = \partial_j D_i = \partial_i D_j = \partial_i d_j$ for every $i, j \in f$, the Poincaré lemma implies the existence of a function $h^f(p^f)$ such that $d_i(p^f) = -\partial_i h^f(p^f)$ for every $i \in f$.

For every $g \in \mathcal{F} \setminus \{f\}$ and $k, m \in g$, we have that

$$\partial^2_{km}\delta_i(p^{-f}) = \partial^2_{km} D_i = \partial^2_{ki} D_m = 0,$$

where the second equality follows by Slutsky symmetry. We can therefore apply Lemma B.4 to the $\mathcal{C}^{|\mathcal{F}|}$ function δ_i (with partition $\mathcal{F} \setminus \{f\}$): this function must take the form

$$\delta_i(p^{-f}) = \alpha_i + \sum_{\mathcal{F}'\subseteq\mathcal{F}\setminus\{f\}} \sum_{\iota\in\prod_{g\in\mathcal{F}'} g} \alpha_i(\iota) \prod_{g\in\mathcal{F}'} p_{\iota(g)}.$$

Having done that for every i, we obtain a collection of weights $(\alpha_i, \alpha_i(\iota))$ for every i and ι. For every $\mathcal{F}' \subseteq \mathcal{F}$ and $\iota \in \prod_{g\in\mathcal{F}'} g$, define $\alpha^f(\iota)$ as follows. If $f \in \mathcal{F}'$ and $|\mathcal{F}'| \geq 2$, then let ι' be the element of $\prod_{g\in\mathcal{F}'\setminus\{f\}} g$ such that $\iota'(g) = \iota(g)$ for every $g \in \mathcal{F}' \setminus \{f\}$, and set

$\alpha^f(\iota) \equiv \alpha_{\iota(f)}(\iota')$. If $\mathcal{F}' = \{f\}$, then let $\alpha^f(\iota) \equiv \alpha_{\iota(f)}$. Finally, if $f \notin \mathcal{F}'$, then let $\alpha^f(\iota)$ take any value. Note that, for every $i \in f$,

$$\delta_i = \partial_i \sum_{\mathcal{F}' \subseteq \mathcal{F}} \sum_{\iota \in \prod_{g \in \mathcal{F}'} g} \alpha^f(\iota) \prod_{g \in \mathcal{F}'} p_{\iota(g)}.$$

Let $f' \in \mathcal{F} \setminus \{f\}$ and $(i,k) \in f \times f'$. By Slutsky symmetry, we have that $\partial_k \delta_i = \partial_i \delta_k$, implying that, for every $p \in \mathcal{Q}$,

$$\sum_{\substack{\mathcal{F}' \subseteq \mathcal{F}: \\ f, f' \in \mathcal{F}'}} \sum_{\substack{\iota \in \prod_{g \in \mathcal{F}'} g: \\ \iota(f) = i \text{ and } \iota(f') = k}} \left[\alpha^f(\iota) - \alpha^{f'}(\iota)\right] \prod_{\substack{g \in \mathcal{F}': \\ g \neq f, f'}} p_{\iota(g)} = 0.$$

As the above condition has to hold on an open set of prices, it must be that $\alpha^f(\iota) = \alpha^{f'}(\iota)$ for every $\mathcal{F}'$ such that $f, f' \in \mathcal{F}'$ and $\iota \in \prod_{g \in \mathcal{F}'} g$ such that $\iota(f) = i$ and $\iota(f') = k$. Hence, there exists a function $\alpha(\cdot)$, defined over $\bigcup_{\mathcal{F}' \subseteq \mathcal{F}} \prod_{g \in \mathcal{F}'} g$ such that, for every $f \in \mathcal{F}$, $\mathcal{F}' \subseteq \mathcal{F}$ such that $f \in \mathcal{F}'$, and $\iota \in \prod_{g \in \mathcal{F}'} g$, $\alpha^f(\iota) = \alpha(\iota)$. Moreover, as the weight $\alpha^f(\iota)$ is irrelevant for firm f's demand whenever $\iota \in \prod_{g \in \mathcal{F}'} g$ with $f \notin \mathcal{F}'$, we have that

$$\delta_i = \partial_i \sum_{\mathcal{F}' \subseteq \mathcal{F}} \sum_{\iota \in \prod_{g \in \mathcal{F}'} g} \alpha(\iota) \prod_{g \in \mathcal{F}'} p_{\iota(g)} = \sum_{\substack{\mathcal{F}' \subseteq \mathcal{F}: \\ f \in \mathcal{F}'}} \sum_{\substack{\iota \in \prod_{g \in \mathcal{F}'} g: \\ \iota(f) = i}} \alpha(\iota) \prod_{\substack{g \in \mathcal{F}': \\ g \neq f}} p_{\iota(g)}$$

for every $i \in f$. □

C Proof of Proposition 3

Proof. To prove the "only if" part of the proposition, let $\mathcal{F} = \{f_1, f_2\}$. Suppose that $(D, \mathcal{F})$ admits a log-potential. By Lemma B.2, for every $f, g \in \mathcal{F}$ with $f \neq g$, $i, j \in f$, and $k \in g$, we have that $\partial_k D_i / D_j = 0$ and $\partial^2_{ik} \log D_i / D_k = 0$. As $\partial_k (\partial_i V / \partial_j V) = 0$ for every $f, g \in \mathcal{F}$ with $f \neq g$, $i, j \in f$, and $k \in g$, the function V is weakly separable with respect to the partition $\mathcal{F}$. Hence, by Theorem 2 in Goldman and Uzawa (1964), there exist functions Λ and ϕ^f such that, for every p, we have:

$$V(p) = \Lambda\left(\phi^{f_1}\left(p^{f_1}\right), \phi^{f_2}\left(p^{f_2}\right)\right).$$

Hence, for every $\iota \in \{1, 2\}$ and $i \in f_\iota$, $D_i(p) = -\partial_i \phi^{f_\iota}\left(p^{f_\iota}\right) \partial_\iota \Lambda\left(\phi^{f_1}\left(p^{f_1}\right), \phi^{f_2}\left(p^{f_2}\right)\right)$.

Let $i \in f_1$ and $k \in f_2$. We have:

$$0 = \partial^2_{ik} \log \frac{D_i}{D_k} = \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_k} \log \left[\frac{\partial_i \phi^{f_1}\left(p^{f_1}\right)}{\partial_k \phi^{f_2}\left(p^{f_2}\right)} \frac{\partial_1 \Lambda\left(\phi^{f_1}\left(p^{f_1}\right), \phi^{f_2}\left(p^{f_2}\right)\right)}{\partial_2 \Lambda\left(\phi^{f_1}\left(p^{f_1}\right), \phi^{f_2}\left(p^{f_2}\right)\right)}\right]$$
$$= \frac{\partial \phi^{f_1}\left(p^{f_1}\right)}{\partial p_i} \frac{\partial}{\partial \phi^{f_1}} \frac{\partial \phi^{f_2}\left(p^{f_2}\right)}{\partial p_k} \frac{\partial}{\partial \phi^{f_2}} \log \frac{\partial_1 \Lambda\left(\phi^{f_1}\left(p^{f_1}\right), \phi^{f_2}\left(p^{f_2}\right)\right)}{\partial_2 \Lambda\left(\phi^{f_1}\left(p^{f_1}\right), \phi^{f_2}\left(p^{f_2}\right)\right)}$$

$$= \partial_i \phi^{f_1}\left(p^{f_1}\right) \partial_k \phi^{f_2}\left(p^{f_2}\right) \partial^2_{12} \log \frac{\partial_1 \Lambda(x_1, x_2)}{\partial_2 \Lambda(x_1, x_2)}\bigg|_{x_1=\phi^{f_1}\left(p^{f_1}\right),\ x_2=\phi^{f_2}\left(p^{f_2}\right)}.$$

As $\partial_i \phi^{f_1}\left(p^{f_1}\right) \neq 0$ and $\partial_k \phi^{f_2}\left(p^{f_2}\right) \neq 0$ for every $p \in \mathcal{Q}$, it follows that

$$\partial^2_{12} \log \frac{\partial_1 \Lambda(x_1, x_2)}{\partial_2 \Lambda(x_1, x_2)} = 0$$

for every (x_1, x_2) in the domain of Λ.

We integrated that same partial differential equation in the second half of the proof of Lemma A.4: the solutions take the form $\Lambda(x_1, x_2) = \Psi(h_1(x_1) + h_2(x_2))$ for some functions Ψ, h_1, and h_2. It follows that

$$V(p) = \Psi\left[h_1\left(\phi^{f_1}\left(p^{f_1}\right)\right) + h_2\left(\phi^{f_2}\left(p^{f_2}\right)\right)\right]$$

Defining $h^{f_\iota} \equiv h_\iota \circ \phi^{f_\iota}$ and using Roy's identity proves the proposition. □

D Micro-Foundations for Baskets and Nests: Details and Relegated Proofs

D.1 Baskets

We begin by proving the integrability part of Proposition 6.

Proof of Proposition 6. We need to show that the function V defined in equation (11) is continuous, decreasing, and convex. The first two properties are obvious given Assumption C′. To prove the third property, define $\hat{\Psi}(x) \equiv \Psi(\mathrm{e}^x)$ and

$$\hat{V}(p) \equiv \log\left[\sum_{B \in \mathcal{B}} a(B) \prod_{g \in B} h^g(p^g)\right],$$

and note that $V(p) = \hat{\Psi} \circ \hat{V}(p)$. As each term in the sum in the definition of $\hat{V}$ is log-convex by Assumptions C′-(i), and the sum of log-convex functions is log-convex, it follows that $\hat{V}$ is convex. Moreover, $\hat{\Psi}$ is non-decreasing and convex, as $\hat{\Psi}'(x) = \mathrm{e}^x\, \Psi'(\mathrm{e}^x)$ is non-decreasing by Assumption C′-(ii). Hence, V is convex, as the composition of a convex and non-decreasing function and a convex function. □

Next, we discuss how the lift changes with prices. The derivative of the lift between firms f and g with respect to the price of good j, sold by firm f or g, can be shown to have the same sign as $\ell^{fg} - 1$. Hence, if $\ell^{fg} < 1$, the lift remains below 1, no matter the prices of firms f and g. Conversely, if $\ell^{fg} > 1$, the lift remains above 1, no matter the prices of firms f

and g. Moreover, assuming that $\lim_{p^f \to 0} h^f(p^f) = \infty$ and that there exists a basket B (with $a(B) > 0$) in which firms f and g are both present, the lift converges to 1 as the prices of firm f converge to 0.

We claim that these observations imply that a decrease in firm f's prices tend to make it more "likely" that that firm's products become complements with those of *any other* firm. For simplicity, suppose that Ψ is a power function, implying that its curvature is constant and strictly less than 1. Recall that the products of firms f and g are complements if and only if $\ell^{fg} > \rho$. If $\ell^{fg} > 1$, this complementarity condition is satisfied and, from the above, remains so as firm f's prices decrease. If instead $\ell^{fg} < 1$, then ℓ^{fg} increases with a decrease in firm f's prices, converging to 1 in the limit. Hence, either the complementarity condition initially holds and continues to do so, or else it holds when firm f's prices decrease below some threshold.

D.2 Nests

We begin by completing the description of the micro-foundation for nested demand systems. A consumer who turns down the stage-1 outside option, chooses nest n at stage 2, turns down the nest-level outside option at stage 3, and chooses firm f at stage 4 receives an indirect utility of $\log v^f(p^f) + \varepsilon^f + \varepsilon_1^n + \varepsilon^n + \varepsilon$, where ε is the stage-1 taste shock to the inside goods, ε^n the stage-2 taste shock to nest n, ε_1^n the stage-3 taste shock to the inside goods in nest n, and ε^f the stage-4 taste shock to firm f. A consumer who takes the nest-level outside option at stage 3, after having turned down the outside option at stage 1 and chosen nest n at stage 2, receives utility $\varepsilon_0^n + \varepsilon^n + \varepsilon$, where ε_0^n is the stage-3 taste shock to the nest-level outside option. Finally, a consumer who takes the stage-1 outside option receives utility ε_0. The stage-1 taste shocks $(\varepsilon_0, \varepsilon)$ are distributed according to the measure μ. The conditional indirect utility functions v^f $(f \in \mathcal{F})$ and the measure μ satisfy the same assumptions as in Section 4.1. The stage-3 taste shocks in nest n, $(\varepsilon_0^n, \varepsilon_1^n) \in [-\infty, \infty)^2$, are drawn from a probability measure μ^n that satisfies the same assumptions as μ.

We can now prove Proposition 7:

Proof of Proposition 7. Consider a multi-stage discrete/continuous choice process with nests, as described above and in Section 4.3. We derive the optimal consumer behavior by backward induction.

At stage 5, the conditional demand for product i sold by firm f is $-\partial_i v^f$ by Roy's identity. At stage 4, the Holman-Marley theorem implies that, conditional on having turned down the nest-level outside option, a consumer chooses firm f with probability $s^{f|n} = \mathrm{e}^{v^f(p^f)}/H^n$, where $H^n \equiv \sum_{g \in n} \mathrm{e}^{v^g(p^g)}$. The expected utility from being able to choose a firm from nest n is $\log H^n$. It follows that, at stage 3, a consumer in nest n

forgoes the nest-level outside option if (and only if) $\varepsilon_0^n \leq \log H^n + \varepsilon_1^n$, which arises with probability $K^n(\log H^n) \equiv \mu^n\left(\{(\varepsilon_0^n, \varepsilon_1^n) : \ \varepsilon_0^n \leq \log H^n + \varepsilon_1^n\}\right)$. The mean expected utility from choosing nest n at stage 2 (gross of the taste shocks received in stages 1 and 2) is denoted $\Psi^n(H^n) = \int \max(\log H^n + \varepsilon_1^n, \varepsilon_0^n) d\mu^n$. Differentiating under the integral sign yields $\Psi^{n\prime}(H^n) = K^n(\log H^n)/H^n$ (see the proof of Proposition IX in Nocke and Schutz, 2018b). At stage 2, a consumer who turned down the stage-1 outside option chooses nest n if (and only if) $\Psi^n(H^n) + \varepsilon^n \geq \Psi^m(H^m) + \varepsilon^m$ for every $m \in \mathcal{N}$. Applying again the Holman-Marley theorem (recall that the components of $(\varepsilon_m)_{m\in\mathcal{N}}$ are drawn i.i.d. from a type-I extreme-value distribution with dispersion parameter λ), this arises with probability $s^n = \exp\left(\Psi^n(H^n)/\lambda\right)/H$, where $H \equiv \sum_{m\in\mathcal{N}} \exp\left(\Psi^m(H^m)/\lambda\right)$. Expected consumer utility is $\lambda \log H$. Hence, the mass of consumers turning down the stage-1 outside option is $K(\lambda \log H) \equiv \mu\left(\{(\varepsilon_0, \varepsilon) : \ \varepsilon_0 \leq \lambda \log H + \varepsilon\}\right)$.

Summing up, the demand for product $i \in f \in n$ equals

$$D_i(p) = K(\lambda \log H) \times s^n \times K^n(\log H^n) \times s^{f|n} \times (-\partial_i v^f(p^f)). \tag{32}$$

Consumer surplus is given by $\Psi(H) \equiv \int \max(\lambda \log H + \varepsilon, \varepsilon_0) d\mu$. Differentiating under the integral sign yields $\Psi'(H) = \lambda K(\lambda \log H)/H$ (see again the proof of Proposition IX in Nocke and Schutz, 2018b).

Let $h^f \equiv \exp v^f$ and $\Phi^n \equiv \mathrm{e}^{\Psi^n/\lambda}$. Replacing $K(\lambda \log H)$ by $H\Psi'(H)/\lambda$, $K^n(\log H^n)$ by $H^n \Psi^{n\prime}$, Ψ^n by $\lambda \log \Phi^n$, and v^f by $\log h^f$ in equation (32) and simplifying, we obtain the demand system of equation (8). It is immediate that the functions h^f and Ψ satisfy parts (i) and (ii) of Assumption C″. Moreover, the functions Φ^n satisfy part (iii) of that assumption, as the elasticity of Φ^n is equal to $K^n(\log H^n)/\lambda$, which is strictly positive, non-decreasing, and bounded above by $1/\lambda$.

Conversely, consider a demand system taking the form of equation (8) and satisfying Assumption C″. Then, $\lambda \equiv \left[\max_{n\in\mathcal{N}} \lim_{x\to\infty} x\Phi^{n\prime}(x)/\Phi^n(x)\right]^{-1}$ is finite and strictly positive by Assumption C″. For every $n \in \mathcal{N}$ and $x > 0$, let $\tilde{\Phi}^n(x) \equiv \Phi^n(x)^\lambda$. Then, each $\tilde{\Phi}^n$ is $\mathcal{C}^1$ and strictly positive, with an elasticity that is strictly positive, non-decreasing, and bounded above by 1. The argument in the proof of Proposition IX in Nocke and Schutz (2018b) implies the existence of probability measures μ^n satisfying the assumptions made above and such that

$$\log \tilde{\Phi}^n(x) = \int \max\left(\log x + \varepsilon_1^n, \varepsilon_0^n\right) d\mu^n \text{ and } \frac{\tilde{\Phi}^{n\prime}(x)}{\tilde{\Phi}^n(x)} = \frac{1}{x}\mu^n\left(\{(\varepsilon_0^n, \varepsilon_1^n) : \varepsilon_0^n \leq \log x + \varepsilon_1^n\}\right)$$

for every $n \in \mathcal{N}$ and $x > 0$. Next, let $\tilde{\Psi}(x) \equiv \Psi(x^{1/\lambda})$ for every $x > 0$. As $x\tilde{\Psi}'(x) = x^{1/\lambda}\Psi'(x^{1/\lambda})/\lambda$ is strictly positive and non-decreasing, $\tilde{\Psi}$ satisfies satisfies condition (ii)-(a) of Proposition IX in Nocke and Schutz (2018b). The argument in the proof of that proposition implies the existence of a measure μ such that

$$\tilde{\Psi}(x) = \int \max\left(\log x + \varepsilon, \varepsilon_0\right) d\mu \text{ and } \tilde{\Psi}'(x) = \frac{1}{x}\mu\left(\{(\varepsilon_0, \varepsilon) : \varepsilon_0 \leq \log x + \varepsilon\}\right)$$

for every $x > 0$. Applying the analysis of the first part of the proof to the multi-stage discrete/continuous choice process with measure μ, probability measures μ^n, dispersion parameter λ, and conditional indirect utility functions $v^f \equiv \log h^f$, we obtain that, after simplification, the demand for each product $i \in n \in f$ and consumer surplus are as given in equations (8) and (9).

Finally, we turn to the integrability part of the proposition. The argument in the proof of Proposition 6 above implies that it is enough to show that

$$\log \sum_{n \in \mathcal{N}} \Phi^n \left(\sum_{g \in n} h^g(p^g) \right)$$

is convex. As the sum of log-convex functions is log-convex, a sufficient condition for this is that

$$V^n(p^n) \equiv \log \Phi^n \left(\sum_{g \in n} h^g(p^g) \right)$$

is convex for every n. Let

$$\hat{V}^n(p^n) \equiv \log \sum_{g \in n} h^g(p^g)$$

and $\hat{\Phi}^n(x) \equiv \log \Phi^n(\mathrm{e}^x)$, and note that $V^n(p^n) = \hat{\Phi}^n \circ \hat{V}^n(p^n)$. Then, using again the fact that log-convexity is preserved by summation, $\hat{V}^n$ is convex. Moreover, $\hat{\Phi}^n$ is non-decreasing and convex, as $\hat{\Phi}^{n\prime}(x) = \mathrm{e}^x \, \Phi^{n\prime}(\mathrm{e}^x)/\Phi^n(\mathrm{e}^x)$ is non-decreasing by Assumption C″-(iii). Hence, V^n is convex, as it is the composition of a convex and non-decreasing function and a convex function. □

E Equilibrium Existence for Baskets and Nests: Details and Relegated Proofs

E.1 Preliminaries and Auxiliary Results

In this subsection, we study Ramsey problem (15). We begin by analyzing the behavior of the partial derivative $\partial_i h^f$ as p_i (and perhaps some other prices) tends to infinity:

Lemma E.1. *Suppose that h^f satisfies Assumption C′-(i), and let $\Phi : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ be a $\mathcal{C}^1$ function with a strictly positive and non-decreasing elasticity. Let $f' \subseteq f$ and $\alpha \in [0, 1)$.*

Then, for every $i \in f'$ and $p^{-f'} \in (0,\infty)^{f \setminus f'}$, we have:

$$\lim_{p^{f'} \to \infty} p_i \frac{\partial_i h^f(p^{f'}, p^{-f'}) \Phi' \left(h^f(p^{f'}, p^{-f'}) \right)}{\Phi \left(h^f(p^{f'}, p^{-f'}) \right)^{\alpha}} = 0.$$

Proof. Define $\xi(p^f) = \Phi\left(h^f(p^f)\right)^{1-\alpha}/(1-\alpha)$, and note that for every $i \in f'$, $\partial_i \xi = \partial_i h^f(p^f)\Phi'\left(h^f(p^f)\right)/\Phi\left(h^f(p^f)\right)^{\alpha}$. Thus, let us show that $p_i \partial_i \xi(p^{f'}, p^{-f'}) \underset{p^{f'} \to \infty}{\longrightarrow} 0$. As the function ξ is non-negative and strictly decreasing, it has a finite limit as $p^{f'}$ goes to ∞ (holding fixed $p^{-f'}$). Moreover, ξ is convex in p_i, as its derivative with respect to p_i can be rewritten as

$$\partial_i \xi = -\frac{-\partial_i h^f(p^f)}{h^f(p^f)} \Phi\left(h^f(p^f)\right)^{1-\alpha} \frac{h^f(p^f)\Phi'\left(h^f(p^f)\right)}{\Phi\left(h^f(p^f)\right)}$$

and each of the terms on the right-hand side is positive and non-increasing in p_i.

By the fundamental theorem of calculus, we have

$$\xi(p_i, p_{-i}) - \xi\left(\frac{1}{2}p_i, p_{-i}\right) = \int_{\frac{p_i}{2}}^{p_i} \frac{\partial \xi(t, p_{-i})}{\partial p_i} dt \leq \frac{1}{2} p_i \frac{\partial \xi(p_i, p_{-i})}{\partial p_i} \leq 0,$$

where the first inequality follows as ξ is convex in p_i. As ξ has a finite limit as $p^{f'}$ tends to infinity, we have that $\xi(p_i, p_{-i}) - \xi\left(\frac{1}{2}p_i, p_{-i}\right) \underset{p^{f'} \to \infty}{\longrightarrow} 0$, which implies that $p_i \partial_i \xi(p^{f'}, p^{-f'}) \underset{p^{f'} \to \infty}{\longrightarrow} 0$ by the sandwich theorem. □

The following proposition provides sufficient conditions for Assumptions C′-(i), E1, and E3 to hold:

Proposition E.1. *Let f be a finite and non-empty set and $h^f(p^f) \equiv \Phi^f\left(\sum_{j \in f} h_j(p_j)\right)$ for every $p^f \in \mathbb{R}_{++}^f$, where, for every j, $h_j : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is $\mathcal{C}^1$, strictly decreasing, and log-convex, and $\Phi^f : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is $\mathcal{C}^1$, with an elasticity that is strictly positive and non-decreasing. Then, Assumptions C′-(i), E1, and E3 hold for firm f.*

Proof. It is clear that h^f is strictly positive and strictly decreasing. It is log-convex by Proposition X in Nocke and Schutz (2018b). Hence, Assumption C′-(i) holds for firm f. Moreover, h^f can be extended by continuity to $(0,\infty]^f$ by taking limits term by term inside the function Φ^f; clearly, $h^f(p^f) > 0$ whenever p^f has at least one finite component. Similarly, $\partial_i h^f$ can be extended by continuity to $(0,\infty]^f$ by setting $\partial_i h^f(p^f)$ equal to zero if $p_i = \infty$, and otherwise to $h_i'(p_i)\Phi^{f\prime}\left(\sum_{j \in f} h_j(p_j)\right)$, where limits are again taken term by term inside the function $\Phi^{f\prime}$. Note that the extension of $\partial_i h^f$ is indeed continuous, as $h_i'(p_i)\Phi^{f\prime}\left(\sum_{j \in f} h_j(p_j)\right)$ tends to 0 as p_i (and perhaps some other prices) tend to infinity by Lemma E.1 (applied to the function h^f, with $\alpha = 0$ and Φ being the identity function). Hence, Assumption E1 holds as well. For what follows, it is also useful to extend by continuity the domain of the functions h_j and h'_j to the closed interval $[0,\infty]$. The limits exist by monotonicity; note that $h_j(0)$ may be equal to $+\infty$ and $h'_j(0)$ may be equal to $-\infty$.

We now turn to Assumption E3. Let us show that maximization problem (15) has a solution in $\prod_{j\in f}[c_j,\infty]$ for every $u^f \in (\underline{u}^f, u_0^f)$. Let $(p^n)_{n\geq 0} = ((p_j^n)_{j\in f})_{n\geq 0}$ be a sequence over $(0,\infty]^f$ such that $h^f(p^n) = u^f$ for every n and $\tilde{\pi}^f(p^n) \underset{n\to\infty}{\longrightarrow} \pi^f(u^f)$. For every $j \in f$, the sequence $(p_j^n)_{n\geq 0}$ is either bounded or unbounded. In the former case, we can extract a subsequence that converges to some $p_j^* \in [0,\infty)$. In the latter case, we can extract a subsequence that diverges to $p_j^* = \infty$. Doing so (sequentially) for every $j \in f$, we obtain a subsequence $(p'^n)_{n\geq 0}$ that tends to some limiting price vector $p^* \in [0,\infty]^f$ as n tends to infinity. To ease notation, we relabel $(p'^n)_{n\geq 0}$ as $(p^n)_{n\geq 0}$.

Assume for a contradiction that, for some $i \in f$, $p_i^* = 0$ and $h'_i(0) = -\infty$. Then,

$$\tilde{\pi}^f(p^n) \leq \left[(p_i^n - c_i)(-h'_i(p_i^n)) + \sum_{j\in f\setminus\{i\}} \sup_{p_j\in[c_j,\infty)} (p_j - c_j)(-h'_j(p_j))\right] \Phi^{f\prime}\circ(\Phi^f)^{-1}(u^f) \underset{n\to\infty}{\longrightarrow} -\infty,$$

where we have used the fact that the suprema are finite, as h'_j is continuous and $p_j h'_j(p_j) \underset{p_j\to\infty}{\longrightarrow} 0$ by Lemma E.1, and $\sum_{j\in f} h_j(p_j^n) = (\Phi^f)^{-1}(u^f)$ for all n due to the utility constraint in the Ramsey problem. As $\tilde{\pi}^f(p^n)$ tends to $\pi^f(u^f) > 0$ when n goes to ∞, this is a contradiction. Hence, for every $i \in f$ such that $p_i^* = 0$, $h'_i(0)$ is finite and strictly negative. Moreover,

$$\pi^f(u^f) = \Phi^{f\prime}\circ(\Phi^f)^{-1}(u^f) \sum_{\substack{j\in f:\\ \tilde{p}_j^*<\infty}} (\tilde{p}_j^* - c_j)(-h'_j(p_j^*)).$$

Assume for a contradiction that $f' \equiv \{j \in f :\ p_j^* < c_j\}$ is non-empty. Then, $h'_j(p_j^*) \in (-\infty, 0)$ for every $j \in f'$. Moreover, there exists $k \in f$ such that $p_k^* \in (c_k,\infty)$, for otherwise $\pi^f(u^f)$ would be non-positive. Choose $\eta > 0$ so that $p_j^* - \eta > 0$ for every $j \in f'$ such that $p_j^* > 0$. For every $j \in f'$ such that $p_j^* = 0$, we have shown that $h'_j(0)$ (and thus $h_j(0)$) is finite, and we extend the domain of h_j to $(-\eta,\infty]$ by setting $h_j(x) = h_j(0) + xh'_j(0)$. Note that h_j is then $\mathcal{C}^1$ on $(-\eta,\eta)$. Define the function

$$\xi : (x,p_k) \in (-\eta,\eta)\times\mathbb{R}_{++} \mapsto \sum_{j\in f'} h_j(p_j^* + x) + h_k(p_k) + \sum_{\substack{j\in f\setminus f':\\ j\neq k}} h_j(p_j^*) - u^f. \tag{33}$$

As ξ is $\mathcal{C}^1$, $\xi(0,p_k^*) = 0$, and $\partial\xi/\partial x \neq 0$ at $(0,p_k^*)$, the implicit function theorem implies the existence of an $\eta' \in (0,\eta]$ and a $\mathcal{C}^1$ function $x \in (-\eta',\eta) \mapsto \tilde{p}_k(x)$ such that $\tilde{p}_k(0) = p_k^*$ and $\xi(x,\tilde{p}_k(x)) = 0$ for every $x \in (-\eta',\eta)$. Moreover,

$$\tilde{p}'_k(0) = -\frac{\sum_{j\in f'} h'_j(p_j^*)}{h'_k(p_k^*)}.$$

Let $(x^n)_{n\geq 0}$ be a strictly decreasing sequence such that $x^n \underset{n\to\infty}{\longrightarrow} 0$. For every $n \geq 0$, define the price vector $\tilde{p}^n$ as follows:

$$\tilde{p}^n_j = \begin{cases} \tilde{p}_k(x^n) & \text{if } j = k \\ p^*_j + x^n & \text{if } j \in f' \\ p^*_j & \text{otherwise.} \end{cases}$$

By construction, $\tilde{p}^n$ is feasible for maximization problem (15). Moreover, $(\tilde{p}^n_k)_{n\geq 0}$ is strictly increasing, $(\tilde{p}^n)_{n\geq 0}$ converges to p^* as n tends to infinity, and

$$\lim_{n\to\infty} \frac{\tilde{p}^n_k - p^*_k}{x^n} = \lim_{n\to\infty} \frac{\tilde{p}_k(x^n) - \tilde{p}_k(0)}{x^n} = -\frac{\sum_{j\in f'} h'_j(p^*_j)}{h'_k(p^*_k)}. \tag{34}$$

Since $\tilde{p}^n$ is feasible for maximization problem (15), we have that

$$\Phi^{f'} \circ (\Phi^f)^{-1}(u^f) \sum_{\substack{j\in f:\\ \tilde{p}^n_j<\infty}} (\tilde{p}^n_j - c_j)(-h'_j(\tilde{p}^n_j)) \leq \pi^f(u^f) = \Phi^{f'} \circ (\Phi^f)^{-1}(u^f) \sum_{\substack{j\in f:\\ \tilde{p}^*_j<\infty}} (\tilde{p}^*_j - c_j)(-h'_j(p^*_j)).$$

Rewriting, this means that

$$\begin{aligned} 0 &\leq (\tilde{p}^n_k - c_k)h'_k(\tilde{p}^n_k) - (p^*_k - c_k)h'_k(p^*_k) + \sum_{j\in f'} \left((\tilde{p}^n_j - c_j)h'_j(\tilde{p}^n_j) - (p^*_j - c_j)h'_j(p^*_j)\right) \\ &= (\tilde{p}^n_k - p^*_k)\left(h'_k(\tilde{p}^n_k) + (p^*_k - c_k)\frac{h'_k(\tilde{p}^n_k) - h'_k(p^*_k)}{\tilde{p}^n_k - p^*_k}\right) \\ &\quad + \sum_{j\in f'} (\tilde{p}^n_j - p^*_j)\left(h'_j(\tilde{p}^n_j) + (p^*_j - c_j)\frac{h'_j(\tilde{p}^n_j) - h'_j(p^*_j)}{\tilde{p}^n_j - p^*_j}\right). \end{aligned}$$

Dividing by x^n, this implies that

$$\begin{aligned} 0 \leq \frac{\tilde{p}^n_k - p^*_k}{x^n} &\left[h'_k(\tilde{p}^n_k) + (p^*_k - c_k) \overbrace{\frac{h'_k(\tilde{p}^n_k) - h'_k(p^*_k)}{\tilde{p}^n_k - p^*_k}}^{\equiv \delta^n_k} \right] \\ &+ \sum_{j\in f'} \left[h'_j(\tilde{p}^n_j) + (p^*_j - c_j) \underbrace{\frac{h'_j(\tilde{p}^n_j) - h'_j(p^*_j)}{\tilde{p}^n_j - p^*_j}}_{\equiv \delta^n_j} \right]. \end{aligned}$$

As the h-functions are decreasing and log-convex, they are convex. Hence, δ^n_k and δ^n_j are non-negative for every j and n. If $(\delta^n_k)_{n\geq 0}$ is unbounded, then we can extract a subsequence that diverges to $+\infty$. Together with equation (34), this implies that the right-hand side of the above inequality tends to $-\infty$ as n goes to infinity along the divergent subsequence, which is a contradiction. The same contradiction obtains if $(\delta_j^n)_{n\geq 0}$ is unbounded for some j in f'. Hence, the sequences $(\delta_k^n)_{n\geq 0}$ and $(\delta_j^n)_{n\geq 0}$ are bounded and we can extract subsequences that converge to some values $\delta_k \geq 0$ and $\delta_j \geq 0$, respectively. Taking limits along the convergent subsequences, using again equation (34), and simplifying yields

$$0 \leq -\frac{\sum_{j\in f'} h_j'(p_j^*)}{h_k'(p_k^*)}(p_k^* - c_k)\delta_k + \sum_{j\in f'}(p_j^* - c_j)\delta_j.$$

As δ_k and δ_j are non-negative for every j, it follows that $\delta_k = 0$ and $\delta_j = 0$ for every j. Yet, the log-convexity of h_k implies that, for every n,

$$\begin{aligned} 0 &\leq \frac{1}{p_k^* - \tilde{p}_k^n}\left(\frac{h_k'(p_k^*)}{h_k(p_k^*)} - \frac{h_k'(\tilde{p}_k^n)}{h_k(\tilde{p}_k^n)}\right) \\ &= \frac{h_k'(p_k^*) - h_k'(\tilde{p}_k^n)}{p_k^* - \tilde{p}_k^n}\frac{1}{h_k(p_k^*)} + h_k'(\tilde{p}_k^n)\frac{1}{p_k^* - \tilde{p}_k^n}\left(\frac{1}{h_k(p_k^*)} - \frac{1}{h_k(\tilde{p}_k^n)}\right) \\ &\underset{n\to\infty}{\longrightarrow} 0 + h_k'(p_k^*)\left.\frac{d}{dp_k}\frac{1}{h_k(p_k)}\right|_{p_k^*} = -\frac{h_k'(p_k^*)^2}{h_k(p_k^*)^2} < 0, \end{aligned}$$

where we have taken the limit along the aforementioned subsequence and used the fact that $\delta_k = 0$. We have thus obtained a contradiction. It follows that $p_j^* \geq c_j$ for every $j \in f$. Hence, maximization problem (15) has a solution in $[\varepsilon, \infty]^f$, where $\varepsilon = \min_{j\in f} c_j$. □

The following lemma gathers some useful properties of the function $\pi^f(\cdot)$:

Lemma E.2. *Let h^f satisfy Assumptions C'-(i), E1, and E3. The function π^f is strictly positive if $u^f \in (\underline{u}^f, u_0^f)$, and equal to zero if $u^f = \underline{u}^f$ or $u^f = u_0^f$. Its restriction to $[\underline{u}^f, u_0^f]$ is continuous. Moreover, for any $\mathcal{C}^1$ function $\Phi : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ with a strictly positive and non-decreasing elasticity, and for every $\alpha \in [0, 1)$,*

$$\frac{\pi^f(u^f)\Phi'(u^f)}{\Phi(u^f)^\alpha} \underset{u^f\to\underline{u}^f}{\longrightarrow} 0.$$

Proof. The first part of the lemma was proven in the main text. The continuity of π^f follows by the theorem of the maximum, as the correspondence $u^f \in [\underline{u}^f, u_0^f] \mapsto \{p^f \in [\varepsilon, \infty]^f : h^f(p^f) = u^f\}$ is continuous and compact-valued.[36]

Next, we turn to the last part of the lemma. Let $(u^n)_{n\geq 0}$ be a sequence over $(\underline{u}^f, u_0^f]$ that converges to $\underline{u}^f$. For every n, let $p^n \in P^f(u^n)$. Let us construct a sequence $(\tilde{p}^n)_{n\geq 0}$ over $\mathbb{R}_{++}^f$ such that for every $n \geq 0$, $h^f(\tilde{p}^n) = u^n$ and

$$\pi^f(u^n) \leq \sum_{j\in f}\tilde{p}_j^n(-\partial_j h^f(\tilde{p}^n)). \tag{35}$$

[36]To see why $[\varepsilon, \infty]^f$ can be treated as a compact set, define the function ϕ which, to every $y \in [0, 1]$ associates $y/(1-y)$ if $y < 1$ and ∞ otherwise, and note that $x \in [0, 1]^f \mapsto (\varepsilon + \phi(x_j))_{j\in f} \in [\varepsilon, \infty]^f$ is a continuous bijection.

Let $n \geq 0$. If the price vector p^n only has finite components, we set $\tilde{p}^n = p^n$, which clearly satisfies condition (35). If instead p^n has some infinite components, then let $f' \subsetneq f$ denote the set of finite components of p^n, and note that this set is non-empty, as $u^n > \underline{u}^f$. For every $\eta \geq 0$ and $K \in (0, \infty]$, define the price vector $p(\eta, K)$ as follows: for every $j \in f$,

$$p_j(\eta, K) = \begin{cases} p_j^n + \eta & \text{if } j \in f' \\ K & \text{otherwise.} \end{cases}$$

Let $\eta > 0$. As $h^f(p(\eta, \infty)) < u^n = h^f(p(0, \infty))$, the continuity of h^f implies the existence of a $\overline{K} \in (0, \infty)$ such that $h^f(p(\eta, K)) < u^n$ for every $K \in [\overline{K}, \infty)$. As $h^f(p(0, K)) > u^n$, the continuity and monotonicity properties of h^f imply that for every $K \geq \overline{K}$, there exists a unique $\eta(K) > 0$ such that $h^f(p(\eta(K), K)) = u^n$. Moreover, $\lim_{K \to \infty} p(\eta(K), K) = p^n$. As $p^n \in P^f(u^n)$, we have that

$$\pi^f(u^n) = \tilde{\pi}^f(p^n) < \sum_{\substack{j \in f \\ p_j^n < \infty}} p_j^n(-\partial_j h^f(p^n)).$$

By continuity of the partial derivatives of h^f and since $\lim_{K \to \infty} p(\eta(K), K) = p^n$, the above inequality implies the existence of a $\hat{K} \geq \overline{K}$ such that

$$\pi^f(u^n) < \sum_{j \in f} p_j(\eta(\hat{K}), \hat{K})(-\partial_j h^f(p(\eta(\hat{K}), \hat{K}))).$$

Thus, condition (35) holds for $\tilde{p}^n \equiv p(\eta(\hat{K}), \hat{K})$.

We have thus constructed a sequence $(\tilde{p}^n)_{n \geq 0}$ of finite price vectors such that, for every $n \geq 0$, $h^f(\tilde{p}^n) = u^n$ and condition (35) holds. Moreover, for every $j \in f$, we must have that $\tilde{p}_j^n \underset{n \to \infty}{\longrightarrow} \infty$, for otherwise $u^n = h^f(\tilde{p}^n)$ could not possibly converge to $\underline{u}^f$ (recall from Assumption E1 that $h^f(p^f) > \underline{u}^f$ whenever p^f has some finite components). It follows that $\tilde{p}^n$ tends to $(\infty, \ldots, \infty)$ as n goes to ∞, and we can apply Lemma E.1:

$$0 \leq \frac{\pi^f(u^n)\Phi'(u^n)}{\Phi(u^n)^\alpha} \leq \sum_{j \in f} \tilde{p}_j^n \frac{-\partial_j h^f(\tilde{p}^n)\Phi'(h^f(\tilde{p}^n))}{\Phi(h^f(\tilde{p}^n))^\alpha} \underset{n \to \infty}{\longrightarrow} 0. \qquad \square$$

E.2 Proof of Theorem 2

Proof. We begin by showing that the pricing game is indeed strategically equivalent to the game of competition in utility space defined in Section 4.2. Note that, for firm f, price vectors p^f such that $\tilde{\pi}^f(p^f) \leq 0$ are strictly dominated (by, e.g, pricing each product j at $c_j + 1$). In particular, price vectors p^f such that $h^f(p^f) \geq u_0^f$ result in $\tilde{\pi}^f(p^f) \leq 0$ and are therefore strictly dominated.[37] Moreover, any price vector p^f such that $u^f \equiv h^f(p^f) \in (\underline{u}^f, u_0^f)$ and $\tilde{\pi}^f(p^f) < \pi^f(u^f)$ is strictly dominated by pricing at some p^* in $P^f(u^f)$. Removing those strictly dominated strategies for all firms, the pricing game becomes strategically equivalent to one in which each firm f chooses $u^f \in (\underline{u}^f, u_0^f)$ with payoff functions defined in equation (17).

To show that O has a global maximizer, we extend its domain to $\prod_{g\in\mathcal{F}}[\underline{u}^g, u_0^g]$ by setting $O(u) = 0$ whenever $u^f = \underline{u}^f$ or $u^f = u_0^f$ for some f. (Clearly, any global maximizer must be in the interior of the domain, as O is strictly positive there but equal to zero at any boundary point.) For every profile of utilities u, let

$$H(u) \equiv \sum_{B\in\mathcal{B}} a(B) \prod_{g\in B} u^g.$$

As the extended domain of O is compact, all we need to do is show that O is continuous. Continuity at any $\hat{u}$ such that $\hat{u} > \underline{u}^g$ for all g follows immediately by Lemma E.2.

Next, let $\hat{u} \in \left(\prod_{g\in\mathcal{F}}[\underline{u}^g, u_0^g]\right)$ such that $\hat{u}^f = \underline{u}^f$ for at least one f. Consider a sequence $(u(t))_{t\geq 0}$ of utility vectors that tends to $\hat{u}$ as t goes to infinity, and let us show that $\lim_{t\to\infty} O(u(t)) = O(\hat{u}) = 0$. As $O(u(t)) = 0$ whenever $u^f(t) = \underline{u}^f$ for at least one f, we extract the subsequence $(u'(t))$ by only keeping those utility vectors $u(t)$ such that $u^g(t) > \underline{u}^g$ for every g. If that subsequence is finite, then we immediately have that $\lim_{t\to\infty} O(u(t)) = 0$. If instead the subsequence is infinite, which we assume in the following, then we need to show that $\lim_{t\to\infty} O(u'(t)) = 0$. We now drop the prime superscript to ease notation.

Let $u_0 \equiv (u_0^g)_{g\in\mathcal{F}}$ and $\underline{u} \equiv (\underline{u}^g)_{g\in\mathcal{F}}$. Assumption C′ implies that

$$0 \leq O(u(t)) = \frac{\prod_{g\in\mathcal{F}} \pi^g(u^g(t))}{H(u(t))} H(u(t))\Psi'(H(u(t))) \leq \frac{\prod_{g\in\mathcal{F}} \pi^g(u^g(t))}{H(u(t))} H(u_0)\Psi'(u_0).$$

Hence, all we need to do is show that the fraction in the above expression, which we denote by $\xi(t)$, tends to 0 as t goes to infinity. If part (ii) of Assumption E2 holds, i.e., $\underline{u}^g > 0$ for every firm g, then $H(u(t)) \geq H(\underline{u}) > 0$, and so

$$\xi(t) \leq \frac{1}{H(\underline{u})} \prod_{g\in\mathcal{F}} \pi^g(u^g(t)) \underset{t\to\infty}{\longrightarrow} \frac{1}{H(\underline{u})} \prod_{g\in\mathcal{F}} \pi^g(\hat{u}^g) = 0,$$

where we have used Lemma E.2 and the fact that $\hat{u}^f = \underline{u}^f$ for at least one f.

[37] As aggregate surplus is highest under marginal cost pricing, we have that

$$h^f(p^f) + \tilde{\pi}^f(p^f) \leq h^f(c^f) + \tilde{\pi}^f(c^f) = u_0^f.$$

This implies that $\tilde{\pi}^f(p^f) \leq 0$ whenever $h^f(p^f) \geq u_0^f$.

If part (ii) of Assumption E2 is not satisfied, then part (i) must hold, i.e., $a(\{g\}) > 0$ for every g. We then have that

$$\xi(t) = \prod_{g \in \mathcal{F}} \frac{\pi^g(u^g(t))}{H(u(t))^{\frac{1}{|\mathcal{F}|}}} \leq \prod_{g \in \mathcal{F}} \underbrace{\frac{\pi^g(u^g(t))}{(a(\{g\})u^g(t))^{\frac{1}{|\mathcal{F}|}}}}_{\equiv \xi^g(t)}.$$

By Lemma E.2, as t goes to infinity, $\xi^g(t)$ tends to $\pi^g(\hat{u}^g)/(a(\{g\})\hat{u}^g)^{1/|\mathcal{F}|}$ if $\hat{u}^g > \underline{u}^g$, and to 0 if $\hat{u}^g = \underline{u}^g$. Since $\hat{u}^f = \underline{u}^f$ for at least one f, it follows that $\xi(t) \underset{t\to\infty}{\longrightarrow} 0$. □

E.3 Proof of Theorem 3

Proof. The approach is the same as in the proof of Theorem 2. The reasoning laid out there implies that the pricing game is formally equivalent to the game of competition in utility space of Section 4.3. Let $H(u) \equiv \sum_{n \in \mathcal{N}} \Phi^n \left(\sum_{f \in n} u^f \right)$ for every utility vector $u \in \prod_{g \in \mathcal{F}} (\underline{u}^g, u_0^g]$. We extend again the domain of O to $\prod_{g \in \mathcal{F}} [\underline{u}^g, u_0^g]$ by setting $O(u) = 0$ whenever $u^f = \underline{u}^f$ or $u^f = u_0^f$ for some f. The continuity of O at any utility vector $\hat{u}$ such that $\hat{u}^g > \underline{u}^g$ for every g follows again by Lemma E.2.

Next, consider a $\hat{u} \in \prod_{g \in \mathcal{F}} [\underline{u}^g, u_0^g]$ such that $\hat{u}^f = \underline{u}^f$ for at least one f, and let us show that $O(u) \underset{u\to\hat{u}}{\longrightarrow} 0$. Let $(u(t))_{t \geq 0}$ be a sequence of utility vectors that tends to $\hat{u}$, with $u^g(t) > \underline{u}^g$ for every t and g. (Recall that $O(u(t))$ would be equal to 0 if $u^f(t)$ were equal to $\underline{u}^f$ for some f.) As $H\Psi'(H)$ is non-decreasing, we have that

$$0 \leq O(u(t)) \leq \frac{\prod_{n \in \mathcal{N}} \left(\Phi^{n\prime} \left[\sum_{g \in n} u^g(t) \right] \prod_{g \in n} \pi^g(u^g(t)) \right)}{H(u(t))} H(u_0)\Psi'(H(u_0)),$$

where $u_0 = (u_0^g)_{g \in \mathcal{F}}$. Hence, all we need to do is show that the fraction in the above expression, which we denote by $\xi(t)$, tends to 0 as t goes to infinity. We have:

$$\xi(t) = \prod_{n \in \mathcal{N}} \frac{\Phi^{n\prime} \left[\sum_{g \in n} u^g(t) \right]}{H(u(t))^{\frac{1}{|\mathcal{N}|}}} \prod_{g \in n} \pi^g(u^g(t)) \leq \prod_{n \in \mathcal{N}} \underbrace{\frac{\Phi^{n\prime} \left[\sum_{g \in n} u^g(t) \right]}{\Phi^n \left[\sum_{g \in n} u^g(t) \right]^{\frac{1}{|\mathcal{N}|}}} \prod_{g \in n} \pi^g(u^g(t))}_{\equiv \xi^n(t)}.$$

Clearly, if nest n is such that $\hat{u}^g > \underline{u}^g$ for every $g \in n$, then $\xi^n(t)$ has a finite limit as t tends to ∞. Consider instead a nest n such that $\hat{u}^f = \underline{u}^f$ for some $f \in n$ (which exists by assumption). If $|n| = 1$, then $|\mathcal{N}| \geq 2$ (as there are at least two firms in the industry), and

$$\xi^n(t) = \frac{\Phi^{n\prime}(u^f(t))}{\Phi^n(u^f(t))^{\frac{1}{|\mathcal{N}|}}} \pi^f(u^f(t)),$$

which tends to 0 as t goes to infinity by Lemma E.2. Suppose instead that $|n| \geq 2$, and let M be an upper bound on the elasticity of Φ^n (which exists by Assumption C″-(iii)). Then,

$$\xi^n(t) \leq M \frac{\Phi^n \left[\sum_{g \in n} u^g(t)\right]^{1-\frac{1}{|\mathcal{N}|}}}{\sum_{g \in n} u^g(t)} \prod_{g \in n} \pi^g(u^g(t)) \leq M \Phi^n \left[\sum_{g \in n} u_0^g\right]^{1-\frac{1}{|\mathcal{N}|}} \prod_{g \in n} \frac{\pi^g(u^g(t))}{(u^g(t))^{\frac{1}{|n|}}}.$$

For every $g \in n$, either $\hat{u}^g = \underline{u}^g$ and so $\lim_{t \to \infty} \pi^g(u^g(t))/(u^g(t))^{1/|n|} = 0$ by Lemma E.2, or $\hat{u}^g > \underline{u}^g$ and $\pi^g(u^g(t))/(u^g(t))^{1/|n|}$ has a finite limit as $t \to \infty$. Since $\hat{u}^f = \underline{u}^f$ for at least one $f \in n$, this implies that $\xi^n(t) \underset{t \to \infty}{\longrightarrow} 0$. It follows that $\prod_{n \in \mathcal{N}} \xi^n(t) \underset{t \to \infty}{\longrightarrow} 0$, and that $\lim_{t \to \infty} \xi(t) = 0$. □

F Equilibrium Uniqueness for Baskets and Nests: Further Results and Proofs

F.1 Basket Demand

Fix a multiproduct-firm pricing game with a basket structure, as defined in Section 4.2. Let $\widehat{\Pi}^f(t) \equiv \Pi^f\left(\left(\mathrm{e}^{-t^g}\right)_{g \in \mathcal{F}}\right)$ be firm f's profit in the game in which each firm g chooses t^g. As mentioned in Section 5, we prove Theorem 4 under a weaker version of Assumption U1. We assume throughout that, for every firm $f \in \mathcal{F}$, the function h^f takes the form

$$h^f(p^f) = \Phi^f\left(\sum_{j \in f} h_j(p_j)\right) \tag{36}$$

for some functions Φ^f and h_j from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$. We further assume that the demand system is smooth and can be derived from multi-stage discrete/continuous choice:

Assumption C‴. *For every firm f, the function h^f takes the form of equation* (36)*. Moreover:*

(i) *Each h_j is $\mathcal{C}^3$, strictly positive, strictly decreasing, and log-convex.*

(ii) *Ψ is $\mathcal{C}^2$ and $H \mapsto H\Psi'(H)$ is strictly positive and non-decreasing.*

(iii) *Each Φ^f is $\mathcal{C}^2$, with an elasticity, K^f, that is strictly positive and non-decreasing.*

We require the following notation: for every $j \in \mathcal{N}$, $\gamma_j(p_j) \equiv (h_j'(p_j))^2/h_j''(p_j)$, $\iota_j(p_j) = p_j h_j''(p_j)/(-h_j'(p_j))$, $\bar{\mu}_j \equiv \lim_{p_j \to \infty} \iota_j(p_j)$, and $\bar{\mu}^f \equiv \max_{j \in f} \bar{\mu}_j$ whenever these expressions are well defined. We make the following assumption:

Assumption U1′. *The following conditions holds for every firm f:*

(i) *The curvature of the function Φ^f, denoted ϑ^f, is non-negative and non-decreasing.*

(ii) *For every $j \in f$, ι_j is non-decreasing.*

(iii) *At least one of the following conditions holds:*

(a) $\min_{j\in f} \inf_{p_j>0} \frac{h_j(p_j)}{\gamma_j(p_j)} \geq \max_{j\in f} \sup_{p_j>0} \frac{h'_j(p_j)}{\gamma'_j(p_j)}$.

(b) $\mu^f \leq \mu^*(\simeq 2.78)$, *and for every $j \in f$, $\bar{\mu}_j = \bar{\mu}^f$, $\lim_{p_j\to\infty} h_j(p_j) = 0$, and h_j/γ_j is non-decreasing.*

(c) *There exists a function $\tilde{h}^f$, a marginal cost level $c^f > 0$, and a collection of quality shifters $(a_j)_{j\in f} \in \mathbb{R}^f_{++}$ such that $h_j = a_j\tilde{h}^f$ and $c_j = c^f$ for every $j \in f$. In addition, $\tilde{h}^f/\tilde{\gamma}^f$ is non-decreasing.*

Let us show that Assumption U1 does indeed imply Assumption U1′ as well as parts (i) and (iii) of Assumption C‴. As the conditions imposed on the functions Φ^f are the same in both sets of assumptions, all we need to do is study the conditions imposed on the functions h_j. Suppose first that part (i) of Assumption U1 holds for firm f. Then, Assumption C‴-(i) clearly holds. Moreover, routine calculations show that, for every $j \in f$, $\iota_j(p_j) = \sigma^f$ and $\gamma_j(p_j) = \frac{\sigma^f-1}{\sigma^f} h_j(p_j)$, so that parts (ii) and (iii)-(a) of Assumption U1′ also hold. Suppose instead that part (ii) of Assumption U1 holds for firm f. Again, Assumption C‴-(i) holds. Moreover, $\iota_j(p_j) = p_j/\lambda_j$ and $\gamma_j(p_j) = h_j(p_j)$, implying again parts (ii) and (iii)-(a) of Assumption U1′.

We are now in a position to state a more general version of Theorem 4:

Theorem F.1. *Consider a multiproduct-firm pricing game with basket demand satisfying Assumptions C‴, E2, U1′, and U2. The exact potential function of the logged pricing game in which each firm f chooses t^f is strictly concave on a rectangle that includes all the pure strategies that are not strictly dominated. Therefore, the pricing game has a unique pure-strategy Nash equilibrium (no matter what the transformation of payoffs), which corresponds to the unique potential maximizer. If, in addition, Assumption U3 holds, then the potential maximizer also corresponds to the unique correlated equilibrium of the logged pricing game.*

The existence part follows by Theorem 2, as Assumption C‴ implies Assumptions E1 and E3 by Proposition E.1. The rest of the proof proceeds in several steps. As mentioned in the main text, we establish the strict concavity of $E(t)$ by showing that each function $\log \pi^f(\mathrm{e}^{-t^f})$ is strictly concave and that the term $\log \Psi'\left(\sum_B a(B) \prod_{f\in B} \mathrm{e}^{-t^f}\right)$ is concave. Once strict concavity is established, the uniqueness of the pure-strategy equilibrium follows from the argument given in the main text. We then show that strategies with a very high t^f are strictly dominated, which allows us to compactify action sets and apply Theorem 2 in

Neyman (1997) to obtain the uniqueness of the correlated equilibrium in the logged pricing game.

Strict concavity of $\log \pi^f(\mathbf{e}^{-t^f})$**.** Our analysis of the Ramsey problem for firm f relies on concepts, results, and terminology from Nocke and Schutz (2018a), which we summarize now. Given Assumptions C‴-(i) and U1′-(ii), the function $\nu_i : p_i \in (c_i, \infty) \mapsto \frac{p_i - c_i}{p_i} \iota_i(p_i)$ is $\mathcal{C}^1$ with strictly positive derivative and range $(0, \bar{\mu}_i)$. Its inverse function, r_i, defined over $(0, \bar{\mu}_i)$, is also $\mathcal{C}^1$ and strictly increasing, with range (c_i, ∞). We refer to $\nu_i(p_i)$ as the ι-markup on product i, and to r_i as the pricing function of product i. We extend the domain of r_i to $\mathbb{R}_{++}$ by setting $r_i(\mu) = \infty$ whenever $\mu \geq \bar{\mu}_i$. We say that a profile of prices for firm f, p^f, satisfies the common ι-markup property if there exists $\mu^f \in (0, \bar{\mu}^f)$ such that $p_j = r_j(\mu^f)$ for every $j \in f$.

The following lemma characterizes the solution to firm f's Ramsey problem:

Lemma F.1. *Suppose that Assumptions C‴-(i), C‴-(iii), and U1′-(ii) hold for firm f. Then, firm f's Ramsey problem with utility target $u^f \in (\underline{u}^f, u_0^f)$ has a unique solution, which satisfies the common ι-markup property. The optimal common ι-markup, $\mu^f(u^f)$, is the unique solution to equation $\sum_{j \in f} h_j(r_j(\mu^f)) = (\Phi^f)^{-1}(u^f)$. The function $\mu^f(\cdot)$ is continuous and strictly decreasing, with range $(0, \bar{\mu}^f)$.*

Proof. We drop the firm superscript to ease notation. By Proposition E.1, the Ramsey problem has a solution. Let $p^* = (p_j^*)_{j \in f}$ be such a solution, and let us show that p^* satisfies the common ι-markup property. Let $i \in f$ such that $p_i^* < \infty$ and $\mu^* \equiv \nu_i(p_i^*)$. Let us show that $\nu_j(p_j^*)$ is also equal to μ^* for every $j \in f \setminus \{i\}$ such that $p_j^* < \infty$. If no such j exists, then there is nothing to prove. Otherwise, fix such a j, and consider the problem of setting p_i and p_j to maximize the same Ramsey objective subject to the same utility constraint, but holding fixed p_k at p_k^* for every $k \neq i, j$. Clearly, (p_i^*, p_j^*) is a solution to that problem, and so the first-order conditions

$$-h_i'(p_i^*) - (p_i^* - c_i)h_i''(p_i^*) + \lambda h_i'(p_i^*) = 0$$

and

$$-h_j'(p_j^*) - (p_j^* - c_j)h_j''(p_j^*) + \lambda h_j'(p_j^*) = 0$$

must hold for some Lagrange multiplier λ.[38] Dividing the first condition by $h_i'(p_i^*)$ and the

[38] These first-order conditions were derived from the equivalent maximization problem

$$\max_{(p_i, p_j)} \sum_{\substack{\ell \in \{i,j\}: \\ p_\ell < \infty}} (p_\ell - c_\ell)(-h_\ell'(p_\ell)) \quad \text{s.t.} \quad h_i(p_i) + h_j(p_j) + \sum_{k \in f \setminus \{i,j\}} h_k(p_k^*) = \Phi^{-1}(u).$$

second condition by $h_j'(p_j^*)$, simplifying, and rearranging terms, we obtain that

$$\nu_i(p_i^*) = 1 - \lambda = \nu_j(p_j^*),$$

which does imply that all the products that have a finite price have the same ι-markup, μ^*.

Next, let $j \in f$ such that $p_j^* = \infty$ (if such a j exists), and let us show that $r_j(\mu^*) = \infty = p_j^*$. Assume for a contradiction that $r_j(\mu^*) < \infty$, or, equivalently, $\bar{\mu}_j > \mu^*$. As

$$h_j(c_j) + h_i(p_i^*) + \sum_{k \in f \setminus \{i,j\}} h_k(p_k^*) > \Phi^{-1}(u),$$

we have that

$$h_j(c_j) + h_i(p_i^* + \varepsilon) + \sum_{k \in f \setminus \{i,j\}} h_k(p_k^*) > \Phi^{-1}(u),$$

for some $\varepsilon > 0$. This implies that

$$h_j(c_j) + h_i(p_i) + \sum_{k \in f \setminus \{i,j\}} h_k(p_k^*) < \Phi^{-1}(u)$$

for every $p_i \in [p_i^*, p_i^* + \varepsilon]$. Moreover, as

$$h_j(\infty) + h_i(p_i) + \sum_{k \in f \setminus \{i,j\}} h_k(p_k^*) \leq \Phi^{-1}(u)$$

for every $p_i \in [p_i^*, p_i^* + \varepsilon]$, the intermediate value theorem and the monotonicity of h_j imply the existence of a unique $\hat{p}_j(p_i) \in [c_j, \infty]$ such that

$$h_j(\hat{p}_j(p_i)) + h_i(p_i) + \sum_{k \in f \setminus \{i,j\}} h_k(p_k^*) = \Phi^{-1}(u).$$

Moreover, $\hat{p}_j(\cdot)$ is continuous on $[p_i^*, p_i^* + \varepsilon]$ and, by the implicit function theorem, $\mathcal{C}^1$ on $(p_i^*, p_i^* + \varepsilon]$, with derivative $-h_i'(p_i)/h_j'(\hat{p}_j(p_i))$. Let $\tilde{\pi}(p_i)$ be the value of the Ramsey objective when product i is priced at p_i, product j is priced at $\hat{p}_j(p_i)$, and the other products are priced at $(p_k^*)_{k \in f \setminus \{i,j\}}$. Then, $\tilde{\pi}$ is continuous on $[p_i^*, p_i^* + \varepsilon]$, and $\mathcal{C}^1$ on $(p_i^*, p_i^* + \varepsilon]$ with derivative

$$\begin{aligned}
\tilde{\pi}'(p_i) &= \left[-h_i'(p_i) - (p_i - c_i) h_i''(p_i) + \frac{d\hat{p}_j}{dp_i} \left(-h_j'(\hat{p}_j) - (\hat{p}_j - c_j) h_j''(\hat{p}_j) \right) \right] \Phi' \circ \Phi^{-1}(u) \\
&= \left(-h_i'(p_i) \right) \left[\nu_j(\hat{p}_j(p_i)) - \nu_i(p_i) \right] \Phi' \circ \Phi^{-1}(u).
\end{aligned}$$

As p_i tends to p_i^*, the term inside the square brackets on the second line tends to $\bar{\mu}_j - \mu^* > 0$, implying that $\tilde{\pi}'(p_i) > 0$ for p_i sufficiently close to p_i^*. It follows that $\tilde{\pi}$ is strictly increasing in the neighborhood of p_i^*, so that $\tilde{\pi}(p_i) > \tilde{\pi}(p_i^*)$ for some $p_i > p_i^*$ close enough to p_i^*. Thus, the vector of prices where good i is priced at p_i, good j is priced at $\hat{p}_j(p_i)$, and the other goods are priced at $(p_k^*)_{k\in f\setminus\{i,j\}}$ is feasible for the Ramsey problem. Yet, it gives rise to a strictly greater value of the Ramsey objective than the supposed solution p^*, a contradiction.

It follows that $r_j(\mu^*) = \mu^*$ for every $j \in f$. Moreover, as p^* must be feasible for the Ramsey problem, μ^* must be a solution to equation $\sum_{j\in f} h_j(r_j(\mu)) = \Phi^{-1}(u)$. The monotonicity properties of the functions h_j and r_j imply that this solution is unique. The fact that $\mu(\cdot)$ is continuous and strictly decreasing with range $(0, \bar{\mu}^f)$ then follows from standard arguments. $\square$

Next, we obtain the derivative of the value of the Ramsey problem. Note that the assumptions made above imply that $\lim_{p_j\to\infty} \gamma_j(p_j) = 0$ for every j (see Lemma A in Nocke and Schutz, 2018a). We thus define $\gamma_j(\infty) \equiv 0$. We have:

Lemma F.2. *Suppose that Assumptions C‴-(i), C‴-(iii), and U1′-(ii) hold for firm f. The value of firm f's Ramsey problem with utility target $u^f \in (\underline{u}^f, u_0^f)$ satisfies*

$$\pi^f(u^f) = \mu^f(u^f) \sum_{j\in f} \gamma_j(r_j(\mu^f(u^f)))\Phi^{f\prime} \circ (\Phi^f)^{-1}(u^f).$$

Moreover, $\pi^f(\cdot)$ is $\mathcal{C}^1$, with derivative

$$\pi^{f\prime}(u^f) = \mu^f(u^f) - 1 - \mu^f(u^f)\frac{\sum_{j\in f}\gamma_j(r_j(\mu^f(u^f)))}{\sum_{j\in f} h_j(r_j(\mu^f(u^f)))}\vartheta^f \circ (\Phi^f)^{-1}(u^f). \tag{37}$$

Proof. We drop the firm superscript to ease notation. To obtain the formula for $\pi(u)$, note that

$$\begin{aligned}
\pi(u) &= \sum_{\substack{j\in f:\\ r_j(\mu(u))<\infty}} (r_j(\mu(u)) - c_j)\,(-h_j'(r_j(\mu(u))))\Phi' \circ \Phi^{-1}(u) \qquad (38)\\
&= \sum_{\substack{j\in f:\\ r_j(\mu(u))<\infty}} \frac{r_j(\mu(u)) - c_j}{r_j(\mu(u))}\iota_j(r_j(\mu(u)))\gamma_j(r_j(\mu(u)))\Phi' \circ \Phi^{-1}(u)\\
&= \mu(u)\sum_{j\in f}\gamma_j(r_j(\mu(u)))\Phi' \circ \Phi^{-1}(u).
\end{aligned}$$

As $\mu(u)$ is the unique solution to equation $\sum_{j\in f} h_j(r_j(\mu^f)) = (\Phi^f)^{-1}(u^f)$, the implicit function theorem implies that $\mu(\cdot)$ is locally $\mathcal{C}^1$ at every u such that $\mu(u) \neq \bar{\mu}_j$ for every $j \in f$, with derivative

$$\mu'(u) = \frac{1}{\Phi' \circ \Phi^{-1}(u) \sum_{\substack{j\in f:\\ r_j(\mu(u))<\infty}} h_j'(r_j(\mu(u)))r_j'(\mu(u))}$$

At any such u, $\pi(\cdot)$ is locally $\mathcal{C}^1$, and its derivative can be obtained by differentiating equation (38):

$$\pi'(u) = \frac{\Phi'' \circ \Phi^{-1}(u)}{\Phi' \circ \Phi^{-1}(u)}\mu(u)\sum_{j\in f}\gamma_j(r_j(\mu(u)))$$

$$+ \Phi' \circ \Phi^{-1}(u)\mu'(u) \sum_{\substack{j\in f:\\ r_j(\mu(u))<\infty}} r_j'(\mu(u)) \left[-h_j'(r_j(\mu(u))) - (r_j(\mu(u)) - c_j)h_j''(r_j(\mu(u)))\right]$$

$$= \frac{\Phi^{-1}(u) \times \Phi'' \circ \Phi^{-1}(u)}{\Phi' \circ \Phi^{-1}(u)} \mu(u) \frac{\sum_{j\in f} \gamma_j(r_j(\mu(u)))}{\sum_{j\in f} h_j(r_j(\mu(u)))}$$

$$+ \Phi' \circ \Phi^{-1}(u)\mu'(u)(\mu(u) - 1) \sum_{\substack{j\in f:\\ r_j(\mu(u))<\infty}} r_j'(\mu(u))h_j'(r_j(\mu(u)))$$

$$= -\vartheta \circ \Phi^{-1}(u)\mu(u) \frac{\sum_{j\in f} \gamma_j(r_j(\mu(u)))}{\sum_{j\in f} h_j(r_j(\mu(u)))} + \mu(u) - 1.$$

As $\pi(\cdot)$ is continuous and $\pi'(\cdot)$ has a limit at every u such that $\mu(u) = \bar{\mu}_j$ for some j, it follows that π is $\mathcal{C}^1$ on $(\underline{u}, u^0)$. □

Next, we show that strategies with a high utility level are strictly dominated for firm f:

Lemma F.3. *Suppose that Assumptions C‴, U1′, and U2 hold. Then, equation $\pi^{f\prime}(u^f) = 0$ has a unique solution on $(\underline{u}^f, u_0^f)$, denoted u_1^f. Moreover, $\mu^f(u_1^f) \geq 1$, $\pi^{f\prime}(u^f) > 0$ for every $u^f \in (\underline{u}^f, u_1^f)$, and the strategies in (u_1^f, u_0^f) are strictly dominated for firm f.*

Proof. We drop the firm superscript to ease notation. As $\pi(\underline{u}) = \pi(u_0)$ and $\pi(\cdot)$ is continuous on $[\underline{u}, u_0]$ and differentiable on $(\underline{u}, u_0)$, equation $\pi'(u) = 0$ has a solution in $(\underline{u}, u_0)$ by Rolle's lemma. Let us show that the solution is unique.

Let $\hat{u}$ be the unique solution to $\mu(u) = 1$. We see from equation (37) that $\pi'(u) < 0$ for every $u > \hat{u}$, so equation $\pi'(\hat{u}) = 0$ has no solution on $(\hat{u}, u_0)$.

Suppose first that $\pi'(\hat{u}) = 0$. Then, we see from equation (37) that $\vartheta \circ \Phi^{-1}(\hat{u}) = 0$. By monotonicity of Φ and ϑ, this implies that $\vartheta \circ \Phi^{-1}(u) = 0$ for every $u < \hat{u}$, so that $\pi'(u) > 0$ for every such u. It follows that $\hat{u}$ uniquely solves equation $\pi'(u) = 0$, and that π is strictly increasing on $(\underline{u}, \hat{u}]$, and strictly decreasing on $[\hat{u}, u_0)$.

Next, suppose instead that $\pi'(\hat{u}) < 0$. For every $u < \hat{u}$, we have that $\pi'(u) = (\mu(u) - 1)A(u)$, where

$$A(u) \equiv 1 - \frac{\mu(u)}{\mu(u) - 1} \frac{\sum_{j\in f} \gamma_j(r_j(\mu(u)))}{\sum_{j\in f} h_j(r_j(\mu(u)))} \vartheta \circ \Phi^{-1}(u).$$

Hence, $A(u)$ has the same sign as $\pi'(u)$ for $u < \hat{u}$. Given Assumptions C‴-(i), U1′-(ii), and U1′-(iii), we can apply Lemmas VII, VIII, and IX in Nocke and Schutz (2018b) to obtain that the function

$$\mu \in (1, \bar{\mu}^f) \mapsto \frac{\mu}{\mu - 1} \frac{\sum_{j\in f} \gamma_j(r_j(\mu))}{\sum_{j\in f} h_j(r_j(\mu))}$$

is strictly decreasing. As $\mu(\cdot)$ is strictly decreasing and ϑ and Φ are non-decreasing, this implies that $A(u)$ is locally strictly decreasing in u whenever $\vartheta \circ \Phi^{-1}(u) > 0$. As any solution to equation $A(u)=0$ on $(\underline{u},\hat{u})$ must be such that $\vartheta\circ\Phi^{-1}(u)>0$, this implies that $A(u)$ is locally strictly decreasing at any such solution. Hence, the solution, u_1, is unique. Moreover, π is strictly increasing on $(\underline{u},u_1)$ and strictly decreasing on (u_1,u_0).

To see why the strategies with $u^f\geq u_1^f$ are strictly dominated for firm f, note that, regardless of what the other firms do, firm f's profit is strictly decreasing in u^f on $[u_1^f,u_0^f)$, as an increase in u^f strictly lowers $\pi^f(u^f)$ and weakly lowers $\Psi'\left(\sum_B a(B)\prod_{g\in B}u^g\right)$, as $\Psi''\leq 0$. □

Given Lemma F.3, restricting the action set of each firm f to $(\underline{u}^f,u_1^f]$ does not affect the set of (correlated) equilibria of the pricing game (regardless of what strictly increasing transformation is applied to profits). For the same reason, we now assume that each firm f chooses t^f in $[t_1^f,\underline{t}^f)$, where $t_1^f\equiv-\log u_1^f$ and $\underline{t}^f\equiv-\log\underline{u}^f$.

We can now conclude on the strict concavity of $\log\pi^f(\mathrm{e}^{-t^f})$:

Lemma F.4. *Suppose that Assumptions C′′′, U1′, and U2 hold. Then,* $t^f\in[t_1^f,\underline{t}^f)\mapsto\log\pi^f\left(e^{-t^f}\right)$ *is strictly concave.*

Proof. We drop the firm superscript to ease notation. As

$$\frac{d}{dt}\log\pi(\mathrm{e}^{-t})=-\,\mathrm{e}^{-t}\,\frac{\pi'(\mathrm{e}^{-t})}{\pi(\mathrm{e}^{-t})},$$

the strict log-concavity of $\pi(\mathrm{e}^{-t})$ is equivalent to $\pi(\cdot)$ having a strictly decreasing elasticity. Using the formulas in Lemma F.2, we obtain that, for every $u<u_1$, the elasticity of π is given by

$$\begin{aligned}\frac{u\pi'(u)}{\pi(u)}&=\frac{\mu(u)-1}{\mu(u)}\frac{\Phi\circ\Phi^{-1}(u)}{\Phi'\circ\Phi^{-1}(u)\sum_{j\in f}\gamma_j(r_j(\mu(u)))}\left[1-\frac{\mu(u)}{\mu(u)-1}\frac{\sum_{j\in f}\gamma_j(r_j(\mu(u)))}{\sum_{j\in f}h_j(r_j(\mu(u)))}\vartheta\circ\Phi^{-1}(u)\right]\\&=\frac{\mu(u)-1}{\mu(u)}\frac{\sum_{j\in f}h_j(r_j(\mu(u)))}{\sum_{j\in f}\gamma_j(r_j(\mu(u)))}\frac{1}{K\circ\Phi^{-1}(u)}\left[1-\frac{\mu(u)}{\mu(u)-1}\frac{\sum_{j\in f}\gamma_j(r_j(\mu(u)))}{\sum_{j\in f}h_j(r_j(\mu(u)))}\vartheta\circ\Phi^{-1}(u)\right].\end{aligned}$$

By Lemma F.3, for every $u<u_1$, $\mu(u)>1$ and the term inside square brackets is strictly positive. Moreover, our assumptions and Lemmas VII, VIII, and IX in Nocke and Schutz (2018b) imply that the term inside the square bracket is non-increasing in u and the term in factor in front of the square bracket is strictly decreasing in u □

Concavity of the $\log\Psi'$ term.

Lemma F.5. *Suppose that Assumptions C′′′-(ii) and U2 hold. Then,*

$$t\in\mathbb{R}^{\mathcal{F}}\mapsto\log\Psi'\left(\sum_B a(B)\prod_{f\in B}e^{-t^f}\right)$$

is concave.

Proof. Define

$$V(t) \equiv \log\left(\sum_B a(B) \prod_{f\in B} \mathrm{e}^{-t^f}\right)$$

and $\chi(X) \equiv \log \Psi'(\mathrm{e}^X)$, and notice that

$$\log \Psi'\left(\sum_B a(B) \prod_{f\in B} \mathrm{e}^{-t^f}\right) = \chi(V(t)).$$

As log-convexity is preserved by multiplication and summation, V is convex. Moreover, as

$$\chi'(X) = \mathrm{e}^X \, \frac{\Psi''(\mathrm{e}^X)}{\Psi'(\mathrm{e}^X)},$$

we have that χ is concave and non-increasing.

It follows that, for every $\lambda \in [0,1]$ and every t, t',

$$\chi(V(\lambda t + (1-\lambda)t')) \geq \chi\left(\lambda V(t) + (1-\lambda)V(t')\right) \geq \lambda\chi(V(t)) + (1-\lambda)\chi(V(t')),$$

where the first inequality follows as V is convex and χ is non-increasing and the second inequality follows by the concavity of χ. Hence, $\chi \circ V$ is concave, as stated. □

Combining Lemmas F.3 and F.5, we obtain that $E(t)$ is strictly concave on the rectangle $\prod_{f\in\mathcal{F}}[t_1^f, \underline{t}^f)$, which contains all the strategies that are not strictly dominated.

Uniqueness of the correlated equilibrium.

Lemma F.6. *Suppose that Assumptions C‴, U1′, U2, and U3 hold. Then, for every firm f, there exists $\bar{t}^f \in (t_1^f, \underline{t}^f)$ such that the strategies $t^f > \bar{t}^f$ are strictly dominated.*

Proof. Let $H(u) \equiv \sum_B a(B) \prod_{g\in B} u^g$ for every u. We have that $\underline{\psi} \equiv \Psi'\left(H\left((u_1^g)_{g\in\mathcal{F}}\right)\right)$ is finite and strictly positive and $\Psi'(H(u)) \geq \underline{\psi}$ for every $u \in \prod_{g\in\mathcal{F}}[\underline{u}^g, u_1^g]$ by monotonicity of Ψ'. Let $\overline{\psi} \equiv \lim_{H\to H((\underline{u}^g)_{g\in\mathcal{F}})} \Psi'(H)$. The limit exists and is finite by monotonicity of Ψ' and Assumption U3. Moreover, $\Psi'(H(u)) \leq \overline{\psi}$ for every $u \in \prod_{g\in\mathcal{F}}[\underline{u}^g, u_1^g]$.

When firm f sets t_1^f, it receives a profit of at least $\pi^f(\mathrm{e}^{-t_1^f})\underline{\psi} > 0$. Note that, for every t,

$$\widehat{\Pi}^f(t) \leq \pi^f(\mathrm{e}^{-t^f})\overline{\psi} \underset{t^f\to\underline{t}^f}{\longrightarrow} 0.$$

Hence, there exists $\bar{t}^f \in (t_1^f, \underline{t}^f)$ such that $\widehat{\Pi}^f(t) < \pi^f(\mathrm{e}^{-t_1^f})\underline{\psi}$ for every t such that $t^f > \bar{t}^f$. It follows that strategies $t^f > \bar{t}^f$ are strictly dominated by t_1^f. □

Lemmas F.3 and F.6 imply that in any correlated equilibrium of the pricing game, firm f puts no weight outside the compact set $[t_1^f, \bar{t}^f]$. As the exact potential E is strictly concave on $\prod_{g\in\mathcal{F}}[t_1^g, \bar{t}^g]$, Theorem 2 in Neyman (1997) implies that the logged pricing game has a unique correlated equilibrium.

F.2 Nested Demand

Fix a multiproduct-firm pricing game with nested demand, as defined in Section 4.3. The approach consists again in showing that, with the change of variable $t^f = -\log u^f$, the log-potential

$$E(t) \equiv \sum_{f\in\mathcal{F}} \log \pi^f(\mathrm{e}^{-t^f}) + \sum_{n\in\mathcal{N}} \log \Phi^{n\prime}\left(\sum_{f\in n} \mathrm{e}^{-t^f}\right) + \log \Psi'\left[\sum_{n\in\mathcal{N}} \Phi^n\left(\sum_{f\in n} \mathrm{e}^{-t^f}\right)\right]$$

is strictly concave on a rectangle that includes all the pure strategies that are not strictly dominated. We continue to denote by $\widehat{\Pi}^f(t)$ the profit of firm f in that game. As in the case of baskets, we assume that h^f is derivable from multi-stage discrete/continuous choice and thus takes the form of equation (36), and that the functions h_j, Φ^f, and Ψ satisfy Assumptions C‴, U1′, and U2. We assume in addition that the functions Φ^n are well behaved in the following sense:

Assumption U4. *For every nest n, the function Φ^n is $\mathcal{C}^2$. Its elasticity, denoted K^n, is strictly positive and non-decreasing. Its curvature, denoted ϑ^n, is non-negative and non-decreasing.*

We can now state our uniqueness result for the case of nested demand:

Theorem F.2. *Consider a multiproduct-firm pricing game with nested demand satisfying Assumptions C‴, U1′, U2, and U4. The exact potential function of the logged pricing game in which each firm f chooses t^f is strictly concave on a rectangle that includes all the pure strategies that are not strictly dominated. Therefore, the pricing game has a unique pure-strategy Nash equilibrium (no matter what the transformation of payoffs), which corresponds to the unique potential maximizer. If, in addition, Assumption U3 holds, then the potential maximizer also corresponds to the unique correlated equilibrium of the logged pricing game.*

Theorem F.2 generalizes Theorem III in Nocke and Schutz (2018b) in two ways by allowing for a larger class of functions h^f and by establishing the uniqueness of the correlated equilibrium.

The approach of Section F.1 extends readily to the case of nested demand. It is clear that Lemmas F.1 and F.2 continue to hold, as the properties of firm f's Ramsey problem are unaffected by whether demand has a basket structure or a nested structure. For the same reason, the part of Lemma F.3 that is concerned with the properties of π^f and the definition of u_1^f continues to hold. Moreover, the strategies in (u_1^f, u_0^f) remain strictly dominated by u_1^f for firm $f \in n$. This holds because, regardless of how the other firms behave, Π^f is strictly decreasing in u^f on that interval: as u^f increases, $\pi^f(u^f)$ decreases, and so do $\Phi^{n\prime}\left(\sum_{g\in n} u^g\right)$ and $\Psi'\left[\sum_{m\in\mathcal{N}} \Phi^m\left(\sum_{g\in m} u^g\right)\right]$ by Assumptions U2 and U4. Hence, we can again restrict the action set of each firm f to $[t_1^f, \underline{t}^f)$. Given its focus on the properties of π^f, Lemma F.4 holds as well.

Next, we state and prove the counterpart of Lemma F.5 for the case of nested demand:

Lemma F.7. *Suppose that Assumptions C‴-(ii), U2, and U4 hold. Then, the functions*

$$t \in \mathbb{R}^{\mathcal{F}} \mapsto \log \Psi' \left[\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} e^{-t^g} \right) \right]$$

and, for every $n \in \mathcal{N}$,

$$t \in \mathbb{R}^n \mapsto \log \Phi^{n\prime} \left(\sum_{g \in n} e^{-t^g} \right)$$

are concave.

Proof. The concavity of $\log \Phi^{n\prime} \left(\sum_{g \in n} \mathrm{e}^{-t^g} \right)$ follows from the argument in the proof of Lemma F.5 (replacing Ψ' by $\Phi^{n\prime}$). To prove the concavity of $\log \Psi' \left[\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} \mathrm{e}^{-t^g} \right) \right]$, note that

$$\log \Psi' \left[\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} \mathrm{e}^{-t^g} \right) \right] = \chi \left(\log \left[\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} \mathrm{e}^{-t^g} \right) \right] \right),$$

where $\chi(X) \equiv \log \Psi'(\mathrm{e}^X)$. The result follows, as χ is non-decreasing and concave by Assumption U4 and $\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} \mathrm{e}^{-t^g} \right)$ is log-convex (see the argument at the end of the proof of Proposition 7). □

Combining the above lemmas, we obtain that $E(t)$ is strictly concave on $\prod_{g \in \mathcal{F}} [t_1^f, \underline{t}^f)$, which establishes the uniqueness of the pure-strategy Nash equilibrium. Next, we prove the counterpart of Lemma F.6 for the case of nested demand:

Lemma F.8. *Suppose that Assumptions C‴, U1′, U2, U3, and U4 hold. Then, for every firm f, there exists $\overline{t}^f \in (t_1^f, \underline{t}^f)$ such that the strategies $t^f > \overline{t}^f$ are strictly dominated.*

Proof. Using the same arguments as in the proof of Lemma F.6, the limits

$$\underline{\psi} \equiv \lim_{u \to (u_1^g)_{g \in \mathcal{F}}} \Psi' \left[\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} u^g \right) \right] \quad \text{and} \quad \overline{\psi} \equiv \lim_{u \to (\underline{u}^g)_{g \in \mathcal{F}}} \Psi' \left[\sum_{m \in \mathcal{N}} \Phi^m \left(\sum_{g \in m} u^g \right) \right]$$

exist and are finite and strictly positive. As $\Phi^{n\prime}$ and Ψ' are non-increasing, we have that when firm $f \in n$ sets t_1^f, it earns at least

$$\underline{\Pi}^f \equiv \pi^f(\mathrm{e}^{-t_1^f}) \Phi^{n\prime} \left(\sum_{g \in n} \mathrm{e}^{-t_1^g} \right) \underline{\psi} > 0.$$

Moreover, for every t,

$$\widehat{\Pi}^f \leq \pi^f(\mathrm{e}^{-t^f})\Phi^{n\prime}(\mathrm{e}^{-t^f})\overline{\psi},$$

which tends to 0 as t^f tends to $\underline{t}^f$ by Lemma E.2. Hence, there exists $\bar{t}^f \in (t_1^f, \underline{t}^f)$ such that $\widehat{\Pi}^f(t) < \underline{\Pi}^f$ for every t such that $t^f > \bar{t}^f$. It follows that strategies $t^f > \bar{t}^f$ are strictly dominated by t_1^f. □

We can then apply Neyman (1997)'s Theorem 2 as in the previous subsection.

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