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Approval vs. Participation Quorums

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Approval vs. Participation Quorums*

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Abstract

Using a pivotal costly voting model of elections between a status quo and a challenger alternative, we compare participation and approval quorum requirements in terms of how they shape voter incentives to cast votes, and how they ultimately impact voter turnout, election outcomes, and welfare. We first show that approval and participation quorum restrictions of equal strictness result in at most two types of stable non-trivial equilibria: "abstention," in which status quo supporters strategically abstain from voting, and "coordination," in which they vote with positive probability. While abstention equilibria are always identical in the two quorum settings, coordination equilibria may differ, but only when the cost of voting is sufficiently low and status quo support among voters is neither extremely high or low, nor is it close to the degree of support for the challenger. We show that, in those cases, the difference in the outcomes of interest between approval and participation quorum settings is quantitatively small. The main difference between the two settings therefore arises from the fact that, under an approval quorum, coordination equilibrium exists for a narrower range of status quo support levels than under a participation quorum. We discuss the implications of these findings for designing optimal quorum restrictions, suggesting that choosing an approval quorum over a participation quorum and setting its strictness close to half of the number of voters, or setting no quorum restrictions at all, are often welfare maximizing choices.

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Keywords: voting, participation quorum, approval quorum

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1 Introduction

Various types of real-life elections including both large (e.g., national level referenda) and small settings (e.g., parliaments, meetings of shareholders and boards of directors, societies, committees, etc.) often involve quorum requirements. There are two main types of quorum requirements: participation and approval. A participation requirement mandates that a status quo can be overruled only if the number of votes cast is above a certain threshold. An approval requirement requires a minimum number of votes to be cast in favor of a challenger alternative to overrule a status quo. In this paper, we directly compare the two types of quorums to understand how they shape voter incentives to participate in costly voting, and how they affect voter turnout, outcomes of elections, and welfare.

Quorum requirements of both types are an integral part of voting procedures and electoral legislation at various levels of elections. For example, the European Council will adopt a proposal only if at least 55% of member states, representing at least 65% of the EU population, vote in favor.[1] The General Assembly of the United Nations requires a two-thirds majority on key issues like international peace and security[2]. National referenda in some European countries have participation requirements, e.g., the Netherlands (30%), Bulgaria, Italy, Poland, Portugal, Romania, Slovenia, Slovakia (50% in each). Constitutional amendments in Denmark require a 40% approval quorum, while Hungary sets a minimal approval threshold of 25%. Other EU countries do not have quorum requirements for national referenda.[3]

In smaller elections, quorum restrictions are also widespread, particularly in meetings of shareholders and boards of directors in private companies. In Spain, legislation requires the 25% meeting attendance of joint-stock company shareholders or their representatives. In Switzerland, certain decisions in stock corporations require two thirds of shareholder votes. In Australia, both boards of directors and shareholder meetings require the presence of at least two members unless otherwise specified in the company's constitution. In Austria, the presence of one single shareholder is sufficient for shareholders' meetings in stock corporations, and most resolutions are passed by a simple majority of votes cast, although some important decisions require 75% of votes to be passed.[4]

The adoption of quorum restrictions is typically motivated by the willingness to assure that collective decisions over political, social, and business issues actually represent voter preferences. Yet, despite the prevalence of quorum requirements in real-life elections

[1] http://www.consilium.europa.eu/en/council-eu/voting-system/qualified-majority/

[2] https://www.un.org/en/model-united-nations/general-assembly

[3] See, for example, Aguiar-Conraria, Magalhaes and Vanberg (2016) or Hizen (2021) for extensive reviews of quorum rules in different countries.

[4] https://www.dlapiperintelligence.com/goingglobal/corporate/index.html?t=32-quorum-requirements

and substantial variation in their forms, there are still gaps in our understanding of how various types of quorums and levels of their strictness affect voter participation decisions and what impact quorums ultimately have on the outcomes of elections and welfare.

The literature usually studies participation and approval quorums separately, primarily focusing on their effects on voter turnout. The research on the effects of participation quorum requirements on turnout is fairly extensive and generally agrees that, although sometimes quorum requirements may provide incentives for status quo opponents to coordinate (Charlety, Fagart, and Souam, 2019; Matveenko, Valei, and Vorobyev, 2022), they are more likely to decrease voter participation than to stimulate it, because they create incentives for status quo supporters to abstain strategically. Such adverse effects of participation quorum requirements are theoretically derived by, for example, Corte-Real and Pereira (2004), Hizen and Shinmyo (2010), Herrera and Mattozzi (2010), Laruelle and Valencioano (2011), Aguiar-Conraria and Magalhaes (2010a), Flamand and Troumpounis (2014), and Maniquet and Morelli (2015) and are supported by both experimental (Aguiar-Conraria, Magalhaes and Vanberg, 2016, 2020; Hizen, 2021) and empirical evidence (Uleri, 2002; Aguiar-Conraria and Magalhaes, 2010b; Kouba and Haman, 2021). Some attention has also been given to the welfare consequences of participation quorums from both voter and candidate perspectives. In particular, Matveenko et. al (2022) study the effects of different participation quorum requirements on election outcomes and voter welfare, and characterize socially optimal quorum rules as a function of the candidates' levels of support. They show that stricter quorum requirements do not necessarily benefit a status quo candidate, and that an absence of quorum requirements is never socially optimal.

While there is a fair amount of literature on the effects of participation quorums, research on approval quorums is scarcer. As in the case of participation quorums, studies of approval quorums almost exclusively focus on the impact of restrictions on voter turnout. Laruelle and Valenciano (2011) and Aguiar-Conraria et. al. (2016, 2020) use theoretical models to argue that approval quorums generate fewer abstention incentives than participation quorums. Other models suggest that the two types of quorums could similarly reduce turnout (Herrera and Mattozzi, 2010), or that, while both quorum types are likely to reduce turnout, their relative impacts may vary due to the different nature of abstention incentives they generate (Aguiar-Conraria and Magalhaes, 2010a). Rare empirical (Aguiar-Conraria and Magalhaes, 2010b) and experimental (Aguiar-Conraria et. al., 2016 and 2020) evidence suggests that approval quorums decrease turnout less than participation quorums, and may even have no significant impact on participation rates.

In this paper, we explicitly study the consequences of both participation and approval quorum requirements, focusing not only on turnout, but also on election outcomes and voter welfare, and make an explicit comparison of the two types of quorums in terms of the equilibria they generate. We analyze elections with quorum requirements in a pivotal costly voting framework, in which voters value a status quo and a challenger alternative differently, have identical participation costs, and decide only whether to abstain or to vote for their preferred alternative based on the probability that their votes will be decisive for election outcomes. Similar costly private value voting models have been pioneered by Palfrey and Rosenthal (1983, 1985) and Ledyard (1984), and have been widely used to study voter turnout as well as electoral policies and institutions by, for example, Borgers (2004), Krasa and Polborn (2009), Ghosal and Lockwood (2009), Taylor and Yildirim (2010a,b), Bognar et al. (2015), Kartal (2015), Krishna and Morgan (2015), Arzumanyan and Polborn (2017), Grillo (2017), Chakravartya, Kaplan, and Myles (2018), Baghdasaryan, Iannantuoni, and Maggian (2019), and more recently by Gersbach, Mamageishvili, and Tejada (2021), Durazzo and Turchick (2022), Vorobyev (2022), and Mamageishvili and Tejada (2023). In models of this type, the exact numbers of voters supporting each alternative are typically unknown. Instead, a voter may support a candidate with a commonly known probability, which thus is a measure of the ex-ante support for alternatives. The ex-ante support is the central component of our model, as we explore how the effects of approval and participation quorum requirements of different strictness on turnout, election outcomes, and voter welfare vary with support levels.[5]

First, we identify cases in which participation and approval quorums of quantitatively equal strictness produce identical sets of equilibria, and therefore have identical impacts on voter behavior and election outcomes. Second, we closely explore those cases in which the equilibria in the two quorum settings are different. Specifically, we show that this happens when the cost of voting is sufficiently large, or when the levels of support for the alternative are either extreme or close to equal. Then, we explore cases in which participation and approval quorums do not produce identical equilibria, and compare them in terms of equilibrium voter turnout, the ex-ante likelihood of the status quo to win, and ex-ante voter welfare. We show that both types of quorum restrictions result in at most two types of stable non-trivial equilibria: "abstention," in which status quo supporters strategically abstain, and "coordination," in which they participate with positive probability. While abstention equilibria are always identical in the two quorum settings, coordination equilibria differ, though the difference in the outcomes of interest is quantitatively small. The main difference between the two settings therefore comes from the fact that, under an approval quorum, coordination equilibrium exists for a narrower

[5]Aguiar-Conraria and Magalhaes (2010a) are the first to study the effect of approval and participation quorums on voter turnout in a pivotal costly voting framework, although they use a heterogenous cost setting, in contrast to our homogenous cost setting. However, their analysis is limited to voter participation for several specific values of the ex-ante support. Therefore, although the focus of our project is welfare and the efficiency of the policies, not participation per se, our analysis can be also seen as a generalization of their work with respect to the effect of quorum requirements on turnout.

range of status quo support levels than under a participation quorum. We discuss the implications of these findings for designing optimal quorum restrictions, suggesting that choosing an approval quorum over a participation quorum and setting its strictness close to half of the number of voters, or setting no quorum restriction at all, are likely to be welfare maximizing choices.

In terms of our approach and methodology, we closely follow Matveenko et. al. (2022), who study the effects of participation quorum requirements on turnout, elections outcomes, and welfare – the same equilibrium characteristics that we analyze in this paper. Using the same methodology and focusing on the same characteristics of equilibria allows us to directly compare our results to theirs, and to ultimately refine their findings on the optimal design of quorum restrictions.

In terms of its focus, the closest to our work is Maniquet and Morelli (2015). They also directly compare participation and approval quorums of quantitatively equal exogenous strictness by their ability to generate socially desirable outcomes, though in a different theoretical framework, and find that an approval quorum is superior to a participation quorum. One element of their model that is crucial to their main result is the absence of participation costs, which removes all incentives for voters to abstain under an approval quorum, whereas under a participation quorum, status quo supporters may strategically choose not to vote.

The contribution of our paper is therefore twofold. First, we extend the existing limited work on approval quorums by comprehensively studying how their presence affects voter incentives to participate, and what impact they ultimately have on turnout, election outcomes, and voter welfare, particularly highlighting the role of support levels for competing alternatives and voting costs for our findings. Second, we make an explicit comparison between approval and participation quorum restrictions and their impacts on various outcomes of interest in a more general setting than any existing research has done so far, finding a high degree of equivalence between the two settings and new implications for optimal design of quorum restrictions.

The rest of the paper is organized as follows. In the next section, we describe the model setup. Then, we derive conditions for equilibrium voter participation for the approval quorum setting and for the participation quorum setting separately. We proceed to compare the equilibria in the two settings, identifying when the equilibria under approval and participation quorum restrictions are equivalent and when they are distinct. We derive some general results, and then, because the model cannot be solved analytically in general, we numerically solve it for various sets of parameters, and present some consistent findings. We define our main characteristics of interest, including voter turnout, winning probabilities of the alternatives, and voter welfare, and we explore how they differ when equilibria under approval and participation restrictions do not coincide. We conclude with a discussion on the optimal design of quorum restrictions.

2 The Model

Consider two alternatives (candidates), A (status quo) and B (challenger), and $N \geq 3$ voters. Voters have preferences for alternatives but the exact numbers of voters supporting each alternative are unknown. Instead, a voter may support alternative A with the commonly known probability $\alpha \in [0, 1]$ or alternative B with probability $1 - \alpha$. We will refer to a voter supporting alternative A as "an A-supporter", "a status quo supporter" or "a type A voter" throughout the paper. Likewise, we will refer to a voter supporting alternative B as 'a 'B-supporter", "a challenger supporter" or "a type B voter". Parameter α is the key element of our study: we will analyze how equilibria and their characteristics change in response to changes in α which is, in fact, a measure of the status quo alternative ex-ante support, while $1 - \alpha$ represents the ex-ante support for the challenger alternative.

While the alternatives do not take any action, voters decide whether to vote or to abstain, and, if they decide to vote, they also decide which alternative to cast their vote for. Voting is costly: it costs $c > 0$ for each voter to cast a vote. If a voter's preferred alternative wins, the voter gains utility 1 if she did not vote, and $1 - c$ otherwise. If her preferred alternative loses, the voter gains utility 0 if she abstained, and $-c$ if she voted. Elections are run under majority rule and a tie is resolved with a coin flip.

Elections are characterized by an exogenously set quorum requirement $Q \in [1, ..., N]$.[6] The quorum requirement may be either participation or approval, which we refer to as a quorum type or a setting. A participation quorum requirement is the minimum number of votes which must be cast to validate the election. An approval quorum requirement is the minimum number of votes which must be cast for the winning alternative to validate the election. If the quorum requirement is not met, the status quo wins regardless of the number of votes cast for each alternative.

Following the typical approach for pivotal costly voting models with homogenous costs, we solve the model for within-group symmetric mixed-strategy Bayesian Nash equilibria, where all voters supporting the same alternative adopt the same voting strategy, casting their votes with probability $p_T, T \in \{A, B\}$.[7] Further, we focus only on stable equilibria in our model. Suppose there is an arbitrary small perturbation in the equilibrium actions of the type $T \in \{A, B\}$ voters: they vote with probability $\bar{p_T} = p_T^* + \epsilon$

[6]When Q=0, i.e., no quorum requirement is set, our model is identical to that of Taylor and Yildirim (2010b) and Matveenko et. al. (2022). Therefore, all of their results hold in our model with Q=0 as well.

[7]A similar approach is used in a large body of pivotal costly voting models (see, for example, Taylor and Yildirim, 2010b, Arzumanyan and Polborn, 2017, Gersbach, Mamageishvili and Tejada, 2021, Durazzo and Turchik, 2022, and Mamageishvili and Tejada, 2023). A popular alternative approach would assume voter-specific costs and allow for a pure strategy equilibrium characterized by type-specific cost thresholds such that a voter votes if her cost is below the threshold and abstains otherwise. While being marginally more intuitive, such an approach increases the dimensionality of the problem and requires additional assumptions on the properties of the distribution of costs across voters.

where $p_T^* \in [0,1)$ is equilibrium participation and $\epsilon > 0$. Anticipating this, every voter will expect to obtain either a strictly positive or a strictly negative benefit. If the benefit is positive, voters will have incentives to increase their participation, further departing from the equilibrium. If it is negative, voters will have incentives to decrease their participation, eventually returning to the equilibrium level p_T^*. In the former case, we refer to the equilibrium as stable. In the latter case, we refer to it as unstable. A similar concept of the stability of a voting equilibrium is used in, for example, Maniquet and Morelli (2015) and Matveenko et. al. (2022).

Analysis of voter behavior should begin with the observation that, conditional on voting, a voter's weakly dominant strategy is to vote for her preferred alternative; thus, we focus only on participation decisions. A voter decides to vote if and only if the participation cost c does not exceed her expected benefit from participation. The benefit depends on the probability that her vote will be pivotal (decisive). We denote this probability as Π. Hence, for every voter, Π is a function of a pair of strategies, (p_A, p_B), which are the probabilities that a type A voter and a type B voter cast votes, respectively.

Suppose that A-supporters adopt voting strategy p_A and B-voters adopt strategy p_B. Denote $P_i^j(k) = \binom{j}{i} k^i (1-k)^{j-i}$ for shorter notation. Consider a type A voter. Then, the probability that there are a A-types among other $N-1$ voters is $P_a^{N-1}(\alpha)$. The probability that l of them will participate in elections is $P_l^a(p_A)$. With these probabilities, one can construct voters' pivotal probabilities, which, however differ between the approval and participation quorum settings.

3 Approval Quorum Equilibrium

Under an approval quorum requirement, an A-supporter is pivotal only when, without her vote, the number of B-participants is equal to or exceeds the number of A-participants, and the quorum requirement is satisfied, which is possible only if $Q \leq \frac{N}{2}$. In this case, an A-supporter increases her expected utility by 0.5 when she participates, because her participation either turns a draw into an A victory or an A loss into a draw. We denote the probability of such an event as Π_A^1.

$$\Pi_A^1(p_A, p_B) = 1_{\{Q \leq \frac{N}{2}\}} \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) \left(\sum_{l=Q}^{a} P_l^a(p_A) P_l^{N-a-1}(p_B) + \sum_{l=\max\{0,Q-1\}}^{a} P_l^a(p_A) P_{l+1}^{N-a-1}(p_B) \right). \tag{1}$$

A B-supporter is pivotal in two cases. First, a B-supporter is pivotal whenever, without her vote, the number of A-participants is equal to or exceeds the number of B-participants by 1 and the quorum requirement is satisfied. In this case, a B-supporter increases her expected utility by 0.5 when she casts a vote, because her participation either turns a draw into a B victory, or a B loss into a draw. The probability of such an event is denoted by Π_B^1. Second, a B-supporter is pivotal whenever, without her vote, the quorum requirement Q is unfulfilled by 1 and alternative B ties or beats alternative A by any number of votes. In such a case, a B-supporter's participation turns B's loss due to the lack of a quorum into a victory, increasing her utility by 1. The probability of this case is labeled as Π_B^2.

$$\Pi_B^1(p_A, p_B) = \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) \left(\sum_{l=Q}^{a} P_l^a\left(p_A\right) P_{l-1}^{N-a-1}\left(p_B\right) + \sum_{l=Q}^{a} P_l^a\left(p_A\right) P_l^{N-a-1}\left(p_B\right) \right). \tag{2}$$

$$\Pi_B^2(p_A, p_B) = \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) \sum_{l=0}^{Q-1} P_l^a\left(p_A\right) P_{Q-1}^{N-a-1}\left(p_B\right). \tag{3}$$

Equilibrium values of p_A and p_B must then solve the following system:

$$\begin{cases} B_A(p_A, p_B) \equiv 0.5\Pi_A^1(p_A, p_B) = c; \\ B_B(p_A, p_B) \equiv 0.5\Pi_B^1(p_A, p_B) + \Pi_B^2(p_A, p_B) = c, \end{cases} \tag{4}$$

when $0 < p_A < 1$ and $0 < p_B < 1$. When $p_T = 0$, it must be that $B_T < c$, and when $p_T = 1$, it must be that $B_T > c$, $T \in \{A, B\}$.

4 Participation Quorum Equilibrium

Under the participation quorum requirement, an A-supporter is pivotal in three cases. First, an A-supporter is pivotal whenever, without her vote, the number of B-participants is equal to or exceeds the number of A-participants by 1, and the quorum requirement is satisfied. In this case, an A-supporter increases her expected utility by 0.5 when she participates, because her participation either turns a draw into an A victory, or an A loss into a draw. We denote the probability of this event as Π_A^1. Second, an A-supporter is pivotal when, without her vote, the quorum requirement Q falls short by 1 and alternative A loses to alternative B by 1 vote. In this case, an A-supporter's participation turns A's victory due to the lack of a quorum into a draw, decreasing her utility by 0.5. Note that such a case may arise only if Q is even and not less than 2. We denote the probability of this event as Π_A^2. Finally, an A-supporter is pivotal whenever, without her vote, the quorum requirement Q is not reached by 1 and alternative A loses more than 1 vote to alternative B. In such a case, an A-supporter's participation turns A's victory due to the lack of a quorum into a loss, decreasing her utility by 1. This is possible only if Q is at least 3. We label the probability of such an event as Π_A^3.

$$\Pi_A^1(p_A, p_B) = \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) \left(\sum_{l=\lceil \frac{Q}{2} \rceil}^{a} P_l^a\left(p_A\right) P_l^{N-a-1}\left(p_B\right) + \sum_{l=\lceil \frac{Q-1}{2} \rceil}^{a} P_l^a\left(p_A\right) P_{l+1}^{N-a-1}\left(p_B\right) \right). \tag{5}$$

$$\Pi_A^2(p_A, p_B) = 1_{\{Q \geq 2\}} 1_{\{Q \mod 2 = 0\}} \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) P_{\frac{Q}{2}-1}^a\left(p_A\right) P_{\frac{Q}{2}}^{N-a-1}\left(p_B\right). \tag{6}$$

$$\Pi_A^3(p_A, p_B) = 1_{\{Q \geq 3\}} \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) \sum_{l=0}^{\lfloor \frac{Q-3}{2} \rfloor} P_l^a\left(p_A\right) P_{Q-1-l}^{N-a-1}\left(p_B\right). \tag{7}$$

Now, consider a type B voter. A B-supporter is also pivotal in three cases. First, a B-supporter is pivotal whenever, without her vote, the number of A-participants is equal to or exceeds the number of B-participants by 1, and the quorum requirement is satisfied. In this case, a B-supporter increases her expected utility by 0.5 when she casts a vote, because her participation either turns a draw into a B victory, or a B loss into a draw. The probability of this event is denoted by Π_B^1. Second, a B-supporter is pivotal when, without her vote, the quorum requirement Q is unfulfilled by 1, and alternative B loses 1 vote to alternative A. In such a case, a B-supporter's participation turns B's loss due to the lack of a quorum into a draw, increasing her utility by 0.5. Note that this case can arise only if Q is even and not less than 2. We label the probability of this case Π_B^2. Finally, a B-supporter is pivotal whenever, without her vote, the quorum requirement Q is unfulfilled by 1 and alternative B ties or beats alternative A by any number of votes. In such a case, a B-supporter's participation turns B's loss due to the lack of a quorum into victory, increasing her utility by 1. We label the probability of this Π_B^3.

$$\Pi_B^1(p_A, p_B) = \sum_{a=0}^{N-1} P_a^{N-1}(\alpha)\left(\sum_{l=\lceil \frac{Q}{2}\rceil}^{a} P_l^a(p_A)\, P_l^{N-a-1}(p_B) + \sum_{l=\lceil \frac{Q+1}{2}\rceil}^{a} P_l^a(p_A)\, P_{l-1}^{N-a-1}(p_B)\right). \tag{8}$$

$$\Pi_B^2(p_A, p_B) = 1_{\{Q\geq 2\}} 1_{\{Q \mod 2=0\}} \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) P_{\frac{Q}{2}}^a(p_A)\, P_{\frac{Q}{2}-1}^{N-a-1}(p_B). \tag{9}$$

$$\Pi_B^3(p_A, p_B) = \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) \sum_{l=0}^{\lfloor \frac{Q-1}{2}\rfloor} P_l^a(p_A)\, P_{Q-1-l}^{N-a-1}(p_B). \tag{10}$$

Equilibrium values of p_A and p_B must then solve the following system:

$$\begin{cases} B_A(p_A, p_B) \equiv 0.5\Pi_A^1(p_A, p_B) - 0.5\Pi_A^2(p_A, p_B) - \Pi_A^3(p_A, p_B) = c; \\ B_B(p_A, p_B) \equiv 0.5\Pi_B^1(p_A, p_B) + 0.5\Pi_B^2(p_A, p_B) + \Pi_B^3(p_A, p_B) = c, \end{cases} \tag{11}$$

when $0 < p_A < 1$ and $0 < p_B < 1$. When $p_T = 0$, it must be that $B_T < c$, and when $p_T = 1$, it must be that $B_T > c$, $T \in \{A, B\}$.

5 Equilibrium Comparison

Our ultimate goal is to compare approval and participation quorum settings in terms of the properties of equilibria they generate. Specifically, we explore how the equilibrium characteristics of interest, including voter turnout, the winning probabilities of each alternative, and voter welfare differ in the two settings for the same values of N, Q, α and c. Because these characteristics are fully defined by the equilibrium participation probabilities of A-type and B-type voters, firstly, we compare equilibrium participation rates in approval and participation quorum settings. We begin with a series of general results.

First, when $Q = 1$, formula (5) is identical to formula (1), formula (8) is identical to formula (2), and formula (10) is identical to formula (3). As a result, system (4) and system (11), which define equilibria, are identical, implying that the sets of equilibria under a participation quorum and an approval quorum coincide.[8]

Second, note that if A-type voters abstain with certainty, the pivotal probabilities of B-types (and therefore their expected benefit functions and equilibrium participation) in the participation quorum setting and in the approval quorum setting must be the same for any Q, because in such a case there must be at least Q B-types participating in both settings to allow alternative B to win. As a result, in any equilibrium in which A-type voters abstain, the participation of B-type voters is the same in both quorum settings. The following proposition states this formally.

Proposition 1. *For any Q, the sets of equilibria under a participation quorum or an approval quorum coincide when A-type voters abstain in equilibrium.*

Proof: When $p_A = 0$, the expected benefit of B-types under a participation quorum is the same as the benefit under an approval quorum. Under a participation quorum, the first two elements of the benefit function, given by formula (8) and formula (9), are equal to zero when $p_A = 0$, and therefore

$$B_B(0, p_B) = \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) P_{Q-1}^{N-a-1}(p_B).$$

Likewise, under an approval quorum, the first element of the benefit function, given by formula (2), is 0, and the second component, given by formula (3), equals the above expression. Because the benefit functions that define the equilibrium participation of B-type voters under a participation quorum and under an approval quorum are identical, the sets of equilibria under the two quorum settings must coincide.□

Third, when $Q = N$, A-type voters do not have incentives to participate under either an approval quorum or under a participation quorum, because the abstention of one A-supporter guarantees that alternative A wins in both settings. Therefore, according to Proposition 1, equilibria under the two quorum settings are the same when $Q = N$.

Fourth, for all $2 \leq Q \leq N$, in both the participation and the approval quorum settings, there exists a stable equilibrium in which no voter votes ($p_A = 0$, $p_B = 0$). This is because, if all the voters abstain, a deviation to voting by a single voter is never profitable, because a quorum requirement $Q \geq 2$ (participation or approval) cannot be met. Therefore, abstention is the best response. We refer to this equilibrium as "full abstention" equilibrium throughout the paper. While we always keep its existence in mind, we do not focus on this equilibrium and its characteristics due to the triviality of its nature and properties.

Finally, according to formula (1), for $N/2 < Q \leq N$, the benefit function of A-types under an approval quorum is 0, implying that in any equilibrium $p_A = 0$.

[8] See Matveenko et. al. (2022) for proof of the existence of a solution to systems (4) and (11) when $Q = 1$.

With these general observations, we proceed with numerical calculations to show that various combinations of voting cost c and support measure α result in several classes of equilibria under approval and participation quorum rules. These classes of equilibria are illustrated on Figure 1 and Figure 2 for different values of Q.

In the outer area (white, above the dotted line and any solid line), i.e., when the participation cost is large or when the level of status quo support α is extremely high, the full abstention equilibrium is the only stable equilibrium under both the approval and participation settings. In the area A (grey, below the dotted line), i.e., when voting costs are either sufficiently small or when α is not too close to 1, under both quorum rules there is a stable equilibrium in which A-types always abstain, while B-types vote with strictly positive probability. We will refer to this equilibrium as an "abstention" equilibrium, as opposed to "full abstention" defined above. According to Proposition 1, the participation rates of B-types in abstention equilibria are the same in the two settings. Therefore, in terms of the equilibria they produce, approval and participation quorum rules are equivalent in areas A and B, that is, when either the voting cost is large enough or none of the two alternatives is sufficiently strong ex-ante.

In other areas, there are additional stable equilibria that we refer to as "coordination". In area B (pink), i.e., when voting cost is low and α is sufficiently far from the extreme values of 0 and 1, both types of voters always vote under both quorum rules, and therefore the rules are equivalent. When the voting cost is small and support levels of the alternatives are reasonably unbalanced, i.e., α is neither close to 0.5 nor it is close to 0 or 1, there can be equilibria which differ across the two quorum rules.

In area C_1 (dark green), under both quorum rules, A-types always vote, while B-types vote with probability $0 < p_b < 1$ which differs across the two settings. Likewise, in area D_1 (dark blue), B-types always vote and A-types vote with setting-specific probability $0 < p_a < 1$, while in area E_1 (dark red), both types vote with probabilities strictly between 0 and 1. In areas C_2 (light green), D_2 (light blue) and E_2 (light red), this sort of equilibria exist only under a participation quorum, while under an approval quorum, the equilibrium in which A-types abstain is the only stable equilibrium beyond the full abstention.

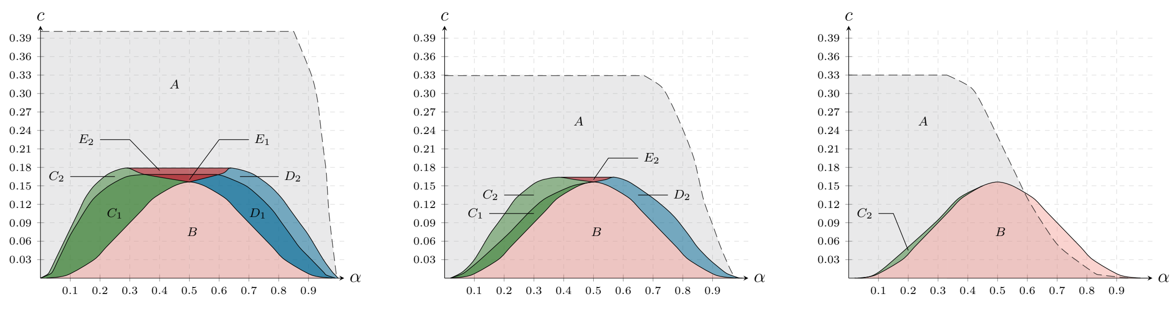

Figure 1: Equilibria for $N = 7$ and $Q = 2$ (left), $Q = 3$ (center), $Q = 5$ (right).

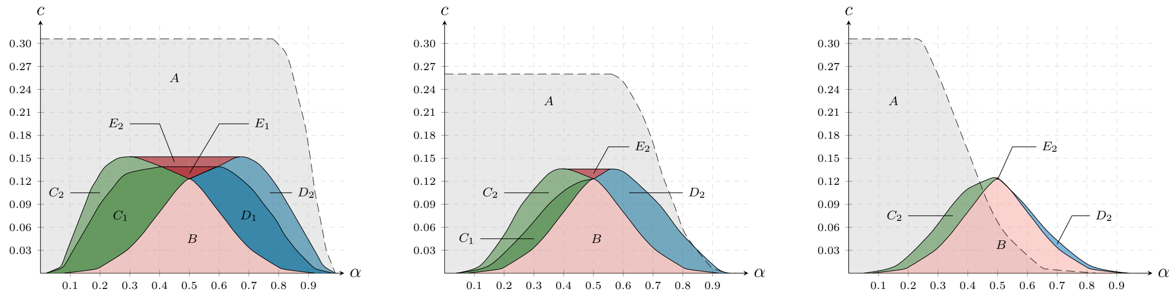

Figure 2: Equilibria for $N = 10$ and $Q = 3$ (left), $Q = 5$ (center), $Q = 8$ (right).

Several important observations with respect to how the above areas change with changes in N and Q can be made.[9] First, with higher values of N and Q, the sum of the areas B, C_1, C_2, D_1, D_2, E_1 and E_2 shrink; that is, the area in which A-type voters abstain (white and grey) expands. In other words, the larger the number of voters and the stricter quorum requirements are, the smaller are the set of combinations of α and c for which approval and participation quorums produce different equilibria and therefore different outcomes.

Second, area B (pink), in which both types of voters always vote, resulting in the same equilibria in the two quorum settings, does not depend on Q and shrinks with larger N. This result can be established generally, and is formalized in the following proposition.

Proposition 2. *For any N, there exists a unique cost threshold $\bar{c}(\alpha)$ which decreases in N, such that there is a stable equilibrium for all $2 \leq Q < N$ under a participation quorum and for all $2 \leq Q \leq N/2$ under an approval quorum, in which all voters vote ($p_A = 1$, $p_B = 1$) if and only if $c \leq \bar{c}(\alpha)$. $\bar{c}(\alpha)$ is such that $\bar{c}'(\alpha) > 0$ for $\alpha < 0.5$, $\bar{c}'(\alpha) < 0$ for $\alpha > 0.5$, $\bar{c}(0) = \bar{c}(1) = 0$.*

Proof: Full participation is an equilibrium if the expected benefits of both A-type and B-type voters are at least as high as the participation cost at ($p_A = 1$, $p_B = 1$). For a participation quorum, plugging ($p_A = 1$, $p_B = 1$) into expressions (11) yields the following benefit functions:

$$\begin{cases} B_A(1,1) = 0.5\left(1_{\{N \mod 2=0\}} P^{N-1}_{N/2-1}(\alpha) + 1_{\{N \mod 2=1\}} P^{N-1}_{\frac{N-1}{2}}(\alpha)\right); \\ B_B(1,1) = 0.5\left(1_{\{N \mod 2=0\}} P^{N-1}_{N/2}(\alpha) + 1_{\{N \mod 2=1\}} P^{N-1}_{\frac{N-1}{2}}(\alpha)\right). \end{cases} \tag{12}$$

Therefore, ($p_A = 1$, $p_B = 1$) is an equilibrium under a participation quorum if $c \leq \bar{c}(\alpha)$, where $\bar{c}(\alpha)$ is the minimum of $B_A(1,1)$ and $B_B(1,1)$:

$$\bar{c} = \begin{cases} 0.5\left(1_{\{N \mod 2=0\}} P^{N-1}_{N/2-1}(\alpha) + 1_{\{N \mod 2=1\}} P^{N-1}_{\frac{N-1}{2}}(\alpha)\right) & \text{if } \alpha > 0.5, \\ 0.5\left(1_{\{N \mod 2=0\}} P^{N-1}_{N/2}(\alpha) + 1_{\{N \mod 2=1\}} P^{N-1}_{\frac{N-1}{2}}(\alpha)\right) & \text{if } \alpha \leq 0.5. \end{cases} \tag{13}$$

[9] We present graphs for a limited set of values of N and Q to illustrate these observations. However, we solve our model for a much broader set of parameters, and obtain results fully consistent with those presented. These results are available upon request.

Under an approval quorum, the voters' benefit functions at ($p_A = 1$, $p_B = 1$) are also given by formula (12), but only for $2 \leq Q \leq N/2$. Therefore for $2 \leq Q \leq N/2$, the cost threshold that guarantees that ($p_A = 1$, $p_B = 1$) is an equilibrium under an approval quorum is also $\bar{c}(\alpha)$ defined above. For larger Q, the expected benefit of A-types is always 0, and hence ($p_A = 1$, $p_B = 1$) cannot be an equilibrium.

Finally, from the properties of a binomial function, it follows that for both odd and even N, $\bar{c}$ given by formula (13) is strictly decreasing in N, strictly increasing in α for $\alpha < 0.5$, strictly decreasing in α for $\alpha > 0.5$, and is equal to 0 when $\alpha = 0$ or $\alpha = 1$. $\square$

Third, each of the areas C_1, C_2, D_1, D_2, E_1 and E_2, i.e., those in which approval and participation quorums produce different sets of equilibria, shrinks with larger Q, but not necessarily with larger N.

Together, the above results suggest that the set of the combinations of c and α for which approval and participation quorum restrictions of quantitatively the same strictness result in different equilibria is very limited. Non-equivalence of the two quorum settings in more likely to arise when the participation cost is relatively low, ex-ante support levels for the alternatives are sufficiently but not extremely unbalanced, and the quorum requirement is not too strict. We next explore the properties of equilibria under both approval and participation quorum settings, focusing particularly on those cases in which the equilibria are different across the two settings. We consider three key characteristics of equilibria: voter turnout, the outcome of elections measured as the winning probability of the status quo, and voter welfare, and study how they are affected by the choice of the quorum setting.

5.1 Voter Turnout

Given equilibrium voter participation probabilities p_A and p_B, expected voter turnout as a function of α, p_A and p_B can be calculated as:

$$E[T] = \alpha p_A + (1 - \alpha) p_B. \tag{14}$$

Figures 3–5 illustrate voter turnout as functions of α in the approval and the participation quorum settings for various Q and cost levels.[10] The cost levels are chosen to illustrate all possible scenarios of equilibria existence described in the previous section. For example, for $N = 10$ and $Q = 3$, we show turnout (Figure 3) for three cost levels: $c = 0.01$, $c = 0.13$, $c = 0.15$. For $c = 0.01$, under both quorum rules and for almost all the values of α except the very extreme ones, there are equilibria in which both types of voters vote with strictly positive probability (that is, the line corresponding to $c = 0.01$ almost entirely lies in area B on Figure 2, left panel). For $c = 0.13$, the range of such α

[10] Again, for the sake of space, we present graphs for a limited set of parameters to illustrate our findings, while we have solved our model for a much broader set of parameters, and obtained results fully consistent with those presented.

is narrower, and for its intermediate values, under both rules, the equilibrium participation of both types of voters is strictly between 0 and 1 (line $c = 0.13$ goes through area E_1). For $c = 0.15$, this coordination equilibrium exists only in the participation quorum setting, while in the approval setting, the abstention equilibrium, i.e., the equilibrium in which none of A-type voters participate, is the only equilibrium (except for the trivial full abstention case) (line $c = 0.15$ goes above E_1 but through E_2).

Expected voter turnout in equilibria with strictly positive participation of both types of voters is illustrated by solid lines for the participation quorum setting and by dashed lines for the approval quorum setting whenever such equilibria exist. Note that, in the approval quorum setting, such an equilibrium exists only for $Q \leq N/2$. Turnout levels in the abstention equilibria (such equilibria coincide in the two settings) are illustrated by dotted lines.

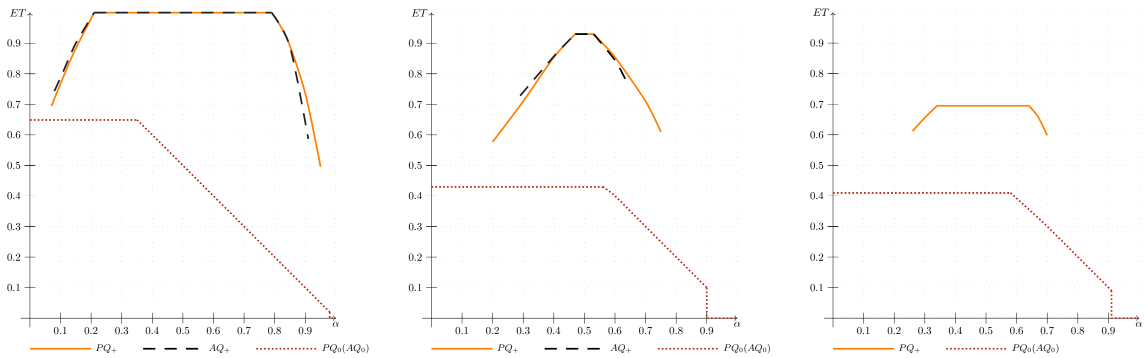

Figure 3: Voter turnout for $N = 10$, $Q = 3$ and $c = 0.01$ (left), $c = 0.13$ (center), $c = 0.15$ (right).

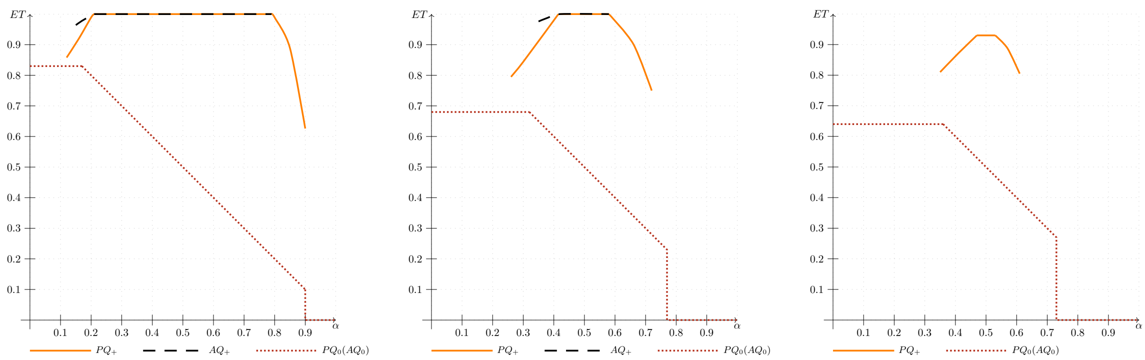

Figure 4: Voter turnout for $N = 10$, $Q = 5$ and $c = 0.01$ (left), $c = 0.09$ (center), $c = 0.13$ (right).

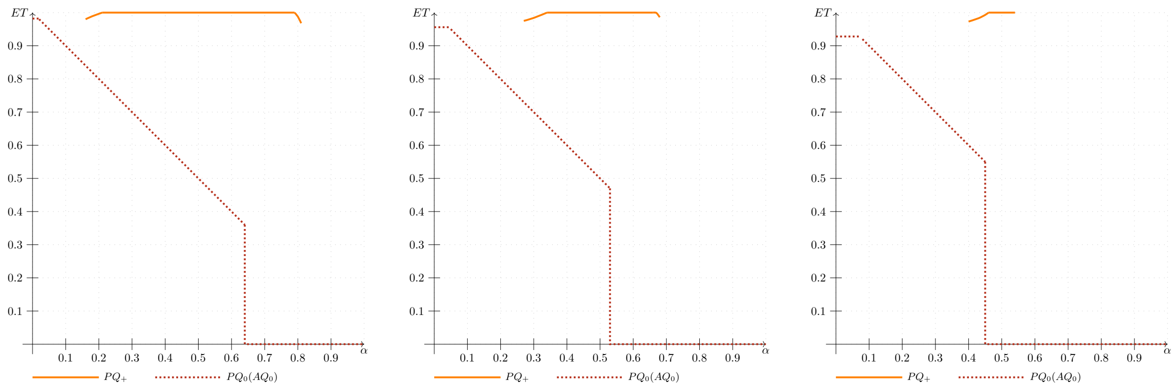

Figure 5: Voter turnout for $N = 10$, $Q = 8$ and $c = 0.01$ (left), $c = 0.05$ (center), $c = 0.11$ (right).

Several patterns are evident in the above figures. First, average turnout is naturally higher in equilibria in which A-type voters participate (solid and dashed lines) than when they abstain (dotted lines), regardless of the quorum setting. Second, when the coordination equilibrium exists in both settings (i.e., when $Q \leq N/2$), expected turnout is at least as low under an approval quorum as under a participation quorum when $\alpha < 0.5$, while the result is reversed when $\alpha \geq 0.5$. Yet, when it exists, the difference in voter turnout across the two settings is quantitatively small. Third, for the values of α in some neighbourhood of 0.5 (i.e., for the values of α which correspond to areas B or $E1$ on Figure 3) where both types of voters participate with probability 1 or both types of voters participate with probabilities strictly between 0 and 1), the two settings result in the same levels of expected turnout. Finally, the expected turnout in each type of equilibria is naturally decreasing with participation cost of c regardless of the quorum setting and the type of equilibrium.

These patterns imply that, if the planner's objective is to maximize voter turnout, an approval quorum is the best choice only for very specific conditions: when the quorum restriction is mild and status quo support is low but not extremely low (i.e., when α is in a particular narrow range within the $(0, 0.5)$ interval). In all other cases, a participation quorum results in the same or strictly higher voter participation.

Overall, our results suggest that the effect of quorum requirements of both types on voter participation is complex and crucially depends not only on the ex-ante support for the alternatives, but also on the type of equilibrium achieved. While in the abstention equilibrium in both quorum settings, voter participation is non-increasing in α, in the participation equilibrium, turnout is maximized around $\alpha = 0.5$ and asymmetrically decreases towards its more extreme values. To understand the intuition behind this result as well as the further results which are direct consequences of equilibrium voter participation, consider the voters' incentives to cast a vote in each type of the equilibria.

In the abstention equilibria, in which status quo supporters strategically abstain, aiming at victory for the status quo due to the lack of quorum, small α implies two things.

First, a type B voter expects that the other voters are likely to be type B as well, and thus all challenger supporters have relatively strong incentives to coordinate on participation in order to meet an approval or participation quorum threshold. Second, small α means that there are likely to be more B-supporters among the voters, which, together with the high participation probability of type B voters, implies that overall expected turnout is relatively high. On the contrary, when α is large, B-supporters expect that it is unlikely that there are many B supporters among the other voters, hence, that it is unlikely that the quorum requirement will be met, and so their participation incentives are low. The incentives are stronger when stricter quorum requirements are in place. As a result, in an abstention equilibrium, voter turnout is non-increasing in status quo support level α and it decreases faster with stricter quorum requirements.

In the participation equilibria, in which status quo supporters do not rely on winning via failure to meet quorum requirements, the intuition is more complex. For intermediate values of α, depending on the level of cost c, either both types of voters vote with certainty in both quorum settings, or both participate with probabilities strictly between 0 and 1, such that the overall turnout is balanced and constant. Smaller α implies that most voters are expected to be type B. As a result, A-types coordinate and always vote in order to deliver a sufficient number of votes for the status quo, while B-types do not need full mobilization to outweigh the votes for the status quo, and participate at a lower rate. Moreover, given the constant participation of A-types, meeting the approval quorum threshold requires more votes for the challenger than meeting the participation quorum threshold of the same value, B-types vote at a higher rate under an approval quorum than under a participation quorum. The lower the α the lower the participation of B-types, and therefore the lower the overall turnout. When α is too small, however, even full participation of A-types cannot give them a reasonable chance of winning, and the participation equilibrium collapses. When α is sufficiently above 0.5, the logic is reversed. Because most voters are likely to be supporters of the status quo, B-types must participate with certainty, while A-types can afford lower participation rate, which decreases further with higher α, decreasing the overall turnout as well, until α becomes too high to induce any participation among B-types This then leads to a collapse of the participation equilibrium. The stricter the quorum requirement, the less extreme is the value of α needed to cause the participation equilibrium to collapse.

These effects of quorum restrictions and levels of voter support for alternatives on voter participation incentives and consequently on turnout are central to understanding how approval and participation quorum requirements affect election outcomes and voter welfare, which we study in the following sections.

5.2 Election Outcome

Further, we compare the approval and the participation quorum settings in terms of the ex-ante probabilities of each alternative to win an election, which could also be thought of as the expected welfare of the candidates if alternatives are considered to be agents whose utility depends solely on whether they win an election. Because the winning probabilities of the two alternatives must add up to 1, the expected outcome of an election can be fully described with a single function, such as the ex-ante probability that the status quo alternative will win the election. We denote this probability as W_A.

Whether an alternative wins the election first depends on whether the quorum requirement is met, and second, on the number of votes cast in its favor. Under an approval quorum, alternative A wins first when fewer than Q B-supporters participate, so that the quorum requirement is not met, and second, when enough B-types participate, but A received more votes, or the numbers of votes for each alternative are equal, but A wins the coin flip. Therefore, the probability that the status quo wins an election with N voters under an approval quorum requirement Q can be expressed as:

$$W_A^{AQ} = \sum_{a=0}^{N} P_a^N(\alpha) \left(\sum_{l=0}^{a} P_l^a(p_A) \sum_{m=0}^{Q-1} P_m^{N-a}(p_B) + \frac{1}{2} \sum_{l=Q}^{\lfloor N/2 \rfloor} P_l^a(p_A) P_l^{N-a}(p_B) + \sum_{l=Q+1}^{a} P_l^a(p_A) \sum_{m=Q}^{l-1} P_m^{N-a}(p_B) \right). \quad (15)$$

Under a participation quorum, if the quorum requirement is met (the first component of the formula below), A wins when she receives more votes than B, or when the numbers of votes for both alternatives are equal and A wins the coin flip. If the quorum requirement is not met (the second component of the formula below), A always wins. Therefore, given voters' equilibrium strategies p_A and p_B, the probability that the status quo will win an election with N voters and participation quorum requirement Q is given by the following expression:

$$W_A^{PQ} = \sum_{a=\lceil \frac{Q}{2} \rceil}^{N} P_a^N(\alpha) \sum_{l=\lceil \frac{Q}{2} \rceil}^{a} P_l^a(p_A) \left(\sum_{m=\max\{0,Q-l\}}^{l-1} P_m^{N-a}(p_B) + \frac{1}{2} P_l^{N-a}(p_B) \right) + \\ + \sum_{a=0}^{N} \sum_{l=0}^{\min\{a,Q-1\}} \sum_{m=0}^{\max\{0,Q-l-1\}} P_a^N(\alpha) P_l^a(p_A) P_m^{N-a}(p_B). \quad (16)$$

Figures 6–8 illustrate how A's winning probabilities as functions of α in the approval and participation quorum settings differ for various values of Q and cost levels. As with the figures for turnout in the previous section, we choose the cost levels to illustrate all possible scenarios of equilibria existence. Status quo winning probabilities in coordination equilibria with strictly positive participation of both types of voters are illustrated with solid lines for the participation quorum setting and with dashed lines for the approval quorum setting, whenever such equilibria exist. Recall that, in the approval quorum setting, such an equilibrium exists only for $Q \leq N/2$. The winning probabilities in abstention equilibria, i.e., in which A-type voters abstain (such equilibria coincide under the two settings), are illustrated with dotted lines.

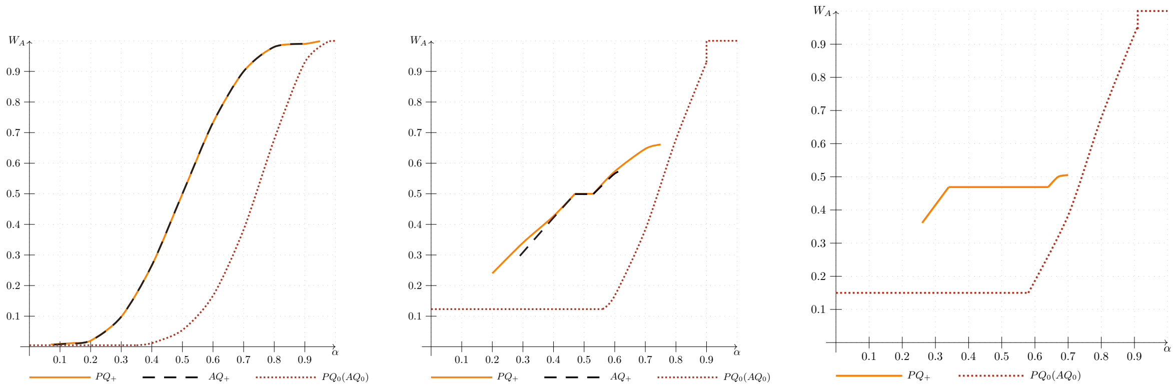

Figure 6: Status quo winning probability for $N = 10$, $Q = 3$ and $c = 0.01$ (left), $c = 0.13$ (center), $c = 0.15$ (right).

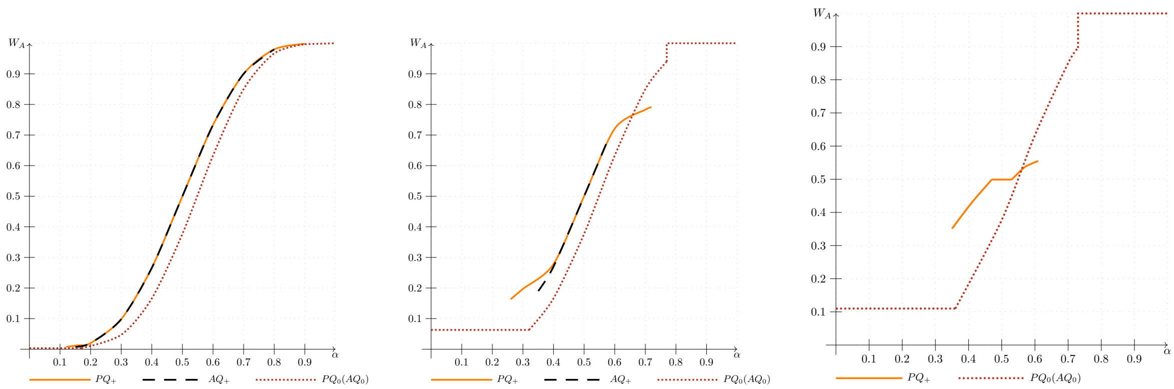

Figure 7: Status quo winning probability for $N = 10$, $Q = 5$ and $c = 0.01$ (left), $c = 0.09$ (center), $c = 0.13$ (right).

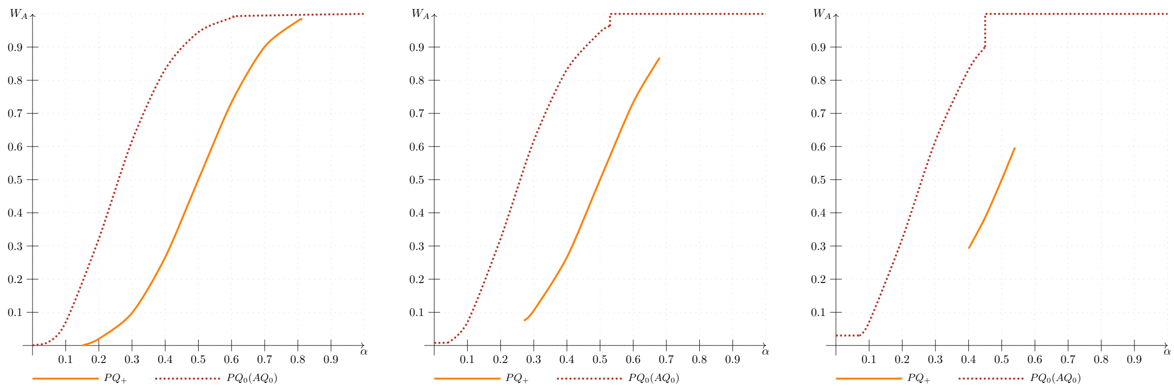

Figure 8: Status quo winning probability for $N = 10$, $Q = 8$ and $c = 0.01$ (left), $c = 0.05$ (center), $c = 0.11$ (right).

The figures highlight several important observations. First, when $Q \leq N/2$, under approval quorum, the status quo is better off in the equilibrium in which his supporters participate with positive probability than in the equilibrium in which they abstain, whenever the former exists. This is also almost true for the participation quorum setting, although when $Q = N/2$ there is a narrow range of the values of α for which the status quo's winning probability is lower in coordination equilibrium than in abstention equilibrium. For $Q > N/2$, the situation is reversed: the status quo alternative is more likely to win in the abstention equilibrium. The latter comparison, however, makes sense only in the participation quorum setting, because under an approval quorum, status quo supporters always abstain when $Q > N/2$. This result implies that, under a participation quorum $Q > N/2$, it is always in the interest of the status quo to discourage the participation of status quo supporters.

Second, when equilibria with positive participation of the status-quo supporters exist in both the approval and participation quorum settings, a participation quorum results in at least as high chances for the status-quo to win as in an approval quorum, although the difference is quantitatively rather small. Also, note that the range of the values of α for which such an equilibrium exists is larger under a participation quorum than under an approval quorum.

These two results imply that, when the quorum requirement is relatively mild (specifically, when $Q \leq N/2$), the status quo benefits more when the quorum restriction is of the participation type than when it is the approval type: either status quo supporters abstain, and therefore the equilibrium is the same in both settings, resulting in the same chances of the status quo to win, or, if they participate with positive probability in equilibrium, status quo chances to win under a participation quorum are at least as low as under an approval quorum, being strictly higher for some values of α. On the contrary, when the quorum requirement is strict ($Q > N/2$), an approval quorum guarantees that abstention equilibrium (i.e., the best equilibrium from the status quo perspective) is the only stable equilibrium, while under a participation quorum, a worse equilibrium may exist as well.

Finally, for values of α sufficiently close to 0.5, in equilibria in which participation by both types of voters is strictly between 0 and 1 (e.g., areas E_1 and E_2 on Figure 2), status quo's winning probability is exactly 0.5 in both quorum settings. That is, if the equilibrium participation probabilities p_A and p_B are not restricted and are both within the $(0, 1)$ interval, they perfectly balance the chances of the two alternatives to win. This is the usual "neutrality" result for standard pivotal costly voting models with no restrictions such as quorums (see, for example, Goeree and Grosser, 2007; Krasa and Polborn, 2009; Taylor and Yildirim, 2010b; Durazzo and Turchik, 2022). We therefore show that this result extends to cases with quorum requirements, both approval and participation.

5.3 Voter Welfare

We now evaluate the approval and participation quorum settings in terms of the equilibrium expected voter welfare. When voting is costly, voter participation implies a tradeoff between the quality of the aggregation of voter preferences and participation costs. Higher participation increases the probability that the alternative actually preferred by the majority will be elected, but at the same time it implies larger total costs borne by voters. Therefore, a quorum setting is welfare-superior to the alternative when it ex-ante more effectively balances the costs associated with voting and the likelihood of electing the alternative that would be supported by the majority of voters ex-post. In either quorum setting, expected per voter welfare can be calculated as

$$E[W]^Q = \alpha E[W_A]^Q + (1-\alpha) E[W_B]^Q, \tag{17}$$

where $E[W_T]^Q$ is the expected welfare under quorum requirement Q of a voter who supports alternative $T \in \{A, B\}$. $E[W_T]^Q$ can be expressed as follows:

$$E[W_T]^Q = p_T^*(u_T^Q - c) + (1-p_T^*) v_T^Q, \tag{18}$$

where v_T^Q is the probability that the voter's preferred alternative will win (i.e., voter's expected benefit) if the voter abstains; u_T^Q is the probability that the voter's preferred alternative will win if she participates, and p_T^* is the voter's equilibrium participation probability.

Because v_A^Q is effectively the probability that alternative A will win an election with $N-1$ voters of ex-ante unknown type, the expression for v_A^Q in the approval and participation quorum settings can be obtained by substituting N with $N-1$ in formulas (15) and (16) respectively:

$$v_a^{AQ} = \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) \left(\sum_{l=0}^{a} P_l^a(p_A) \sum_{m=0}^{Q-1} P_m^{N-a-1}(p_B) + \frac{1}{2} \sum_{l=Q}^{\lfloor (N-1)/2 \rfloor} P_l^a(p_A) P_l^{N-a-1}(p_B) + \sum_{l=Q+1}^{a} P_l^a(p_A) \sum_{m=Q}^{l-1} P_m^{N-a-1}(p_B) \right). \tag{19}$$

$$v_A^{PQ} = \sum_{a=\lceil \frac{Q}{2} \rceil}^{N-1} P_a^{N-1}(\alpha) \sum_{l=\lceil \frac{Q}{2} \rceil}^{a} P_l^a(p_A) \left(\sum_{m=\max\{0,Q-l\}}^{l-1} P_m^{N-a-1}(p_B) + \frac{1}{2} P_l^{N-a-1}(p_B) \right) +$$

$$+ \sum_{a=0}^{N-1} \sum_{l=0}^{\min\{a,Q-1\}} \sum_{m=0}^{\max\{0,Q-l-1\}} P_a^{N-1}(\alpha) P_l^a(p_A) P_m^{N-1-a}(p_B). \tag{20}$$

Then, u_A^Q is the probability that alternative A will win when there are $N-1$ other voters in the election, and there is already one vote cast for A. u_A^Q can therefore be expressed as:

$$u_A^{AQ} = \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) \left(\sum_{l=0}^{a} P_l^a(p_A) \sum_{m=0}^{Q-1} P_m^{N-a-1}(p_B) + \frac{1}{2} \sum_{l=Q-1}^{a} P_l^a(p_A) P_{l+1}^{N-a-1}(p_B) + \sum_{l=Q-1}^{a} P_l^a(p_A) \sum_{m=Q}^{l} P_m^{N-a-1}(p_B) \right). \tag{21}$$

$$u_A^{PQ} = \sum_{a=\lceil \frac{Q-1}{2} \rceil}^{N-1} P_a^{N-1}(\alpha) \left(\sum_{l=\lceil \frac{Q-1}{2} \rceil}^{\max\{a,1\}} P_l^a \left(p_A\right) \left(\sum_{m=\max\{0,Q-l-1\}}^{l} P_m^{N-a-1}\left(p_B\right) + \frac{1}{2} P_{l+1}^{N-a-1}\left(p_B\right) \right) \right) +$$

$$+ \mathbb{1}\{Q=2\} \frac{1}{2} \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) P_0^a \left(p_A\right) P_1^{N-a-1}\left(p_B\right) +$$

$$+ \mathbb{1}\{Q>1\} \sum_{a=0}^{N-1} \sum_{l=0}^{\min\{a,Q-2\}} \sum_{m=0}^{\max\{0,Q-l-2\}} P_a^{N-1}(\alpha) P_l^a \left(p_A\right) P_m^{N-1-a}\left(p_B\right). \tag{22}$$

Similarly, one can construct the expected welfare for supporters of the challenging alternative B, anticipating that $v_B^Q = 1 - v_A^Q$ and u_B^Q can be calculated as follows:

$$u_B^{PQ} = \sum_{a=0}^{N-1} P_a^{N-1}(\alpha) \left(\frac{1}{2} \sum_{l=\lceil \frac{Q}{2} \rceil}^{a} P_l^a \left(p_A\right) P_{l-1}^{N-a-1}\left(p_B\right) + \sum_{l=0}^{a} P_l^a \left(p_A\right) \sum_{m=\max\{l,Q-l-1\}}^{N-a-1} P_m^{N-a-1}\left(p_B\right) \right). \tag{23}$$

Figures 9–11 illustrate voter welfare as functions of α in the approval and anticipation quorum settings for various Q and cost levels. As for the figures of turnout and status quo winning probability in the previous sections, we have chosen cost levels to illustrate all possible scenarios of equilibria existence (see Figure 2). Note that in full abstention equilibria, which exist in both settings for any $Q > 1$, expected voter welfare is 0.5.

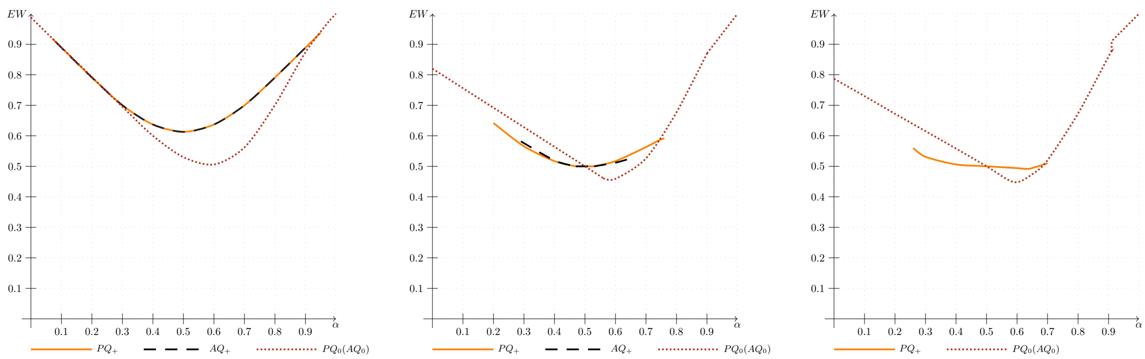

Figure 9: Voter welfare for $N = 10$, $Q = 3$ and $c = 0.01$ (left), $c = 0.13$ (center), $c = 0.15$ (right).

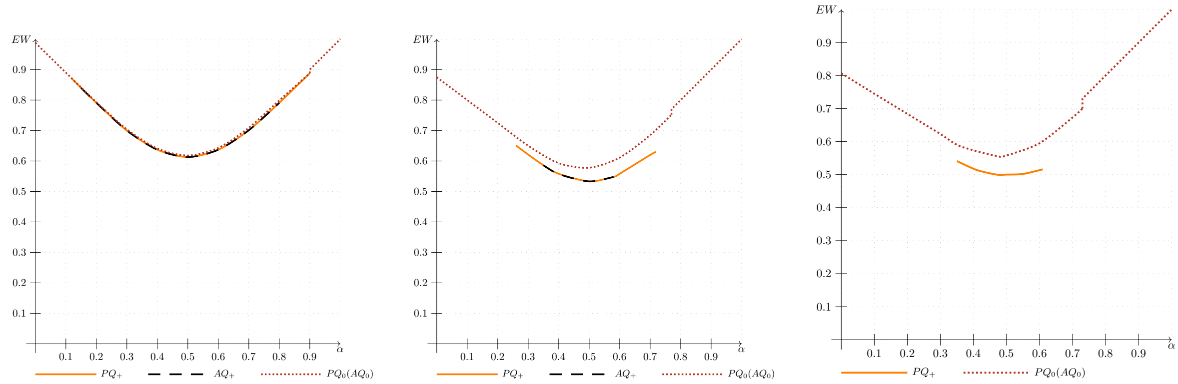

Figure 10: Voter welfare for $N = 10$, $Q = 5$ and $c = 0.01$ (left), $c = 0.09$ (center), $c = 0.13$ (right).

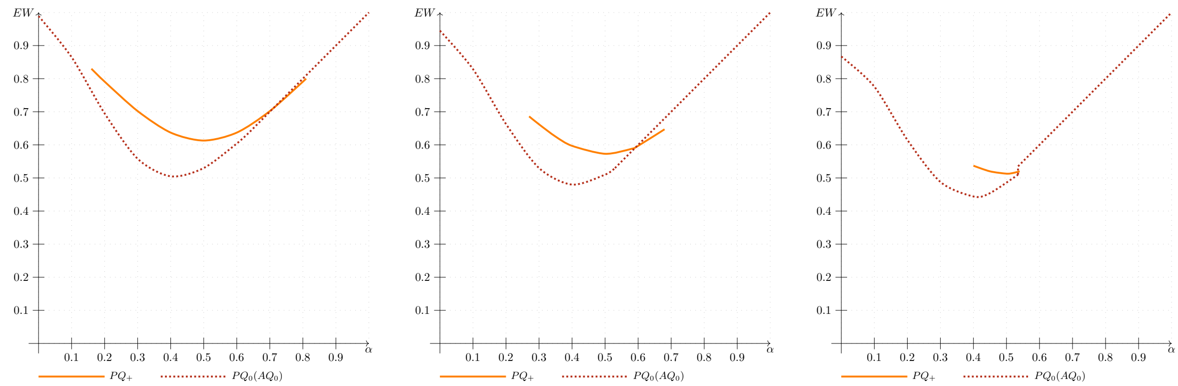

Figure 11: Voter welfare for $N = 10$, $Q = 8$ and $c = 0.01$ (left), $c = 0.05$ (center), $c = 0.11$ (right).

The above figures illustrate that when $\alpha \leq 0.5$ and $Q \leq N/2$, an approval quorum weakly dominates a participation quorum from the voter welfare perspective. This result arises from two observations. First, when coordination equilibrium exists for both quorums, voter welfare is higher under an approval quorum than under a participation quorum in such equilibrium . Second, when coordination equilibrium exists only in the participation quorum setting, welfare in coordination equilibrium is always lower than in an abstention equilibrium, which is the unique equilibrium in the approval setting. As a result, when $\alpha \leq 0.5$ and $Q \leq N/2$, an approval quorum either guarantees that the welfare maximizing abstention equilibrium is achieved due to its uniqueness, or, at least for some values of α, it provides higher welfare than a participation quorum in coordination equilibrium.

For other combinations of Q and α, the relationship between the two quorum settings is more complex. In general, for $Q > N/2$ a participation quorum outperforms an approval quorum for all α except in a very narrow range of values. For most of the values of α for which there is a coordination equilibrium under a participation quorum, voter welfare in such equilibrium exceeds the welfare under an abstention equilibrium (which is the only one under an approval quorum). For $Q \leq N/2$ and sufficiently large α, the abstention equilibrium is the only equilibrium under both approval and participation quorums, implying an equivalence between the two settings.

6 Optimal Quorum Design

Thus far, we have discussed whether quorum restrictions of a given strictness Q should be of the participation or of the approval type, while from a policy perspective the relevant question should be framed as "What is the optimal (t, Q), where t is the type of quorum restriction and Q is its strictness?". In this section, we address this question.

An important observation about our results is that changes in Q for the same quorum type lead to relatively large changes in equilibrium characteristics of interest (turnout, status quo winning probability, and voter welfare) only for the type of equilibrium in which A-types abstain. For the equilibrium in which both types participate with positive probability, changes in Q have a very moderate impact on equilibrium characteristics. Likewise, switches between quorum type t given the same level of strictness Q lead to small changes in characteristics of equilibria of the same type. We illustrate this observation in Figure 12 on the example of voter welfare.

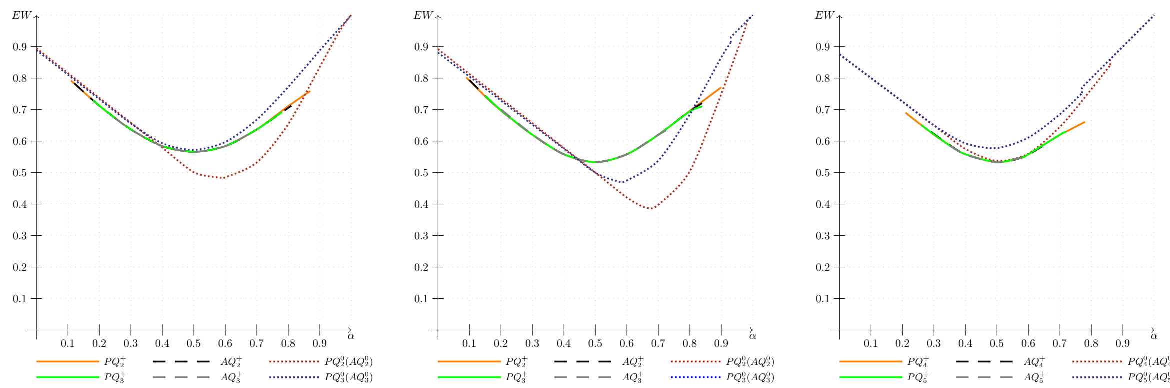

Figure 12: Voter welfare for $N = 7$, $Q = 2$ and $Q = 3$ (left), $N = 10$, $Q = 2$ and $Q = 3$ (center), $N = 10$, $Q = 4$ and $Q = 5$(right); $c = 0.09$.

The above figure implies that the primary question in the discussion of an optimal quorum rule is what type of equilibrium is achieved when more than one equilibrium exists. If A-type voters are likely able to coordinate and therefore to achieve the equilibrium with positive participation of both types whenever it exists, the difference between participation and approval quorums is very small for a given Q. Moreover, changes in Q have little impact on welfare for both types of quorum settings in such an equilibrium.

On the contrary, if status quo supporters fail to coordinate on participation, changes in Q have substantial impact on welfare. Recalling that when A-type voters abstain, approval and participation equilibrium result in identical equilibria, we illustrate in Figure 13 how turnout, status quo winning probability, and voter welfare change under a participation quorum with various levels of Q in equilibrium in which status quo supporters abstain.

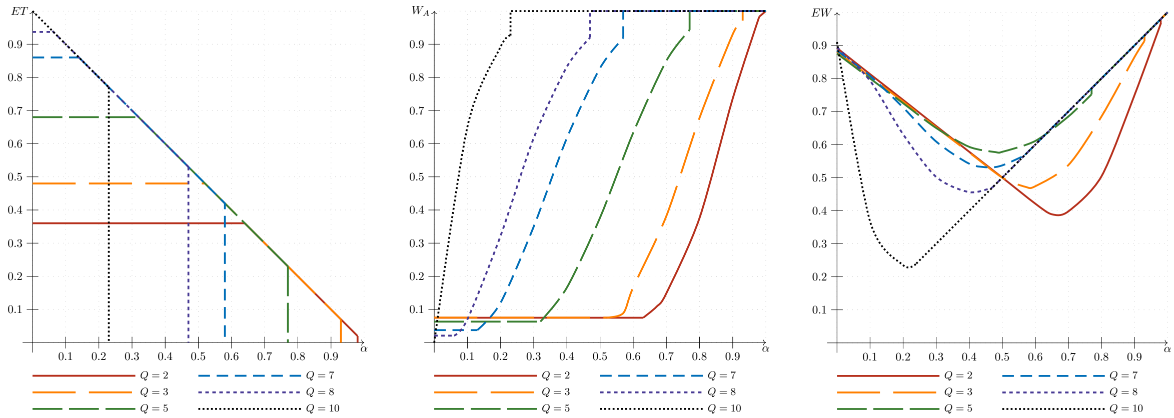

Figure 13: Expected turnout (left), status quo winning probability (center) and voter welfare (center) for $N = 10$.

From the above figure, it is clear that the optimal value of Q strongly depends on α, and a particular level of strictness could be best for one range of values of α while being the worst for another one.[11] Therefore, knowing the exact value of α becomes crucial for a policy maker choosing an optimal quorum restriction. In reality, however, conditioning the choice of quorum requirements on a specific value of α may not be an easy task, because levels of support for alternatives are likely to be unknown, and can vary from one election to another. Suppose that a particular quorum requirement must be adopted for a sequence of decisions on various issues: a board of directors regularly votes on company matters, a legislature votes on laws in every session, etc. While the voting procedure, including the type and the strictness of the quorum requirements applied, remains the same from one election to another, the levels of support for alternatives vary. What would an optimal quorum requirement look like in such a case? While the answer primarily depends on the distribution of potential realizations of support levels α and on the objective function of the planner, alternative criteria for choosing the optimal quorum requirement may be applied. One possible approach is to design a requirement that would result in the least harm to welfare under the worst possible outcome.

Studying the effects of participation quorums using the same framework as in this paper, Matveenko et. al. (2022) argue that, according to such a criterion, a participation quorum restriction equal to half of the number of voters, or no restrictions at all, are likely to be reasonable choices. In the previous section, we show (see Figure 10) that when $Q = N/2$, there is virtually no difference in voter welfare in participation and approval quorum settings in an equilibrium in which both types of voters participate

[11] Matveenko et. al. (2022) provide a detailed discussion of the effects of the strictness of a participation quorum (and therefore of approval one, when status quo supporters abstain) and additional figures illustrating the patterns we discuss here.

with positive probability. Importantly, in both settings, in such an equilibrium welfare is lower than in an equilibrium where A-types abstain, and the range of the values of α for which the equilibrium with abstaining A-types in the only stable equilibrium is wider for an approval quorum than for a participation quorum. These observations imply that an approval quorum with $Q = N/2$ is more likely to result in equilibrium in which the most efficient equilibrium is unique, while being essentially identical to a participation quorum of the same strictness in all other dimensions. Therefore, we can refine the main finding of Matveenko et. al. (2022) with respect to optimality of quorum restrictions from the perspective of voter welfare by saying that an approval quorum with strictness equal to half of the number of voters, or no restriction at all, is likely to be optimal choice.

7 Concluding Remarks

In this paper, we theoretically compare participation and approval quorum requirements in elections in terms of their impact on voters' participation incentives and consequently on turnout, election outcomes, and voter welfare. Analyzing a pivotal costly voting model with quantitatively equal approval and participation quorum requirements, we first find that, although quorum requirements of both types have complex non-monotone effects on voter participation decisions, in many cases, these effects are identical for the two types of quorums, resulting in identical equilibrium voter behavior and therefore identical outcomes. Nevertheless, for certain combinations of participation cost and ex-ante support levels for the alternatives, approval and participation quorums produce different equilibria. By exploring these cases, we show that both types of quorum restrictions result in at most two types of stable non-trivial equilibria: in one, status quo supporters strategically abstain, while in the other, they participate with positive probability. While equilibria of the first type are always identical in the two quorum settings, equilibria of the latter type differ, though the difference in the outcomes of interest is quantitatively small. The main difference between the two quorum settings therefore arises from different ranges of status quo support levels for which equilibria with participating status-quo supporters exist.

We next compare equilibrium turnout, the likelihood of a status quo alternative to win in elections, and voter welfare under approval and participation quorums, showing that the results of the comparisons crucially depend on the ex-ante levels of support for a status quo and a challenger, and voting costs. Finally, we discuss optimal design of quorum requirements in terms of their type and strictness from the voter welfare perspective, and argue that although there is no quorum rule that would be optimal for all levels of support for competing alternatives and costs of voting, in reality, when exact support levels for the alternatives are unknown or can vary from one election to another, choosing approval quorum requirements with strictness close to half of the number of voters, or setting no quorum requirements at all, may be the most reasonable choices.

Although our results have clear policy implications, we do not suggest ubiquitous adoption of particular quorum requirements in real life elections in response to the findings in this paper. Rather, we stress that quorum requirements should be set with care, because their effects on election outcomes and welfare are significant and complex, and crucially depend on things like the ex-ante levels of support for status quo and challenger alternatives, and voting costs. Therefore, with this paper, we primarily aim to draw attention to the fact that both the type and the strictness of quorum requirements can have substantial impacts on election outcomes and welfare, and we encourage further theoretical and empirical research into the role of quorums in various electoral settings.

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