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Financial Stability and Financial Regulation under Diagnostic Expectations

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Abstract

This paper studies the implications of "diagnostic" expectations—an empirically relevant form of non-rational extrapolation in expectation formation (Bordalo, Gennaioli and Shleifer (2018))—for financial stability and the appropriate conduct of financial regulation. We find that interactions between diagnostic expectations and financial frictions intensify instability in financial markets relative to the rational expectations benchmark. We also find that diagnostic expectations command tighter financial regulation, regardless of the degree of diagnostic expectations of the regulator.

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1 Introduction

"*There will be manias. The manias will be followed by panics.*"
[Akerlof and Shiller (2010)]

Recent empirical findings (Bordalo, Gennaioli and Shleifer (2018); Bordalo et al. (2019); Greenwood et al. (2022)) support the view that cognitive misperceptions about future macroeconomic developments significantly influence systemic risk in financial markets. Most studies interested in macro-financial stability, however, have omitted those misperceptions. Rather, the literature has usually examined the implications of financial frictions for the stability of the financial system under the classic assumption of full-information rational expectations (FIRE). (See the literature review for details.)

This paper contributes to filling this gap. Specifically, we consider an environment with non-rational, extrapolative expectations and external financing frictions to examine the joint implications of those elements for financial stability and the appropriate conduct of financial regulation. We find that interactions between the elements intensify instability in the financial system relative to the rational-expectations (RE) benchmark. We also find that non-rational expectations command tighter financial regulation, regardless of the degree of rationality of the regulator.

Our analysis builds on a standard macroeconomic model with a financial sector (à la Gertler and Kiyotaki (2010), Gertler and Karadi (2011), and Brunnermeier and Sannikov (2014)). To that modeling framework—following Maxted (2023)—we add an extrapolative process of expectation formation, in the spirit of Bordalo, Gennaioli and Shleifer (2018). Specifically, relative to the RE benchmark, agents tend to assign a greater likelihood to future events that are reminiscent of those realized in the recent past. These expectations—which are usually referred to as "diagnostic"—are particularly relevant in macro-finance studies for the following two reasons. First, the expectations are grounded in sound psychological theory, notably the theory of the "representativeness heuristic" (Tversky and Kahneman (1983)). Second, they generate forecast errors on asset returns that are consistent with those empirically measured (Bordalo, Gennaioli and Shleifer (2018); Bordalo et al. (2019)).

Our modeling environment—in contrast to Maxted (2023)—features two production technologies that differ in productivity and riskiness. In particular, one of the technologies is more productive but is also riskier. Because agents are risk neutral, if expectations were rational, the productive technology would be regarded as the most attractive. By contrast, under diagnostic expectations, the unproductive technology can temporarily be perceived in such a manner, when in the recent past, sufficiently adverse aggregate disturbances hit the technologies. These cognitive misperceptions then interact with standard frictions in credit markets that limit by net worth the capacity of some financial experts (i.e., *financiers*) to take leveraged positions on a productive asset and allocate the asset to the productive technology. The aggregate net worth of financiers together with the cognitive misperceptions then determine financial leverage, the asset allocation, and the asset price in equilibrium. Moreover, the resiliency of the financiers' net worth to aggregate disturbances provides a natural measure of the stability of the financial system.

Our first main finding is that diagnostic expectations intensify financial instability relative to the RE benchmark. This happens for the following two reasons.

First, diagnostic expectations intensify the usual two-way loop between fluctuations in financial net worth and in asset prices that endogenously arises in standard macro-finance models with rational expectations. In line with those models, when adverse disturbances hit the technologies, financiers lose net worth, and to meet binding collateral requirements, they are forced to sell assets at distressed prices, which creates a vicious circle of increasingly decreasing prices, aggregate net worth losses and forced asset sales. Under diagnostic expectations, however, because agents become relatively more pessimistic about the prospects of the technologies following adverse disturbances, asset prices fall further, intensifying financial distress.

Second, diagnostic expectations tend to generate negative forecast errors on asset returns during booms—when financiers as a whole are well capitalized—and positive forecast errors during busts—when those agents are instead poorly capitalized. The reason is that past disturbances positively affect both the perceived returns of the technologies and the aggregate net worth of financiers. Because they tend to shape high asset valuations relative to underlying fundamentals, negative forecast errors tend to erode the future net worth of financiers, while positive forecast errors tend to have the opposite effects. The adverse effects have a larger impact on the dynamics of net worth, nonetheless, because the higher the aggregate share of the asset allocated to riskier technologies, the larger the impact of forecast errors on asset valuations and asset returns.

In a second part of the paper, we derive normative implications of diagnostic expectations for financial regulation. Specifically, we analytically characterize a socially optimal allocation and compare its properties with the (competitive) equilibrium allocation. Two contrasting elements stand out. First, the socially optimal allocation internalizes the collective effects of individual decisions on aggregate variables and aggregate fluctuations, whereas individual decisions in the competitive equilibrium do not. Second, the socially optimal allocation is inherently characterized by the social value of the asset, whereas individual decisions in the equilibrium are inherently taken with respect to the asset price.

These differences naturally motivate an active role for financial regulation, even under rational expectations. Under diagnostic expectations, however, we find restrictions on financial risk-taking are tighter, regardless of the regulator's degree of diagnostic expectations (which is our second main finding). The nature of the restrictions does depend on the degree of diagnosticity of the planner. Specifically, a benevolent planner (i.e., a planner whose expectations are equally diagnostic to those of private agents) restricts the allocation of the asset to the productive technology during recoveries from busts. By contrast, a paternalistic planner (i.e., a planner with rational expectations in an environment in which private agents have diagnostic expectations) restricts it during booms.

Related Literature. This paper is primarily related to two strands of literature. A first strand investigates positive implications of diagnostic expectations on financial markets and/or macroeconomic outcomes.

Within this literature (e.g., Maxted (2023); Krishnamurthy and Li (2020); Bianchi, Ilut and Saijo (2021); L'Huillier, Singh and Yoo (2021); L'Huillier, Phelan and Wieman (2022)), the closest study to ours is Maxted (2023), who also incorporates diagnostic expectations into an otherwise standard macro-finance model. A key difference from that study is that in our model, diagnostic expectations intensify financial instability relative to the RE benchmark. This difference occurs essentially because our model features multiple production technologies that in equilibrium induce both highly pro-cyclical asset prices and pro-cyclical exposure of the overall economy to fundamental risk. Our results are thus closely in line with the view that non-rational extrapolation in expectation formation is a key generator of vulnerabilities and instability in financial markets (Minsky (1977); Kindleberger (1978)). Another important difference from Maxted (2023) is that our study considers more general diagnostic processes of expectation formation, in which the underlying diagnostic variable can differ from the exogenous disturbance. This more general process allows us to consider other relevant specifications in which, for instance, agents form diagnostic expectations tied to past asset returns, as in Barberis et al. (2015).

The second strand of literature focuses on the normative implications of non-rational expectations for financial regulation. Fontanier (2022) considers a general environment with financial frictions and non-rational expectations, in which new externalities may arise relative to the rational benchmark when the non-rational component of expectations is tied to asset prices. To conduct the welfare analysis, Fontanier (2022) restricts attention to the socially optimal allocation derived by a paternalistic planner. Rather, our study considers socially optimal allocations for both paternalistic and benevolent planners as well as contrasts the properties of those allocations. Dávila and Walther (2021) also study the optimal design of financial regulation when private agents have distorted beliefs relative to a planner, but they focus on implications for the optimal regulation of differences in beliefs between investors and creditors. Finally, Farhi and Werning (2020) study an economy with diagnostic expectations in which social inefficiencies may arise from aggregate demand externalities. Their study focuses on the implications of extrapolative expectations for the coordination of monetary policy and macro-prudential policy.

Layout. The paper is structured as follows. Section 2 describes the model and characterizes its equilibrium. Section 3 conducts the positive analysis, and section 4 explores additional properties under alternative model specifications. Section 5 presents the normative analysis. Section 6 concludes.

2 The Model

The model is a production economy with financial frictions, in which agents form expectations over random events diagnostically rather than rationally. Of particular interest are the joint implications of diagnostic expectations and financial frictions on the equilibrium outcome (section 3 and section 4) and allocative efficiency (section 5).

2.1 Environment

Time $t \in \mathbb{R}_+$ is continuous and unbounded. There is a single real asset $k_t \geq 0$ and a single output good $y_t \geq 0$. Preferences over consumption of the good are linear. Thus, agents only derive utility from its present discounted value.

Technologies. There are two production technologies, both of which use the asset to produce the output good according to

$$y_{j,t} = A_j k_{j,t} \geq 0 \; , \tag{1}$$

where $A_j > 0$ is the productivity of technology $j \in \{1, 2\}$ and $k_{j,t} \geq 0$ are the units of the asset allocated to that technology. The technologies allow for internal reinvestment at a standard rate of return, $\mathcal{I}_j(\iota_{j,t})k_{j,t}dt$, which satisfies $\mathcal{I}_j(0) = 0$, $\mathcal{I}'_j(\cdot) > 0$, and $\mathcal{I}''_j(\cdot) < 0$, where $\iota_{j,t} \in [0, A_j]$ is the reinvestment rate per unit of the asset. We assume one of the technologies (i.e., $j = 1$) is more productive but it is also riskier. Formally, $A_1 \geq A_2$, $\mathcal{I}_1(\cdot) \geq \mathcal{I}_2(\cdot)$, and $\sigma_1 > \sigma_2 \geq 0$, with

$$\frac{dk_{j,t}}{k_{j,t}} = \mathcal{I}_j(\iota_{j,t})dt + \sigma_j dZ_t \ , \tag{2}$$

where $dZ_t \sim_{i.i.d.} \mathcal{N}(0, dt)$ is a Brownian disturbance common to the technologies. The disturbance can be interpreted as an aggregate shock to the productive quality of the asset, or in other words, as a *quality shock*.

Perceptions towards Technologies. If expectations were rational, agents would correctly forecast $\hat{E}_t[dZ_t] = E_t[dZ_t] = 0$, where hat variables indicate perceptions. Thus, agents would correctly perceive the evolution of the asset under the relatively *productive technology* (i.e., $j = 1$) as the most profitable of the two. By contrast, under diagnostic expectations, agents tend to misestimate average $E_t[dZ_t]$ because they think recently realized disturbances positively affect the likelihood of future disturbances. As a consequence, if recent disturbances are sufficiently negative, agents can incorrectly regard the unproductive technology (i.e., $j = 2$) as the one with the most profitable asset evolution.

Formally—following Maxted (2023)—we assume agents synthesize *information* about past disturbances as

$$\omega_t \equiv \int_0^t e^{-\delta(t-s)} dZ_s \ , \tag{3}$$

where $\delta > 0$ indicates the memory rate of decay at which past realizations are discounted. Agents then use recent information $\omega_t \in \mathbb{R}$ to forecast future disturbances according to

$$\hat{E}_t[dZ_t] \equiv E_t[d\hat{Z}_t], \text{ with } d\hat{Z}_t \equiv \hat{\mu}\omega_t dt + dZ_t, \tag{4}$$

where parameter $\hat{\mu} > 0$ is the diagnostic weight of information on expectation formation. Thus, forecast $\hat{E}_t[dZ_t] = \hat{\mu}\omega_t dt$ positively depends on information ω_t, and forecast errors $-\hat{E}_t[dZ_t] \neq 0$ are possible. Moreover, if information $\omega_t < 0$ is sufficiently negative, the following perceptions about the evolution of the asset under the technologies hold: $\hat{E}_t[dk_2/k_2] > \hat{E}_t[dk_1/k_1]$. Based on these results—and because past disturbances do not have predictive power over future disturbances—in what follows, we interpret ω_t as *sentiment*.[1]

Returns on Technologies. The asset can be traded in spot markets at a price $q_t > 0$. We postulate that the price evolves according to

$$\frac{dq_t}{q_t} = \mu_{q,t} dt + \sigma_{q,t} dZ_t \ , \text{ with } \ \sigma_{q,t} \geq 0 \ , \tag{5}$$

where $\mu_{q,t} \in \mathbb{R}$ and $\sigma_{q,t} \geq 0$ are endogenous drift and diffusion processes to be determined in equilibrium and dZ_t is the aggregate disturbance introduced in (2). Let $dR_{j,t} \in \mathbb{R}$ denote the rate of return to allocate the asset to technology j. Then,

$$dR_{j,t} \equiv \frac{A_j - \iota_{j,t}}{q_t} dt + \frac{d\left(q_t k_{j,t}\right)}{q_t k_{j,t}} = \left[\frac{A_j - \iota_{j,t}}{q_t} + \mu_{q,t} + \mathcal{I}_j(\iota_{j,t}) + \sigma_{q,t}\sigma_j\right] dt + (\sigma_{q,t} + \sigma_j) dZ_t \ , \tag{6}$$

where the first term on the RHS of the definition is the dividend yield from operating the technology, the second term is the percentage change in the market value of the asset holdings, and the equality follows from Ito's product rule. Moreover,

$$\hat{E}_t[dR_{j,t}] = \left[\frac{A_j - \iota_{j,t}}{q_t} + \mu_{q,t} + \mathcal{I}_j(\iota_{j,t}) + \sigma_{q,t}\sigma_j + (\sigma_{q,t} + \sigma_j)\, \hat{\mu}\omega_t\right] dt \ , \tag{7}$$

where the last term on the RHS reflects the influence of diagnostic expectations over the perceptions of price risk $\sigma_{q,t} dZ_t$ and quality risk $\sigma_j dZ_t$. Thus, if sentiment ω_t is sufficiently low, diagnostic perceptions towards risks are sufficiently strong to render $\hat{E}_t[dR_{2,t}] > \hat{E}_t[dR_{1,t}]$.

Agents. The economy is populated by a continuum of households and financiers of unit size. All of the agents have the same expectations and behave competitively.

Financial Frictions. Households can only operate the unproductive technology while financiers can only operate the productive technology. Both types of agents can issue debt to take leveraged positions on the asset, but only financiers are subject to a collateral constraint. This constraint is motivated by a standard agency problem in credit markets

[1] Note that if $\hat{\mu} = 0$ expectations would be rational.

that allows financiers to walk away with a fraction of their assets immediately after issuing debt. The constraint restricts asset holdings of financiers to satisfy $q_t k_{1,t} \leq \lambda n_t$, where $n_t \geq 0$ is their net worth and parameter $\lambda - 1 > 0$ is the upper limit on their debt-to-net-worth ratio. For simplicity, we assume debt is short term and non-contingent, meaning that debt issued at time t matures at time $t + dt$ and promises a fixed rate of return regardless of realization dZ_t. Based on linearity in preferences, we postulate that the interest rate on debt is given by rdt, where $r > 0$ is the subjective time discount rate of agents.

Portfolio Problems. Households maximize the present discounted value of consumption subject to the law of motion of their wealth. Formally, they solve

$$\max_{c_t, \iota_{2,t}, k_{2,t} \geq 0} \hat{E}_t \int_t^{+\infty} e^{-r(s-t)} c_s ds \ , \tag{8}$$

subject to

$$dw_s = dR_{2,s} q_s k_{2,s} + r(w_s - q_s k_{2,s})ds - c_s ds + \tau_s ds \ , \tag{9}$$

where $c_t \geq 0$ is consumption, $w_t \in \mathbb{R}$ is wealth, and $\tau_t \in \mathbb{R}$ are net transfers from financiers (which are specified below). As in Gertler and Kiyotaki (2010), Gertler and Karadi (2011), and Maggiori (2017), financiers do not consume, but rather each of them pays out dividends to a unique household. They do so according to an exogenous Poisson process with arrival rate $\theta > 0$. When they pay out, financiers transfer their entire net worth to their associated household, and immediately afterwards, they are replaced by an identical newcomer whose starting net worth is specified below. Financiers maximize the present discounted value of dividend payouts

$$\max_{\iota_{1,s}, k_{1,s} \geq 0} \hat{E}_t \int_t^{+\infty} \theta e^{-(r+\theta)(s-t)} n_s ds \ , \tag{10}$$

subject to the law of motion of net worth

$$dn_s = dR_{1,s} q_s k_{1,s} - r(q_s k_{1,s} - n_s)ds \ , \tag{11}$$

and the collateral constraint

$$q_s k_{1,s} \leq \lambda n_s \ . \tag{12}$$

Equilibrium. A competitive equilibrium is an allocation $\{\iota_{1,t}, k_{1,t}, \iota_{2,t}, k_{2,t}, c_t\}$ and an asset price process $\{q_t, \mu_{q,t}, \sigma_{q,t}\}$ such that (i) the allocation solves portfolio problems (8)-

(9) and (10)-(12) given the price process; (*ii*) the markets for the good, the asset, and debt clear.

2.2 Solving the Equilibrium

First, we derive the optimal choices of households and financiers. Then, we combine those choices with market clearing to obtain a set of conditions that analytically characterize the equilibrium. Finally, we restrict attention to a Markov equilibrium, which allows us to reduce the conditions to a tractable system of second-order partial differential equations (PDEs).

2.2.1 Households' Problem.

The below proposition characterizes the optimal choices of households.

Proposition 1. *At any given time t, households are indifferent among any consumption rate c_t. Moreover, they choose reinvestment rate $\iota_{2,t}$ and asset holding $k_{2,t}$ as follows:*

$$\mathcal{I}_2'(\iota_{2,t}) = \frac{1}{q_t} \, , \tag{13}$$

and

$$q_t k_{2,t} \left[\begin{array}{ll} = 0 & \text{if } \alpha_{2,t} < 0 \\ \in [0, +\infty) & \text{if } \alpha_{2,t} = 0 \end{array} \right. \, , \tag{14}$$

where the estimated risk-adjusted excess return to allocate the asset to the unproductive technology over holding debt, that is, $\alpha_{2,t} \leq 0$, is given by

$$\alpha_{2,t} \equiv \frac{1}{dt} \hat{E}_t \left[dR_{2,t} \right] - r \leq 0 \, . \tag{15}$$

Proof. See the Appendix. □

The intuition behind the proposition is as follows. Households are indifferent among any consumption rate because the interest rate on debt equals their subjective time discount rate. When $\alpha_{2,t} < 0$, households strictly prefer holding debt securities to allocating the asset to the unproductive technology. Thus, $k_{2,t} = 0$ is optimal. By contrast, when

$\alpha_{2,t} = 0$, households are indifferent between the two investment opportunities, and therefore, any $k_{2,t} \geq 0$ is optimal. Excess return $\alpha_{2,t} \leq 0$ cannot be positive in equilibrium. Otherwise, households would take unbounded leveraged positions on the asset, since they are not subject to financing constraints. Reinvestment rule (13) indicates that reinvestment positively depends on asset price q_t.

2.2.2 Financiers' Problem.

Let $V_t \geq 0$ be the value function associated with problem (10)-(12). We postulate that the value is linear in net worth. Formally, $V_t \equiv v_t n_t$, where marginal value of net worth $v_t \geq 1$ is endogenous but independent of individual choices. In addition, we postulate that marginal value v_t evolves stochastically over time, according to an Ito process with disturbance dZ_t and endogenous drift and diffusion $\mu_{v,t} \in \mathbb{R}$ and $\sigma_{v,t} \leq 0$, respectively. The below proposition characterizes the optimal choices of financiers.

Proposition 2. *At any given time t, financiers choose reinvestment rate $\iota_{1,t}$ and asset holding $k_{1,t}$ as follows:*

$$\mathcal{I}_1'(\iota_{1,t}) = \frac{1}{q_t} \ , \tag{16}$$

and

$$\frac{q_t k_{1,t}}{n_t} \begin{bmatrix} = 0 & \text{if } \alpha_{1,t} < 0 \\ \in [0, \lambda] & \text{if } \alpha_{1,t} = 0 \\ = \lambda & \text{if } \alpha_{1,t} > 0 \end{bmatrix} , \tag{17}$$

where the estimated risk-adjusted excess return to allocate the asset to the productive technology over holding debt, namely, $\alpha_{1,t} \in \mathbb{R}$, is given by

$$\alpha_{1,t} \equiv \frac{1}{dt} \hat{E}_t \left[dR_{1,t} \right] - r + \left(\sigma_{q,t} + \sigma_1 \right) \sigma_{v,t} \ . \tag{18}$$

The marginal value of net worth, v_t, satisfies

$$0 = \alpha_{1,t} \frac{q_t k_{1,t}}{n_t} + \mu_{v,t} + \hat{\mu} \omega_t \sigma_{v,t} + \frac{\theta}{v_t} - \theta \ . \tag{19}$$

Proof. See the Appendix. □

When $\alpha_{1,t} > 0$, financiers expect a positive excess return to allocating the asset to the productive technology. Thus, they take leveraged positions on the asset until they hit their limit on debt. When $\alpha_{1,t} = 0$, financiers are willing to take any position on the asset, because they are indifferent between the two investment alternatives. Lastly, when $\alpha_{1,t} < 0$, financiers do not acquire the asset, because they expect a higher return from holding debt. The last term in (18) is a compensation for holding quality risk $\sigma_1 dZ_t$ and price risk $\sigma_{q,t} dZ_t$. This term exists because financiers are subject to a collateral constraint. This constraint renders financiers concerned with co-movements between the return of their investments and the rate of change in the marginal value of net worth. Reinvestment rule (16) is analogous to reinvestment rule (13).

Condition (19) expresses marginal value v_t as a present discounted value of expected rents $\alpha_{1,t} q_t k_{1,t}/n_t \geq 0$. These rents are the profits earned by financiers from operating the productive technology. If financiers never earn any rent—and thus the collateral constraint is always slack—then $v_t = 1$. By contrast, if $\alpha_{1,t} > 0$ at least occasionally, $v_t \geq 1$.

2.2.3 Equilibrium Characterization.

We postulate that excess returns $\alpha_{1,t} = 0$ and $\alpha_{2,t} = 0$ cannot simultaneously be null almost surely (a.s.). That is, households and financiers cannot simultaneously be marginal buyers of the asset in events with positive probability. The below proposition then characterizes the equilibrium as well as describes key equilibrium objects.

Proposition 3. *Let $\eta_t \equiv n_t/q_t k_t \in [0,1]$ be the aggregate net worth of financiers as a share of total wealth and let $\kappa_t \equiv k_{1,t}/k_t \in [0,1]$ be the aggregate share of the asset allocated to the productive technology. Then, the equilibrium outcome can be partitioned into the following three regimes,*

$$\begin{array}{llll} \text{1. Financially unconstrained regime:} & \kappa_t = 1 \leq \lambda\eta_t\ , & \alpha_{1,t} = 0\ , & \alpha_{2,t} < 0\ ; \\ \text{2. Financially constrained regime:} & \kappa_t = \lambda\eta_t \in [0,1]\ , & \alpha_{1,t} > 0\ , & \alpha_{2,t} = 0\ ; \\ \text{3. Precautionary regime:} & \kappa_t = 0\ , & \alpha_{1,t} < 0\ , & \alpha_{2,t} = 0\ ; \end{array} \tag{20}$$

the equilibrium allocation can be summarized as $\{\iota_{1,t}, \iota_{2,t}, \kappa_t\}$, and the equilibrium can be characterized by $\{(13), (15), (16), (18), (19), (20)\}$. The equilibrium utility of households *per unit of the asset, namely, $u_t > 0$, satisfies*

$$0 = \kappa_t \left\{A_1 - \iota_{1,t} + \left[\mathcal{I}_1(\iota_{1,t}) + \sigma_1 \hat{\mu}\omega_t\right] u_t\right\} + \\ + (1-\kappa_t)\left\{A_2 - \iota_{2,t} + \left[\mathcal{I}_2(\iota_{2,t}) + \sigma_2 \hat{\mu}\omega_t\right] u_t\right\} + \hat{E}_t\left[du_t\right] - r u_t. \tag{21}$$

Proof. See the Appendix. □

The notation does not distinguish between individual and aggregate variables because representative agents exist. A representative household does so because individual households are identical and a representative financier exists because the behavior of individual financiers is linear in net worth.

The equilibrium regimes directly follow from combining optimality conditions in propositions 1 and 2 with market clearing for the asset. In equilibrium, consumption per unit of the asset c_t/k_t is given by net output flows $y_t/k_t = (A_1 - \iota_{1,t})\,\kappa_t + (A_2 - \iota_{2,t})\,(1-\kappa_t)$. Utility u_t is the present discounted value of consumption per unit of the asset. The utility is interpreted as the *social value* of the asset.

2.2.4 Markov Equilibrium.

For tractability, we restrict attention to a Markov equilibrium, which allows to reduce the equilibrium conditions to a system of second-order PDEs. As is common practice, we no longer report the time subscript.

Definition 1. *A Markov equilibrium is a set of state variables $\{\eta, \omega, k\}$ and a set of mappings $\{q, v\}$ defined over states $\{\eta, \omega\}$, such that (i) the mappings satisfy conditions $\{(13), (15), (16), (18), (19), (20)\}$ and (ii) the states evolve according to laws of motion consistent with the conditions.*

Wealth share η indicates the tightness of the collateral constraint while sentiment ω indicates the degree of distortions in expectations relative to the RE benchmark. The aggregate quantity of the asset is also a state variable, but it is not relevant for the derivations, because the equilibrium is scale invariant with respect to k. Mappings $\{q, v\}$ alone suffice to characterize the equilibrium because any other endogenous variable can be expressed as a function of those mappings or their partial derivatives with respect to the states.

Regarding the laws of motion, the aggregate quantity of the asset evolves endogenously, according to

$$\frac{dk}{k} = \mu_k dt + \sigma_k dZ \ , \tag{22}$$

with

$$\mu_k = \kappa \mathcal{I}_1(\iota_1) + (1-\kappa)\,\mathcal{I}_2(\iota_2) \ , \tag{23}$$

$$\sigma_k = \kappa\sigma_1 + (1-\kappa)\,\sigma_2 \ . \tag{24}$$

The wealth share of financiers evolves endogenously as well, according to

$$\frac{d\eta}{\eta} = \mu_\eta dt + \sigma_\eta dZ \ , \tag{25}$$

with

$$\begin{aligned} \mu_\eta &= \left[\frac{A_1-\iota_1}{q} + \mathcal{I}_1(\iota_1) + \sigma_q\sigma_1\right]\phi + (\mu_q - r)\,(\phi-1) - \mu_k \qquad (26) \\ &\quad -\sigma_q\sigma_k + (\sigma_q+\sigma_k)\left[(\sigma_q+\sigma_k) - \phi\,(\sigma_q+\sigma_1)\right] - \left(\theta - \frac{\gamma}{\eta}\right) \ , \\ \sigma_\eta &= \phi\,(\sigma_q+\sigma_1) - (\sigma_q+\sigma_k) \ , \qquad (27) \end{aligned}$$

where $\phi \equiv qk_1/n \geq 0$ is the leverage multiple of financiers—which is common across them (Proposition 2)—and the last term in μ_η is the net transfers from financiers to households.[2] The first term of the transfers is the aggregate dividend payout, and the second term is the starting endowment of newcomers. As in Gertler and Kiyotaki (2010) and Gertler and Karadi (2011), each newcomer receives $\gamma/\theta > 0$ units of the asset from a unique household.

Lastly, sentiment evolves exogenously, according to

$$d\omega = -\delta\omega dt + dZ \ . \tag{28}$$

Proposition 4. *The Markov equilibrium can be analytically characterized as the solution to a system of second-order PDEs for* $\{q, v\}$ *in* $\{\eta, \omega\}$.

Proof. See the Appendix. □

[2] This law of motion follows from applying Ito's quotient rule to $\eta = n/qk$ and then subtracting from the resulting expression the net transfers from financiers to households, $\theta - \gamma/\eta$. Note then that $\tau = (\theta\eta - \gamma)qk$.

To conduct the positive and the normative analysis—when needed—we solve the PDEs numerically using spectral methods. To do so, we parametrize return functions $\mathcal{I}_j(\cdot)$ and assign numerical values to the parameters, as detailed in the next subsection.

2.3 Parametrization and Parameter Values

We consider a riskless unproductive technology without reinvestment opportunities to ease exposition. Formally, $\mathcal{I}_2(\cdot) = 0$ and $\sigma_2 = 0$. We interpret such a technology as unproductive storage. In section 4, we investigate the robustness of our results to alternative technological specifications. Throughout the analysis, as is common in the literature (e.g., Brunnermeier and Sannikov (2014); Phelan (2016); He and Krishnamurthy (2019)), we consider quadratic costs for reinvestment. That is, $\mathcal{I}_1(\iota_1) = \chi_1\sqrt{\iota_1}$, where $\chi_1 > 0$ is a parameter.

Table 1 reports parameter values for the baseline case. The time frequency is annual. Parameters for technologies and agents are either taken from the literature or are set to match unconditional averages in an economy with rational expectations and financial frictions. To compute the averages, we use the limiting probability density function of the state. This function measures the share of time the economy spends on average at each state point over a sufficiently long (i.e., infinite) time horizon.

Table 1: Parameter Values

Description	Parameter	Value	Target / Source
Panel A. Technologies			
Productivity gap	$A_1 - A_2$	0.35	Av. credit spread (1%)
Quality risk	σ_1	3%	Av. volatility of output (4%)
Return on reinvestment	χ_1	1.9%	Av. investment-output ratio (20%)
Panel B. Agents			
Subjective time discount rate	r	2%	Interest rate
Limit on debt	$\lambda - 1$	3	Av. leverage multiple (3.7)
Frequency of dividend payouts	θ	10%	Av. life span of financiers (10 years)
Starting endowment of financiers	γ/θ	15%	Av. wealth share of financiers (25%)
Panel C. Expectation formation			
Memory decay rate	δ	0.85	Corr. sentiment-wealth share (0.71)
Expectation weight to information	$\hat{\mu}$	0.2	Output bias (0.75%)

Notes: The table reports the parameter values in the baseline specification of the model. The time frequency is annual.

The productivity of the productive technology, $A_1 = 1$, is normalized to 1. The following parameters are calibrated to match either standard targets or the targets in Gertler and Kiyotaki (2010) and Gertler and Karadi (2011). In particular, productivity gap $A_1 - A_2 = 0.35$ targets an excess return to operating the productive technology of $E[dR_1] - r = 1\%$. The investment return $\chi_1 = 1.9\%$ targets an average ratio of reinvestment to output of $E[\iota/[A_1\kappa + A_2(1-\kappa)]] = 20\%$, whereas volatility $\sigma_1 = 3\%$ targets an average volatility of output of $Var[A_1\kappa + A_2(1-\kappa)] = 4\%$. The subjective time discount rate is consistent with a standard value for the real interest rate of $r = 2\%$. The debt limit of financiers, $\lambda - 1 = 3$, targets an average leverage multiple of financiers of $E[\phi] = 3.7$. The average frequency of dividend payouts is $\theta = 10\%$, and the endowment of financiers satisfies $\gamma/\theta = 15\%$. The former value implies an average lifespan of financial firms of 10 years, whereas the latter value targets an average share of wealth of financiers of $E[\eta] = 25\%$.

Finally, diagnostic weight $\hat{\mu} = 0.2$ targets an output bias of 0.75% for a standard deviation in sentiment, as reported by Bordalo et al. (2020). The persistence parameter $\delta = 0.85$ targets a correlation between sentiment and wealth share of $Corr[\omega, \eta] = 71\%$, as in Maxted (2023).

We perform robustness analyses in section 4, where we consider alternative specifications for the formation of diagnostic expectations, the collateral constraint, and differences across the technologies.

3 Positive Analysis

We can now derive the equilibrium outcome and examine its positive properties. To do so, we proceed gradually considering the following four economies.

3.1 Rational Expectations and No Financial Frictions

Consider first an economy with rational expectations and without financial frictions. Formally, expectation weight $\hat{\mu} = 0$ is null, financiers' net worth $n \in \mathbb{R}$ can be negative, and leverage limit $\lambda = +\infty$ is unbounded.[3] The below proposition describes the equilibrium outcome.

[3] If the net worth of financiers could not be negative, boundary condition $(\sigma_q + \sigma_1) \to 0$ as $\eta \to 0$ would be required to ensure $n \geq 0$—as in Maggiori (2017). The reason is that external financing is limited to non-contingent debt and the asset is risky. The possibility of $n < 0$ can then be interpreted as the possibility of issuing risky debt (i.e., debt whose rate of return depends on shock dZ) or of issuing equity.

Proposition 5. *In the economy with rational expectations and without financial frictions, neither sentiment ω nor wealth share η affect the equilibrium outcome. The asset price is a constant that satisfies*

$$\alpha_1 = 0 \Leftrightarrow \frac{A_1 - \iota_1}{q} + \mathcal{I}_1(\iota_1) - r = 0 \ , \ \text{with } \ \mathcal{I}_1'(\iota_1) = \frac{1}{q} \ . \tag{29}$$

Value $v = 1$ is also a constant. The aggregate quantity of the asset is allocated to the productive technology, that is, $\kappa = 1$.

Proof. See the Appendix. □

In this economy, only the first equilibrium regime in expression (20) occurs. Although sentiment and the wealth share do not affect the equilibrium outcome, the outcome fluctuates, because of fluctuations in aggregate quantity k. In other terms, no cyclical deviations from the trend occur, but the trend fluctuates. The social value of the asset equals the asset price. That is, $u = q$.

3.2 Diagnostic Expectations but No Financial Frictions

Consider next an economy with diagnostic expectations and without financial frictions. That is, weight $\hat{\mu} > 0$ is positive, net worth $n \in \mathbb{R}$ can be negative, and limit $\lambda = +\infty$ is unbounded.

Proposition 6. *In the economy with diagnostic expectations and without financial frictions, sentiment ω is the only relevant state that affects the equilibrium outcome. A threshold state $\bar{\omega} < 0$ exists such that*

$$\begin{array}{ll} \text{if } \omega < \bar{\omega} \Rightarrow & \kappa = 0 \ , \ \alpha_1 < 0 \ , \ \alpha_2 = 0 \ ; \\ \text{if } \omega > \bar{\omega} \Rightarrow & \kappa = 1 \ , \ \alpha_1 = 0 \ , \ \alpha_2 < 0 \ ; \end{array} \ . \tag{30}$$

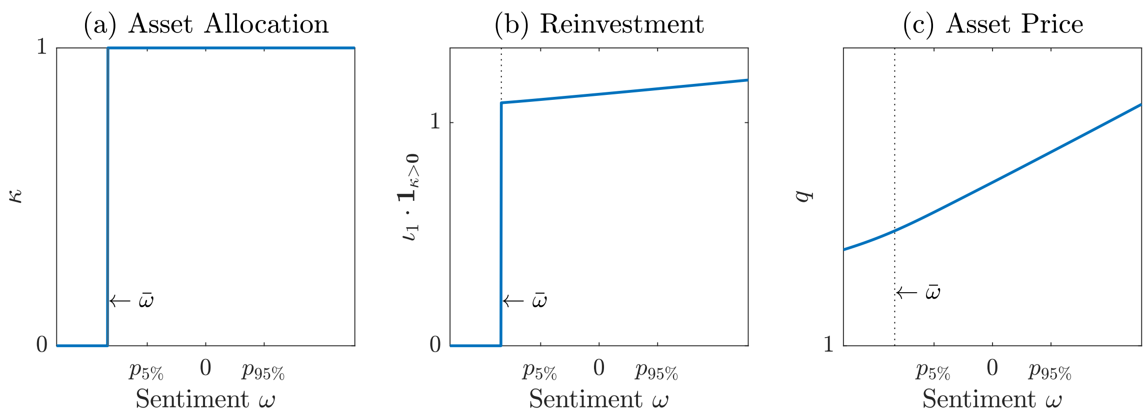
The threshold state $\bar{\omega} < 0$ is the solution to

$$\alpha_1 = \alpha_2 = 0 \Rightarrow \frac{A_1 - \iota_1 - A_2}{q} + \mathcal{I}_1(\iota_1) + (\sigma_q + \hat{\mu}\omega)\,\sigma_1 = 0 \ . \tag{31}$$

Proof. See the Appendix. □

The equilibrium outcome features two well-demarcated regimes. When $\omega < \bar{\omega}$, the economy operates in a *precautionary* regime, in which the aggregate quantity of the asset is allocated to the unproductive technology and the asset is priced according to $\alpha_2 = 0$. In this regime, households are the marginal buyers of the asset, whereas financiers strictly prefer to acquire debt rather than to operate the productive technology. Output flows $y/k = A_2 < A_1$ are low, as are asset price q, aggregate growth rate $\mu_k = 0$, and aggregate risk $\sigma_k = 0$. By contrast, when $\omega > \bar{\omega}$, the economy operates in a *non-precautionary* regime, in which the aggregate quantity of the asset is allocated to the productive technology, financiers are the marginal buyers of the asset, and households strictly prefer to hold debt to operate the unproductive technology. Formally, $\kappa = 1$ and $\alpha_1 = 0$. In this case, aggregate output flows $y/k = A_1$, asset price q, aggregate growth rate $\mu_k = \mathcal{I}_1(\iota_1)$, and aggregate risk $\sigma_k = \sigma_1$ are instead high. The outcome repeatedly alternates between these two regimes according to the law of motion (28). This law of motion generates a stationary distribution of sentiment of $\omega \sim \mathcal{N}[0, 1/(2\delta)]$.[4]

FIGURE 1: DIAGNOSTIC EXPECTATIONS BUT NO FINANCIAL FRICTIONS



Notes: The figure plots the allocation of the asset between the technologies (panel a), the reinvestment rate conditional on a positive allocation of the asset to the productive technology (panel b), and the asset price (panel c) as a function of the relevant state of the economy. Variables are normalized by their respective value in the economy in section 3.1. Threshold $\bar{\omega} < 0$ separates the precautionary and the non-precautionary regimes. Point $p_{x\%}$ in the x-axis, with $x \in \{5; 95\}$, indicates the $x\%$-percentile of the limiting distribution of sentiment.

[4] The regimes correspond to the first and third, respectively, in expression (20). The threshold state that separates the two regimes is negative because (i) the asset price is positively related to sentiment—that is, $\sigma_q > 0$—and (ii) the productive technology is riskier but yields higher dividend returns than the unproductive technology.

This economy then exhibits recurrent boom-bust cycles in aggregate output, asset prices, and economic growth rates. The driver of the cycles is sentiment. Price q, growth rate μ_k, and aggregate risk σ_k are pro-cyclical. Forecast errors $-\hat{E}[dZ] = -\hat{\mu}\omega \neq 0$ are instead counter-cyclical. Because of these errors, the persistence of sentiment and the tails of its stationary distribution are perceived to be larger than they actually are. Put formally, under diagnostic expectations, sentiment is perceived to fluctuate according to $d\omega = -(\delta - \hat{\mu})\,\omega dt + dZ$ and $\omega \sim \mathcal{N}\left[0, 1/\left(2\left(\delta - \hat{\mu}\right)\right)\right]$. Lastly, the pro-cyclicality of the asset price strengthens a positive interaction between the price and reinvestment when sentiment is sufficiently high. This effect contributes to generate higher asset prices and reinvestment rates relative to their corresponding levels in the first economy (Figure 1, panel c).[5]

3.3 Financial Frictions but Rational Expectations

Now consider an economy with rational expectations and financial frictions. That is, expectation weight $\hat{\mu} = 0$ is null, net worth $n \geq 0$ cannot be negative, and leverage limit $\lambda < +\infty$ is bounded.

Proposition 7. *In the economy with rational expectations and financial frictions, wealth share η is the only relevant state that affects the equilibrium outcome. A threshold state $\bar{\eta} \in (0,1)$ exists such that*

$$\begin{array}{llll} \text{if } \eta < \bar{\eta} \Rightarrow & \kappa = \lambda\eta < 1\,, & \alpha_1 > 0\,,\ \alpha_2 = 0\,; \\ \text{if } \eta > \bar{\eta} \Rightarrow & \kappa = 1\,, & \alpha_1 = 0\,,\ \alpha_2 < 0\,; \end{array} . \tag{32}$$

The threshold state $\bar{\eta} \in (0,1)$ is the solution to

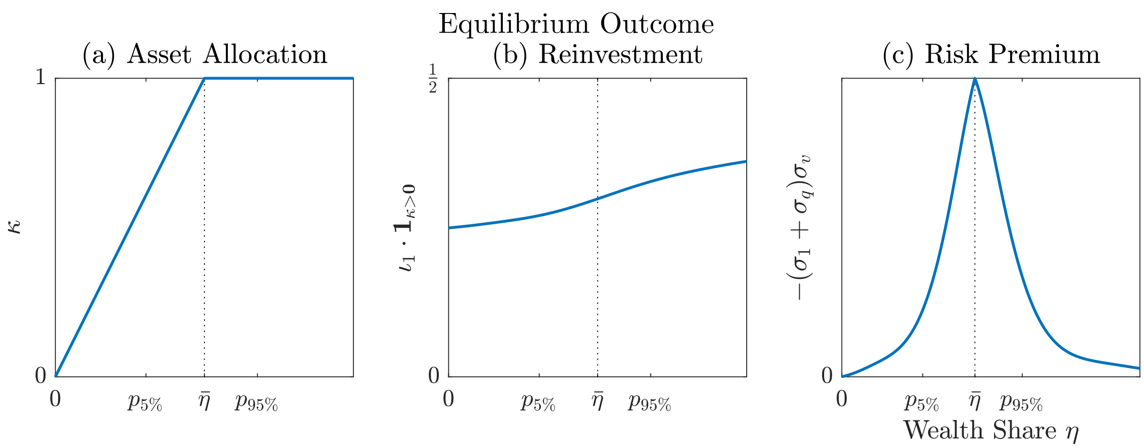
$$\lambda\eta = 1 \Leftrightarrow \eta = \frac{1}{\lambda}\,. \tag{33}$$

[5] In Figure 1, the price and reinvestment also exceed their levels in the first economy when sentiment is low. This effect is a consequence of an asymmetric effect of sentiment on the price. High sentiment exerts upward pressure on the price, whereas low sentiment exerts downward pressure. These pressures are not only exerted on impact, but are also exerted throughout the state space. The reason is that the price is forward-looking. Nonetheless, the upward pressure is relatively stronger, because when sentiment is low, the aggregate quantity of the asset is allocated to the unproductive technology, which eliminates the exposure of the asset to quality risk as well as the direct negative effect of low sentiment on the price. Under the baseline parameter values, the upward pressure is sufficiently strong to support a price above its level in the first economy throughout the grid of sentiment used in the numerical solution.

Proof. See the Appendix. □

As in the economy in subsection 3.2, the equilibrium outcome features two well-demarcated regimes. In contrast to that economy, however, the regimes are inherently determined by the financial capacity of financiers to acquire assets and operate the productive technology. Specifically, when $\lambda\eta < 1$, the economy operates in a *financially constrained* regime in which financiers are constrained by their leverage limit to acquire assets. Accordingly, households hold the remnant share of the asset and are the marginal buyers. That is, $\kappa = \lambda\eta < 1$ and $\alpha_2 = 0 < \alpha_1$ (Figure 2, panel (a)). By contrast, when $\lambda\eta \geq 1$, the economy operates in a *financially unconstrained* regime in which financiers hold the aggregate quantity of the asset, households only hold debt issued by financiers, and financiers are the marginal buyers.

Figure 2: Rational Expectations and Financial Frictions



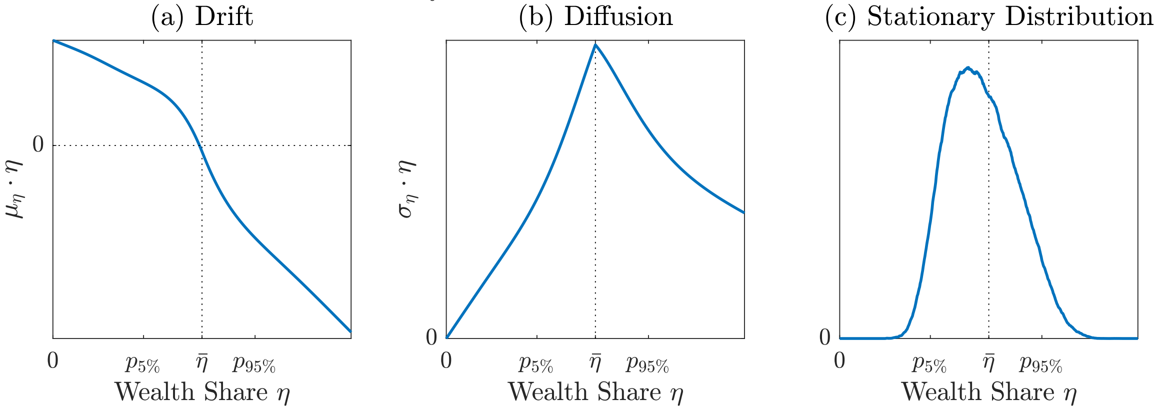
Notes: The figure plots the allocation of the asset between the technologies (panel a), the reinvestment rate conditional on a positive allocation of the asset to the productive technology (panel b), and the price of risk (panel c) as a function of the relevant state of the economy. The variables in the figure (except the risk premium) are normalized by their values in the economy in section (3.1). Threshold $\bar{\eta}$ separates the financially constrained and the financially unconstrained regimes. Point $p_{x\%}$ indicates the x%-percentile of the limiting distribution of the wealth share of financiers.

Because financiers operate the productive technology, aggregate output and the asset price are increasing in the wealth share, as is investment rate ι_1. Value v is instead decreasing, because the rents from operating the productive technology are inversely related to the wealth share. Specifically, the rents are positive $\alpha_1\lambda > 0$ when $\eta < 1/\lambda$, and null otherwise.

The decreasing value creates a negative risk-premium term in (18), $(\sigma_q + \sigma_1)\,\sigma_v \leq 0$, which reflects financiers' effective risk aversion in the presence of financial constraints (Figure 2, panel (c)).

FIGURE 3: RATIONAL EXPECTATIONS AND FINANCIAL FRICTIONS

Dynamics of Wealth Share



Notes: The figure plots the drift (panel a), the diffusion (panel b), and the limiting distribution of the wealth share (panel c) as a function of the share. Threshold $\bar{\eta}$ separates the financially constrained and the financially unconstrained regimes. Point $p_{x\%}$ indicates the x%-percentile of the limiting distribution of the wealth share of financiers.

The equilibrium outcome repeatedly alternates between the two financial regimes (Figure 3) according to the law of motion (25). Fluctuations display two properties. First, fluctuations are mean reverting around a stochastic steady state (i.e., η such that $\mu_\eta \eta = 0$—panel a), because of the counter-cyclicality of rents $\alpha_1 \lambda$ and the a-cyclicality of dividend payouts. Second, fluctuations are stochastic (panel b) because of a positive interaction between net-worth risk $\sigma_\eta = (\phi - 1)\,\sigma_q + (1 - \eta)\,\phi\sigma_1$ and price risk $\sigma_q = \varepsilon_q \sigma_\eta$, where $\varepsilon_q \equiv (\partial q / \partial \eta)\,(\eta / q) \geq 0$ is the elasticity of the asset price with respect to the wealth share.[6] Combining these two formulae, in effect, one gets the following endogenous financial amplification factors of disturbances:

$$\frac{\sigma_\eta}{\sigma_1} = \frac{\phi - \phi\eta}{1 - (\phi - 1)\,\varepsilon_q} \geq 0 \quad \text{and} \quad \frac{\sigma_q}{\sigma_1} = \frac{(\phi - \phi\eta)\,\varepsilon_q}{1 - (\phi - 1)\,\varepsilon_q} \geq 0, \text{ with } \phi = \min\left\{\frac{1}{\eta}, \lambda\right\}, \tag{34}$$

which peak around threshold state $\bar{\eta}$, when the collateral constraint is locally occasionally binding.

[6]Formula $\sigma_q = \varepsilon_q \sigma_\eta$ follows from Ito's Lemma.

All in all, like its counterpart in subsection 3.2, this economy exhibits recurrent boom-bust cycles in aggregate output, asset prices, and economic growth rates. Price q, growth rate $\mu_k = \kappa \mathcal{I}_1(\iota_1)$, and aggregate risk $\sigma_k = \kappa\sigma_1$ are also pro-cyclical. By contrast, though, sentiment does not influence economic cycles, and forecasts are not subject to systematic errors. Rather, the wealth share of financiers is the driver of the cycles, and conditional forecast errors on average are null. Another key difference from the previous economy is that the cycles feature endogenous financial amplification and positive risk premia. These elements are time-varying and peak when the collateral constraint is locally occasionally binding.

3.4 Diagnostic Expectations and Financial Frictions

Finally, consider the whole economy presented in section 2, with diagnostic expectations and financial frictions. The below corollary describes the equilibrium outcome.

Corollary 1. *In the economy with diagnostic expectations and financial frictions, both sentiment ω and wealth share η affect the equilibrium outcome. Two threshold states, $\bar{\omega} < 0$ and $\bar{\eta} \in (0,1)$, partition the state space as follows:*

$$\begin{aligned} &\text{if } \omega < \bar{\omega} \Rightarrow && \kappa = 0\,, && \alpha_1 < 0\,,\ \alpha_2 = 0\,; \\ &\text{if } \omega > \bar{\omega} \text{ and } \eta < \bar{\eta} \Rightarrow && \kappa = \lambda\eta\,, && \alpha_1 > 0\,,\ \alpha_2 = 0\,; \\ &\text{if } \omega > \bar{\omega} \text{ and } \eta > \bar{\eta} \Rightarrow && \kappa = 1\,, && \alpha_1 = 0\,,\ \alpha_2 < 0\,; \end{aligned} \tag{35}$$

Threshold state $\bar{\omega}$ is the solution to

$$\alpha_1 = \alpha_2 = 0 \Rightarrow \frac{A_1 - \iota_1 - A_2}{q} + \mathcal{I}_1(\iota_1) + (\sigma_q + \hat{\mu}\omega)\,\sigma_1 + (\sigma_q + \sigma_1)\,\sigma_v = 0\,. \tag{36}$$

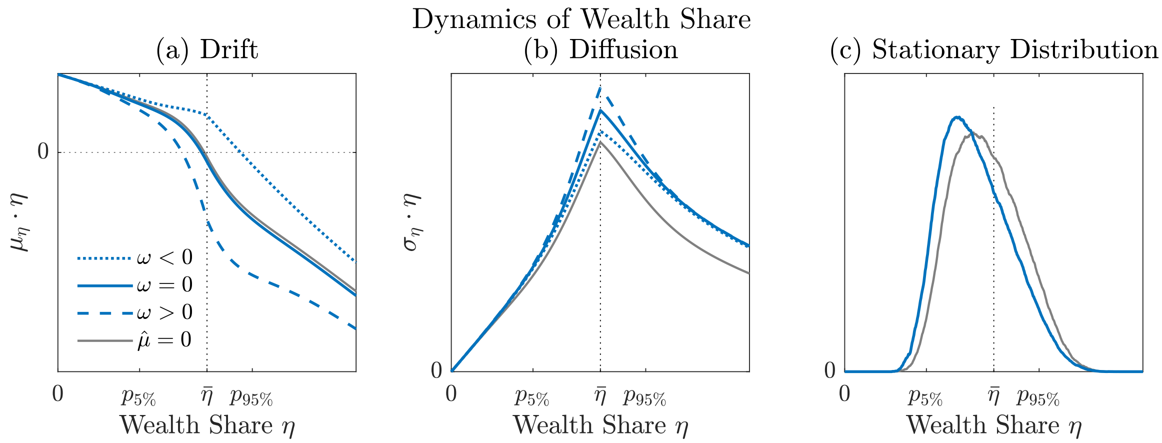
Threshold state $\bar{\eta}$ is the solution to

$$\lambda\eta = 1 \Leftrightarrow \eta = \frac{1}{\lambda}\,. \tag{37}$$

Proof. See the Appendix. □

This corollary directly follows from propositions 3, 6, and 7. With financial frictions, the characterization of the sentiment threshold, $\bar{\omega}$, includes the risk-premium term $(\sigma_q + \sigma_1)\,\sigma_v \leq 0$. Everything else being the same, this term reduces the perceived relative value of the productive technology, because operating that technology exposes de facto risk-averse financiers to aggregate risk. The equilibrium outcome repeatedly alternates among the aforementioned three regimes—precautionary, financially constrained, and financially unconstrained—according to the laws of motion (25) and (28). In these cycles, sentiment ω and wealth share η co-move positively, because current disturbances positively affect both the likelihood of future disturbances—as shown by (28)—and the wealth share of financiers—as shown by (34).

Figure 4: Diagnostic Expectations and Financial Frictions



Notes: The figure plots the drift (panel a), the diffusion (panel b), and the limiting marginal distribution of the wealth share (panel c) as a function of the share. Grey lines refer to the economy with rational expectations (i.e., section 3.3), whereas blue lines refer to the economy with diagnostic expectations (i.e., section 3.4). For the latter economy, variables are plotted with respect to three different values of sentiment.

Figure (4) highlights the key takeaways from this economy. Notably, diagnostic expectations intensify financial instability relative to the RE benchmark, as measured by a leftward shift in the stationary marginal distribution of the wealth share of financiers (Figure 4, panel c). This additional instability results from the following two interactions between diagnostic expectations and financial frictions.

First, relative to a world with rational expectations, the positive co-movement between ω and η strengthens the positive interaction between risks σ_η and σ_q. Following adverse disturbances, for instance, the fall in sentiment further depress the asset price, which further deteriorates the wealth share η, thus intensifying the interaction (panel b).[7]

[7] Put more formally, $dZ < 0$ exerts downward pressure on ω on impact, which reduces the first term in $\sigma_q = \varepsilon_{q,\omega}/\omega + \varepsilon_{q,\eta}\sigma_\eta$, with $\varepsilon_{q,\omega} \equiv (\partial q/\partial \omega)(\omega/q) \geq 0$ and $\varepsilon_{q,\eta} \equiv (\partial q/\partial \eta)(\eta/q) \geq 0$. This reduction, in turn, intensifies the interaction between the first term in $\sigma_\eta = (\phi - 1)\sigma_q + \phi\sigma_1 - \sigma_k$ and σ_q.

Second, everything else being the same, when both states ω and η are high, negative forecast errors $-\hat{E}\left[dZ|\cdot\right] = -\hat{\mu}\omega$ exert upward pressure on the asset price. The higher price then deteriorates conditional average return $E\left[dR_1|\cdot\right]$, which eventually hurts the profitability of financiers—as measured by conditional average growth rate $\mu_\eta = E\left[d\eta/\eta|\cdot\right]$. The opposite naturally happens when both states ω and η are instead low. Because forecast errors and aggregate risk $\sigma_k = \kappa\sigma_1$ are inversely related, however, the former effects dominate (panel a). This asymmetry then exerts leftward pressure on the stationary marginal distribution of the wealth share relative to the RE benchmark.

Overall, this economy features both systematic forecast errors and financial amplification effects. Relative to the economy with rational expectations, because of counter-cyclical forecast errors and pro-cyclical risk-taking, financial instability intensifies and economic cycles are more volatile and unstable.

4 Alternative Specifications

The key takeaway of the positive analysis is that diagnostic expectations intensify financial instability relative to the RE benchmark. In this section, we show this finding also holds under alternative processes of diagnostic expectation formation (subsections 4.1 and 4.2) and under other types of collateral constraints (subsection 4.3). These alternative specifications are interesting on their own because they identify additional channels of interaction between diagnostic expectations and financial frictions. Lastly, in subsection 4.4, we investigate the extent to which differences in technologies contribute to the amplification effects of diagnostic expectations on financial instability.

4.1 Generic Processes in Diagnostic Expectation Formation

In the baseline model, agents observe the entire history of past disturbances and use a weighted average of those disturbances to form expectations about future disturbances. In addition, agents use those expectations to estimate any moment of every other future random variable. Let's now consider a more general process of diagnostic expectation formation, in which agents rely on a generic Ito path $\{dX_s\}_{s<t}$ to form expectations about a generic Ito variable dY_t.

Proposition 8. *If agents rely on Ito path* $\{dX_s\}_{s<t}$ *to form diagnostic expectations about*

Ito variable dY_t, the implied diagnostic expectation operator over disturbance dZ_t is

$$\hat{E}_t\left[dZ_t\right] = \hat{\mu}\frac{\omega_t}{\sigma_{Y,t}Y_t}dt \ , \tag{38}$$

where $\sigma_{Y,t} \in \mathbb{R}$ is the diffusion of the variable and where sentiment $\omega_t \in \mathbb{R}$ is given by

$$\omega_t = \int_0^t e^{-\delta(t-s)}dX_s \ . \tag{39}$$

Proof. See the Appendix. □

This proposition shows how the more general expectations map into a forecast operator for disturbances and an Ito process for sentiment. This specification thus allows us to accommodate alternative scenarios in which agents use past portfolio returns to form expectations about future investment returns, as in Barberis et al. (2015).

In what follows, we restrict attention to an intuitive case in which agents rely on past forecast errors to form expectations about future returns. Formally, let $dX_s = \frac{dR_{1,s}-\hat{E}_s[dR_{1,s}]}{Std_s[dR_{1,s}]}$ be a process for risk-adjusted forecast errors (RAFE) and let $dY_t = dR_{1,t}$ be given by the return of the productive technology. Then, applying results from Proposition 8, one gets the following forecast operator for disturbances and the following law of motion of sentiment.

Corollary 2. *If $dX_s = \frac{dR_{1,s}-\hat{E}_s[dR_{1,s}]}{Std_s[dR_{1,s}]}$ and $dY_t = dR_{1,t}$, then*

$$\hat{E}_t\left[dZ_t\right] = \hat{\mu}\omega_t dt \ , \tag{40}$$

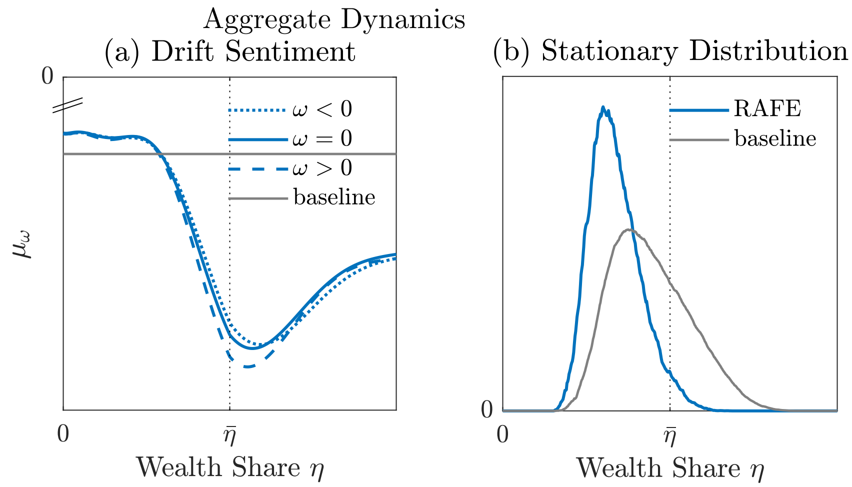
and

$$d\omega_t = \left(-\delta + \frac{\hat{\mu}}{\sigma_{q,t}+\sigma_1}\right)\omega_t dt + dZ_t \ . \tag{41}$$

Proof. See the Appendix. □

The operator in this corollary is the same as in the baseline specification because forecast errors are deflated by risk. The law of motion of sentiment is also mean-reverting as in the baseline, but only under the RAFE specification is the drift term of sentiment endogenous. In particular, the drift is inversely related to the volatility of the asset price. This relationship creates an additional interaction between diagnostic expectations and financial frictions. Specifically, everything else being the same, the greater the volatility of the asset price, the more unstable extreme values of sentiment are (in the sense that extreme sentiment tends to revert quicker to the unconditional mean). Numerical simulations in Figure 5 show this relationship further contributes to financial instability. This happens because asset price volatility tends to positively co-move with sentiment and the wealth share (panel a).

Figure 5: RAFE-based Specification for Diagnostic Expectations



Notes: The figure plots the drift of sentiment as a function of the wealth share for different values of sentiment (panel a) and the limiting marginal distribution of the wealth share (panel b). Grey lines refer to the economy with the baseline specification for diagnostic expectations (i.e., section 3.4), whereas blue lines refer to the economy with the specification based on risk-adjusted forecast errors (RAFE). In the RAFE specification, we set $\delta = 7.5$ to match the same correlation between sentiment and the wealth share as in the baseline.

4.2 Non-linearity in Diagnostic Expectation Formation

In the baseline model, expectation operator $\hat{E}_t\left[dZ_t\right]/dt = \hat{\mu}\omega_t$ is linear in sentiment ω_t. Recent empirical studies (i.e., Da, Huang and Jin (2020) and Egan, MacKay and Yang (2020)), however, find extrapolation in expectation formation is non-linear and tends to be stronger following negative asset returns. Let's then incorporate this non-linearity in expectation formation and examine its implications for financial stability. To do so, we consider the following modification to the expectation operator:

$$\hat{E}_t[dZ_t] = \left[(\hat{\mu} - \Delta)\mathbf{1}_{\omega_t<0} + (\hat{\mu} + \Delta)\mathbf{1}_{\omega_t>0}\right]\omega_t dt,$$

where parameter $\Delta \in [-\hat{\mu}, \hat{\mu}]$ indicates non-linear effects of sentiment on expectation formation. Specifically, if $\Delta < 0$, expectations diverge more from the rational benchmark when sentiment is negative, which seems to be the empirically relevant case, whereas the opposite happens if $\Delta > 0$. Accordingly, we refer to the former case as *diagnostic pessimism* and to the latter as *diagnostic optimism*.

The non-linear effects on diagnostic expectation formation influence both the interaction between forecasts errors $-\hat{E}_t[Z_t]$ and aggregate risk $\sigma_{k,t}$ and the sensitivity of the asset price to sentiment. Specifically, under diagnostic pessimism, both channels are weakened for high values of the states relative to the baseline specification, whereas they are amplified for low values. The opposite naturally happens under diagnostic optimism. Seemingly paradoxically then, because of counter-cyclical forecast errors and pro-cyclical aggregate risk, diagnostic pessimism improves financial stability relative to the baseline specification, whereas diagnostic optimism worsens it, as shown in Figure 6.

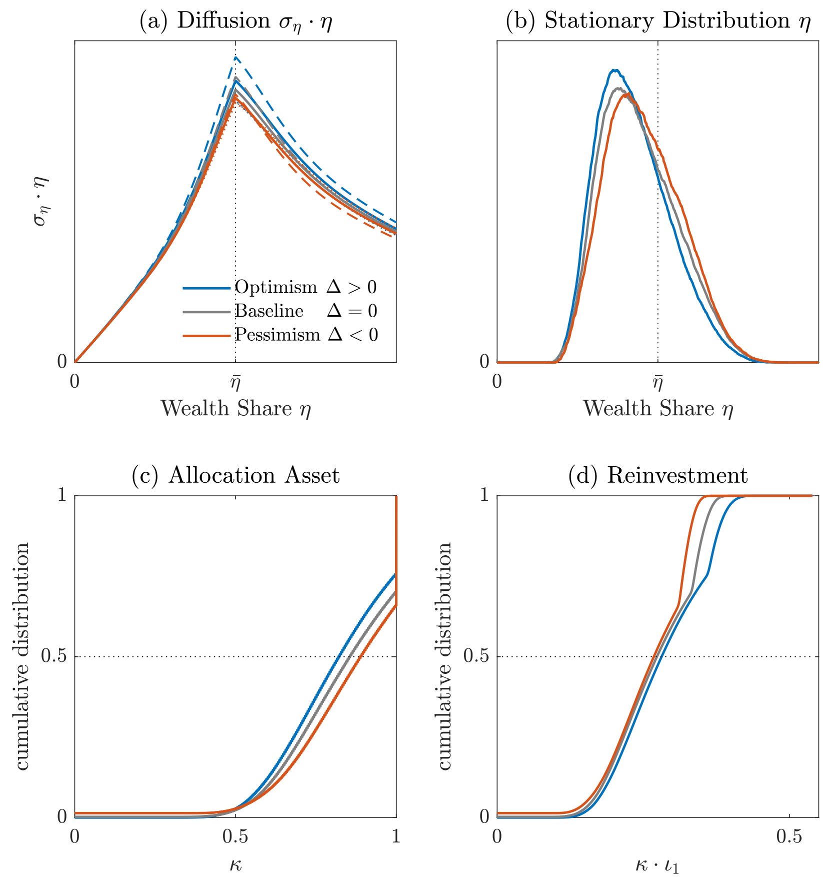
4.3 Endogenous Limit on Leverage

In the baseline model, the collateral constraint creates an exogenous limit on leverage, $\phi_t \leq \lambda$. The literature—notably, Gertler and Kiyotaki (2010) and Gertler and Karadi (2011)—has instead considered a constraint that generates an endogenous leverage limit $\phi_t \leq \nu v_t$, where $\nu \geq 1$ is a parameter and $v_t \geq 1$ is the marginal value of net worth. This constraint is derived from an agency problem, where upon default, financiers lose access to their company, whose value is V_t. In this subsection, we consider this alternative specification and examine its possible implications for financial stability.

An endogenous leverage limit νv_t creates an additional interaction between diagnostic expectations and financial frictions. In particular, everything else being the same, higher sentiment ω_t improves perceived rents $\alpha_{1,t}\phi_t > 0$, increases marginal value v_t, and thus relaxes leverage constraint $\phi_t \leq \nu v_t$. Lower sentiment naturally does the opposite. This interaction then shapes two additional direct effects of sentiment on the allocation of the asset. First, in the financially constrained regime, higher sentiment increases the share of the asset allocated to the productive technology. Second, outside the precautionary regime, higher sentiment reduces the threshold state that separates the two financial regimes.

To study how these effects shape the equilibrium outcome and influence financial stability, we set $\nu = 2.7$, to target for comparability the same average leverage as in the baseline specification. Table 2 compares key equilibrium moments between the two speci-

Figure 6: Non-linear Specification for Diagnostic Expectations



Notes: The figure plots the diffusion of the wealth share (panel a), the limiting marginal distribution of the share (panel b), the limiting cumulative distribution of the allocation of the asset betweeen the technologies (panel c), and the same function of aggregate reinvestment per unit of the asset (panel d). Grey lines refer to the symmetric baseline specification with diagnostic expectations, whereas blue and red lines report outcomes for diagnostic optimism and pessimism, respectively.

fications. The main takeaway is that endogenous leverage limit νv_t does not significantly change the equilibrium outcome or affect financial stability, mainly because rents $\alpha_{1,t}\phi_t$ are weakly correlated with sentiment ω_t and because the average value of the rents is low. These two features combined render value v_t not sensitive to sentiment, as measured by average elasticity $E[\varepsilon_{v,\omega}] = -0.4\%$, where $\varepsilon_{v,\omega} \equiv \frac{\partial v}{\partial \omega}\frac{\omega}{v}$. These results thus reveal the amplifying effects of diagnostic expectations on financial instability primarily operate through sentiment-based fluctuations in asset prices, not through sentiment-driven variations in leverage limits.

Table 2: Exogenous vs. Endogenous Collateral Constraint

Average Moment	Exogenous $\phi_t \leq \lambda$		Endogenous $\phi_t \leq \nu v_t$	
	$\hat{\mu} = 0$	$\hat{\mu} > 0$	$\hat{\mu} = 0$	$\hat{\mu} > 0$
Allocation Asset κ	0.879	0.830	0.864	0.820
Reinvestment $\kappa \cdot \iota_1$	0.255	0.272	0.255	0.254
Price q	0.543	0.568	0.541	0.554
Leverage ϕ	3.765	3.828	3.704	3.695
Wealth share η	0.238	0.221	0.240	0.226
Elasticity value $\varepsilon_{v,\omega}$	0	-0.4%	0	-0.4%
Precautionary regime	0	0.002	0	0.003
Constrained regime	0.583	0.689	0.801	0.861
Unconstrained regime	0.417	0.309	0.199	0.046

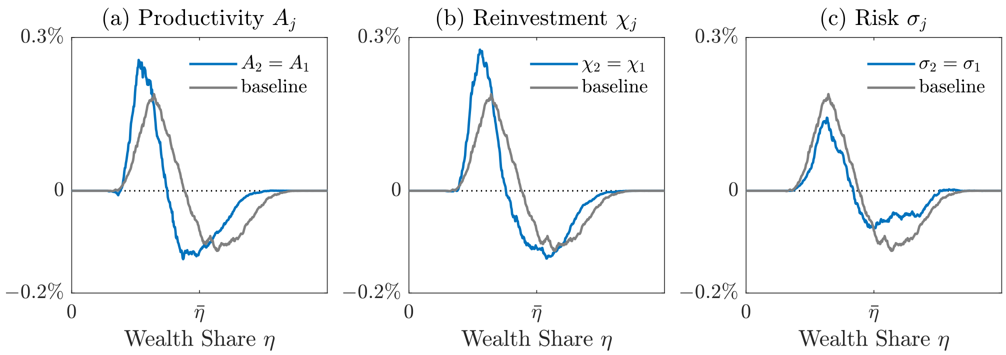
Notes: The table reports selected unconditional average moments for the baseline specification with exogenous collateral constraint and for the specification with endogenous constraint as in Gertler and Kiyotaki (2010) and Gertler and Karadi (2011)

4.4 Differences between Technologies

Lastly, the baseline model proposes three sources of differences between the technologies: productivity A_j, reinvestment opportunity $\mathcal{I}_j(\iota_j)$, and exposure to capital-quality risk σ_j. Let's then investigate the individual contribution of each source to the amplifying effects of diagnostic expectations on financial instability. To do so, we derive the equilibrium for three alternative specifications that each shut down one of these differences (Figure 7).

Closing either the productivity gap or the reinvestment gap, that is, $A_2 = A_1$ or $\chi_2 = \chi_1$, does not substantially affect the contribution of diagnostic expectations to financial instability. By contrast, closing the gap in risk exposure, that is, $\sigma_2 = \sigma_1$, eliminates most of it. These results thus further highlight the importance of pro-cyclical aggregate

Figure 7: Differences between Technologies



Notes: The figure plots the difference between the limiting marginal distributions of the wealth share under diagnostic expectations and rational expectations. Grey lines refer to the economy with the baseline specification (i.e., section 3.4), whereas blue lines refer to the economies with the alternative specifications presented in section 4.4.

risk—in conjunction with counter-cyclical forecast errors—for the positive contribution of diagnostic expectations to financial instability.

5 Normative Analysis

The positive analysis highlighted how diagnostic expectations intensify financial instability relative to the RE benchmark. We now study the normative implications of the additional instability. To do so, we characterize a socially optimal allocation and compare its properties with the equilibrium allocations presented in section 3.

5.1 The Socially Optimal Allocation

To characterize this allocation, we consider a social welfare problem that is consistent with the incentive constraints of private agents and satisfies the resource constraints of the competitive equilibrium. In addition, for tractability, we impose the following three restrictions on the problem. First, the problem has a Markov structure with the same state variables as in the competitive equilibrium. Second, the social planner who determines the allocation does not have commitment, meaning social welfare is maximized state by state, taking the future paths of the socially optimal allocation as given. Lastly, the planner evaluates social welfare using an expectation weight $\tilde{\mu}$ that lies in the interval

$[0, \hat{\mu}]$. Intuitively, under the first two restrictions, the planner can solve any coordination problem among private agents that may arise in a (Markov) competitive equilibrium within a time instant or a short time period, but she cannot solve coordination problems that may arise at more distant time horizons. The last restriction imposes reasonable expectations on the planner, in the sense that those expectations are a convex combination of the rational and the diagnostic.

The below definition formally specifies the socially optimal allocation.

Definition 2. *The socially optimal allocation is the solution to the optimization problem in the following dynamic program:*

$$\begin{aligned} r\tilde{u} \;=\; & \max_{\{\iota_1,\kappa\}} \Big\{ \kappa \left\{ A_1 - \iota_1 + \left[\mathcal{I}_1(\iota_1) + \sigma_1 \tilde{\mu}\omega \right] \tilde{u} \right\} + (1-\kappa) A_2 + \frac{\partial \tilde{u}}{\partial \omega} \left(-\delta\omega + \tilde{\mu}\omega + \kappa\sigma_1 \right) \\ & + \frac{\partial \tilde{u}}{\partial \eta} \left(\mu_\eta \eta + \sigma_\eta \eta \tilde{\mu}\omega + \sigma_\eta \eta \kappa \sigma_1 \right) + \frac{1}{2} \frac{\partial^2 \tilde{u}}{(\partial\omega)^2} + \frac{\partial^2 \tilde{u}}{\partial\omega\partial\eta} \sigma_\eta \eta + \frac{1}{2} \frac{\partial^2 \tilde{u}}{(\partial\eta)^2} \left(\sigma_\eta \eta \right)^2 \Big\}, \end{aligned} \tag{42}$$

with

$$\iota_1 \in [0, A_1] \;\; \text{and} \;\; \kappa \in [0, \min\{\lambda\eta, 1\}], \tag{43}$$

where drift μ_η and diffusion σ_η are given by

$$\begin{aligned} \mu_\eta \;=\; & \frac{1}{1 - (\frac{\kappa}{\eta} - 1)\varepsilon_{q,\eta}} \Big\{ \left[\frac{A_1 - \iota_1}{q} + \mathcal{I}_1(\iota_1) + \sigma_q \sigma_1 \right] \frac{\kappa}{\eta} - \kappa \mathcal{I}_1(\iota_1) - \sigma_q \kappa \sigma_1 + \\ & + \frac{1}{q} \left[-\frac{\partial q}{\partial \omega} \delta\omega + \frac{1}{2} \frac{\partial^2 q}{(\partial\omega)^2} + \frac{\partial^2 q}{\partial\omega\partial\eta} \sigma_\eta \eta + \frac{1}{2} \frac{\partial^2 q}{(\partial\eta)^2} (\sigma_\eta \eta)^2 - rq \right] \left(\frac{\kappa}{\eta} - 1 \right) + \\ & + (\sigma_q + \kappa\sigma_1) \left[(\sigma_q + \kappa\sigma_1) - \frac{\kappa}{\eta} (\sigma_q + \sigma_1) \right] - \left(\theta - \frac{\gamma}{\eta} \right) \Big\}, \end{aligned} \tag{44}$$

$$\sigma_\eta = \frac{\frac{\kappa}{\eta} \left(\frac{1}{q} \frac{\partial q}{\partial \omega} + \sigma_1 \right) - \left(\frac{1}{q} \frac{\partial q}{\partial \omega} + \kappa \sigma_1 \right)}{1 - \left(\frac{\kappa}{\eta} - 1 \right) \varepsilon_{q,\eta}}, \tag{45}$$

respectively, with

$$\sigma_q = \frac{\left(\frac{\kappa}{\eta} - \kappa \right) \varepsilon_{q,\eta} \sigma_1 + \frac{1}{q} \frac{\partial q}{\partial \omega}}{1 - \left(\frac{\kappa}{\eta} - 1 \right) \varepsilon_{q,\eta}}. \tag{46}$$

Mapping q is consistent with the following three mutually exclusive relationships:

$$
\begin{array}{lllll}
\textit{Relationship \#1:} & \lambda\eta \geq 1 \ , & \kappa = 1 \ , & \alpha_1 = 0 \ , & \alpha_2 < 0 \ ; \\
\textit{Relationship \#2:} & \lambda\eta \geq 1 \ , & \kappa \in [0,1) \ , & & \alpha_2 = 0 \ ; \\
\textit{Relationship \#3:} & \lambda\eta < 1 \ , & \kappa \in [0,\lambda\eta] \ , & & \alpha_2 = 0 \ ;
\end{array} \quad , \tag{47}
$$

with

$$
\alpha_1 \equiv \frac{1}{dt}\hat{E}\left[dR_1|\cdot\right] - r + \left(\sigma_q + \sigma_1\right)\sigma_v \ , \quad \alpha_2 \equiv \frac{1}{dt}\hat{E}\left[dR_2|\cdot\right] - r \ . \tag{48}
$$

Mapping v satisfies

$$
0 = \alpha_1 \frac{\kappa}{\eta} + \mu_v + \hat{\mu}\omega\sigma_v + \frac{\theta}{v} - \theta \ . \tag{49}
$$

In the above notation, mapping $\tilde{u} \geq 0$ is the present discounted value of consumption per unit of the asset under expectation weight $\tilde{\mu} \in [0, \hat{\mu}]$. Mappings $\{q, v\}$ and objects $\{\mu_\eta, \sigma_\eta, \sigma_q, \mu_v, \sigma_v, \alpha_1, \alpha_2\}$ are "shadow" variables. Namely, in a decentralization of the socially optimal allocation as a competitive equilibrium, they would map into the variables that their notation denotes.[8]

In the optimization problem, controls $\{\iota_1, \kappa\}$ are set state by state, to maximize the utility of households—as measured by the expectation weight of the planner. Restrictions (43) follow from the resource constraints and the collateral constraint. Mappings $\{\tilde{u}, q, v\}$ and their partial derivatives with respect to the states are taken as given. The reason is that those objects are inherently determined by the future paths of the allocation and thus are not influenced by the current allocation. By contrast, in the dynamic program, the mappings and their derivatives are endogenous. Expression (47) for mapping q determines three mutually exclusive relationships between share κ and private valuations α_1 and α_2. These relationships feature two differences relative to those in expression (20). First, the planner can set $\kappa > 0$ while $\alpha_1 < 0$, which means the asset can be allocated to the productive technology even when the expected excess return of that technology is negative according to private valuations. Second, the planner can also set $\kappa < \lambda\eta$ while $\alpha_1 > 0$, which implies the collateral constraint can be slack even when the expected excess return of the productive technology is positive according to private valuations.

[8]The socially optimal allocation can be decentralized as a competitive equilibrium using Pigouvian taxes or subsidies on reinvestment and the productive technology.

Proposition 9. *The socially optimal reinvestment rate solves*

$$\mathcal{I}_1'(\iota_1) = \frac{1 + \frac{1}{1-(\phi-1)\varepsilon_{q,\eta}} \frac{1}{q} \frac{\partial \tilde{u}}{\partial \eta}}{\tilde{u} + \frac{1-\eta}{1-(\phi-1)\varepsilon_{q,\eta}} \frac{\partial \tilde{u}}{\partial \eta}} \ . \tag{50}$$

The socially optimal share κ is the candidate value that maximizes the RHS in (42). The candidate values are $\kappa = 0$, $\kappa = \min\{\lambda\eta, 1\}$, and any interior $\kappa \in (0, \min\{\lambda\eta, 1\})$ that solves

$$\begin{aligned} 0 \quad = \quad & \left[\frac{A_1 - \iota_1 - A_2}{\tilde{u}} + \mathcal{I}_1(\iota_1) + (\sigma_{\tilde{u}} + \tilde{\mu}\omega)\,\sigma_1\right] + \varepsilon_{\tilde{u},\eta}\left[\frac{\partial \mu_\eta}{\partial \kappa} + (\tilde{\mu}\omega + \kappa\sigma_1)\frac{\partial \sigma_\eta}{\partial \kappa}\right] + \\ & + \frac{1}{\tilde{u}}\left(\frac{\partial^2 \tilde{u}}{(\partial\eta)^2}\sigma_\eta \eta + \frac{\partial^2 \tilde{u}}{\partial\eta\partial\omega}\right)\frac{\partial \sigma_\eta}{\partial \kappa}\eta \ , \end{aligned} \tag{51}$$

where $\frac{\partial \mu_\eta}{\partial \kappa}$ and $\frac{\partial \sigma_\eta}{\partial \kappa}$ are the partial derivatives of μ_η and σ_η with respect to κ, respectively.

Proof. See the Appendix for details. □

These optimality conditions follow from the first-order derivatives in (42) with respect to ι_1 and κ. Comparing these conditions with their counterparts of the competitive equilibrium—that is, (16) and (20)—highlights the following two differences between the socially optimal and the equilibrium allocation. First, the social planner internalizes the collective contribution of individual decisions to aggregate variables and aggregate dynamics, whereas individual agents in the competitive equilibrium do not. In the conditions for reinvestment, for instance, this difference is reflected by the second terms in the numerator and the denominator of the RHS in (50), which are not present in (16). Second, the social planner inherently bases her decisions on the social value of the asset as perceived by her own expectations—namely, value $\tilde{u}$—whereas private agents in the competitive equilibrium take decisions based on asset price q.

All in all, the optimality conditions in the proposition together with definition 2 analytically characterize the socially optimal allocation and its associated mappings $\{\tilde{u}, v, q\}$. The characterization can be reduced to a system of second-order PDEs for the mappings in states $\{\omega, \eta\}$. We solve the PDEs numerically using spectral methods when needed.

Proposition 10. *The socially optimal allocation and its associated mappings $\{\tilde{u}, v, q\}$ can be analytically characterized by a system of second-order PDEs for the mappings in the state $\{\omega, \eta\}$.*

Proof. See the Appendix. □

5.2 The Equilibrium Outcome under the Socially Optimal Allocation

We now contrast the socially optimal allocation with the equilibrium allocation. We report results gradually as in section 3, but we omit the first economy, because its equilibrium is already first-best. To ease exposition, we only consider planners with expectation weights $\tilde{\mu} = 0$ and $\tilde{\mu} = \hat{\mu}$, who can be regarded as *paternalistic* and *benevolent*, respectively. Planners with intermediate degrees of diagnostic expectations naturally favor allocations in between the ones those two planners prefer.

5.2.1 Diagnostic Expectations but No Financial Frictions

Consider first the economy in subsection 3.2. The below proposition describes the socially optimal allocation.

Proposition 11. *In the economy with diagnostic expectations and without financial frictions, the socially optimal allocation is first-best efficient according to the expectation weight of the planner. If the planner is benevolent, the socially optimal allocation is the same as the equilibrium allocation. If the planner is paternalistic, the socially optimal allocation is the same as the equilibrium allocation of the economy presented in subsection 3.1.*

Proof. See the Appendix. □

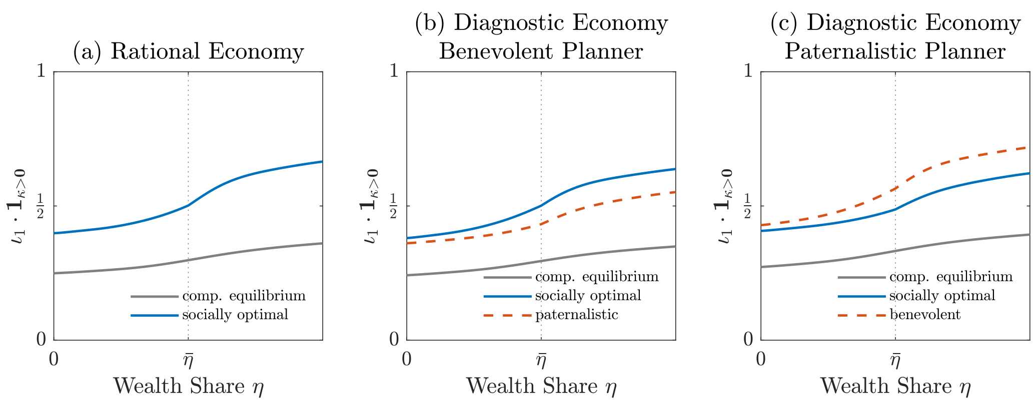
In this economy, the planner can implement the allocation that, according to her own expectations, attains the first-best. This is because the economy has no frictions. Naturally, for a benevolent planner, the equilibrium allocation with sentiment-driven economic cycles is already first-best efficient. By contrast, for a paternalistic planner, the equilibrium allocation of the economy with rational expectations is first-best efficient.

Under paternalism, the socially optimal allocation is thus insulated from sentiment. Put differently, the aggregate quantity of the asset is allocated to the productive technology even when sentiment $\omega < \bar{\omega}$ is low, and reinvestment is invariant to fluctuations in sentiment even when sentiment $\omega \geq \bar{\omega}$ is high. Therefore, the socially optimal allocation eliminates fluctuations in aggregate output, reinvestment, and economic growth rates, but relative to the competitive equilibrium, it intensifies fluctuations in the asset price. The latter happens because the sensitivity of the price to sentiment is larger if the asset is always exposed to the shock—as under the socially optimal allocation—than what it is if the asset is exposed only when sentiment is high—as under the equilibrium allocation.

5.2.2 Financial Frictions but Rational Expectations

Consider now the economy in subsection 3.3. Because of financial frictions, the planner cannot implement the allocation that, according to her own expectations, attains the first-best. Notwithstanding, in general, the planner can improve social welfare over the competitive equilibrium. This is because the collateral constraint together with non-contingent debt depress the asset price—and thus also reinvestment—excessively relative to what is socially desirable (Figure 8).

FIGURE 8: SOCIALLY OPTIMAL REINVESTMENT RATES



Notes: The figure plots the reinvestment rates of the competitive equilibrium allocation (grey lines) and the socially optimal allocation (blue lines) for the economy with rational expectations (panel a), the economy with diagnostic expectations and a benevolent planner (panel b), and the economy with diagnostic expectations and a paternalistic planner (panel c). For the latter two economies, the socially optimal reinvestment rates of the paternalistic and the benevolent planner, respectively, are also plotted. All of the reinvestment rates are deflated by the first-best value in the corresponding economy.

Specifically, relative to the competitive equilibrium, the socially optimal allocation features higher reinvestment throughout the cycle. These higher rates speeds up the average recapitalization of financiers (i.e., drift $\mu_\eta \eta$), which increases the relative frequency of larger wealth shares in the stationary distribution. However, at least under the baseline parameter values, the socially optimal allocation does not alter the allocation of the asset between the technologies. That is, $\kappa = \min\{\lambda\eta, 1\}$ is optimal for the planner.[9] The latter result is

[9] Brunnermeier and Sannikov (2014) also find null to negligible welfare gains from altering the asset

modified, nonetheless, once we allow for diagnostic expectations.

5.2.3 Diagnostic Expectations and Financial Frictions

Finally, consider the economy with both diagnostic expectations and financial frictions presented in section 2. In this economy, the socially optimal allocation shares the key properties of its counterparts in subsections 5.2.1 and 5.2.2. Notably, as in subsection 5.2.1, the allocation depends on the expectations of the planner. Moreover, as in subsection 5.2.2, the allocation in general improves social welfare over the competitive equilibrium, but without attaining the first best.

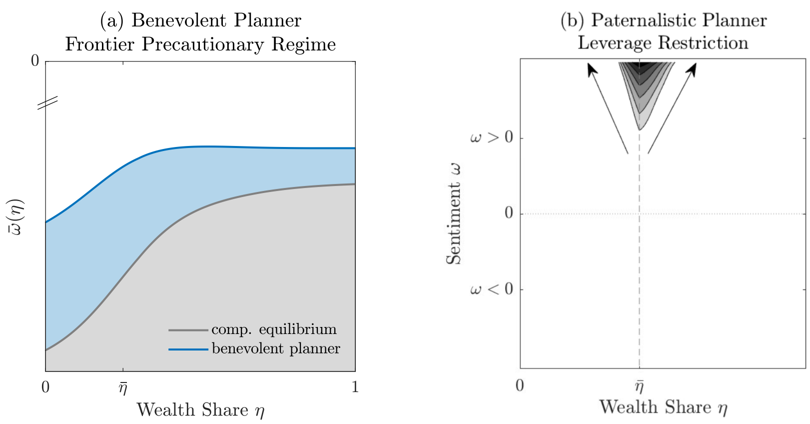
These two properties are reflected in the socially optimal reinvestment rate (Figure 8). Relative to the competitive equilibrium, reinvestment is higher under both paternalism and benevolence, but in neither case does reinvestment attain the first-best. When the wealth share is sufficiently high, moreover, reinvestment is higher under benevolence than under paternalism. The reason is that a benevolent planner perceives sentiment as fundamental information for setting the allocation, whereas a paternalistic planner seeks to insulate the allocation from sentiment.

The interplay between diagnostic expectations and financial frictions creates additional considerations for the socially optimal allocation. Specifically, relative to the competitive equilibrium, the share of the asset allocated to the productive technology is lower in some regions of the state space (Figure 9). These regions depend, in turn, on whether the planner is benevolent or paternalistic. In particular, if the planner is benevolent, the share is lower when sentiment is moderately low. Put formally, the precautionary regime expands. The reason is that in that region, the planner believes allocating the asset to the productive technology excessively deteriorates the expected recovery rate of the wealth share. If the planner is paternalistic, by contrast, the share is lower when financial amplification effects peak (i.e., around threshold state $\bar{\eta}$ and when sentiment is moderately high). This happens because in that region, the planner is particularly concerned with the stronger financial amplification effects arising from the interactions between diagnostic expectations and financial frictions.

Overall, the interplay between financial frictions and diagnostic beliefs motivates addi-

allocation relative to the competitive equilibrium in a similar economy to the one in this subsection. By contrast, Van der Ghote (2021) finds large welfare gains, but his economy features concavity in preferences over consumption, which creates gains from reducing consumption volatility and smoothing consumption over time.

Figure 9: Socially Optimal Asset Allocation



Notes: This figure illustrates socially optimal restrictions of the allocation of the asset to the productive technology relative to the competitive equilibrium presented in section 3.4. Panel (a) reports the occurrence of the precautionary regime for the competitive equilibrium (grey area) and the socially optimal allocation when the planner is benevolent (blue area). Panel (b) reports restrictions implemented by a paternalistic planner in the non-precautionary regime. A darker shade means the planner imposes stronger restrictions on the share κ relative to the upper bound $\min\{\lambda\eta, 1\}$ that applies in a competitive equilibrium. The white color means no reduction in the share below $\kappa = \min\{\lambda\eta, 1\}$.

tional restrictions on financial risk-taking relative to an economy with rational expectations. The nature of the restrictions depends on the degree of diagnosticity in the expectations of the planner.

6 Conclusion

This paper examines the joint implications of diagnostic expectations and external financing frictions for financial stability and financial regulation. We find that interactions between these two elements exacerbate instability in financial markets relative to the rational expectations benchmark. As a consequence, the socially optimal regulation imposes additional restrictions on financial risk-taking relative to those derived in an economy under rational expectations, regardless of the degree of diagnosticity in the expectations of the planner. This analysis has considered an expectation deviation from the FIRE benchmark. Investigating the effects of imperfect information on financial stability and financial regulation remains for future research.

References

Akerlof, George A, and Robert J Shiller. 2010. *Animal spirits: How human psychology drives the economy, and why it matters for global capitalism.* Princeton university press.

Barberis, Nicholas, Robin Greenwood, Lawrence Jin, and Andrei Shleifer. 2015. "X-CAPM: An extrapolative capital asset pricing model." *Journal of financial economics*, 115(1): 1–24.

Bianchi, Francesco, Cosmin L Ilut, and Hikaru Saijo. 2021. "Diagnostic Business Cycles." National Bureau of Economic Research Working Paper 28604.

Bordalo, Pedro, Nicola Gennaioli, and Andrei Shleifer. 2018. "Diagnostic expectations and credit cycles." *The Journal of Finance*, 73(1): 199–227.

Bordalo, Pedro, Nicola Gennaioli, Rafael La Porta, and Andrei Shleifer. 2019. "Diagnostic expectations and stock returns." *The Journal of Finance*, 74(6): 2839–2874.

Bordalo, Pedro, Nicola Gennaioli, Yueran Ma, and Andrei Shleifer. 2020. "Overreaction in macroeconomic expectations." *American Economic Review*, 110(9): 2748–82.

Brunnermeier, Markus K., and Yuliy Sannikov. 2014. "A macroeconomic model with a financial sector." *American Economic Review*, 104(2): 379–421.

Dávila, Eduardo, and Ansgar Walther. 2021. "Prudential policy with distorted beliefs." National Bureau of Economic Research.

Da, Zhi, Xing Huang, and Lawrence J. Jin. 2020. "Extrapolative beliefs in the cross-section: What can we learn from the crowds?" *Journal of Financial Economics.*

Egan, Mark L, Alexander MacKay, and Hanbin Yang. 2020. "Recovering Investor Expectations from Demand for Index Funds." National Bureau of Economic Research Working Paper 26608.

Farhi, Emmanuel, and Iván Werning. 2020. "Taming a Minsky Cycle." *Working paper. Harvard University.*

Fontanier, Paul. 2022. "Optimal Policy for Behavioral Financial Crises." *Working paper. Yale University.*

Gertler, Mark, and Nobuhiro Kiyotaki. 2010. "Financial intermediation and credit policy in business cycle analysis." In *Handbook of Monetary Economics.* Vol. 3, 547–599. Elsevier.

Gertler, Mark, and Peter Karadi. 2011. "A model of unconventional monetary policy." *Journal of Monetary Economics*, 58(1): 17–34.

Greenwood, Robin, Samuel G Hanson, Andrei Shleifer, and Jakob Ahm Sørensen. 2022. "Predictable financial crises." *The Journal of Finance*, 77(2): 863–921.

He, Zhiguo, and Arvind Krishnamurthy. 2019. "A Macroeconomic Framework for Quantifying Systemic Risk." *American Economic Journal: Macroeconomics*, 11(4): 1–37.

Kindleberger, Charles P. 1978. *Manias, panics, and crashes: A history of financial crises.* Palgrave Macmillan.

Krishnamurthy, Arvind, and Wenhao Li. 2020. "Dissecting mechanisms of financial crises: Intermediation and sentiment." National Bureau of Economic Research.

L'Huillier, Jean-Paul, Gregory Phelan, and Hunter Wieman. 2022. "Technology shocks and predictable Minsky cycles." *Available at SSRN 4298974*.

L'Huillier, Jean-Paul, Sanjay R Singh, and Donghoon Yoo. 2021. "Incorporating diagnostic expectations into the New Keynesian framework." *Available at SSRN 3910318*.

Maggiori, Matteo. 2017. "Financial intermediation, international risk sharing, and reserve currencies." *American Economic Review*, 107(10): 3038–71.

Maxted, Peter. 2023. "A Macro-Finance Model with Sentiment." *The Review of Economic Studies*.

Minsky, Hyman P. 1977. "The financial instability hypothesis: An interpretation of Keynes and an alternative to "standard" theory." *Challenge*, 20(1): 20–27.

Phelan, Gregory. 2016. "Financial intermediation, leverage, and macroeconomic instability." *American Economic Journal: Macroeconomics*, 8(4): 199–224.

Tversky, Amos, and Daniel Kahneman. 1983. "Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment." *Psychological review*, 90(4): 293.

Van der Ghote, Alejandro. 2021. "Interactions and coordination between monetary and macroprudential policies." *American Economic Journal: Macroeconomics*, 13(1): 1–34.

Online Appendix

Financial Stability and Financial Regulation under Diagnostic Expectations

The Appendix has two parts. The first part proves the propositions and corollaries stated in the text as well as derives the planner's problem in Definition 2. The second part describes the numerical method used to solve the PDEs.

1 Proofs of Propositions and Corollaries

Proposition 1 *At any given time t, households are indifferent among any consumption rate c_t. Moreover, they choose reinvestment rate $i_{2,t}$ and asset holding $k_{2,t}$ as follows:*

$$\mathcal{I}_2'(\iota_{2,t}) = \frac{1}{q_t} , \tag{1}$$

and

$$q_t k_{2,t} \left[\begin{array}{ll} = 0 & \text{if } \alpha_{2,t} < 0 \\ \in [0, +\infty) & \text{if } \alpha_{2,t} = 0 \end{array} \right. , \tag{2}$$

where the estimated risk-adjusted excess return to allocate the asset to the unproductive technology over holding debt, that is, $\alpha_{2,t} \leq 0$, is given by

$$\alpha_{2,t} \equiv \frac{1}{dt} \hat{E}_t \left[dR_{2,t} \right] - r \leq 0 . \tag{3}$$

Proof. Households maximize the present discounted value of consumption

$$B_t \equiv \max_{c_t, \iota_{2,t}, k_{2,t} \geq 0} \hat{E}_t \int_t^{+\infty} e^{-r(s-t)} c_s ds \ , \tag{4}$$

subject to the law of motion of wealth,

$$dw_s = dR_{2,s} q_s k_{2,s} + r(w_s - q_s k_{2,s}) ds - c_s ds + \tau_s ds . \tag{5}$$

Let's postulate that

$$B_t = w_t - e^{rt} \int_0^t e^{-rs} \tau_s ds . \tag{6}$$

Substituting (4) into (6) and rearranging, one gets the following condition:

$$e^{-rt}w_t + \int_0^t e^{-rs}\left(c_s - \tau_s\right) ds = \hat{E}_t \int_0^{+\infty} e^{-rs} c_s ds \ . \tag{7}$$

The RHS in this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. From applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rw_t = \max_{c_t, \iota_{2,t}, k_{2,t} \geq 0} \left\{ c_t - \tau_t + \left[\frac{1}{dt}\hat{E}_t\left[dR_{2,t}\right] - r\right] q_t k_{2,t} + rw_t - c_t + \tau_t \right\} , \tag{8}$$

Note that any c_t is optimal. The optimal $\iota_{2,t}$ and $k_{2,t}$ are

$$\mathcal{I}_2'(\iota_{2,t}) = \frac{1}{q_t} , \tag{9}$$

and

$$k_{2,t} \left[\begin{array}{ll} = +\infty & \text{if } \frac{1}{dt}\hat{E}_t\left[dR_{2,t}\right] - r > 0 \\ \in [0, +\infty) & \text{if } \frac{1}{dt}\hat{E}_t\left[dR_{2,t}\right] - r = 0 \\ = 0 & \text{if } \frac{1}{dt}\hat{E}_t\left[dR_{2,t}\right] - r < 0 \end{array} \right. . \tag{10}$$

The HJB equation thus reduces to

$$0 = \left[\frac{1}{dt}\hat{E}_t\left[dR_{2,t}\right] - r\right] q_t k_{2,t} , \tag{11}$$

where $\iota_{2,t}$ and $k_{2,t}$ are given by (9) and (10), respectively—which under restriction $\alpha_{2,t} \leq 0$, verifies the postulate. ■

Proposition 2 *At any given time t, financiers choose reinvestment rate $\iota_{1,t}$ and asset holding $k_{1,t}$ as follows:*

$$\mathcal{I}_1'(\iota_{1,t}) = \frac{1}{q_t} , \tag{12}$$

and

$$\frac{q_t k_{1,t}}{n_t} \left[\begin{array}{ll} = 0 & \textit{if } \alpha_{1,t} < 0 \\ \in [0, \lambda] & \textit{if } \alpha_{1,t} = 0 \\ = \lambda & \textit{if } \alpha_{1,t} > 0 \end{array} \right. , \tag{13}$$

where the estimated risk-adjusted excess return to allocate the asset to the productive tech-

nology over holding debt, namely, $\alpha_{1,t} \in \mathbb{R}$, *is given by*

$$\alpha_{1,t} \equiv \frac{1}{dt}\hat{E}_t\left[dR_{1,t}\right] - r + \left(\sigma_{q,t} + \sigma_1\right)\sigma_{v,t} \ . \tag{14}$$

The marginal value of net worth, v_t, *satisfies*

$$0 = \alpha_{1,t}\frac{q_t k_{1,t}}{n_t} + \mu_{v,t} + \hat{\mu}\omega_t\sigma_{v,t} + \frac{\theta}{v_t} - \theta \ . \tag{15}$$

Proof. Financiers maximize the present discount value of dividend payouts

$$V_t \equiv \max_{\iota_{1,s}, k_{1,s} \geq 0} \hat{E}_t \int_t^\infty \theta e^{-(r+\theta)(s-t)} n_s ds \ , \tag{16}$$

subject to the law of motion of net worth,

$$dn_s = dR_{1,s} q_s k_{1,s} - r(q_s k_{1,s} - n_s)ds \ , \tag{17}$$

and collateral constraint $q_s k_{1,s} \leq \lambda n_s$, with $n_s \geq 0$.

Note that value $V_t = v_t n_t$ satisfies

$$e^{-(r+\theta)t} v_t n_t + \int_0^t \theta e^{-(r+\theta)s} n_s ds = \hat{E}_t \int_0^\infty \theta e^{-(r+\theta)s} n_s ds \ . \tag{18}$$

The RHS of this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. Applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following HJB equation:

$$\begin{aligned}(r+\theta)\, v_t \quad &= \quad \max_{\iota_{1,t}, \phi_t \geq 0} \left\{\theta + \left[\mu_{v,t} + \hat{\mu}\omega_t\sigma_{v,t} + \frac{1}{dt}\hat{E}_t\left[dR_{1,t}\right]\phi_t - r\left(\phi_t - 1\right) + \sigma_{v,t}\left(\sigma_{q,t} + \sigma\right)\phi_t\right] v_t\right\} \\ &\quad\quad \text{subject to} : \phi_t \leq \lambda \ .\end{aligned} \tag{19}$$

where $\phi_t \equiv q_t k_{1,t}/n_t$. The optimal $\iota_{1,t}$ and ϕ_t are

$$\mathcal{I}_1'\left(\iota_{1,t}\right) = \frac{1}{q_t} \ . \tag{20}$$

and

$$\phi_t \left[\begin{array}{ll} = \lambda & \text{if } \alpha_{1,t} > 0 \\ \in [0, \lambda] & \text{if } \alpha_{1,t} = 0 \\ = 0 & \text{if } \alpha_{1,t} < 0 \end{array} \right. . \tag{21}$$

Substituting (20) and (21) into (19), one gets the following equation:

$$\alpha_{1,t}\phi_t + \mu_{v,t} + \hat{\mu}\omega_t\sigma_{v,t} + \frac{\theta}{v_t} - \theta = 0 . \tag{22}$$

■

Proposition 3 *Let $\eta_t \equiv n_t/q_t k_t \in [0,1]$ be the aggregate net worth of financiers as a share of total wealth and let $\kappa_t \equiv k_{1,t}/k_t \in [0,1]$ be the aggregate share of the asset allocated to the productive technology. Then, the equilibrium outcome can be partitioned into the following three regimes,*

$$\begin{array}{llll} \text{1. Financially unconstrained regime:} & \kappa_t = 1 \leq \lambda\eta_t , & \alpha_{1,t} = 0 , & \alpha_{2,t} < 0 ; \\ \text{2. Financially constrained regime:} & \kappa_t = \lambda\eta_t \in [0,1] , & \alpha_{1,t} > 0 , & \alpha_{2,t} = 0 ; \\ \text{3. Precautionary regime:} & \kappa_t = 0 , & \alpha_{1,t} < 0 , & \alpha_{2,t} = 0 ; \end{array} \tag{23}$$

The equilibrium allocation can be summarized as $\{\iota_{1,t}, \iota_{2,t}, \kappa_t\}$, and can be characterized by $\{(1), (3), (12), (14), (22), (23)\}$. The equilibrium utility of households per unit of the asset, namely, $u_t > 0$, satisfies

$$\begin{aligned} 0 \;=\;& \kappa_t \{A_1 - \iota_{1,t} + [\mathcal{I}_1(\iota_{1,t}) + \sigma_1\hat{\mu}\omega_t]\, u_t\} + \\ & + (1-\kappa_t)\{A_2 - \iota_{2,t} + [\mathcal{I}_2(\iota_{2,t}) + \sigma_2\hat{\mu}\omega_t]\, u_t\} + \hat{E}_t[du_t] - r u_t. \end{aligned} \tag{24}$$

Proof. Expressions $\{(1), (3)\}$ characterize the optimality conditions of households and expressions $\{(12), (14), (22)\}$ characterize the optimality conditions of financiers. Expression (23) ensures that market clearing for the asset is consistent with individual optimality. Specifically, if $\alpha_{2,t} < 0$, then $\alpha_{1,t} = 0$ must hold, which requires $\kappa_t = 1$. If $\alpha_{2,t} = 0$, then either $\alpha_{1,t} < 0$ or $\alpha_{1,t} > 0$ must hold. In the first case, $\kappa_t = 0$ is required, while in the second, $\kappa_t = \lambda\eta_t$ is required. Variables $\{\iota_{1,t}, \iota_{2,t}, \kappa_t\}$ together with $c_t/k_t = (A_1 - \iota_{1,t})\,\kappa_t + (A_2 - \iota_{2,t})\,(1-\kappa_t)$ ensure that market clearing for the good holds. Market clearing for debt automatically holds because of Walras Law.

Let U_t be the utility of households under the equilibrium allocation. Then,

$$U_t \equiv \hat{E}_t \int_t^{+\infty} e^{-r(s-t)} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \ . \tag{25}$$

Utility $\hat{U}_t$ can be expressed as

$$\begin{aligned} & e^{-rt}U_t + \int_0^t e^{-rs} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds = \\ & = \hat{E}_t \int_0^{+\infty} e^{-rs} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \ . \end{aligned} \tag{26}$$

The RHS in this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. Applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following HJB equation:

$$0 = \kappa_t \left(A_1 - \iota_{1,t}\right) + \left(1-\kappa_t\right)\left(A_2 - \iota_{2,t}\right) + \hat{E}_t \left[dU_t\right] - rU_t \ . \tag{27}$$

We postulate that $U_t = u_t k_t$, where $u_t > 0$ is an Ito process with disturbance dZ_t. The above HJB equation can then be reduced to

$$\begin{aligned} & 0 = \kappa_t \left\{A_1 - \iota_{1,t} + \left[\mathcal{I}_1(\iota_{1,t}) + \sigma_1 \hat{\mu}\omega_t\right] u_t\right\} + \\ & + \left(1-\kappa_t\right)\left\{A_2 - \iota_{2,t} + \left[\mathcal{I}_2(\iota_{2,t}) + \sigma_2 \hat{\mu}\omega_t\right] \ u_t\right\} + \hat{E}_t \left[du_t\right] - ru_t \ . \end{aligned} \tag{28}$$

■

Proposition 4 *The Markov equilibrium can be analytically characterized as the solution to a system of second-order PDEs for* $\{q, v\}$ *in* $\{\eta, \omega\}$*.*

Proof. This section derives the system of partial differential equations (PDEs) that analytically characterizes the Markov equilibrium. To do so, we consider more general specifications for diagnostic expectations, the collateral constraint, and the production technologies than the baseline specification in Section 2.3 of the paper. Specifically, (i) drift μ_ω can be any function of the state space that does not depend on drift μ_η; (ii) diffusion σ_ω can be any exogenous function of the state space; (iii) expectation weight $\hat{\mu}$ can also be any exogenous function of the state space; (iv) leverage limit λ can be either a parameter or a linear function of value v; and (v) return function $\mathcal{I}_2(\iota_2) \geq 0$ or volatily $\sigma_2 \geq 0$ can be positive. This more general specification suffices to characterize the Markov equilibrium in all of the specifications in the paper. In the remainder of the section, we omit time subscript t.

The equation that determines price q is

$$\begin{array}{ll} \alpha_1 = 0 & \text{if } \kappa = 1 \\ \alpha_2 = 0 & \text{otherwise} \end{array} , \tag{29}$$

or equivalently,

$$\begin{array}{ll} \frac{A_1-\iota_1}{q} + \mu_q + \mathcal{I}_1\left(\iota_1\right) + \left(\sigma_q + \sigma_1\right)\hat{\mu}\omega + \sigma_q\sigma_1 - r + \left(\sigma_q + \sigma_1\right)\sigma_v = 0 & \text{if } \omega \geq \bar{\omega} \text{ and } \eta \geq \bar{\eta} \\ \frac{A_2-\iota_2}{q} + \mu_q + \mathcal{I}_2(\iota_2) + \left(\sigma_q + \sigma_2\right)\hat{\mu}\omega + \sigma_q\sigma_2 - r = 0 & \text{otherwise} \end{array} . \tag{30}$$

The equation that determines value v is

$$\alpha_1\phi + \mu_v + \hat{\mu}\omega\sigma_v + \frac{\theta}{v} - \theta = 0 \; . \tag{31}$$

Ito's Lemma implies that for $x \in \{q, v\}$

$$\mu_x = \frac{1}{x}\left[\frac{\partial x}{\partial\omega}\mu_\omega\omega + \frac{\partial x}{\partial\eta}\mu_\eta\eta + \frac{1}{2}\frac{\partial^2 x}{\left(\partial\omega\right)^2}\left(\sigma_\omega\omega\right)^2 + \frac{\partial^2 x}{\partial\omega\partial\eta}\sigma_\omega\omega\sigma_\eta\eta + \frac{1}{2}\frac{\partial^2 x}{\left(\partial\eta\right)^2}\left(\sigma_\eta\eta\right)^2\right] , \tag{32}$$

$$\sigma_x = \frac{1}{x}\left[\frac{\partial x}{\partial\omega}\sigma_\omega\omega + \frac{\partial x}{\partial\eta}\sigma_\eta\eta\right] , \tag{33}$$

where recall that

$$\begin{aligned} \mu_\eta &= \left[\frac{A_1 - \iota_1}{q} + \mathcal{I}_1(\iota_1) + \sigma_q\sigma_1\right]\phi + \left(\mu_q - r\right)\left(\phi - 1\right) - \mu_k \qquad (34) \\ &\quad -\sigma_q\sigma_k + \left(\sigma_q + \sigma_k\right)\left[\left(\sigma_q + \sigma_k\right) - \phi\left(\sigma_q + \sigma_1\right)\right] - \left(\theta - \frac{\gamma}{\eta}\right) , \\ \sigma_\eta &= \phi\left(\sigma_q + \sigma_1\right) - \left(\sigma_q + \sigma_k\right) , \qquad (35) \end{aligned}$$

with

$$\mu_k = \kappa\mathcal{I}_1(\iota_1) + \left(1 - \kappa\right)\mathcal{I}_2(\iota_2) , \tag{36}$$

$$\sigma_k = \kappa\sigma_1 + \left(1 - \kappa\right)\sigma_2 \; . \tag{37}$$

According to (32) and (33), objects $\{\mu_q, \sigma_q\}$ depend on $\{\mu_\eta, \sigma_\eta\}$, but according to (34) and (35), objects $\{\mu_\eta, \sigma_\eta\}$ in turn depend on $\{\mu_q, \sigma_q\}$. To eliminate this circularity, we substitute (34) and (35) into (32) and (33). We obtain

$$\begin{aligned}\mu_\eta = & \frac{1}{1-(\phi-1)\,\varepsilon_{q,\eta}}\left\{\left[\frac{A_1-\iota_1}{q}+(1-\eta)\left[\mathcal{I}_1(\iota_1)+\sigma_q\sigma_1\right]\right]\phi+\right. \\ & +\left(\mu_\omega\varepsilon_{q,\omega}+\xi_{q,\eta/\omega}-r\right)(\phi-1)-(1-\phi\eta)\left[\mathcal{I}_2(\iota_2)-\sigma_k\sigma_2\right]+ \\ & \left.-\left[\sigma_q+\sigma_1\phi\eta+(1-\phi\eta)\sigma_2\right]\left[(\phi-1)\,\sigma_q+(1-\eta)\,\sigma_1\phi\right]-\left(\theta-\frac{\gamma}{\eta}\right)\right\},\end{aligned} \tag{38}$$

$$\sigma_\eta = \frac{(\phi-1)\,\sigma_\omega\varepsilon_{q,\omega}+(1-\eta)\,\sigma_1\phi-(1-\phi\eta)\sigma_2}{1-(\phi-1)\,\varepsilon_{q,\eta}}\,, \tag{39}$$

where

$$\varepsilon_{q,\eta} \equiv \frac{\partial q}{\partial \eta}\frac{\eta}{q}\,, \quad \varepsilon_{q,\omega} \equiv \frac{\partial q}{\partial \omega}\frac{\omega}{q}\,, \tag{40}$$

$$\xi_{q,\eta/\omega} \equiv \frac{1}{2}\frac{\partial^2 q}{(\partial\omega)^2}(\sigma_\omega\omega)^2+\frac{\partial^2 q}{\partial\eta\partial\omega}\sigma_\eta\eta\sigma_\omega\omega+\frac{1}{2}\frac{\partial^2 q}{(\partial\eta)^2}(\sigma_\eta\eta)^2\,. \tag{41}$$

Thus, the system of equations,

$$\{(30)\,,\ (31)\,,\ (32)\,,\ (33)\,,\ (37)\,,\ (38)\,,\ (39)\}\,, \tag{42}$$

with

$$\iota_j = \mathcal{I}_j'^{-1}\left(\frac{1}{q}\right)\,, \tag{43}$$

$$\kappa = \phi\eta \text{ with } \phi = \min\left\{\lambda, \frac{1}{\eta}\right\}\mathbf{1}_{\omega\geq\bar{\omega}}\,, \tag{44}$$

$$\bar{\omega}(\eta) = \{\omega < 0 : \alpha_1(\omega,\eta) = \alpha_2(\omega,\eta) = 0\}\,, \tag{45}$$

$$\bar{\eta}(\omega) = \{\eta \in [0,1] : \lambda(\omega,\eta)\,\eta = 1\}\,, \tag{46}$$

determines a second-order PDEs for $\{q, v\}$ in $\{\omega, \eta\}$.

We impose the following boundary conditions to the PDEs:

$$\lim_{\eta\to1}\sigma_q = 0\,, \quad \lim_{\eta\to1}\frac{\partial\sigma_q}{\partial\eta} = 0\,, \quad \lim_{\eta\to1}\sigma_v = 0\,, \quad \lim_{\eta\to1}\frac{\partial\sigma_v}{\partial\eta} = 0\,. \tag{47}$$

These conditions ensure that diffusions σ_q and σ_v vanish smoothly as the aggregate net worth of financiers approaches total wealth. ■

Proposition 5 *In the economy with rational expectations and without financial frictions, neither sentiment ω nor wealth share η affect the equilibrium outcome. The asset price is a constant that satisfies*

$$\alpha_1 = 0 \Leftrightarrow \frac{A_1 - \iota_1}{q} + \mathcal{I}_1(\iota_1) - r = 0 \ , \ \text{with} \ \mathcal{I}_1'(\iota_1) = \frac{1}{q} \ . \tag{48}$$

Value $v = 1$ is also a constant. The aggregate quantity of the asset is allocated to the productive technology, that is, $\kappa = 1$.

Proof. Under RE, the productive technology yields higher return than the unproductive one, accordingly the Equilibrium relationship #3 from (23) cannot occur. In the absence of financial frictions, the allocation of the asset to the productive technology is not restricted by the collateral constraint, hence the Equilibrium relationship #2 cannot occur. Accordingly, the conditions Equilibrium relationship #1 characterize the equilibrium, that is $\alpha_1 = 0$ with $\kappa = 1$. It derives that $v = 1$, since financiers earn no rent on the asset, and

$$\alpha_1 = \frac{1}{dt} E\left[dR_1\right] - r = 0.$$

Finally, the price of the asset q is constant and satisfies

$$\frac{A_1 - \iota_1}{q} + I_1(\iota_1) - r = 0,$$

where ι_1 satisfies (20). ■

Proposition 6 *In the economy with diagnostic expectations and without financial frictions, sentiment ω is the only relevant state that affects the equilibrium outcome. A threshold state $\bar{\omega} < 0$ exists such that*

$$\begin{array}{ll} \text{if } \omega < \bar{\omega} \Rightarrow & \kappa = 0 \ , \ \ \alpha_1 < 0 \ , \ \ \alpha_2 = 0 \ ; \\ \text{if } \omega > \bar{\omega} \Rightarrow & \kappa = 1 \ , \ \ \alpha_1 = 0 \ , \ \ \alpha_2 < 0 \ ; \end{array} \ . \tag{49}$$

The threshold state $\bar{\omega} < 0$ is the solution to

$$\alpha_1 = \alpha_2 = 0 \Rightarrow \frac{A_1 - \iota_1 - A_2}{q} + \mathcal{I}_1(\iota_1) + (\sigma_q + \hat{\mu}\omega)\,\sigma_1 = 0 \ . \tag{50}$$

Proof. In the absence of financial frictions, Equilibrium relationship #2 cannot occur. Accordingly, the economy alternates between Equilibrium relationships #1 and #3, depending on the value of sentiment ω, i.e., depending on the perceived relative returns to each technology, as indicated in (49). The sentiment threshold state $\bar{\omega}$ is such that the perceived return to each technology is the same, i.e., $\alpha_1 = \alpha_2$. Using (3) and (14), one gets (50). ■

Proposition 7 *In the economy with rational expectations and financial frictions, wealth share η is the only relevant state that affects the equilibrium outcome. A threshold state $\bar{\eta} \in (0,1)$ exists such that*

$$\begin{array}{lll} \text{if } \eta < \bar{\eta} \Rightarrow & \kappa = \lambda\eta < 1 \,, & \alpha_1 > 0 \,,\ \alpha_2 = 0 \,; \\ \text{if } \eta > \bar{\eta} \Rightarrow & \kappa = 1 \,, & \alpha_1 = 0 \,,\ \alpha_2 < 0 \,; \end{array} . \tag{51}$$

The threshold state $\bar{\eta} \in (0,1)$ is the solution to

$$\lambda\eta = 1 \Leftrightarrow \eta = \frac{1}{\lambda} \,. \tag{52}$$

Proof. Under rational expectations, only Equilibrium relationships #1 and #2 can occur, since the productive technology is correctly perceived as providing higher returns. Accordingly, the economy alternates between the financially constrainted and financially unconstrained regime, depending on the wealth share η of financiers. The cut-off value $\bar{\eta}$ naturally satisfies (52). ■

Corollary 1 *In the economy with diagnostic expectations and financial frictions, both sentiment ω and wealth share η affect the equilibrium outcome. Two threshold states, $\bar{\omega} < 0$ and $\bar{\eta} \in (0,1)$, partition the state space as follows:*

$$\begin{array}{llll} \text{if } \omega < \bar{\omega} \Rightarrow & \kappa = 0 \,, & \alpha_1 < 0 \,,\ \alpha_2 = 0 \,; \\ \text{if } \omega > \bar{\omega} \text{ and } \eta < \bar{\eta} \Rightarrow & \kappa = \lambda\eta \,, & \alpha_1 > 0 \,,\ \alpha_2 = 0 \,; \\ \text{if } \omega > \bar{\omega} \text{ and } \eta > \bar{\eta} \Rightarrow & \kappa = 1 \,, & \alpha_1 = 0 \,,\ \alpha_2 < 0 \,; \end{array} \tag{53}$$

Threshold state $\bar{\omega}$ is the solution to

$$\alpha_1 = \alpha_2 = 0 \Rightarrow \frac{A_1 - \iota_1 - A_2}{q} + \mathcal{I}_1(\iota_1) + (\sigma_q + \hat{\mu}\omega)\,\sigma_1 + (\sigma_q + \sigma_1)\,\sigma_v = 0 \,. \tag{54}$$

Threshold state $\bar{\eta}$ is the solution to

$$\lambda\eta = 1 \Leftrightarrow \eta = \frac{1}{\lambda} \,. \tag{55}$$

Proof. With both diagnostic expectations and financial frictions, the economy alternates between the three Equilibrium relationships. The characterization of cut-off states $\bar{\omega}$ and $\bar{\eta}$ follows from the proofs of Propositions 6 and 7. ■

Proposition 8 *If agents rely on Ito path $\{dX_s\}_{s<t}$ to form diagnostic expectations about Ito variable dY_t, the implied diagnostic expectation operator over disturbance dZ_t is*

$$\hat{E}_t\left[dZ_t\right] = \hat{\mu}\frac{\omega_t}{\sigma_{Y,t}Y_t}dt \ , \tag{56}$$

where $\sigma_{Y,t} \in \mathbb{R}$ is the diffusion of the variable and where sentiment $\omega_t \in \mathbb{R}$ is given by

$$\omega_t = \int_0^t e^{-\delta(t-s)}dX_s \ . \tag{57}$$

Corollary 2 *If $dX_s = \frac{dR_{1,s}-\hat{E}_s[dR_{1,s}]}{Std_s[dR_{1,s}]}$ and $dY_t = dR_{1,t}$, then*

$$\hat{E}_t\left[dZ_t\right] = \hat{\mu}\omega_t dt \ , \tag{58}$$

and

$$d\omega_t = \left(-\delta + \frac{\hat{\mu}}{\sigma_{q,t}+\sigma_1}\right)\omega_t dt + dZ_t \ . \tag{59}$$

Proof. Let ω_t be a sentiment operator tied to a Ito process $\{dX_s\}$, i.e.,

$$\omega_t = \int_0^t e^{-\delta(t-s)}dX_s. \tag{60}$$

A diagnostic operator over a generic Ito process dY_t is defined as:

$$\hat{E}_t\left[dY_t\right] \equiv E_t\left[d\hat{Y}_t\right], \quad \text{with} \quad d\hat{Y}_t \equiv \hat{\mu}\omega_t dt + dY_t \tag{61}$$

Let's define the expectation operator $\check{E}_t[dZ_t]$ as

$$\check{E}_t[dZ_t] \equiv E_t[d\hat{Z}_t], \text{ with } d\hat{Z}_t \equiv \hat{\mu}\frac{\omega_t}{\sigma_{Y,t}Y_t}dt + dZ_t \ . \tag{62}$$

Then

$$\check{E}_t[dY_t] = \check{E}_t[\mu_{Y,t}Y_t dt + \sigma_{Y,t}Y_t dZ_t] = \mu_{Y,t}Y_t dt + \sigma_{Y,t}Y_t\check{E}_t[dZ_t] = \left(\mu_{Y,t}Y_t + \hat{\mu}\omega_t\right)dt \ . \tag{63}$$

Accordingly, $\check{E}_t[dY_t] = \hat{E}_t[dY_t]$, and the implied diagnostic expectation operator over dZ_t is

$$\hat{E}_t\left[dZ_t\right] = \check{E}_t[dZ_t] = \hat{\mu}\frac{\omega_t}{\sigma_{Y,t}Y_t}dt\ . \tag{64}$$

Applying these results to the specific case $dX_s = \frac{dR_{1,s}-\hat{E}_s[dR_{1,s}]}{Std_s[dR_{1,s}]}$ and $dY_t = dR_{1,t}$, one gets the expressions presented in Corollary 2. ■

Proposition 9 *The socially optimal reinvestment rate solves*

$$\mathcal{I}_1'(\iota_1) = \frac{1+\frac{1}{1-(\phi-1)\varepsilon_{q,\eta}}\frac{1}{q}\frac{\partial\tilde{u}}{\partial\eta}}{\tilde{u}+\frac{1-\eta}{1-(\phi-1)\varepsilon_{q,\eta}}\frac{\partial\tilde{u}}{\partial\eta}}\ . \tag{65}$$

The socially optimal share κ is the candidate value that maximizes the RHS in (3). The candidate values are $\kappa = 0$, $\kappa = \min\{\lambda\eta, 1\}$, and any interior $\kappa \in (0, \min\{\lambda\eta, 1\})$ that solves

$$\begin{aligned} 0 \quad = \quad & \left[\frac{A_1-\iota_1-A_2}{\tilde{u}}+\mathcal{I}_1(\iota_1)+\left(\sigma_{\tilde{u}}+\tilde{\mu}\omega\right)\sigma_1\right]+\varepsilon_{\tilde{u},\eta}\left[\frac{\partial\mu_\eta}{\partial\kappa}+\left(\tilde{\mu}\omega+\kappa\sigma_1\right)\frac{\partial\sigma_\eta}{\partial\kappa}\right]+ \\ & +\frac{1}{\tilde{u}}\left(\frac{\partial^2\tilde{u}}{(\partial\eta)^2}\sigma_\eta\eta+\frac{\partial^2\tilde{u}}{\partial\eta\partial\omega}\right)\frac{\partial\sigma_\eta}{\partial\kappa}\eta\ , \end{aligned} \tag{66}$$

where $\frac{\partial\mu_\eta}{\partial\kappa}$ and $\frac{\partial\sigma_\eta}{\partial\kappa}$ are the partial derivatives of μ_η and σ_η with respect to κ, respectively.

Proof. This proof lay outs and solves the problem of the planner. The present discounted value of consumption under expectation weight $\tilde{\mu} \in [0, \hat{\mu}]$ is

$$\tilde{U}_t \equiv \tilde{E}_t\int_t^{+\infty} e^{-r(s-t)}\left[\kappa_s\left(A_1-\iota_{1,s}\right)+\left(1-\kappa_s\right)\left(A_2-\iota_{2,s}\right)\right]k_s ds\ , \tag{67}$$

where expectation operator $\tilde{E}_t\left[\cdot\right]$ is

$$\tilde{E}_t[dZ_t] \equiv E_t[d\tilde{Z}_t], \text{ with } d\tilde{Z}_t \equiv \tilde{\mu}\omega_t dt + dZ_t\ . \tag{68}$$

Note that the term in brackets in the integrand follows from resource constraint

$$c_t = y_t = \kappa_t\left(A_1-\iota_{1,t}\right)+\left(1-\kappa_t\right)\left(A_2-\iota_{2,t}\right)\ . \tag{69}$$

Utility $\tilde{U}_t$ can be expressed as

$$e^{-rt}\tilde{U}_t + \int_0^t e^{-rs}\left[\kappa_s\left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \tag{70}$$
$$= \tilde{E}_t \int_0^{+\infty} e^{-rs}\left[\kappa_s\left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \ .$$

The RHS in this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. Applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following HJB equation:

$$0 = \kappa_t\left(A_1 - \iota_{1,t}\right) + \left(1-\kappa_t\right)\left(A_2 - \iota_{2,t}\right) + \tilde{E}_t\left[d\tilde{U}_t\right] - r\tilde{U}_t \ . \tag{71}$$

We postulate that $U_t = u_t k_t$, where $u_t > 0$ is an Ito process with disturbance dZ_t. The above HJB equation can then be reduced to

$$0 = \kappa_t\left\{A_1 - \iota_{1,t} + \left[\mathcal{I}_1(\iota_{1,t}) + \sigma_1\tilde{\mu}\omega_t\right]\tilde{u}_t\right\} + \tag{72}$$
$$+\left(1-\kappa_t\right)\left\{A_2 - \iota_{2,t} + \left[\mathcal{I}_2(\iota_{2,t}) + \sigma_2\tilde{\mu}\omega_t\right]\tilde{u}_t\right\} + \tilde{E}_t\left[d\tilde{u}_t\right] - r\tilde{u}_t \ .$$

In what follows, we restrict attention to a Markov structure with same state variables as in the competitive equilibrium. Thus, we omit time subscript $_t$ from now on. In addition, we consider the parametrization of the baseline specification (Section 2.3 of the paper). Equation (72) can then be expressed as

$$\begin{aligned} r\tilde{u} \quad = \quad & \kappa\left\{A_1 - \iota_1 + \left[\mathcal{I}_1(\iota_1) + \sigma_1\tilde{\mu}\omega\right]\tilde{u}\right\} + \left(1-\kappa\right)A_2 + \frac{\partial\tilde{u}}{\partial\omega}\left(-\delta\omega + \tilde{\mu}\omega + \kappa\sigma_1\right) + \\ & + \frac{\partial\tilde{u}}{\partial\eta}\left(\mu_\eta\eta + \sigma_\eta\eta\tilde{\mu}\omega + \sigma_\eta\eta\kappa\sigma_1\right) + \frac{1}{2}\frac{\partial^2\tilde{u}}{\left(\partial\omega\right)^2} + \frac{\partial^2\tilde{u}}{\partial\omega\partial\eta}\sigma_\eta\eta + \frac{1}{2}\frac{\partial^2\tilde{u}}{\left(\partial\eta\right)^2}\left(\sigma_\eta\eta\right)^2\right\} \ , \end{aligned} \tag{73}$$

where

$$\mu_\eta = \frac{1}{1-(\frac{\kappa}{\eta}-1)\varepsilon_{q,\eta}}\left\{\left[\frac{A_1-\iota_1}{q}+\mathcal{I}_1(\iota_1)+\sigma_q\sigma_1\right]\frac{\kappa}{\eta}-\kappa\mathcal{I}_1(\iota_1)-\sigma_q\kappa\sigma_1+\right. \tag{74}$$
$$+\frac{1}{q}\left[-\frac{\partial q}{\partial\omega}\delta\omega+\frac{1}{2}\frac{\partial^2 q}{(\partial\omega)^2}+\frac{\partial^2 q}{\partial\omega\partial\eta}\sigma_\eta\eta+\frac{1}{2}\frac{\partial^2 q}{(\partial\eta)^2}(\sigma_\eta\eta)^2-rq\right]\left(\frac{\kappa}{\eta}-1\right)+$$
$$\left.+(\sigma_q+\kappa\sigma_1)\left[(\sigma_q+\kappa\sigma_1)-\frac{\kappa}{\eta}(\sigma_q+\sigma_1)\right]-\left(\theta-\frac{\gamma}{\eta}\right)\right\},$$

$$\sigma_\eta = \frac{\frac{\kappa}{\eta}\left(\frac{1}{q}\frac{\partial q}{\partial\omega}+\sigma_1\right)-\left(\frac{1}{q}\frac{\partial q}{\partial\omega}+\kappa\sigma_1\right)}{1-\left(\frac{\kappa}{\eta}-1\right)\varepsilon_{q,\eta}}, \tag{75}$$

with

$$\sigma_q = \frac{\left(\frac{\kappa}{\eta}-\kappa\right)\varepsilon_{q,\eta}\sigma_1+\frac{1}{q}\frac{\partial q}{\partial\omega}}{1-\left(\frac{\kappa}{\eta}-1\right)\varepsilon_{q,\eta}}. \tag{76}$$

The above formulae for $\{\mu_\eta, \sigma_\eta, \sigma_q\}$ follow from evaluating $\{(33), (38), (39)\}$ at the baseline parametrization.

The problem of the planner is then

$$r\tilde{u} = \max_{\{\iota_1,\kappa\}}\left\{\kappa\{A_1-\iota_1+[\mathcal{I}_1(\iota_1)+\sigma_1\tilde{\mu}\omega]\tilde{u}\}+(1-\kappa)A_2+\frac{\partial\tilde{u}}{\partial\omega}(-\delta\omega+\tilde{\mu}\omega+\kappa\sigma_1)+\right. \tag{77}$$
$$\left.+\frac{\partial\tilde{u}}{\partial\eta}(\mu_\eta\eta+\sigma_\eta\eta\tilde{\mu}\omega+\sigma_\eta\eta\kappa\sigma_1)+\frac{1}{2}\frac{\partial^2\tilde{u}}{(\partial\omega)^2}+\frac{\partial^2\tilde{u}}{\partial\omega\partial\eta}\sigma_\eta\eta+\frac{1}{2}\frac{\partial^2\tilde{u}}{(\partial\eta)^2}(\sigma_\eta\eta)^2\right\},$$

with

$$\iota_1 \in [0, A_1] \text{ and } \kappa \in [0, \min\{\lambda\eta, 1\}], \tag{78}$$

where $\{\mu_\eta, \sigma_\eta, \sigma_q\}$ are given by $\{(74), (75), (76)\}$.

The first-order condition with respect to ι_1 implies that

$$\mathcal{I}_1'(\iota_1) = \frac{1+\frac{1}{1-(\phi-1)\varepsilon_{q,\eta}}\frac{1}{q}\frac{\partial\tilde{u}}{\partial\eta}}{\tilde{u}+\frac{1-\eta}{1-(\phi-1)\varepsilon_{q,\eta}}\frac{\partial\tilde{u}}{\partial\eta}}. \tag{79}$$

Note that the problem is concave in ι_1.

The first-order condition with respect to κ implies that

$$\left[\frac{A_1-\iota_1-A_2}{\tilde{u}}+\mathcal{I}_1(\iota_1)+(\sigma_{\tilde{u}}+\hat{\mu}\omega)\,\sigma_1\right]+\varepsilon_{\tilde{u},\eta}\left[\frac{\partial\mu_\eta}{\partial\kappa}+(\hat{\mu}\omega+\kappa\sigma_1)\,\frac{\partial\sigma_\eta}{\partial\kappa}\right]+$$
$$+\frac{1}{\tilde{u}}\left(\frac{\partial^2\tilde{u}}{\partial\omega\partial\eta}+\frac{\partial^2\tilde{u}}{(\partial\eta)^2}\sigma_\eta\eta\right)\frac{\partial\sigma_\eta}{\partial\kappa}\eta\gtreqless 0\,, \tag{80}$$

with $\kappa=0$ if inequality "<" holds and $\kappa=\min\{\lambda\eta,1\}$ if the other inequality does so.

Note that $\frac{\partial\mu_\eta}{\partial\kappa}$ and $\frac{\partial\sigma_\eta}{\partial\kappa}$ are the partial derivatives with respect to κ of the RHS on expressions (74) and (75), respectively. Diffusion $\sigma_{\tilde{u}}$ is

$$\sigma_{\tilde{u}}=\frac{1}{\tilde{u}}\left[\frac{\partial\tilde{u}}{\partial\omega}+\frac{\partial\tilde{u}}{\partial\eta}\sigma_\eta\eta\right]. \tag{81}$$

■

Proposition 10 *The socially optimal allocation and its associated mappings $\{\tilde{u},v,q\}$ can be analytically characterized by a system of second-order PDEs for the mappings in the state $\{\omega,\eta\}$.*

Proof. The equations that determine price q and value v are

$$\begin{array}{ll}\frac{A_1-\iota_1}{q}+\mu_q+\mathcal{I}_1(\iota_1)+(\sigma_q+\sigma_1)\,\hat{\mu}\omega+\sigma_q\sigma_1-r+(\sigma_q+\sigma_1)\,\sigma_v=0 & \text{if } \kappa=1\\ \frac{A_2}{q}+\mu_q+\sigma_q\hat{\mu}\omega-r=0 & \text{otherwise}\end{array}\,. \tag{82}$$

and

$$\alpha_1\frac{\kappa}{\eta}+\mu_v+\hat{\mu}\omega\sigma_v+\frac{\theta}{v}-\theta=0\,, \tag{83}$$

respectively.

The system of equations,

$$\{(32)\,,\ (33)\,,\ (86)\,,\ (74)\,,\ (75)\,,\ (79)\,,\ (80)\,,\ (82)\,,\ (83)\}\,, \tag{84}$$

thus determines a second-order PDEs for $\{\tilde{u},q,v\}$ in $\{\omega,\eta\}$.

We impose the following boundary conditions to the PDEs:

$$\lim_{\eta\to1}\sigma_x=0\,,\quad \lim_{\eta\to1}\frac{\partial\sigma_x}{\partial\eta}=0\,. \tag{85}$$

for $x\in\{\tilde{u},q,v\}$. ■

Proposition 11 *In the economy with diagnostic expectations and without financial frictions, the socially optimal allocation is first-best efficient according to the expectation weight of the planner. If the planner is benevolent, the socially optimal allocation is the same as the equilibrium allocation. If the planner is paternalistic, the socially optimal allocation is the same as the equilibrium allocation of the economy presented in subsection 3.1.*

Proof. If the planner is benevolent, the socially optimal allocation is the same as the equilibrium allocation.

Let's postulate that $\partial \tilde{u}/\partial \eta = 0$. Thus,

$$r\tilde{u} = \max_{\{\iota_1,\kappa\}} \left\{ \kappa \left\{ A_1 - \iota_1 + \left[\mathcal{I}_1(\iota_1) + \sigma_1 \hat{\mu} \omega \right] \tilde{u} \right\} + (1-\kappa) A_2 + \right. \\ \left. + \frac{\partial \tilde{u}}{\partial \omega} \left(-\delta\omega + \tilde{\mu}\omega + \kappa\sigma_1 \right) + \frac{1}{2} \frac{\partial^2 \tilde{u}}{(\partial \omega)^2} \right\}. \tag{86}$$

Let's also postulate that $\tilde{u} = q$. Then, the equilibrium allocation solves the optimization problem, which verifies the postulates.

If the planner is paternalistic, the socially optimal allocation is the same as the equilibrium allocation of the economy presented in subsection 3.1.

Let's postulate that $\partial \tilde{u}/\partial \eta = \partial \tilde{u}/\partial \omega = 0$. Thus,

$$r\tilde{u} = \max_{\{\iota_1,\kappa\}} \left\{ \kappa \left\{ A_1 - \iota_1 + \mathcal{I}_1(\iota_1)\tilde{u} \right\} + (1-\kappa) A_2 \right\}.$$

Let's also postulate that $\tilde{u}$ equals the asset price of the economy presented in subsection 3.1. Then, the equilibrium allocation of that economy solves the optimization problem, which verifies the postulates. ■

2 Numerical Solution Method

To solve the PDEs we use spectral methods. Specifically, we interpolate $\{q, v\}$ or $\{\tilde{u}, q, v\}$ with linear combinations of Chebyshev polynomials of the first kind. We evaluate the interpolation at the Chebyshev nodes. We use a nonlinear solver to find the coefficients associated with the polynomials in the linear combination. As initial guess for the solver, we use the values of $\{q, v\}$ or $\{\tilde{u}, q, v\}$ in the economy of Section 3.1 of the paper.