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Existence of a Non-Stationary Equilibrium in Search-And-Matching Models: TU and NTU

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Existence of a Non-Stationary Equilibrium in Search-and-Matching Models: TU and NTU*

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Abstract

This paper proves the existence of a non-stationary equilibrium in the canonical search-and-matching model with heterogeneous agents. Non-stationarity entails that the number and characteristics of unmatched agents evolve endogenously over time. An equilibrium exists under minimal regularity conditions and for both paradigms considered in the literature: transferable and non-transferable utility. We suggest that our proof strategy applies more broadly to a class of continuous-time, infinite-horizon models with a continuum of heterogeneous agents, also referred to as mean-field games, that evolve deterministically over time.

Keywords equilibrium existence, search and matching, non-stationary, mean-field game

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1 Introduction

The steady state is the focal point of theoretical research on dynamic heterogeneous agent models. This assumption entails that the aggregate state does not evolve; individual expectations over the future remain unchanged as time goes on. Stationarity makes complex models more tractable, but the assumption is not without loss. As is well documented,[1] aggregate fluctuations affect quantitatively individual decisions and, more importantly, change the nature of intertemporal trade-offs in many environments. One critical insight is that aggregate fluctuations can amplify idiosyncratic risk.[2] Embedding these trade-offs within a non-stationary general equilibrium model remains a persistent challenge, and for most models of interest, we do not even know whether a time-dependent equilibrium exists (Achdou et al. (2014)).

This paper establishes preconditions for the analysis of non-stationary dynamics: we build tools for proving the existence of a non-stationary equilibrium in environments where the aggregate state evolves deterministically over time. We view this as the natural generalization of the steady state, whose aggregate dynamics are, by construction, also deterministic.

Our focus is the canonical search-and-matching model. This model has been widely used to study productive and social interactions.[3] As in the pioneering work by Shimer and Smith (2000), a continuum of heterogeneous agents engage in a time-consuming and haphazard search for one another and exit the search pool upon forming a match. Following the two dominant paradigms in the literature, match payoffs can be transferable (TU), i.e., there is Nash bargaining over match surplus, or non-transferable (NTU), i.e., match payoffs are exogenously given. In this model, fluctuations arise naturally, e.g., due to a seasonal thick market externality, gradual market clearing or the business cycle. Known equilibrium existence results, however, apply only in the stylized stationary environment where entry and exit into the search pool is balanced at all moments in time (Burdett and Coles (1997), Shimer and Smith (2000), Smith (2006), Lauermann et al. (2020)).[4]

A non-stationary equilibrium resolves a complex feedback loop between a time-moving ag-

[1] Ngai and Tenreyro (2014) show that the US and UK housing market experience seasonal fluctuations in prices and transactions. Bloom et al. (2018) document that firm uncertainty increases sharply in recessions, leading to more cautious hiring practices.

[2] In a non-stationary Aiyagari (1994) model (see Achdou et al. (2022)), a looming rise in interest rates makes consecutive negative income shocks more costly, contributing to greater precautionary savings. In a growth model, the anticipation of future industry consolidation can dampen investment in long-run quality in favor of greater short-term intangible investment (see de Ridder (2022)). In a companion paper (see Bonneton and Sandmann (2022)), we show how the downside risk of not matching while the market is thick can impede positive assortative matching.

[3] Notable applications include sorting in the labor and the marriage market (see Chade et al. (2017) for a review) and foundations of Walrasian equilibrium (Rubinstein and Wolinsky (1985), Gale (1987), Lauermann (2013)).

[4] Relatedly, Lauermann and Nöldeke (2015) and Manea (2017) prove the existence of a stationary equilibrium when there are finitely many types.

gregate state and individual decisions. In the search-and-matching economy, the endogenous variables are: the distribution of agents' characteristics in the search pool, agents' value-of-search, and thereby determined matching decisions and transfers. Aggregate population dynamics and the individual decision problem are coupled; when the search pool evolves and therefore, future match prospects evolve, so do optimal matching decisions, and hence the rate at which agents exit the search pool. The interplay between aggregate dynamics and the individual decision problem is shared with virtually all dynamic general equilibrium models under rational expectations. Lasry and Lions (2007) refer to this class of models as mean-field games.

We prove equilibrium existence in three steps. As in general equilibrium theory, existence will depend on the application of a topological fixed point theorem. In Section 4, we establish a non-trivial adaptation of the Schauder (1930) fixed point theorem, which imposes few constraints on the model. This theorem translates abstract concepts, notably compactness in function spaces, into premises that can be interpreted economically. In Section 5.1, we construct a value-of-search operator whose fixed points correspond to a non-stationary equilibrium. In Sections 5.2 and 5.3, we prove that the operator satisfies the premises of our fixed point theorem. To that end, we construct bounds on the value-of-search across individuals that are derived from two revealed preferences arguments.

We first establish a fixed point theorem (Theorem 3). Due to its potential appeal to other models, we present it in a self-contained section. The domain of this fixed point theorem is the space of tuples $(F^1, ..., F^N) \in \mathcal{F}^N$ of measurable mappings $F^n : [0,1] \times \mathbb{R}_+$ endowed with a pseudo-semimetric. In search-and-matching models, $N = 2$ is the number of populations, e.g., workers and firms, and $F^n(x,t)$ is the value-of-search of agent type x from population n at time t. We prove that *an operator* $H : \mathcal{F}^N \to \mathcal{F}^N$ *admits a fixed point if it is (i) continuous with respect to the pseudo-seminorm, and (ii) maps into a function space whose (two-dimensional) total variation norm is uniformly bounded.* Premise (i) is the familiar continuity premise from Schauder. Both premises are sufficiently general to allow the value function to fluctuate endogenously over time and to be discontinuous with respect to time and type.[5]

The key step in proving our fixed point theorem is the construction of a sequence of approximate equilibria. By confining the value function profile to a smaller function space, the space of k-Lipschitz functions, the fixed point operator is guaranteed to be compact-valued. Since the operator is also continuous due to Premise (i), Schauder's theorem guarantees the existence of a fixed point. This fixed point corresponds to an approximate equilibrium with vanishing

[5]In NTU search-and-matching models, it is known that block segregation (see McNamara and Collins (1990), Smith (2006) and references therein) prohibits continuity of the value-of-search: agents can cluster according to classes so that any two agents match upon meeting if and only if they belong to the same class. Similarly, discontinuities in the value function across types arise naturally under informational pooling as in Akerlof (1970).

approximating error as we increase the constant k. We then prove that the sequence of approximating equilibria converges. This is the consequence of a generalized multidimensional (time and type) Helly selection theorem which establishes that Premise (ii) implies sequential compactness of $H(\mathcal{F}^N)$.[6]

Second, we construct a value-of-search operator whose fixed points correspond to a non-stationary equilibrium. Under this operator, agents take others' value-of-search, hence matching decisions, as given to compute their own discounted expected future match payoff. The operator can be interpreted as the out-of-equilibrium value-of-search in that the value-of-search ascribed to other agents of equal type need not coincide with their own.

Third, we prove that the operator satisfies the premises of our fixed point theorem: continuity and uniformly bounded variation. This holds true for general primitives of the economy. The central assumptions are Lipschitz continuity and linear boundedness of entry and meeting rates. In particular, we allow both rates to depend generally on time and the current size and composition of the search pool, which relaxes assumptions considered in the literature.

We circumvent the tractability issues that come with non-stationary dynamics by constructing tight bounds on the difference in the value of search between two agents. Those bounds follow from two revealed preference arguments (NTU and TU) coined *mimicking arguments* whose underlying idea is to let one agent replicate someone else's matching decisions. In the TU paradigm, we establish bounds in terms of time-invariant output rather than time-varying payoffs by employing an inductive reasoning over the mimicking argument that gives rise to a 'matryoshka doll'-like proof. These bounds are also key to studying sorting in non-stationary equilibrium (see Bonneton and Sandmann (2022) (NTU), and Bonneton and Sandmann (2021) (TU)).

Related Work. This paper contributes to the theoretical literature on search and matching, see Chade et al. (2017) for an excellent review. To date, all equilibrium existence results derive conditions on the primitives of the model for which a steady state exists (Burdett and Coles (1997), Shimer and Smith (2000), Smith (2006), Lauermann and Nöldeke (2015), Manea (2017) and Lauermann et al. (2020)). Many economic phenomena, however, are inherently non-stationary, including time-variant entry as in a seasonal housing market (see Ngai and Tenreyro (2014)), and a gradually clearing job market (by which, e.g., academic economists have organized the junior job market for Ph.D. hires).

Questioning what happens outside the steady state is at the heart of burgeoning literature at the intersection of continuous-time macroeconomics and mean field games. Yet for many

[6] Relatedly, Smith (2006) makes use of the Helly selection theorem in dimension one (type) to establish sequential compactness of the value function space.

models of interest no one even knows whether an equilibrium exists when the economy is not assumed to be in the steady state (see Achdou et al. (2014)). Difficulties include the fact that it is usually impossible to characterize the value-of-search in closed form. Smith (2011) quipped that "the simplest non-stationary models can be notoriously intractable."

Our Theorem 3 relates to Jovanovic and Rosenthal (1988) who also propose a topological approach to prove the existence of non-stationary (and stationary) equilibria in a general class of models coined anonymous games; These can be viewed as mean field games in discrete time.[7,8] Observe however that their critical assumption on the continuity of individual expected utilities do not hold in search-and-matching models; match opportunities can give rise to discontinuities in the agent's own match decision and across types. This has been extensively explored in the context of block segregation (refer to McNamara and Collins (1990) for the original result; also see Smith (2006) and references therein). Our fixed point theorem is sufficiently general to allow the value function to fluctuate endogenously over time and to be discontinuous with respect to time and type.

Recent work by Balbus et al. (2022) (on supermodular anonymous games) and Pröhl (2023) (on the non-stationary Aiyagari (1994) model with aggregate uncertainty) have established non-topological existence results that rely on monotonicity conditions. Where such monotonicity exists equilibria can be ranked or are unique. The interactive nature of search-and-matching models rule out their structural assumptions.[9,10]

The mean field game literature has made strides as of late by allowing for aggregate uncertainty under the probabilistic approach (see Carmona and Delarue (2018) and Bilal (2023)). Mathematically, aggregate noise is a convenient tool, for it smoothes the value function across states, allowing the researcher to leverage PDE techniques. Conceptually, our approach is different since, like in the steady state, the aggregate dynamics we consider are, by construction, deterministic.

[7]In search-and-matching, discrete time may insufficiently discipline the model, leading, e.g., to pathological coordination failures across period (as reported in Damiano et al. (2005)).

[8]Bergin and Bernhardt (1995) investigate the complementary case where there is aggregate uncertainty.

[9]Supermodularity posits incremental and monotone effect of others' actions on expected utility. Supermodularity is not satisfied in our context where there are threshold acceptance strategies in the same way that Bertrand competition cannot be modelled as a supermodular game.

[10]Aiyagari (1994) builds on a sufficient statistic approach whereby individual decisions aggregate into a single variable such as the interest rate. Match acceptance decisions of heterogeneous agents do not admit such aggregation.

2 The Search-and-Matching Economy

We develop a continuous-time, infinite-horizon matching model in which agents engage in time-consuming and haphazard search for one another. When two agents meet they observe each other's type. If both agents give their consent they permanently exit the search pool and consumee their respective match payoffs. Otherwise they continue waiting for a more suitable partner. Each agent maximizes their expected present value of payoffs, discounted at rate $\rho > 0$.

2.1 Set-up

Agents

There are two distinct populations denoted X and Y, each containing a continuum of agents that seek to match with someone from the other population. Each agent is characterized by a type which belongs to the unit interval $[0, 1]$. We usually denote by x a type of an agent from population X, and y a type of an agent from population Y. The distribution of types in the search pool at time t is characterized by a function $\mu_t = (\mu_t^X, \mu_t^Y)$ so that the mass of types $x \in U \subseteq [0, 1]$ is $\int_U \mu_t^X(x)dx$.[11] The initial distribution at time 0 is given by μ_0.[12]

Our assumptions on primitives of the model make use of the L^1 seminorm. Define

$$N(\mu_t', \mu_t'') \equiv \max\Big\{ \int_0^1 \big|\mu'^X_t(x) - \mu''^X_t(x)\big|dx, \int_0^1 \big|\mu'^Y_t(y) - \mu''^Y_t(y)\big|dy \Big\}.$$

Search

Over time agents randomly meet each other. Meetings follow an (inhomogeneous) Poisson point process. Such a process is characterized by the time-variant (Poisson) meeting rate $\lambda = (\lambda^X, \lambda^Y)$ where $\lambda_t^X(y|x)$ is agent type x's time-t meeting rate with an agent type y. In the simplest case, the meeting rate is proportional to the search pool population so that $\lambda_t^X(y|x) = \mu_t^X(x)$, as in Shimer and Smith (2000) and Smith (2006). More generally, we take λ to be a function of the underlying state variable μ_t and time t. Then the subindex t is short-hand for dependence on both the prevailing time t and state μ_t, i.e., $\lambda_t^X(y|x) \equiv \lambda^X(t, \mu_t)(y|x)$.

The meeting rates λ_t^X and λ_t^Y are not arbitrary but intricately linked. Coherence of the

[11] $x \mapsto \mu_t^X(x)$ satisfies all the properties of a probability density function, except, it need not integrate to one. Typically $\int_0^1 \mu_t^X(x)dx \neq 1$, as will surely be the case in a gradually clearing search pool.

[12] We usually pursue the construction from the point of view of population X; symmetric constructions apply for agent types y from population Y. Moreover, we impose but never mention explicitly that all functions introduced are measurable; e.g., $\mu_0^X(x)$ is measurable in x. As will be seen, all equilibrium objects will be continuous in time, thereby implying joint measurability in type and time.

model demands that the number of meetings of agent types x with agent types y must be equal to the number of meetings of agent types y with agent types x:

$$\lambda_t^X(y|x)\mu_t^X(x) = \lambda_t^Y(x|y)\mu_t^Y(y).$$

We then assume that higher types meet other agents at a weakly faster rate.

Assumption 1 (hierarchical search)**.** *Higher types meet other agents at a weakly faster rate; that is,* $\lambda_t^X(y|x_2) \geq \lambda_t^X(y|x_1)$ *for* $x_2 > x_1$ *and* $\lambda_t^Y(x|y_2) \geq \lambda_t^Y(x|y_1)$ *for* $y_2 > y_1$*.*[13]

We further require that the meeting rate is Lipschitz continuous and linearly bounded in the following sense.[14]

Assumption 2 (regularity of meetings)**.** *There exists* $L^\lambda > 0$ *such that for all* x, y *and* z

(i) $\lambda_t^X(y|x) \leq L^\lambda\big(1 + \mu_t^Y(y)\big)$ *and* $\lambda_t^Y(x|y) \leq L^\lambda\big(1 + \mu_t^X(x)\big)$*;*

(ii) $N(\lambda(t, \mu_t')(\cdot|z), \lambda(t, \mu_t'')(\cdot|z)) \leq L^\lambda N(\mu_t', \mu_t'')$*.*

Population Dynamics

Population dynamics are governed by entry and exit. Any two agents x and y of opposite populations that meet and mutually consent to form a match exit the search pool. The rate at which an individual agent type x matches and exits the market at time t—the hazard rate—is

$$\int_0^1 m_t(x, y)\lambda_t^X(y|x)dy;$$

$m_t(x, y) \in \{0, 1\}$, determined in equilibrium, denotes the time-t match indicator. This is equal to one if agent types x and y match upon meeting and zero otherwise.

Entry is characterized by a time-variant rate $\eta = (\eta^X, \eta^Y)$. In the simplest case, the search pool ebbs and flows due to time-dependent entry rates that fluctuate over time such as in hot and cold housing markets. Similarly, in a gradually clearing search pool without entry, i.e., $\eta_\tau^X(x) = 0$, the steady state corresponds to the trivial case where no agent is left in the search

[13] Under symmetric populations, i.e., $\mu_t^X(x) = \mu_t^Y(x)\ \forall x \in [0, 1]$, it is easy to prove that coherency of the model demands that any hierarchical search technology is in fact anonymous. This means that all types meet others at the same rate: for all y_1, y_2, $\lambda_t^X(y|x_2) = \lambda_t^X(y|x_1)$.

[14] Assumptions 1 and 2 relax a proportionality assumption in Lauermann et al. (2020). In order to prove the fundamental matching lemma in the steady state (see their condition 32) they assume that $\lambda^X(t, \mu_t)(y|x)$ is proportional to $\mu_t^Y(y)$. Our non-stationary analysis does not require this.

pool. More generally, we take η to be a function of the underlying state variable μ_t and time t. Then $\eta_t^X(x) \equiv \eta^X(t,\mu_t)(x)$ is agent type x's time-t entry rate.[15,16]

Assumption 3 (regularity of entry)**.** *There exists $L^\eta > 0$ such that for all x, y and t*

(i) $\eta_t^X(x) \leq L^\eta$ *and* $\eta_t^Y(y) \leq L^\eta$*;*

(ii) $N(\eta(t,\mu'_t),\eta(t,\mu''_t)) \leq L^\eta N(\mu'_t,\mu''_t)$.

The economy can be non-stationary in that entry and exit need not be equal, leading to a time-variant state $\mu_t = (\mu_t^X, \mu_t^Y)$:[17]

$$\mu_{t+h}^X(x) = \mu_t^X(x) + \int_t^{t+h} \left\{ -\mu_\tau^X(x) \int_0^1 \lambda_\tau^X(y|x) m_\tau(x,y)dy + \eta_\tau^X(x) \right\} d\tau. \tag{1}$$

Proposition 1. *System (1) admits a unique solution for any $(\mu_0, \lambda, \eta, m)$ satisfying Assumptions 2 and 3.*

The proof of this result is adapted from the well-known Cauchy-Lipschitz-Picard-Lindelöf theorem that establishes the existence of a unique solution of a system of finite-dimensional ODEs (see Appendix A.2).[18]

Observe for comparison that the initial search pool population cannot be taken as arbitrary in the steady state. The timeless state μ_0 is to be deduced for a given matching rate m so that exit and inflow (generated by exogenous match destruction) into the search pool exactly balance each other. The unique existence of such a state μ_0 is the "hardest step" to prove the existence of a stationary equilibrium.[19] Our non-stationary analysis will grapple with different difficulties that we will soon introduce.

[15]The entry rate η encompasses several natural entry rates such as no entry and constant flows of entry (as in Burdett and Coles (1997)). In addition, entry may be generated by exogenous match destruction (as in Shimer and Smith (2000) and Smith (2006)). Exogenous match destruction entails a time-invariant distribution of agents $\ell(x)dx$, whether they are matched or unmatched. $\big(\ell(x) - \mu_t(x)\big)dx$ is the measure of matched agents at time t, existing matches are destroyed according to an exogenous rate $\delta \in (0,1)$. The entry rate is $\eta(\mu_t)(x) = \delta\big(\ell(x) - \mu_t(x)\big)$.

[16]Assumption 3 ensures that $L^\eta(1+\mu_t^X(x))$ is an upper bound on the entry rate. Assuming maximal entry and no exit, we obtain a simple ODE, $d\overline{\overline{\mu}}_t/dt = L^\eta(1+\overline{\overline{\mu}}_t)$ with initial condition $\overline{\overline{\mu}}_0 = \max\{\sup_x \mu_0^X(x), \sup_y \mu_0^Y(y)\}$, that bounds any solution to (1). In particular, the state can grow at most exponentially so that $\mu_t^X(x) \leq \overline{\overline{\mu}}_t \equiv (1+\overline{\overline{\mu}}_0)e^{tL^\eta} - 1$.

[17]Our formulation is that of a system of integral equations rather than differential equations, because the left- and right time derivative of $\mu_t^X(x)$ do not always coincide as will be the case if $z \mapsto \int_0^z m_t(x,y)dy$ is discontinuous.

[18]There is one difference to the classical result. Owing to our focus on a continuum of types the system is infinite-dimensional. We draw on the more general treatment by Dieudonné (2013) (see Chapter 10.4) to deal with the dimensionality of our problem. What is key when passing from the finite to the infinite is the mean-field property embedded in Assumptions 2 and 3.

[19]This result is also known as the fundamental matching lemma. Shimer and Smith (2000) proved it when there is a quadratic search technology and endogenous match destruction. More recently, Lauermann et al. (2020) generalized the result to encompass a broader class of search technologies.

Value-of-Search

Any given agent's experience in the search pool is characterized by random encounters with other agents. Presented with a match opportunity, an agent must weigh the immediate match payoff against the option value-of-search, the discounted expected future match payoff were one to continue one's search. Denote agent type x's time-t value-of-search $V_t^X(x)$ and $\pi_t^X(y|x)$ the one-time match payoff when matching with y. Naturally, the optimal matching decision is to accept to match with another agent whenever the payoff exceeds the option value-of-search:

$$\pi_t^X(y|x) \geq V_t^X(x). \tag{OS}$$

Knowledge of the value-of-search uniquely determines the match indicator:

$$m_t(x,y) = \begin{cases} 1 & \text{if } \pi_t^X(y|x) \geq V_t^X(x) \text{ and } \pi_t^Y(x|y) \geq V_t^Y(y), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Our definition of the value-of-search is recursive: agents form beliefs about future match probabilities and payoffs. Future match probabilities depend jointly on the Poisson rate λ and match outcomes upon meeting m—which depends on the value-of-search. Reflecting optimality of individual strategies, we define the value-of-search to be the solution to

$$V_t^X(x) = \int_t^\infty e^{-\rho(\tau-t)} \int_0^1 \pi_\tau^X(y|x)\, p_{t,\tau}^X(y|x) dy\, d\tau, \tag{3}$$

where $p_{t,\tau}^X(y|x)$ is the density of future matches with y at time τ conditional on x being unmatched at time t. This is a standard object and is characterized by the matching rate (see Appendix A.1):

$$p_{t,\tau}^X(y|x) = \Lambda_\tau^X(y|x) \exp\Big\{ - \int_t^\tau \int_0^1 \Lambda_r^X(z|x) dz dr \Big\} \text{ where } \Lambda_\tau^X(y|x) = \lambda^X(\tau, \mu_\tau)(y|x) m_\tau(x,y).$$

Payoffs: NTU and TU

We consider the two most studied paradigms for defining match payoffs: *non-transferable utility (NTU)* and *transferable utility (TU)*.

In the NTU paradigm, match payoffs are exogenously given and time-invariant. We denote $\pi^X(y|x) = \pi_t^X(y|x)$ and normalize payoffs, i.e., $\pi^X(y|x) \in [0,1]$. This paradigm precludes individualized price-setting and bargaining.

Assumption 4 (NTU-increasing match payoffs). *Match payoffs are strictly increasing in the partner's type. There exists* $\Delta > 0$ *such that* $\pi^X(y_2|x) - \pi^X(y_1|x) > \Delta(y_2 - y_1)$ *and* $\pi^Y(x_2|y) - \pi^Y(x_1|y) > \Delta(x_2 - x_1)$ *for all* $y_2 > y_1$ *and* $x_2 > x_1$*.*

Alternatively, the TU paradigm takes as its primitive the match output $f(x, y) \in [0, 1]$, generated when agent types x and y match with one another. Any division of output is conceivable. As in the Diamond-Mortensen-Pissarides model, we use Nash bargaining as a solution concept for the bargaining problem in which agents can claim their value-of-search $V_t^X(x)$ as a threat point. Surplus $f(x, y) - V_t^X(x) - V_t^Y(y)$ is shared according to bargaining weights α^X and α^Y (where $\alpha^X + \alpha^Y = 1$). Formally,

$$\pi_t^X(y|x) = V_t^X(x) + \alpha^X[f(x, y) - V_t^X(x) - V_t^Y(y)]. \tag{4}$$

It follows that match decisions (2) are intratemporally efficient: $m_t(x, y) = 1$ if and only if $f(x, y) - V_t^X(x) - V_t^Y(y) \geq 0$.

Assumption 5 (TU-increasing differences). *Differential match output is strictly increasing in types. There exists* $\Delta > 0$ *such that* $f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1) > \Delta(x_2 - x_1)(y_2 - y_1)$ *for all* $y_2 > y_1$ *and* $x_2 > x_1$*.*

Becker (1973) shows that Assumption 5, also known as strict supermodularity, and Assumption 4 guarantee positive assortative matching in the core allocation for the frictionless TU and NTU paradigm respectively. We need those assumptions to impose structure on match decisions when proving equilibrium existence.

We moreover require regularity conditions regarding match payoff and output:

Assumption 6 (NTU). $x \mapsto \pi^X(y|x)$ *and* $y \mapsto \pi^Y(x|y)$ *admit a uniform bound* L^π *on total variation.*

Similarly, we require the same condition for output in the TU paradigm:

Assumption 7 (TU). $x \mapsto f(x, y)$ *and* $y \mapsto f(x, y)$ *admit a uniform bound* L^f *on total variation.*

These assumptions are weaker than requiring that $\pi^X(y|x)$, $\pi^Y(x|y)$ and $f(x, y)$ are continuously differentiable (as in Smith (2006) and Shimer and Smith (2000)).[20]

We will make use of these assumptions to prove bounded variation of the value-of-search (Proposition 6).

[20] Discontinuities in payoffs can arise naturally, e.g., when agents differ in discrete attributes such as workers' professional degrees, location or export focus of a firm, number of bedrooms in the rental market (see Glaeser and Luttmer (2003)), or political affiliation in the marriage market.

3 Equilibrium

An equilibrium jointly determines the evolution of the endogenous variables of the search-and-matching economy: the distribution of agents' characteristics in the search pool, agents' continuation values of search, matching decisions and transfers (under bargaining in the TU paradigm). None of those can be determined in isolation. Agents compute their value-of-search given their beliefs about the economy at large. In equilibrium, each individual correctly anticipates future match opportunities and payoffs. This generates a feedback loop between the population dynamics and the value-of-search.

Definition 1. *An equilibrium of the search-and-matching economy of given initial search pool population μ_0 is a triple $(\boldsymbol{\mu}, \mathbf{V}, \mathbf{m})$, solution to (1),(2) and (3), where (NTU) payoffs are exogenously given, or (TU) determined via Nash bargaining (4).*

The interplay between aggregate dynamics and the individual decision problem is a feature shared with virtually all dynamic general equilibrium models under rational expectations.

The main result of this paper is to show that an equilibrium exists, both in the NTU and TU paradigm.

Theorem 1. *An equilibrium of the NTU search-and-matching economy $(\mu_0, \lambda, \eta, \rho, \pi)$ satisfying Assumptions 1, 2, 3, 4, 6 exists.*

Theorem 2. *An equilibrium of the TU search-and-matching economy $(\mu_0, \lambda, \eta, \rho, \alpha, f)$ satisfying Assumptions 1, 2, 3, 5, 7 exists.*

The proof of both results will be developed jointly in Sections 4 and 5.

4 A Fixed Point Theorem for Non-Stationary Models

In this section, we develop a fixed point theorem that will help us prove the existence of an equilibrium of the search-and-matching economy. It is a non-trivial adaptation of the well known Schauder-Tychonoff fixed point theorem (Schauder (1930) - Tychonoff (1935)). Due to its potential appeal for proving existence in other models, this section is self-contained.

Our theorem applies to continuous-time, infinite-horizon models in which a group of heterogeneous agents, as described by a type $x \in [0, 1]$, take actions that affect others through the aggregate only. This class of models is sometimes referred to as anonymous or mean-field games. Within this class of models, our fixed point theorem is sufficiently general to allow the value function to fluctuate endogenously over time and to be discontinuous with respect to time and type. It is therefore relevant in models that dispense with the steady state assumption or admit pooling behavior. The latter arises in search-and-matching models in the form of block segregation (refer to McNamara and Collins (1990) for the original result; also see Smith (2006) and references therein.) Note that existing applications of fixed point theorems in economic models (see for instance Stokey and Lucas (1989)) often rely on the Arzela-Ascoli theorem, which explicitly rules out discontinuities in the value function.

4.1 Preliminaries and Statement of the Theorem

We would like to establish the existence of a fixed point of the operator $H = (H^1, ..., H^N) : \mathcal{F}^N \to \mathcal{F}^N$ where $\mathcal{F}$ and $\mathcal{F}^N$ are the spaces of measurable mappings $[0,1] \times \mathbb{R}_+ \to [0,1]$ and $[0,1] \times \mathbb{R}_+ \to [0,1]^N$ respectively. In more detail, a fixed point is a mapping $\overline{F} = (\overline{F}^1, ..., \overline{F}^N) \in \mathcal{F}^N$ such that $H[\overline{F}] = \overline{F}$.

To state the theorem we introduce two notions of distance. First, the continuity premise of our fixed point theorem requires the following operator to measure the "distance" between functions.

Definition 2 (pseudo-seminorm). *Define, for all functions $F = (F^1, ..., F^N) \in \mathcal{F}^N$,*

$$||F|| = \max_{n \in \{1,...,N\}} \int_0^\infty \int_0^1 e^{-t} \big| F^n(x,t) \big| dt\, dx.$$

The mapping $(F, \bar{F}) \mapsto ||F - \bar{F}||$ is called a pseudo-metric because it satisfies symmetry and the triangle inequality. Following this terminology, we call $|| \cdot ||$ a pseudo-seminorm.

Second, we introduce the total variation norm for mappings in $\mathcal{F}$. As we shall see, if a set of functions is uniformly bounded in the total variation norm, then it is sequentially compact. Our focus on two-dimensional functions is a special case of the general definition provided by Idczak and Walczak (1994) and Leonov (1996).[21]

Definition 3 (total variation norm). *The total variation norm for functions $F^n \in \mathcal{F}$ and arbitrary bounded time interval $[\underline{t}, \overline{t}]$ is given by*

$$T\mathcal{V}\big(F^n, [0,1] \times [\underline{t}, \overline{t}]\big) = \mathcal{V}_0^1\big(F^n(\cdot, \underline{t})\big) + \mathcal{V}_{\underline{t}}^{\overline{t}}\big(F^n(0, \cdot)\big) + \mathcal{V}_2\big(F^n, [0,1] \times [\underline{t}, \overline{t}]\big)$$

with

$$\mathcal{V}_0^1\big(F^n(\cdot, t_0)\big) = \sup_{\mathcal{P}} \sum_{i=1}^{m} \big| F^n(x_i, t_0) - F^n(x_{i-1}, t_0) \big|$$

[21] Subsequent work by Chistyakov and Tretyachenko (2010) extends the total variation norm to more abstract spaces.

where $\mathcal{P}$ is a partition of $[0,1]$, i.e. $0 = x_0 < x_1 < ... < x_m = 1$,

$$\mathcal{V}_{\underline{t}}^{\overline{t}}\big(F^n(0,\cdot)\big) = \sup_{\mathcal{P}} \sum_{i=1}^{m} \big|F^n(0,t_i) - F^n(0,t_{i-1})\big|$$

where $\mathcal{P}$ is a partition of $[\underline{t},\overline{t}]$, i.e. $\underline{t} = t_0 < t_1 < ... < t_m = \overline{t}$,

$$\mathcal{V}_2\big(F^n,[0,1]\times[\underline{t},\overline{t}]\big) = \sup_{\mathcal{P}} \sum_{i=1}^{m} \big|F^n(x_i,t_i) - F^n(x_i,t_{i-1}) - F^n(x_{i-1},t_i) + F^n(x_{i-1},t_{i-1})\big|$$

where $\mathcal{P}$ is a discrete path in $[0,1]\times[\underline{t},\overline{t}]$, s.t. $\underline{t} = t_0 < t_1 < ... < t_m = \overline{t}$ and $0 = x_0 < x_1 < ... < x_m = 1$.

We can now state our fixed point theorem:

Theorem 3. *Suppose that $H : \mathcal{F}^N \to \mathcal{F}^N$ satisfies*

(i) *for all $\overline{F} = (\overline{F}^1, ..., \overline{F}^N) \in \mathcal{F}^N$ and $\epsilon > 0$ there exists $\delta > 0$ such that $\big|\big|H[\overline{F}] - H[F]\big|\big| < \epsilon$ for all $F = (F^1, ..., F^N) \in \mathcal{F}^N$ such that $\big|\big|\overline{F} - F\big|\big| < \delta$;*

(ii) $\forall T > 0\ \exists C > 0$ *such that* $\forall n \in \{1, ..., N\} : T\mathcal{V}\big(H^n[F], [0,1]\times[0,T]\big) < C$ *for all* $F \in \mathcal{F}^N$.

Then H admits a fixed point.

We will refer to condition (i) as continuity and (ii) as uniformly bounded variation.

4.2 Proof of the Fixed Point Theorem

Outline of the proof: we construct a sequence of operators that approximate the fixed point operator H (Step 1). Each approximate fixed point operator will satisfy all the assumptions of the Schauder fixed point theorem (Step 0, Step 2) and hence admits a fixed point (Step 2). We then show that the sequence of approximate fixed point admits a convergent subsequence (Step 3). To conclude, we prove that H maps the convergent subsequence's limit point into a fixed point of H (Step 4).

To begin with, endow $\mathcal{F}$ and $\mathcal{F}^N$ with the discounted supremum metric. Discounting is what helps us deal with an infinite horizon.

Definition 4 (discounted sup metric)**.** *The discounted sup metric for functions $F = (F^1, ..., F^N) \in \mathcal{F}^N$ and $\overline{F} = (\overline{F}^1, ..., \overline{F}^N) \in \mathcal{F}^N$ is given by*

$$\mathbf{d}^N(F,\overline{F}) = \max_{n\in\{1,...,N\}} \mathbf{d}(F^n,\overline{F}^n) = \max_{n\in\{1,...,N\}} \sup_{x,t} e^{-t}\big|F^n(x,t) - \overline{F}^n(x,t)\big|.$$

Step 0 (Preliminary): A Set of Compact Functions

To apply Schauder's fixed point theorem we require that the fixed point operator maps into a set of compact functions. As a preliminary step, we show that the set of k-Lipschitz functions is compact. (Note however that functions in the image of H need not be k-Lipschitz, let alone continuous.)

A function $F^m : [0,1] \times \mathbb{R}_+ \to [0,1]$ is k-Lipschitz if for any $(x,t),(y,r) \in [0,1] \times \mathbb{R}_+$

$$\left|F^m(x,t) - F^m(y,r)\right| \le k \cdot \max\left\{|x-y|, |t-r|\right\}.$$

Denote $\mathcal{F}_{(k)} \subset \mathcal{F}$ the (convex) subset of k-Lipschitz functions.

Proposition 2. *$(\mathcal{F}_{(k)}, \mathbf{d})$ is compact.*

The proof of this Proposition is deferred to Appendix B.1.

Step 1: Construction of the Approximate Fixed Point Operator

We construct an approximate fixed point operator that is continuous and maps into the set of k-Lipschitz functions. We achieve this via convolution. Since our type set is an interval, we need to specify what happens at the boundary $x \in \{0,1\}$ as well as the origin of time $t = 0$. To avoid mass points we extend the support of functions in $\mathcal{F}$ from $[0,1] \times [0,\infty)$ to $[-1,2] \times [-1,\infty)$.

Denote the approximate operator $H^m_{(k)} : \mathcal{F}^N \to \mathcal{F}$, and define, for any $(x_0,t_0) \in [0,1] \times \mathbb{R}_+$,

$$H^m_{(k)}[F](x_0,t_0) = \int_{-1}^{2} \int_{-1}^{\infty} \hat{H}^m[F](x,t)\delta_{(k)}(x_0 - x, t_0 - t) dx\, dt$$

where, first, $\hat{H}^m[F]$ is the extension of $H^m[F] \in \mathcal{F}$ to a mapping $[-1,2] \times [-1,\infty) \to [0,1]$,

$$\hat{H}^m[F](x,t) = \begin{cases} H^m[F](|x|,|t|) & \text{if } -1 \le x < 0 \\ H^m[F](x,|t|) & \text{if } 0 \le x \le 1 \\ H^m[F](2-x,|t|) & \text{if } 1 < x \le 2, \end{cases}$$

and, secondly, for $b_{(k)} = 4/k$ and $k \ge 4$ we define

$$\delta_{(k)}(x,t) = \frac{1}{(b_{(k)})^2} \quad \text{if } (x,t) \in B_{(k)}(0) \equiv \left\{(x',t') \in \mathbb{R}^2 : \max\{|x'|,|t'|\} \le \frac{b_{(k)}}{2}\right\}, \text{ zero otherwise.}$$

Step 2: Properties of the Approximate Fixed Point Operator

We now show that the approximate fixed point operator satisfies all the necessary properties that allow us to apply the Schauder fixed point theorem: compactness of its image and continuity.

Lemma 1. $H^m_{(k)}[\mathcal{F}^N] \subseteq \mathcal{F}_{(k)}$

Lemma 2. $H^m_{(k)} : (\mathcal{F}^N, \mathbf{d}^N) \to (\mathcal{F}, \mathbf{d})$ *is continuous.*

The proof of both Lemmata is deferred to Appendix B.2.

Proposition 3. $H_{(k)} = (H^1_{(k)}, ..., H^N_{(k)}) : \mathcal{F}^N \to \mathcal{F}^N$ *has a fixed point* $F^*_{(k)}$.

This Proposition is an application of the Schauder fixed point theorem.

Proof. First observe that $(\mathcal{F}^N, \mathbf{d}^N)$ is a complete metric space (see Theorem 43.5. in Munkres (2015)) and that $(\mathcal{F}^N_{(k)}, \mathbf{d}^N)$ is a subset of this space. Moreover, observe that the metric $\mathbf{d}^N$ satisfies the three axioms posited by Schauder (1930).[22] Second, since $(\mathcal{F}^N_{(k)}, \mathbf{d}^N)$ is the finite-dimensional product of compact sets (Proposition 2), it is compact. It is also closed (since $\mathcal{F}^N_{(k)}$ is compact) and convex. Finally, continuity of the component operator $H^n_{(k)}$ (Lemma 2) on the larger space $\mathcal{F}^N$ establishes continuity of $H_{(k)} = (H^1_{(k)}, ..., H^N_{(k)}) : (\mathcal{F}^N_{(k)}, \mathbf{d}^N) \to (\mathcal{F}^N_{(k)}, \mathbf{d}^N)$. Then the Schauder fixed point theorem (see Schauder (1930), Satz I) asserts that *if the continuous operator* $H_{(k)}$ *maps the convex, closed and compact set* $\mathcal{F}^N_{(k)}$ *into itself, then there exists a fixed point* $F^*_{(k)}$, *i.e.,* $H_{(k)}[F^*_{(k)}] = F^*_{(k)}$. □

Step 3: Existence of a Convergent Subsequence of Approximate Fixed Points

By considering all $k \in \mathbb{N}$, Proposition 3 establishes that there exists a sequence of approximate fixed points $(F^*_{(k)})_{k\in\mathbb{N}}$. We now show that this sequence admits a convergent subsequence.

Proposition 4. *The fixed points* $(F^*_{(k)})_{k\in\mathbb{N}}$ *admit an accumulation point* F^* *in* $(\mathcal{F}^N, \mathbf{d}^N)$.

This follows from a higher-dimensional Helly-type selection theorem.

Proof. Idczak and Walczak (1994) and Leonov (1996) (Theorem 4) prove that *if all the elements of a sequence of functions* $(F^n_{(k)})_{k\in\mathbb{N}}$*:* $F^n_{(k)} \in \mathcal{F}$ *satisfy* $T\mathcal{V}\big(F^n_{(k)}, [0,1] \times [0,T]\big) < C$ *for some constant* C*, then the sequence admits a subsequence of functions that converges pointwise on*

[22] Those axioms are 1° $\mathbf{d}^N(F, \bar{F}) = \mathbf{d}^N(F - \bar{F}, 0)$, 2° $\lim_{n\to\infty} \mathbf{d}^N(F_{(n)}, \bar{F}) = \lim_{n\to\infty} \mathbf{d}^N(G_{(n)}, \bar{G}) = 0$ implies $\lim_{n\to\infty} \mathbf{d}^N(F_{(n)} + G_{(n)}, \bar{F} + \bar{G}) = 0$ and 3° for $\{\lambda_n\}$ a sequence of real numbers and $\{F_{(n)}\}$ a sequence in $\mathcal{F}^N$, $\lim_{n\to\infty} \lambda_n = \lambda$, $\lim_{n\to\infty} \mathbf{d}^N(F_{(n)}, \bar{F}) = 0$ implies $\lim_{n\to\infty} \mathbf{d}^N(\lambda_n F_{(n)}, \lambda\bar{F}) = 0$. Those axioms are naturally satisfied if the metric is induced by a norm (which is prohibited by discounting in our case).

$[0,1] \times [0,T]$ *to a function with the same property.* Assumption (ii) of the theorem asserts the existence of such a uniform bound C for all $H^n[F] \in \mathcal{F}$ where $n \in \{1, ..., N\}$ and $F \in \mathcal{F}^N$. Since the uniform bound on the total variation will be preserved under convolution, such bound also obtains for all $H^n_{(k)}[F] \in \mathcal{F}$ where $k \in \mathbb{N}$. In particular, this applies for the sequence $(H^n_{(k)}[F^*_{(k)}])_{k\in\mathbb{N}} = (F^{*,n}_{(k)})_{k\in\mathbb{N}}$. Then, for all $T > 0$ and $n = 1$, the Helly-type selection theorem ensures the existence of a pointwise convergent subsequence $(F^{*,1}_{(k_\ell)})_{k\in\mathbb{N}}$ on $[0,1] \times [0,T]$ where each $F^{*,1}_{(k_\ell)} \in \mathcal{F}$. Then, iterating over n ensures the existence of a pointwise convergent subsequence $(F^*_{(k_\ell)})_{k\in\mathbb{N}}$ in $[0,1] \times [0,T]$ where $F^*_{(k_\ell)} \in \mathcal{F}^N$. Following an identical reasoning, we can find a subsequence of the subsequence which converges pointwise in $[0,1] \times [0,T+1]$. Proceeding by induction then establishes pointwise convergence in $[0,1] \times [0,\infty)$; we denote $F^* \in \mathcal{F}^N$ the limit point. □

Step 4: Conclusion

The preceding steps established the existence of a convergent subsequence of approximate fixed points. Proposition 5 asserts that the image of this limit is a fixed point of H, which concludes the proof of Theorem 3.

Proposition 5. *$H[F^*]$ is a fixed point of $H : \mathcal{F}^N \to \mathcal{F}^N$.*

The proof of this Proposition is deferred to Appendix B.3.

5 Proving Existence

In this section, we use Theorem 3 to prove that the search-and-matching economy admits an equilibrium. In Section 5.1, we construct an operator $\boldsymbol{V} : \mathcal{F}^2 \to \mathcal{F}^2$ that maps a value-of-search profile $v = (v^X, v^Y) \in \mathcal{F}^2$ into a new value-of-search profile. As in the preceding section, $\mathcal{F}^2$ is the space of jointly measurable mappings $[0,1] \times \mathbb{R}_+ \to [0,1]^2$. We will show that there exists an equilibrium of the economy if and only if $\boldsymbol{V}$ admits a fixed point. In light of Theorem 3, $\boldsymbol{V}$ admits a fixed point if it satisfies two conditions: continuity and uniformly bounded variation. As the key technical contribution we prove in Section 5.2 that the operator $\boldsymbol{V}$ satisfies both. Section 5.3 concludes the existence proof.

It is worthwhile to mention that our proof of equilibrium existence implies that continuity and uniformly bounded variation are equilibrium properties of the value-of-search.

5.1 Construction of a Fixed Point Operator

Since our focus is on both the TU and the NTU paradigm, we construct two separate fixed point operators, denoted $\overset{NTU}{\boldsymbol{V}}$ and $\overset{TU}{\boldsymbol{V}}$. For ease of notation, we omit the superscript whenever we prove properties that pertain to both paradigms.

An interpretation may help fathom our construction. One may think of $v \in \mathcal{F}^2$ as a belief about other agents' value-of-search. Under this interpretation $\boldsymbol{V}_t^X[v](x)$ becomes agent type x's time-t out-of-equilibrium value-of-search when he holds the belief v. This means that x expects other agents to match according to v, yet computes his own value-of-search under the rule that he accepts a match whenever it is optimal for him to do so: accept if the offered match payoff exceeds the discounted expected future match payoff.[23] Observe that this is an interpretation only. Our construction attempts to preserve desirable in-equilibrium properties of the value-of-search, not decide what is the most "reasonable" out-of-equilibrium behavior.

5.1.1 Non-Transferable Utility

To compute his value-of-search, an agent must hold a belief over the likelihood of future meetings. This is a function of the underlying state variable μ_t and time. We begin by defining the aggregate population dynamics under the belief v.

Definition 5. *$\overset{NTU}{\boldsymbol{\mu}_t}[v]$ is the unique solution to (1) for given $(\mu_0, \lambda, \eta, \overset{NTU}{m}[v])$, where*

$$\overset{NTU}{m_t}[v](x,y) = \begin{cases} 1 & \text{if } \pi^X(y|x) \geq v_t^X(x) \text{ and } \pi^Y(x|y) \geq v_t^Y(x) \\ 0 & \text{otherwise} \end{cases}$$

is the aggregate probability of matching upon meeting under v.

In contrast, agent type x accepts any match whose payoff exceeds his expected discounted match payoff under v, not the value-of-search $v_t^X(x)$ he ascribes to other agents of identical type x. As in the set-up, this match acceptance rule gives rise to an implicit definition of the value-of-search.

Definition 6. *The out-of-equilibrium value-of-search given v is the solution to*

$$\overset{NTU}{\boldsymbol{V}}{}_t^X[v](x) = \int_t^\infty e^{-\rho(\tau-t)} \int_0^1 \pi^X(y|x) \overset{NTU}{\boldsymbol{p}}{}_{t,\tau}^X[v](y|x)dy\, d\tau, \tag{5}$$

[23]Payoffs in the TU paradigm are computed under x's belief that her threat point will be $\boldsymbol{V}_t^X[v](x)$ whereas her potential partner's threat point is $v_t^Y(y)$.

where x's match acceptance decisions are individually rational,

$$\overset{NTU}{\boldsymbol{m}}_t[v](x,y) = \begin{cases} 1 & \text{if } \pi^X(y|x) \geq \overset{NTU}{\boldsymbol{V}}{}^X_t[v](x) \text{ and } \pi^Y(x|y) \geq v^Y_t(y) \\ 0 & \text{otherwise,} \end{cases}$$

and the probability of meetings is pinned down by aggregate match decisions,

$$\overset{NTU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x) = \lambda^X(\tau, \overset{NTU}{\boldsymbol{\mu}}_\tau[v])(y|x)\overset{NTU}{\boldsymbol{m}}_\tau[v](x,y)\exp\Big\{-\int_t^\tau\int_0^1 \lambda^X(r, \overset{NTU}{\boldsymbol{\mu}}_r[v])(y'|x)\overset{NTU}{\boldsymbol{m}}_r[v](x,y')dy'dr\Big\}.$$

We remark that the operator $\overset{NTU}{\boldsymbol{V}}{}^X_t[v](x)$ in Definition 6 is well-defined. The unique existence of a solution $\overset{NTU}{\boldsymbol{V}}{}^X_t[v](x)$ to equation (5) follows, because the recursively defined value-of-search is the supremum of the right-hand side of equation (5) over the set of match indicators $\boldsymbol{m}_t(x,y)$ satisfying $\boldsymbol{m}_t(x,y) = 0$ if $\pi^Y(x|y) < v^Y_t(y)$.

5.1.2 Transferable Utility

As in the NTU construction, we begin by defining the aggregate population dynamics under the belief v.

Definition 7. *$\overset{TU}{\boldsymbol{\mu}}_t[v]$ is the unique solution to (1) for given $(\mu_0, \lambda, \eta, \overset{TU}{m}[v])$, where*

$$\overset{TU}{m}_t[v](x,y) = \begin{cases} 1 & \text{if } f(x,y) - v^X_t(x) - v^Y_t(y) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is the aggregate probability of matching upon meeting under the value-of-search profile v.

To define the individual value-of-search in the TU paradigm we must also specify future match payoffs. Those are defined implicitly by the Nash bargaining solution. The individual agent believes that her threat point is her actual value-of-search whereas her potential partner's threat point is $v^Y_t(y)$.

Definition 8. *The out-of-equilibrium value-of-search given belief v is the solution to*

$$\overset{TU}{\boldsymbol{V}}{}^X_t[v](x) = \int_t^\infty e^{-\rho(\tau-t)}\int_0^1 \overset{TU}{\boldsymbol{\pi}}{}^X_t[v](y|x)\overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x)dy\,d\tau,$$

where x's subjective match payoffs are

$$\overset{TU}{\boldsymbol{\pi}}{}^X_t[v](y|x) = \overset{TU}{\boldsymbol{V}}{}^X_t[v](x) + \alpha^X\big(f(x,y) - \overset{TU}{\boldsymbol{V}}{}^X_t[v](x) - v^Y_t(y)\big),$$

x's match acceptance decisions are individually rational,

$$\overset{TU}{\boldsymbol{m}}_t[v](x,y) = \begin{cases} 1 & \textit{if } f(x,y) - \overset{TU}{\boldsymbol{V}}{}^X_t[v](x) - v^Y_t(y) \geq 0 \quad \Leftrightarrow \quad \overset{TU}{\boldsymbol{\pi}}{}^X_t[v](y|x) \geq \overset{TU}{\boldsymbol{V}}{}^X_t[v](x) \\ 0 & \textit{otherwise} \end{cases}$$

and the probability of meetings is pinned down by aggregate match decisions,

$$\overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x) = \lambda^X(\tau, \overset{TU}{\boldsymbol{\mu}}_\tau[v])(y|x)\overset{TU}{\boldsymbol{m}}_\tau[v](x,y)\exp\Big\{ -\int_t^\tau \int_0^1 \lambda^X(r, \overset{TU}{\boldsymbol{\mu}}_r[v])(y'|x)\overset{TU}{\boldsymbol{m}}_r[v](x,y')dy'dr\Big\}.$$

Definition 8 is well-posed for identical reasons as in the NTU paradigm.

5.1.3 Equilibrium Characterization

We now establish the purpose of our construction: we show that there exists an equilibrium of the economy if and only if $\boldsymbol{V}$ admits a fixed point. Even though an equilibrium is a triple, the value-of-search encodes all the information needed to recover the match indicator function (through Equation (2)), whence the state μ (through Equation (1) as shown in Proposition 1). It then follows from simple inspection that an equilibrium value-of-search, solution to Equations (1),(2), (3) (NTU) and (1),(2), (3), (4) (TU) is a fixed point of the operator $\boldsymbol{V}$ as defined in Definition 6 (NTU) and 8 (TU). Since the converse is also true, one can recast an equilibrium value-of-search as a fixed-point of the out-of-equilibrium value-of-search.

Remark 1. *For given μ_0, there exists an equilibrium of the NTU search-and-matching economy if and only if $\overset{NTU}{\boldsymbol{V}} : \mathcal{F}^2 \to \mathcal{F}^2$ admits a fixed point. For given μ_0, there exists an equilibrium of the TU search-and-matching economy if and only if $\overset{TU}{\boldsymbol{V}} : \mathcal{F}^2 \to \mathcal{F}^2$ admits a fixed point.*

Many definitions of the out-of-equilibrium value-of-search are conceivable whose fixed points correspond to an equilibrium value-of-search profile. Our construction for the TU paradigm in particular may seem arbitrary. But it is not. It emerges as the one construction which at the same time is continuous in beliefs (Propositions 7 and 8) and satisfies the mimicking argument (Lemma 3 and 4), whence bounded variation (Proposition 6). Those properties will be shown in what follows.

5.2 Properties of the Fixed Point Operator

5.2.1 The Mimicking Argument

A comparison of the value-of-search across types is difficult in non-stationary environments. The law of motion of the economy easily becomes intractable, rendering it impossible to characterize the value-of-search in closed form. To circumvent this problem, we apply a revealed preference argument that we refer to as *the mimicking argument*. In both paradigms, the underlying idea is to let one agent replicate someone else's match acceptance decisions.

Lemma 3 (NTU mimicking argument). *In the NTU paradigm, posit Assumptions 1, 2, 3 and 4. Then, for all $x_2 > x_1$ there exists a non-negative operator $Q_t^X[v](y|x_1)$, with $\int_0^1 Q_t^X[v](y|x_1)dy < 1$, such that*

$$\overset{NTU}{\boldsymbol{V}}{}_t^X[v](x_2) - \overset{NTU}{\boldsymbol{V}}{}_t^X[v](x_1) \geq \int_0^1 \big(\pi^X(y|x_2) - \pi^X(y|x_1)\big) Q_t^X[v](y|x_1)dy.$$

The proof of this Lemma is deferred to Appendix C.2 and relies on payoff monotonicity (Assumption 4): superior types, being more desirable, can exploit their superior match offerings and replicate match outcomes of any inferior type.[24]

A similar result obtains, albeit for a different reason, in the TU paradigm.

Lemma 4 (TU mimicking argument). *In the TU paradigm, posit Assumption 1, 2 and 3. Then for all $x_2 > x_1$ there exists a non-negative operator $Q_t^X[v](y|x_1)$, with $\int_0^1 Q_t^X[v](y|x_1)dy < 1$, such that*

$$\overset{TU}{\boldsymbol{V}}{}_t^X[v](x_2) - \overset{TU}{\boldsymbol{V}}{}_t^X[v](x_1) \geq \int_0^1 \big(f(x_2, y) - f(x_1, y)\big) Q_t^X[v](y|x_1)dy.$$

The proof of this Lemma is deferred to Appendix C.1 and relies on the efficiency of the Nash bargaining sharing rule (Lemma 5). Shimer and Smith (2000) first established this result in the steady state. To establish it in non-stationary environments raises new technical challenges. Joint with the proof of Theorem 3, Lemma 4 is the most elaborate proof in this paper. The issue in non-stationary environments is that future match payoffs determined under Nash bargaining depend on one's own (hitherto analytically unknown) future value-of-search. Hence, individual payoffs are not readily comparable across types. To achieve such a comparison, we make use of an inductive reasoning over the revealed preference argument described above.

We would like to mention that these two Lemmata serve as keystones to derive sufficient conditions for positive assortative matching in the non-stationary search-and-matching economy (see Bonneton and Sandmann (2022) and Bonneton and Sandmann (2021)).

[24]These results require that $\boldsymbol{\mu}[v]$ is well-defined as asserted by Assumptions 2 and 3 under which Proposition 1 holds. Note that these results equally hold under a partial equilibrium analysis that takes the evolution over the state μ as given.

5.2.2 Bounded Variation

We next establish two regularity conditions of the value-of-search. First, we derive Lipschitz continuity in time. Secondly, we use the mimicking argument to deduce that there exists a uniform bound on total variation of the value-of-search in the type dimension. Whereas these results carry little economic content, they will be key to establishing that the value-of-search is of uniformly bounded variation across jointly types and time (see Premise (ii) of Theorem 3).

Proposition 6 (bounded variation of the value-of-search)**.** *In both paradigms:*

(i) *Posit Assumptions 2 and 3. Then the value-of-search is Lipschitz continuous in time; i.e., for all moments in time $T : 0 < T < \infty$ there exists $C > 0$ such that for all $0 \leq t_1 < t_2 \leq T$ and $x \in [0,1]$*

$$\left|\boldsymbol{V}_{t_2}^{X}[v](x) - \boldsymbol{V}_{t_1}^{X}[v](x)\right| \leq C\left|t_2 - t_1\right| \quad \textit{for all } v \in \mathcal{F}^2;$$

(ii) *Posit Assumptions 1, 2, 3 and 4, 6 (NTU) and 7 (TU). Then the value-of-search is of uniformly bounded variation in type; i.e., for all time indices $t \geq 0$, there exists $C > 0$ such that for all partitions of the type interval $[0,1]$*

$$\sum_{i=0}^{m}\left|\boldsymbol{V}_{t}^{X}[v](x_{i+1}) - \boldsymbol{V}_{t}^{X}[v](x_i)\right| \leq C \quad \textit{for all } v \in \mathcal{F}^2.$$

Note that Lipschitz continuity implies a uniform bound on total variation, hence the condition pertaining to time in (i) is a stronger property than the condition pertaining to types in (ii). The proof of this Proposition is deferred to Appendix D: (i) derives from dynamic programming (Lemma 7), (ii) is due to the mimicking arguments, i.e., Lemma 3 and Lemma 4, the matryochka dolls.

5.2.3 Continuity

We then turn to item (i) from Theorem 3: continuity of the operator $v \mapsto \boldsymbol{V}[v]$.

Proposition 7 (NTU)**.** *In the NTU paradigm, posit Assumptions 2, 3 and 4. Then for all $\overline{v} \in \mathcal{F}^2$, $t \in [0,\infty)$: for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\int_0^1 \left|\overset{NTU}{\boldsymbol{V}}{}_{t}^{X}[v](x) - \overset{NTU}{\boldsymbol{V}}{}_{t}^{X}[\overline{v}](x)\right| dx < \epsilon \quad \textit{for all } v : ||v - \overline{v}|| < \delta.$$

A stronger result obtains in the TU paradigm.

Proposition 8 (TU)**.** *In the TU paradigm, posit Assumptions 2, 3 and 5. For all* $\overline{v} \in \mathcal{F}^2$, $t \in [0, \infty)$, $x \in [0, 1]$*: for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left|\overset{TU}{\boldsymbol{V}}{}^X_t[v](x) - \overset{TU}{\boldsymbol{V}}{}^X_t[\overline{v}](x)\right| < \epsilon \quad \textit{for all } v : ||v - \overline{v}|| < \delta.$$

To establish these Propositions we rely on two intuitive preliminary results. In Appendix E we show that match indicator functions $v \mapsto m_t[v]$ are continuous in both paradigms (see Lemmata 8 and 9).[25] Following arguments from differential calculus we then use Grönwall's lemma to track the non-stationary evolution of the state and deduce continuity of $v \mapsto \boldsymbol{\mu}_t[v]$ (see Lemma 10). Unlike the forward-looking population dynamics, the value-of-search is backward-looking in time. The proof of Propositions 7 and 8 (see Appendices E.5 and E.6) encompasses the infinite time-horizon by considering the auxiliary function $v_t^X(x) = e^{\rho t} V_t^X(x)$ (see Corollary 2) that admits a more tractable HJB equation.

The juxtaposition of both Propositions makes apparent a key difference between the NTU and the TU paradigm. In the TU paradigm the operator $v \mapsto \boldsymbol{V}_t^X[v](x)$ is continuous. In the NTU paradigm it need not be. To see this it is instructive to decompose any type x's time-t match opportunities into marginal and inframarginal prospective partner types. Marginal types y are indifferent between accepting and rejecting x, inframarginal types y strictly prefer entering the match. An increase in other agents' time-t value-of-search has two effects. First, in the TU paradigm x matches with inframarginal consumers at reduced payoffs. Continuity is preserved because the decrease in x's payoff is proportional to the increase in y's value-of-search. Second, in both paradigms x ceases to match with marginal types. The loss of marginal types hurts x in the NTU paradigm because marginal types can be strictly profitable to match with. This gives rise to a discontinuity in the value-of-search operator if the set of marginal types is non-negligible. In the TU paradigm the loss of marginal types is inconsequential due to the intratemporal efficiency of Nash Bargaining: if y is indifferent in between matching and not matching with x, then so is x with regard to y.[26]

[25] Related results are proved in Smith (2006) Lemma 8 a) and Shimer and Smith (2000) Lemma 3 in the steady state when the two populations are symmetric.

[26] Note that we did not solve the model by passing to the mean field limit, i.e., by gradually decreasing the scope of individual agents to influence the future evolution of the search pool. Lemma 10 and Propositions 7 and 8 suggest that doing so would not lead to the selection of a different set of equilibria. Suppose that one agent could control the behavior of an interval of agents and thereby exert some non-negligible influence on the evolution of the state. Our results show that as this interval shrinks, such control has an exceedingly vanishing effect on other agents' matching decisions.

5.3 Applying Theorem 3 to the Fixed Point Operator

We are now reaching the confluence of our efforts. Corollary 1 deduces from the results in the preceding subsection 5.2 that the operators $\overset{NTU}{\boldsymbol{V}}$ and $\overset{TU}{\boldsymbol{V}}$ satisfy the premises of Theorem 3, i.e., continuity and uniformly bounded variation. Hence, both operators admit a fixed point. In light of Remark 1, equilibrium existence in both the TU and NTU paradigm follows.

Corollary 1. *In the NTU paradigm, posit Assumptions 1, 2, 3, 4, 6. Then the operator* $\overset{NTU}{\boldsymbol{V}}$ *satisfies the conditions of Theorem 3. In the TU paradigm, posit Assumptions 1, 2, 3, 5, 7. Then the operator* $\overset{TU}{\boldsymbol{V}}$ *satisfies the conditions of Theorem 3.*

Proof. (i) Continuity: In the NTU paradigm, condition (i) of Theorem 3 corresponds to Proposition 7. In the TU paradigm Proposition 8 establishes a stronger notion of continuity that implies condition (i) of Theorem 3.
(ii) Uniformly bounded variation: condition (ii) of Theorem 3 follows from the second item of Proposition 6 that establishes that both $\mathcal{V}_0^1\big(x \mapsto \boldsymbol{V}_t^X[v](x)\big)$ and $\mathcal{V}_{\underline{t}}^{\bar{t}}\big(t \mapsto \boldsymbol{V}_t^X[v](x)\big)$ are uniformly bounded in both paradigms. Moreover, there exists C such that for all $x \in [0,1]$: $\big|\boldsymbol{V}_{t_2}^X[v](x) - \boldsymbol{V}_{t_2}^X[v](x)\big| \leq C\,|t_2 - t_1|$. Hence,

$$\mathcal{V}_2\big((x,t) \mapsto \boldsymbol{V}_t[v](x), [0,1]\times[0,T]\big) \leq \sup_{\mathcal{P}} \sum_{i=1}^{m} \sup_{x\in[0,1]} \big|\boldsymbol{V}_{t_i}^X[v](x) - \boldsymbol{V}_{t_{i-1}}^X[v](x)\big| \leq 2\,C\,T,$$

where $\mathcal{P}$ is any partition of the time interval $[0,T]$. □

6 Conclusion

Although many economic questions in the search-and-matching literature concern non-stationary dynamics (see for instance Lise and Robin (2017)), the theoretical literature has confined itself, with few exceptions,[27] to the steady state. This paper proves the existence of a non-stationary equilibrium for a general class of search-and-matching models, encompassing model specifications in Shimer and Smith (2000), Smith (2006) and Lauermann et al. (2020).

The tools we develop here have scope, however, that goes beyond search-and-matching. Our fixed point theorem, coupled with the economic insight born out by the mimicking arguments, is applicable in many related dynamic general equilibrium models with heterogeneous agents (see Achdou et al. (2014)) where the aggregate state evolves deterministically over time.

[27] See for instance Boldrin et al. (1993), Burdett and Coles (1998), Shimer and Smith (2001).

Appendix

A Set-up: Omitted Proofs

A.1 Derivation of the Match Density

The probability of agent type x not matching during $[t, \tau]$ is $\exp\big\{-\int_t^\tau \int_0^1 \Lambda_r^X(z|x)dzdr\big\}$ (as defined by the inhomogenous Poisson process). By definition of the density of future matches this expression is equal to $1 - \int_t^\tau \int_0^1 p_{t,r}^X(z|x)dzdr$. Then differentiating with respect to time τ implies that $\int_0^1 p_{t,\tau}^X(z|x)dz = \int_0^1 \Lambda_\tau^X(z|x)dz \, \exp\big\{-\int_t^\tau \int_0^1 \Lambda_r^X(z|x)dzdr\big\}$. Since this equation must hold for every time-τ match indicator function $m_\tau(x, y)$, the claimed functional form of $p_{t,\tau}^X(y|x)$ follows from here.

A.2 Proof of Proposition 1

Step 1: We equip the set of possible evolutions of the state μ over a finite time interval with a norm.

Denote $I_\delta(t_0)$ the time interval $[t_0, t_0 + \delta)$. Let M_+ be the set of measurable, bounded and non-negative functions $h : [0, 1] \to \mathbb{R}_+$. Denote M the identical set without the requirement that functions must be non-negative. Equip M with the seminorm, denoted $||\cdot||_1$, i.e., $||h||_1 = \int_0^1 |h(x)|dx$, and, by abuse of notation, identify M and M_+ with the set of equivalence classes where any two functions that agree almost everywhere belong to the same class. It is well-known that $(M, ||\cdot||_1)$ is a Banach space and $(M_+, ||\cdot||_1)$ is complete. Then define $\mathcal{M}_\delta(t_0)$ the set of continuous mappings $\mu : I_\delta(t_0) \to M_+^2$ where $\mu_t^X(x) \leq \overline{\overline{\mu}}_t$ and $\mu_t^Y(y) \leq \overline{\overline{\mu}}_t$. We equip $\mathcal{M}_\delta(t_0)$ with the norm

$$||\mu||_{\mathcal{M}_\delta(t_0)} = \sup_{t \in I_\delta(t_0)} \max\big\{||\mu_t^X||_1, ||\mu_t^Y||_1\big\}.$$

Following standard arguments (see Munkres (2015) Theorem 43.6), $\mathcal{M}_\delta(t_0)$ is complete.

Step 2: Fix a time-and type-dependent match probability $m_t(x, y)$ and initial condition $\mu_{t_0} \in M_+^2$. We define a mapping $T : \mathcal{M}_\delta(t_0) \to \mathcal{M}_\delta(t_0)$ whose fixed points $\mu \in \mathcal{M}_\delta(t_0)$ correspond to the solutions of (1) within time interval $I_\delta(t_0)$:

$$(T^X\mu)_t(x) = \min\big\{\max\big\{\mu_{t_0}^X + \int_{t_0}^t h^X(\tau, \mu_\tau)d\tau; 0\big\}, \overline{\overline{\mu}}_t\big\}$$

where $h = (h^X, h^Y) : I_\delta(t_0) \times M_+^2 \to M^2$ is

$$h^X(t,\mu_t)(x) = -\mu_t^X(x) \int_0^1 \lambda^X(t,\mu_t)(y|x) m_t(x,y) dy + \eta^X(t,\mu_t)(x).$$

Step 3: We show that T is a contraction mapping for δ sufficiently small. Whence by the contraction mapping theorem it admits a unique fixed point. To begin with, consider arbitrary $\mu', \mu'' \in \mathcal{M}_\delta(t_0)$. Then

$$\sup_{t\in I_\delta(t_0)} ||(T^X\mu')_t - (T^X\mu'')_t||_1 \leq \delta \sup_{t\in I_\delta(t_0)} ||h^X(t,\mu'_t) - h^X(t,\mu''_t)||_1.$$

Expanding gives, for all $x \in [0,1]$ and $t \in I_\delta(t_0)$,

$$\begin{aligned}|h^X(t,\mu'_t)(x) - h^X(t,\mu''_t)(x)| \leq |{\mu'}_t^X(x) - {\mu''}_t^X(x)| \int_0^1 \lambda^X(t,\mu''_t)(y|x) m_t(x,y) dy \\ + {\mu'}_t^X(x) \int_0^1 |\lambda^X(t,\mu''_t)(y|x) - \lambda^X(t,\mu'_t)(y|x)| m_t(x,y) dy + |\eta(t,\mu'_t)(x) - \eta(t,\mu''_t)(x)|.\end{aligned}$$

We then make use of Assumptions 2 and 3:

$$||h^X(t,\mu'_t)(x) - h^X(t,\mu''_t)(x)||_1 \leq ||{\mu'}_t^X - {\mu''}_t^X||_1 \, L^\lambda\big(1 + \overline{\overline{\mu}}_t\big) + \big(\overline{\overline{\mu}}_t L^\lambda + L^\eta\big) N({\mu'}_t, {\mu''}_t),$$

whence

$$||T\mu' - T\mu''||_{\mathcal{M}_\delta(t_0)} \leq \delta\big(L^\eta + L^\lambda + 2L^\lambda \overline{\overline{\mu}}_t\big) ||\mu' - \mu''||_{\mathcal{M}_\delta(t_0)}.$$

Then choose δ small enough so that $T : \mathcal{M}_\delta(t_0) \to \mathcal{M}_\delta(t_0)$ is a contraction mapping. The Banach fixed point theorem guarantees the existence of a unique fixed point in $\mathcal{M}_\delta(t_0)$. This fixed point is the solution to (1) in $I_\delta(t_0)$. Finally, the bound $\overline{\overline{\mu}}_t$ allows us to extend this solution to $[0,\infty)$. [28]

[28]Strictly speaking, the proof of Proposition 1 identifies a unique solution μ to the system (1) *within* the equivalence class of states $\mathcal{M}_\infty(0)$. Existence of a unique solution to the system (1) *for a fixed type* x is then established as follows: solve (1) for type x only while maintaining that $\mu_t^X(x')$ for $x' \neq x$ and $\mu_t^Y(y)$ are given by the solution $\mu \in \mathcal{M}_\infty(0)$.

B Theorem 3: Omitted Proofs

B.1 Proof of Proposition 2

Proof of Proposition 2. We show that $(\mathcal{F}_{(k)}, \mathbf{d})$ is complete and totally bounded. This establishes compactness (see for instance Munkres (2015), Theorem 45.1, p. 274).

We focus on completeness first. By abuse of notation, omit superscripts and let $(F_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $(\mathcal{F}_{(k)}, \mathbf{d})$. Then for each $(x,t) \in [0,1] \times [0,\infty)$ the sequence $\big(F_n(x,t)\big)_{n\in\mathbb{N}}$ converges as $n \to \infty$. Denote $F(x,t)$ its pointwise limit and F the thereby obtained function in $\mathcal{F}$. We first show that $F \in \mathcal{F}_{(k)}$. Fix arbitrary $\epsilon > 0$ and $(x,t),(y,r) \in [0,1] \times [0,\infty)$. Due to pointwise convergence there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\max\Big\{\big|F_n(x,t) - F(x,t)\big|, \big|F_n(y,r) - F(y,r)\big|\Big\} < \frac{\epsilon}{2}.$$

It follows from the triangle inequality and k-Lipschitz continuity of F_N that

$$\begin{aligned}\big|F(x,t) - F(y,r)\big| &\leq \big|F(x,t) - F_N(x,t)\big| + \big|F_N(x,t) - F_N(y,r)\big| + \big|F_N(y,r) - F(y,r)\big| \\ &< \epsilon + k \cdot \max\big\{|x-y|, |t-r|\big\}.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary this establishes that $F \in \mathcal{F}_{(k)}$. We then show that $F_n \to F$ in the $\mathbf{d}$-metric. Again fix arbitrary $\epsilon > 0$. If for any given $n \in \mathbb{N}$ the sup is attained for some $t > T$ where $e^{-T} < \epsilon$, clearly $\mathbf{d}(F_n, F) < \epsilon$. Let us then focus our attention on the case $(x,t) \in [0,1] \times [0,T]$. Define $B^{\epsilon}_{(k)}(x,t) = \big\{(y,r) : \max\big\{|x-y|, |t-r|\big\} < \frac{\epsilon}{4k}\big\}$ and let $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|F_n(x,t) - F(x,t)| < \frac{\epsilon}{2}$. Then for any $(y,r) \in B^{\epsilon}_{(k)}(x,t)$

$$\begin{aligned}\big|F_n(y,r) - F(y,r)\big| &\leq \big|F_n(y,r) - F_n(x,t)\big| + \big|F_n(x,t) - F(x,t)\big| + \big|F(x,t) - F(y,r)\big| \\ &< 2k \max\big\{|x-y|, |t-r|\big\} + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

Finally observe that the set $\big\{B^{\epsilon}_{(k)}(x,t) : (x,t) \in [0,1] \times [0,T]\big\}$ forms an open covering of the compact set $[0,1] \times [0,T]$. Whence there exists a finite subcovering of that set, $\big\{B^{\epsilon}_{(k)}(x_j,t_j) : j \in \{1,...,M\}\big\}$. For any $j \in \{1,...,M\}$ let N_j such that for all $n \geq N_j$ we have $\big|F_n(x_j,t_j) - F(x_j,t_j)\big| < \frac{\epsilon}{2}$. Then it follows from the preceding arguments that for all $n \geq N \equiv \max\big\{N_j : j \in \{1,...,M\}\big\}$ we have $\mathbf{d}(F_n, F) < \epsilon$. This establishes completeness.

Let's now focus attention on total boundedness. That is, for every $\epsilon > 0$ there exists a finite number M of functions $F_j \in \mathcal{F}$ such that for all $F \in \mathcal{F}_{(k)}$ we have $\mathbf{d}(F_j, F) < \epsilon$ for some $j \in \{1,...,M\}$. We achieve this by choosing a grid $\mathcal{R}^\epsilon$ on $[0,1]$ as well as a grid $\mathcal{P}^\epsilon$ on $[0,1] \times [0,T]$ for some $T > 0$ such that $e^{-T} < \epsilon$. In particular, let $\mathcal{R}^\epsilon = \big\{0, \epsilon, ..., l^\epsilon \epsilon\big\}$ where $l^\epsilon \epsilon \leq 1 < (l^\epsilon + 1)\epsilon$

and $\mathcal{P}^\epsilon = \left\{ \left(\frac{m}{k}\frac{\epsilon}{2}, \frac{n}{k}\frac{\epsilon}{2} \right) : m, n \in \{0, ..., m^\epsilon\} \times \{0, ...n^\epsilon\} \right\}$ where $\frac{m^\epsilon}{k}\frac{\epsilon}{2} \leq 1 < \frac{m^\epsilon+1}{k}\frac{\epsilon}{2}$ and $\frac{n^\epsilon}{k}\frac{\epsilon}{2} \leq T < \frac{n^\epsilon+1}{k}\frac{\epsilon}{2}$. We then consider the (finite) set of grid functions $\mathcal{G}^\epsilon = \left\{ g : \mathcal{P}^\epsilon \to \mathcal{R}^\epsilon \right\}$. Let g an element in this set. The corresponding function F_g is defined pointwise where $F_g(x,t) = g\left(\frac{m}{k}\frac{\epsilon}{2}, \frac{n}{k}\frac{\epsilon}{2}\right)$ for $(x,t) \in [\frac{m}{k}\frac{\epsilon}{2}, \frac{m+1}{k}\frac{\epsilon}{2}) \times [\frac{n}{k}\frac{\epsilon}{2}, \frac{n+1}{k}\frac{\epsilon}{2})$. Denote $\mathcal{F}^\epsilon_{(k)} \equiv \left\{ F_g \in \mathcal{F} : g \in \mathcal{G}^\epsilon \right\}$ the desired finite set of functions.

ϵ-proximity of $\mathcal{F}_{(k)}$ to $\mathcal{F}^\epsilon_{(k)}$ then follows immediately: for arbitrary $F \in \mathcal{F}_{(k)}$ there exists $g \in \mathcal{G}^\epsilon$ such that for all $(y,\tau) \in \mathcal{P}^\epsilon$ we have $\left| F(y,\tau) - g(y,\tau) \right| \leq \frac{\epsilon}{2}$. Then consider any $(x,t) \in [0,1] \times [0,T]$. Let (x^ϵ, t^ϵ) the greatest element in $\mathcal{P}^\epsilon$ such that $x^\epsilon \leq x$ and $t^\epsilon \leq t$. Then by construction $F_g(x^\epsilon, t^\epsilon) = F_g(x,t)$ and $\max\left\{ |x - x^\epsilon|, |t - t^\epsilon| \right\} \leq \frac{1}{k}\frac{\epsilon}{2}$. Using the triangle inequality and the fact that F is k-Lipschitz continuous we obtain

$$\left| F(x,t) - F_g(x,t) \right| \leq \left| F(x,t) - F(x^\epsilon, t^\epsilon) \right| + \left| F(x^\epsilon, t^\epsilon) - \underbrace{F_g(x,t)}_{=F_g(x^\epsilon,t^\epsilon)} \right| \leq k \max\left\{ |x - x^\epsilon|, |t - t^\epsilon| \right\} + \frac{\epsilon}{2} \leq \epsilon.$$

As (x,t) was arbitrary, this bound holds uniformly across $[0,1] \times [0,T]$. Meanwhile, for $t > T$ ϵ-closeness is satisfied vacuously, whence the result. □

B.2 Proof of Lemmata 1 and 2

Proof of Lemma 1. $H^m_{(k)}[\mathcal{F}^N] \subseteq \mathcal{F}_{(k)}$. Pick arbitrary $F \in \mathcal{F}^N$. Pick arbitrary $(x_1, t_1), (x_0, t_0) \in [0,1] \times [0,\infty)$. We show that

$$\left| H^m_{(k)}[F](x_1,t_1) - H^m_{(k)}[F](x_0,t_0) \right| \leq k \ \max\left\{ |x_1 - x_0|, |t_1 - t_0| \right\} \equiv k\, C.$$

Or, this is vacuously the case if $k\,C > 1$. Thus suppose otherwise that $k\,C \leq 1$. In particular this implies that $C \leq \frac{1}{k} < \frac{2}{k} = b_{(k)}/2$. Then, as figure 1 illustrates,

$$\left| H^m_{(k)}[F](x_1,t_1) - H^m_{(k)}[F](x_0,t_0) \right| \leq \frac{1}{(b_{(k)})^2} \int\limits_{B_{(k)}(x_1,t_1) \triangle B_{(k)}(x_0,t_0)} d(x,t) \leq \frac{1}{(b_{(k)})^2} \int\limits_{B_{(k)}(x_1,t_1) \triangle B_{(k)}(x_1+C,t_1+C)} d(x,t) \leq 2\frac{C\, b_{(k)} + (b_{(k)} - C)C}{(b_{(k)})^2} \leq k\, C,$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference. □

Proof of Lemma 2. Fix $\overline{F} \in \mathcal{F}^N$. Fix $\epsilon > 0$. Let $T \geq 1 : e^{-T} < \epsilon$. Define

$$A_n = \left\{ t \in [0,T] : \int\limits_0^1 \left| H^m[F](x,t) - H^m[\overline{F}](x,t) \right| dx \leq \frac{\epsilon}{18}\frac{1}{T} \quad \forall F \in \mathcal{F}^N : \mathbf{d}^N(F, \overline{F}) < \frac{1}{n} \right\}.$$

By the continuity Assumption (i) of the theorem there exists $N \in \mathbb{N}$ so that for all $n \geq N$

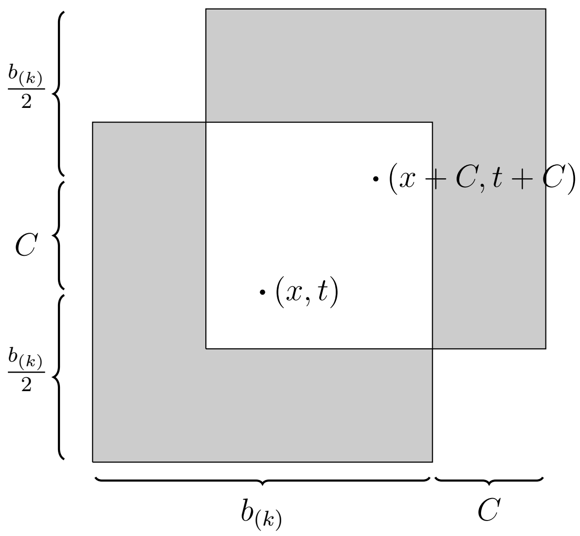

Figure 1: The shaded area corresponds to the measure of $B_{(k)}(x,t) \triangle B_{(k)}(x+C, t+C)$.

and $F : \mathbf{d}^N(F, \overline{F}) < \frac{1}{n}$ the Lebesgue measure of $[0,T] \setminus A_n$ is less than $\frac{\epsilon}{18}(b_{(k)})^2$.[29] Then for all $(x_0, t_0) \in [0,1] \times \mathbb{R}_+$ and $\frac{b_{(k)}}{2} \leq 1$

$$e^{-t}\Big|H^m_{(k)}[F](x_0,t_0) - H^m_{(k)}[\overline{F}](x_0,t_0)\Big| \leq \int\limits_{B_{(k)}(x_0,t_0)} \frac{\big|\hat{H}^m[F](x',t') - \hat{H}^m[\overline{F}](x',t')\big|}{(b_{(k)})^2} d(x',t')$$

$$\leq \int\limits_{-1}^{T+1}\int\limits_{-1}^{2} \big|\hat{H}^m[F](x,t) - \hat{H}^m[\overline{F}](x,t)\big| dt\, dx \leq 9 \int\limits_{0}^{T}\int\limits_{0}^{1} \big|H^m[F](x,t) - H^m[\overline{F}](x,t)\big| dt\, dx$$

$$\leq 9 \int\limits_{A_n}\int\limits_{0}^{1} \big|H^m[F](x,t) - H^m[\overline{F}](x,t)\big|\, dt\, dx + \frac{\epsilon}{2} \leq 9\, T \frac{\epsilon}{18}\frac{1}{T} + \frac{\epsilon}{2} = \epsilon.$$

Since (x_0, t_0) was arbitrary, this bound is uniform, i.e., $\mathbf{d}(H^m_{(k)}[F], H^m_{(k)}[\overline{F}]) < \epsilon$ for all F : $\mathbf{d}^N(F, \overline{F}) < \delta$ where $\delta \leq \frac{1}{N}$. □

B.3 Proof of Proposition 5

Proof of Proposition 5. By abuse of notation denote $(F^*_{(k)})_{k \in \mathbb{N}}$ the pointwise convergent subsequence with limit point $F^* \in \mathcal{F}^N$. This sequence exists due to Propositions 3 and 4. Then, due to the triangle inequality,

$$\big|\big|F^* - H[F^*]\big|\big| \leq \big|\big|F^* - F^*_{(k)}\big|\big| + \big|\big|H_{(k)}[F^*_{(k)}] - H_{(k)}[F^*]\big|\big| + \big|\big|H_{(k)}[F^*] - H[F^*]\big|\big|,$$

[29]The peculiar number $18 = 2 \cdot 9$ is the pertinent bound here because for any point $(x,t) \in [0,1] \times [0,T]$ there could exist a ball containing nine distinct points $(x', t') \in [-1,2] \times [-1,\infty)$ so that the extension $\hat{H}$ interprets (x', t') as if it were (x,t): $\hat{H}[F](x',t') = H[F](x,t)$ for all $F \in \mathcal{F}^N$.

where we have made use of the fact that $F^*_{(k)}$ is a fixed point, i.e., $H_{(k)}[F^*_{(k)}] = F^*_{(k)}$. By construction the first term converges as $k \to \infty$; the third term converges because the convolution with an approximate delta function converges in the seminorm defined on compact sets $[0,1] \times [0,T]$ to the function itself (see for instance Königsberger (2004) 10.1 II). This property extends to $[0,1] \times \mathbb{R}_+$ under the discounted pseudometric.

With regard to the second term, fix arbitrary $\epsilon > 0$. Then there exists $T > 0$ so that $e^{-T} < \epsilon/2$. Therefore, for arbitrary $\frac{b_{(k)}}{2} \leq 1$, $||H_{(k)}[F^*_{(k)}] - H_{(k)}[F^*]||$ is bounded from above by

$$\max_{n\in\{1,\dots,N\}} \int_0^T \int_0^1 \left| \frac{1}{(b_{(k)})^2} \int_{B_{(k)}(x,t)} \left(\hat{H}^n[F^*_{(k)}](x',t') - \hat{H}^n[F^*](x',t')\right) d(x',t') \right| dx\, dt + \frac{\epsilon}{2}.$$

And the first term is bounded by

$$\max_{n\in\{1,\dots,N\}} \int_0^T \int_0^1 \frac{1}{(b_{(k)})^2} \int_{B_{(k)}(x,t)} \left| \hat{H}^n[F^*_{(k)}](x',t') - \hat{H}^n[F^*](x',t') \right| d(x',t')\, dx\, dt$$

$$\leq \max_{n\in\{1,\dots,N\}} \int_{-1}^{T+1} \int_{-1}^{2} \left| \hat{H}^n[F^*_{(k)}](x,t) - \hat{H}^n[F^*](x,t) \right| dx\, dt \leq \max_{n\in\{1,\dots,N\}} 9 \int_0^T \int_0^1 \left| \hat{H}^n[F^*_{(k)}](x,t) - \hat{H}^n[F^*](x,t) \right| dx\, dt$$

$$= \max_{n\in\{1,\dots,N\}} 9 \int_0^T \int_0^1 \left| H^n[F^*_{(k)}](x,t) - H^n[F^*](x,t) \right| dx\, dt.$$

Then recall that Proposition 4 establishes that $F^*_{(k)}$ converges pointwise to F^*. Whence due to the continuity Assumption (i) of the Theorem the expression goes to zero as $k \to \infty$. □

C Mimicking Argument

C.1 Proof of Lemma 4

We introduce a preliminary Lemma.

Lemma 5 (TU intratemporal efficiency)**.** *Under hierarchical search 1: for all $x_2 > x_1$*

$$\overset{TU}{\boldsymbol{V}}{}^X_t[v](x_2) \geq \int_t^\infty e^{-\rho(\tau-t)} \int_0^1 \overset{TU}{\boldsymbol{\pi}}{}^X_t[v](y|x_2) \overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x_1) dy\, d\tau.$$

Proof. Define $u(t) = e^{-\rho t}\left\{ \overset{TU}{\boldsymbol{V}}{}^X_t[v](x_2) - \int_t^\infty e^{-\rho(\tau-t)} \int_0^1 \overset{TU}{\boldsymbol{\pi}}{}^X_t[v](y|x_2) \overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x_1) dy\, d\tau \right\}$.

An identical construction as in Corollary 2 guarantees that for all $T > t$

$$u(T) - u(t) = -\int_t^T e^{-\rho\tau} \int_0^1 \left(\overset{TU}{\boldsymbol{\pi}}{}^X_\tau[v](y|x_2) - \overset{TU}{\boldsymbol{V}}{}^X_\tau[v](x_2)\right)$$
$$\left(\lambda^X(\tau, \overset{TU}{\boldsymbol{\mu}}_\tau[v])(y|x_2)\overset{TU}{\boldsymbol{m}}_\tau[v](x_2,y) - \lambda^X(\tau, \overset{TU}{\boldsymbol{\mu}}_\tau[v])(y|x_1)\overset{TU}{\boldsymbol{m}}_\tau[v](x_1,y)\right) dy\, d\tau.$$

Since $\overset{TU}{\boldsymbol{m}}_\tau[v](x_2, y)$ is intratemporally efficient for given payoffs and search is hierarchical, i.e., Assumption 1 holds, it follows that $u(T) - u(t) \leq 0$. Then noting that $u(T) \leq e^{-\rho T}$ and taking the limit establishes that $u(t) \geq 0$. □

Proof of Lemma 4. Define $\bar{\boldsymbol{p}}^X_{t_0,t_1}[v](x) = \int_0^1 \overset{TU}{\boldsymbol{p}}{}^X_{t_0,t_1}[v](y|x)dy$.

Define for $k = 1, 2, ...$

$$\overset{[k]}{M}{}^X_t[v](y|x_1) = \int_{\tau_0=t}^\infty \int_{\tau_1}^\infty \cdots \int_{\tau_{k-1}}^\infty e^{-\rho(\tau_k - t)}\alpha^X \overset{TU}{\boldsymbol{p}}{}^X_{\tau_{k-1},\tau_k}[v](y|x_1)d\tau_k \,(1-\alpha^X)^{k-1} \prod_{\ell=k-1}^1 \bar{\boldsymbol{p}}^X_{\tau_{\ell-1},\tau_\ell}[v](x_1)d\tau_\ell$$
$$\overset{[k]}{R}{}^X_t[v](x_1,x_2) = \int_{\tau_0=t}^\infty \int_{\tau_1}^\infty \cdots \int_{\tau_{k-1}}^\infty e^{-\rho(\tau_k - t)}\left(\overset{TU}{\boldsymbol{V}}{}^X_{\tau_k}[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^X_{\tau_k}[v](x_1)\right)(1-\alpha^X)^k \prod_{\ell=k}^1 \bar{\boldsymbol{p}}^X_{\tau_{\ell-1},\tau_\ell}[v](x_1)d\tau_\ell.$$

(Note that, due to the order of integration, the product counts downwards from $\ell = k-1$ or $\ell = k$ respectively to 1.) We then prove by induction that

$$\overset{TU}{\boldsymbol{V}}{}^X_t[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^X_t[v](x_1) \geq \int_0^1 \left(f(x_2,y) - f(x_1,y)\right)\sum_{\ell=1}^k \overset{[\ell]}{M}{}^X_t[v](y|x_1)dy + \overset{[k]}{R}{}^X_t[v](x_1,x_2).$$

Base case: due to the preceding Lemma 5

$$\overset{TU}{\boldsymbol{V}}{}^X_t[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^X_t[v](x_1) \geq \int_t^T e^{-\rho(\tau-t)} \int_0^1 \left(\overset{TU}{\boldsymbol{\pi}}{}^X_t[v](y|x_2) - \overset{TU}{\boldsymbol{\pi}}{}^X_t[v](y|x_1)\right)\overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x_1)dyd\tau$$
$$= \int_0^1 \left(f(x_2,y) - f(x_1,y)\right)\int_t^\infty e^{-\rho(\tau-t)}\overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x_1)d\tau dy + \int_t^\infty e^{-\rho(\tau-t)}\left(\overset{TU}{\boldsymbol{V}}{}^X_\tau[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^X_\tau[v](x_1)\right)(1-\alpha^X)\bar{\boldsymbol{p}}^X_{t,\tau}[v](x_1)d\tau$$
$$= \int_0^1 \left(f(x_2,y) - f(x_1,y)\right)\overset{[1]}{M}{}^X_t[v](y|x_1)dy + \overset{[1]}{R}{}^X_t[v](x_1,x_2)$$

Induction step: Suppose that

$$\overset{TU}{\boldsymbol{V}}{}_t^X[v](x_2) - \overset{TU}{\boldsymbol{V}}{}_t^X[v](x_1) \geq \int_0^1 \big(f(x_2,y) - f(x_1,y)\big) \sum_{\ell=1}^{k-1} \overset{[\ell]}{M}{}_t^X[v](y|x_1)dy + \overset{[k-1]}{R}{}_t^X[v](x_1,x_2).$$

We show that $\overset{[k-1]}{R}{}_t^X[v](x_1,x_2) \geq \int_0^1 \big(f(x_2,y) - f(x_1,y)\big)\overset{[k]}{M}{}_t^X[v](y|x_1)dy + \overset{[k]}{R}{}_t^X[v](x_1,x_2)$ from which the claim follows.

To see this, it suffices to note that once more due to the preceding Lemma we have

$$\overset{[k-1]}{R}{}_t^X[v](x_1,x_2) = \int_{\tau_0=t}^{\infty}\int_{\tau_1}^{\infty}\cdots\int_{\tau_{k-2}}^{\infty} e^{-\rho(\tau_{k-1}-t)}\big(\overset{TU}{\boldsymbol{V}}{}_{\tau_{k-1}}^X[v](x_2) - \overset{TU}{\boldsymbol{V}}{}_{\tau_{k-1}}^X[v](x_1)\big)\,(1-\alpha^X)^{k-1}\prod_{\ell=k-1}^{1}\bar{\boldsymbol{p}}{}_{\tau_{\ell-1},\tau_\ell}^X[v](x_1)d\tau_\ell$$

$$\geq \int_{\tau_0=t}^{\infty}\int_{\tau_1}^{\infty}\cdots\int_{\tau_{k-2}}^{\infty} e^{-\rho(\tau_{k-1}-t)}\left[\int_{\tau_{k-1}}^{\infty} e^{-\rho(\tau_k-\tau_{k-1})}\int_0^1\big(\overset{TU}{\boldsymbol{\pi}}{}_{\tau_k}^X[v](y|x_2) - \overset{TU}{\boldsymbol{\pi}}{}_{\tau_k}^X[v](y|x_1)\big)\overset{TU}{\boldsymbol{p}}{}_{\tau_{k-1},\tau_k}^X[v](y|x_1)d\tau_k\right]$$
$$(1-\alpha^X)^{k-1}\prod_{\ell=k-1}^{1}\bar{\boldsymbol{p}}{}_{\tau_{\ell-1},\tau_\ell}^X[v](x_1)d\tau_\ell$$
$$= \int_0^1\big(f(x_2,y)-f(x_1,y)\big)\left[\int_{\tau_0=t}^{\infty}\int_{\tau_1}^{\infty}\cdots\int_{\tau_{k-1}}^{\infty} e^{-\rho(\tau_k-t)}\alpha^X\overset{TU}{\boldsymbol{p}}{}_{\tau_{k-1},\tau_k}^X[v](y|x_1)\,d\tau_k\right.$$
$$\left.(1-\alpha^X)^{k-1}\prod_{\ell=k-1}^{1}\bar{\boldsymbol{p}}{}_{\tau_{\ell-1},\tau_\ell}^X[v](x_1)d\tau_\ell\right]dy$$
$$+\int_{\tau_0=t}^{\infty}\int_{\tau_1}^{\infty}\cdots\int_{\tau_{k-1}}^{\infty} e^{-\rho(\tau_k-t)}\big(\overset{TU}{\boldsymbol{V}}{}_{\tau_k}^X[v](x_2) - \overset{TU}{\boldsymbol{V}}{}_{\tau_k}^X[v](x_1)\big)\,(1-\alpha^X)^k\prod_{\ell=k}^{1}\bar{\boldsymbol{p}}{}_{\tau_{\ell-1},\tau_\ell}^X[v](x_1)d\tau_\ell.$$

Then define $Q_t^X[v](y|x_1) = \sum_{\ell=1}^{k}\overset{[\ell]}{M}{}_t^X[v](y|x_1)$. $Q_t^X[v](y|x_1)$ is arguably non-negative. It remains to verify that its integral over y is less than one. To see this, it suffices to note that $\int_0^1 \overset{[k]}{M}{}_t^X[v](y|x_1)dy \leq \alpha^X(1-\alpha^X)^{k-1}$, whence $\int_0^1 Q_t^X[v](y|x_1)dy = \sum_{\ell=1}^{k}\int_0^1\overset{[k]}{M}{}_t^X[v](y|x_1)dy \leq \sum_{\ell=1}^{k}\alpha^X(1-\alpha^X)^{k-1} = 1.$ □

C.2 Proof of Lemma 3

We prove a slightly stronger result than Lemma 3.[30]

[30] Our companion paper Bonneton and Sandmann (2022) contains another proof of this result.

Lemma 6 (NTU mimicking argument)**.** *Under hierarchical search 1: for all $x_2 > x_1$*

$$\overset{NTU}{\boldsymbol{V}}{}^{X}_{t}[v](x_2) \geq \int_{t}^{\infty} e^{-\rho(\tau-t)} \int_{0}^{1} \pi^X(y|x_2) \overset{NTU}{\boldsymbol{\mathscr{p}}}{}^{X}_{t,\tau}[v](y|x_1) dy\, d\tau.$$

Proof. Define $u(t)$ as in the proof of Lemma 5, but now consider exogenous payoffs and the NTU value-of-search. Then for all $T > t$

$$\begin{aligned} u(T) - u(t) = -\int_{t}^{T} e^{-\rho\tau} \int_{0}^{1} &\big(\pi^X(y|x_2) - \overset{NTU}{\boldsymbol{V}}{}^{X}_{\tau}[v](x_2)\big) \\ &\big(\lambda^X(\tau, \overset{NTU}{\boldsymbol{\mu}}_{\tau}[v])(y|x_2) \underbrace{\overset{NTU}{\boldsymbol{\mathscr{m}}}{}_{\tau}[v](x_2, y)}_{1\{\pi^Y(x_2|y) \geq v^Y_t(y)\}1\{\pi^X(y|x_2) \geq v^X_t(x_2)\}} - \lambda^X(\tau, \overset{NTU}{\boldsymbol{\mu}}_{\tau}[v])(y|x_1) \overset{TU}{\boldsymbol{\mathscr{m}}}_{\tau}[v](x_1, y)\big)\, dy\, d\tau \end{aligned}$$

Note that x_2, being of a superior type, is accepted by a greater number of agents. Formally, $1\{\pi^Y(x_1|y) \geq v^Y_t(y)\} = 1 \Rightarrow 1\{\pi^Y(x_2|y) \geq v^Y_t(y)\} = 1$. It follows that the preceding is weakly smaller than

$$\begin{aligned} -\int_{t}^{T} e^{-\rho\tau} \int_{0}^{1} &\big(\pi^X(y|x_2) - \overset{NTU}{\boldsymbol{V}}{}^{X}_{\tau}[v](x_2)\big) \\ &1\big\{\pi^Y(x_1|y) \geq v^Y_t(y)\big\} \Big(\lambda^X(\tau, \overset{TU}{\boldsymbol{\mu}}_{\tau}[v])(y|x_2) 1\big\{\pi^X(y|x_2) \geq \overset{NTU}{\boldsymbol{V}}{}^{X}_{\tau}[v](x_2)\big\} \\ &\qquad - \lambda^X(\tau, \overset{TU}{\boldsymbol{\mu}}_{\tau}[v])(y|x_1) 1\big\{\pi^X(y|x_1) \geq \overset{NTU}{\boldsymbol{V}}{}^{X}_{\tau}[v](x_1)\big\}\Big) dy\, d\tau. \end{aligned}$$

This expression is less than zero: First, x_2's acceptance threshold is weakly more desirable for x_2 than whichever threshold is instituted by x_1. Secondly, search is hierarchical. We conclude that $u(t) \geq 0$ by letting T converge to infinity. □

D Bounded Variation: Proof of Proposition 6

Proof of Proposition 6 (ii). Consider a generic function $h(x, y)$ short for $\pi^X(y|x)$ or $f(x, y)$ and L^h short for L^π or L^f. Consider an arbitrary partition of the unit interval $[0, 1]$: $0 = x_0 < x_1 < ... < x_m = 1$. Recall the mimicking argument, namely Lemmata 3 and 4. Those assert that in both the NTU and TU paradigm the difference in values of search can be bounded as follows:

$$\boldsymbol{V}^X_t[v](x_i) - \boldsymbol{V}^X_t[v](x_{i-1}) \geq \int_{0}^{1} \big(h(x_i, y) - h(x_{i-1}, y)\big) Q^X_t(y|x_i) dy.$$

Further recall that match payoff or output is normalized, i.e., $h(x, y) \in [0, 1]$. Then

$$
\begin{aligned}
&\sum_{i=1}^{m} \left| \boldsymbol{V}_t^X[v](x_i) - \boldsymbol{V}_t^X[v](x_{i-1}) \right| \\
&= -2 \sum_{i=1}^{m} \min \left\{ \boldsymbol{V}_t^X[v](x_i) - \boldsymbol{V}_t^X[v](x_{i-1}), 0 \right\} + \sum_{i=1}^{m} \left(\boldsymbol{V}_t^X[v](x_i) - \boldsymbol{V}_t^X[v](x_{i-1}) \right) \\
&\leq -2 \sum_{i=1}^{m} \min \Big\{ \int_0^1 \big(h(x_i, y) - h(x_{i-1}, y) \big) Q_t^X(y|x_i) dy,\, 0 \Big\} + 1 \\
&\leq 2 \int_0^1 \sum_{i=1}^{m} \left| (h(x_i, y) - h(x_{i-1}, y)) \right| Q_t^X(y|x_i) dy + 1 \leq 2 \sup_y \sum_{i=1}^{m} \left| (h(x_i, y) - h(x_{i-1}, y) \right| + 1 \leq 2L^h + 1.
\end{aligned}
$$

The last inequality is due to Assumptions 6 and 7 which posit that match payoff and output is of uniformly bounded total variation. □

E Continuity

The proofs of Propositions 7 and 8 make use of the following results:

E.1 Preliminary Results

Lemma 7 (dynamic programming)**.** *In both paradigms:*

$$
\begin{aligned}
\frac{\boldsymbol{V}_{t+h}^X[v](x) - \boldsymbol{V}_t^X[v](x)}{h} &= \rho \boldsymbol{V}_{t+h}^X[v](x) \\
&\quad - \frac{1}{h} \int_t^{t+h} e^{-\rho(\tau - t)} \int_0^1 \big(\pi_\tau^X[v](y|x) - \boldsymbol{V}_\tau^X[v](x) \big) \lambda^X(\tau, \boldsymbol{\mu}_\tau[v])(y|x) \boldsymbol{m}_\tau[v](x, y)\, dy\, d\tau + o(1)
\end{aligned}
$$

where $\pi_\tau^X[v](y|x) = \overset{TU}{\boldsymbol{\pi}}{}_\tau^X[v](y|x)$ in the TU and $= \pi^X(y|x)$ in the NTU paradigm.

The proof is in Appendix E.2.

Corollary 2. *In both paradigms: for $\boldsymbol{v}_t^X[v](x) = e^{-\rho t} \boldsymbol{V}_t^X[v](x)$*

$$
\begin{aligned}
&\frac{\boldsymbol{v}_{t+h}^X[v](x) - \boldsymbol{v}_t^X[v](x)}{h} \\
&= -\frac{1}{h} \int_t^{t+h} e^{-\rho\tau} \int_0^1 \big(\pi_\tau^X[v](y|x) - \boldsymbol{V}_\tau^X[v](x) \big) \lambda^X(\tau, \boldsymbol{\mu}_\tau[v])(y|x) \boldsymbol{m}_\tau[v](x, y)\, dy\, d\tau + o(1)
\end{aligned}
$$

where $\pi_\tau^X[v](y|x) = \overset{TU}{\boldsymbol{\pi}}{}_\tau^X[v](y|x)$ in the TU and $= \pi^X(y|x)$ in the NTU paradigm.

Proof. Observe that

$$\frac{\boldsymbol{v}^X_{t+h}[v](x) - \boldsymbol{v}^X_t[v](x)}{h} = \frac{\boldsymbol{V}^X_{t+h}[v](x) - \boldsymbol{V}^X_t[v](x)}{h} e^{-\rho t} + e^{-\rho t}\frac{e^{-\rho h}-1}{h}\boldsymbol{V}^X_{t+h}[v](x).$$

Then use Lemma 7 to conclude. □

Next, we require, as in the main text (see Definition 2) a notion of distance between arbitrary match indicator functions $m_t(x,y)$. Define $||m|| = \int_0^\infty \int_0^1 \int_0^1 e^{-t}|m_t(x,y)|dx\,dy\,dt$.

Lemma 8 (NTU). *In the NTU paradigm, posit Assumption 4. Then for all beliefs $\overline{v}$ and $t \in [0,\infty)$: for all $\epsilon > 0$ there exists $\delta > 0$ such that $||\overset{NTU}{m}[v] - \overset{NTU}{m}[\overline{v}]|| < \epsilon$ for all $||v - \bar{v}|| < \delta$.*

Lemma 9 (TU). *In the TU paradigm, posit Assumption 5. Then for all $\overline{v}$ and $t \in [0,\infty)$: for all $\epsilon > 0$ there exists $\delta > 0$ such that $||\overset{TU}{m}[v] - \overset{TU}{m}[\overline{v}]|| < \epsilon$ for all $||v - \bar{v}|| < \delta$.*[31]

Continuity of the match indicator functions implies continuity of the state at all times.

Lemma 10. *In both the NTU and TU paradigm, posit Assumptions 2, 3, 4 and 5. Fix $\overline{v}$. Then for all $t \in [0,\infty)$: for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\int_0^1 \left|\boldsymbol{\mu}^X_t[v](x) - \boldsymbol{\mu}^X_t[\overline{v}](x)\right| dx < \epsilon \quad \textit{for all} \quad v : ||v - \bar{v}|| < \delta.$$

E.2 Proof of Lemma 7

We only prove this in the TU paradigm, NTU follows from identical arguments.

Proof of Lemma 7: TU. We make use of the dynamic programming principle:

$$\begin{aligned}\overset{TU}{\boldsymbol{V}}{}^X_t[v](x) &= \int_t^{t+h} e^{-\rho(\tau - t)} \int_0^1 \overset{TU}{\boldsymbol{\pi}}{}^X_\tau[v](y|x)\overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x)dy\,d\tau \\ &\quad + e^{-\rho h}\exp\Big\{-\int_t^{t+h}\int_0^1 \lambda^X(r, \overset{TU}{\boldsymbol{\mu}}_r[v])(y'|x)\overset{TU}{\boldsymbol{m}}_r[v](x,y')dy'dr\Big\}\overset{TU}{\boldsymbol{V}}{}^X_{t+h}(x)[v](y|x).\end{aligned}$$

where we used that $\overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x) = \exp\big\{-\int_t^{t+h}\int_0^1 \lambda^X(r, \overset{TU}{\boldsymbol{\mu}}_r[v])(y'|x)\overset{TU}{\boldsymbol{m}}_r[v](x,y')dy'dr\big\}\overset{TU}{\boldsymbol{p}}{}^X_{t+h,\tau}[v](y|x)$.

Equivalently, we can write

$$\frac{\overset{TU}{\boldsymbol{V}}{}^X_{t+h}[v](x) - \overset{TU}{\boldsymbol{V}}{}^X_t[v](x)}{h} = -\Bigg\{\frac{1}{h}\int_t^{t+h} e^{-\rho(\tau-t)}\int_0^1 \overset{TU}{\boldsymbol{\pi}}{}^X_\tau[v](y|x)\overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x)dy\,d\tau + \frac{e^{-\rho h}-1}{h}\overset{TU}{\boldsymbol{V}}{}^X_{t+h}[v](x)$$

[31]The fact that Lemma 9 holds for the pseudo-seminorm as opposed to the discounted sup metric v makes it a slightly stronger result than Lemma 3 in **?**.

$$+ e^{-\rho h}\frac{\exp\Big\{-\int\limits_t^{t+h}\int\limits_0^1 \lambda^X(r,\overset{TU}{\boldsymbol{\mu}}_r[v])(y'|x)\overset{TU}{\boldsymbol{m}}_r[v](x,y')dy'dr\Big\}-1}{h}\overset{TU}{\boldsymbol{V}}{}^X_{t+h}[v](x)\Bigg\}.$$

The term in the curled brackets is finite, whence $\overset{TU}{\boldsymbol{V}}{}^X_{t+h}[v](x) = \overset{TU}{\boldsymbol{V}}{}^X_t[v](x) + o(1)$.[32] This proves Proposition 6 (i). Further note that $\frac{e^{-\rho h}-1}{h} = -\rho + o(1)$ and

$$\begin{aligned}
\overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x) &= \lambda^X(\tau,\overset{TU}{\boldsymbol{\mu}}_\tau[v])(y|x)\overset{TU}{\boldsymbol{m}}_\tau[v](x,y) + o(1)\\
\frac{\exp\{\cdot\}-1}{h} &= -\int\limits_t^{t+h}\int\limits_0^1 \lambda^X(\tau,\overset{TU}{\mu}_\tau[v])(y|x)\overset{TU}{\boldsymbol{m}}_\tau[v](x,y)dy\,d\tau + o(1).
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\overset{TU}{\boldsymbol{V}}{}^X_{t+h}[v](x) - \overset{TU}{\boldsymbol{V}}{}^X_t[v](x)}{h} &= \rho\overset{TU}{\boldsymbol{V}}{}^X_{t+h}[v](x)\\
&\quad - \frac{1}{h}\int\limits_t^{t+h} e^{-\rho(\tau-t)}\int\limits_0^1 \big(\underbrace{\overset{TU}{\boldsymbol{\pi}}{}^X_\tau[v](y|x) - \overset{TU}{\boldsymbol{V}}{}^X_\tau[v](x)}_{\alpha^X\left(f(x,y)-\overset{TU}{\boldsymbol{V}}{}^X_\tau[v](x)-Z^Y_\tau(y)\right)}\big)\lambda^X(\tau,\overset{TU}{\mu}_\tau[v])(y|x)\overset{TU}{\boldsymbol{m}}_\tau[v](x,y)\,dy\,d\tau + o(1).
\end{aligned}$$

□

E.3 Proof of Lemma 8

Proof of Lemma 8. Observe that $\big|\overset{NTU}{m}_t[v](x,y) - \overset{NTU}{m}_t[\overline{v}](x,y)\big|$ is smaller than

$$\Big|1\big\{\pi(y|x) \geq v^X_t(x)\big\} - 1\big\{\pi(y|x) \geq \overline{v}^X_t(x)\big\}\Big| + \Big|1\big\{\pi(x|y) \geq v^Y_t(y)\big\} - 1\big\{\pi(x|y) \geq \overline{v}^Y_t(y)\big\}\Big|.$$

We bound the double integral of the first term: For any x observe that

$$\begin{aligned}
&\Big|1\big\{\pi^X(y|x) \geq v^X_t(x)\big\} - 1\big\{\pi^X(y|x) \geq \overline{v}^X_t(x)\big\}\Big|\\
&= 1\Big\{y : \min\{v^X_t(x),\overline{v}^X_t(x)\} \leq \pi^X(y|x) < \max\{v^X_t(x),\overline{v}^X_t(x)\}\Big\}.
\end{aligned}$$

Fix some y such that $\pi^X(y|x) \in \big[\min\{v^X_t(x),\overline{v}^X_t(x)\},\max\{v^X_t(x),\overline{v}^X_t(x)\}\big]$.

Case 1: $y : \pi^X(y|x) \leq \overline{v}^X_t(x)$. Denote $\overline{y}$ the greatest y such that $\lim\limits_{y\uparrow\overline{y}}\pi(y|x) \leq \overline{v}^X_t(x)$. Then

$$\Delta(\overline{y}-y) < \pi^X(\overline{y}|x) - \pi^X(y|x) \leq \big|v^X_t(x) - \overline{v}^X_t(x)\big|.$$

[32] The little-o refers to the Landau notation; $o(1)$ means that $\lim\limits_{h\to 0} o(1) = 0$

Case 2: $y: \pi^X(y|x) \geq \overline{v}_t^X(x)$. Denote $\overline{y}$ the smallest y such that $\lim_{y \downarrow \overline{y}} \pi(y|x) \geq \overline{v}_t^X(x)$.

$$\Delta(y - \overline{y}) < \pi^X(y|x) - \pi^X(\overline{y}|x) \leq \left|v_t^X(x) - \overline{v}_t^X(x)\right|.$$

It follows that

$$\int_0^1 \int_0^1 1\Big\{y : \min\{v_t^X(x), \overline{v}_t^X(x)\} \leq \pi^X(y|x) \leq \max\{v_t^X(x), \overline{v}_t^X(x)\}\Big\} dy\, dx \leq \int_0^1 \frac{1}{\Delta}\left|v_t^X(x) - \overline{v}_t^X(x)\right| dx,$$

and we can similarly bound

$$\int_0^1 \int_0^1 1\Big\{x : \min\{v_t^Y(y), \overline{v}_t^Y(y)\} \leq \pi^Y(x|y) \leq \max\{v_t^Y(y), \overline{v}_t^Y(y)\}\Big\} dy\, dx \leq \int_0^1 \frac{1}{\Delta}\left|v_t^Y(y) - \overline{v}_t^Y(y)\right| dy.$$

In effect, we can conclude that

$$||\overset{NTU}{m}[v] - \overset{NTU}{m}[\overline{v}]|| \leq \frac{1}{\Delta} \int_0^\infty e^{-t} \left\{ \int_0^1 \left|v_t^X(x) - \overline{v}_t^X(x)\right| dx + \int_0^1 \frac{1}{\Delta}\left|v_t^Y(y) - \overline{v}_t^Y(y)\right| dy \right\} dt \leq \frac{2}{\Delta} ||v - \overline{v}||.$$

□

E.4 Proof of Lemma 9

Proof of Lemma 9. Step 1: Define, for fixed $\overline{v}$, $J_t^X(\delta; x) = \left\{y : \left|f(x,y) - \overline{v}_t^X(x) - \overline{v}_t^Y(y)\right| < \delta\right\}$. And denote $\beta_t[v](x,y) = 2\max\left\{\left|v_t^X(x) - \overline{v}_t^X(x)\right|, \left|v_t^Y(y) - \overline{v}_t^Y(y)\right|\right\}$. It follows that, for all v,

$$\begin{aligned}
&\left|\overset{TU}{m}_t[v](x,y) - \overset{TU}{m}_t[\overline{v}](x,y)\right| = \left|1\{f(x,y) \geq v_t^X(x) + Z^Y(y)\} - 1\{f(x,y) \geq \overline{v}_t^X(x) + \overline{v}^Y(y)\}\right| \\
&\quad = 1\Big\{y : \min\left\{v_t^X(x) + v_t^Y(y), \overline{v}_t^X(x) + \overline{v}_t^Y(y)\right\} \leq f(x,y) < \max\left\{v_t^X(x) + v_t^Y(y), \overline{v}_t^X(x) + \overline{v}_t^Y(y)\right\}\Big\} \\
&\quad < 1\Big\{(x,y) : \left|f(x,y) - \overline{v}_t^X(x) - \overline{v}_t^Y(y)\right| < 2\max\left\{\left|v_t^X(x) - \overline{v}_t^X(x)\right|, \left|v_t^Y(y) - \overline{v}_t^Y(y)\right|\right\}\Big\} \\
&\quad \leq 1\Big\{y \in J_t^X\big(\beta_t[v](x,y); x\big)\Big\}.
\end{aligned}$$

Step 2: Note that for all $N \in \mathbb{N}$

$$\int_0^1 1\{y \in J_t^X\big(\beta_t[v](x,y); x\big)\} dx = \int_0^{\frac{1}{N}} \sum_{j=0}^{N-1} 1\{y \in J_t^X\big(\beta_t[v](x + \frac{j}{N}, y); x + \frac{j}{N}\big)\} dx,$$

where $\quad \sum_{j=0}^{N-1} 1\{y \in J_t^X\big(\beta_t[v](x + \frac{j}{N}, y); x + \frac{j}{N}\big)\} \leq 1\{y \in \bigcup_{j=1}^{N-1} J_t^X\big(\beta_t[v](x + \frac{j}{N}, y); x + \frac{j}{N}\big)\}$

$$+\sum_{i=0}^{N-2}\sum_{j=i+1}^{N-1} 1\{y \in J_t^X\big(\beta_t[v](x+\frac{i}{N},y);x+\frac{i}{N}\big)\bigcap J_t^X\big(\beta_t[v](x+\frac{j}{N},y);x+\frac{j}{N}\big)\}.$$

Step 3: Denote $\mathcal{A}_t^k[v](x) \subseteq [0,1]$ the set of types y so that there exist at least $k-1$ and at most k unique pairs (i,j) with $j>i$ for which $y \in J_t^X\big(\beta_t[v](x+\frac{i}{N},y);x+\frac{i}{N}\big)\bigcap J_t^X\big(\beta_t[v](x+\frac{j}{N},y);x+\frac{j}{N}\big)$. Counting establishes that there are at most $\sum_{i=0}^{N-2}\sum_{j=i+1}^{N-1} 1 = \frac{N^2-3N+2}{2}$ such pairs. And by construction, if $y', y'' \in \mathcal{A}_t^k[v](x)$ for some $k>1$, there exist $j>i$ so that

$$\begin{aligned}
&f(x+\frac{i}{N},y')-f(x+\frac{j}{N},y')-\bar{v}_t^X(x+\frac{i}{N})+\bar{v}_t^X(x+\frac{j}{N}) < \beta_t[v](x+\frac{i}{N},y')+\beta_t[v](x+\frac{j}{N},y')\\
&f(x+\frac{i}{N},y'')-f(x+\frac{j}{N},y'')-\bar{v}_t^X(x+\frac{i}{N})+\bar{v}_t^X(x+\frac{j}{N}) > -\beta_t[v](x+\frac{i}{N},y'')-\beta_t[v](x+\frac{j}{N},y'').
\end{aligned}$$

Taking differences establishes that $f(x+\frac{i}{N},y')-f(x+\frac{j}{N},y')-f(x+\frac{i}{N},y'')+f(x+\frac{j}{N},y'') < \beta_t[v](x+\frac{i}{N},y')+\beta_t[v](x+\frac{j}{N},y')+\beta_t[v](x+\frac{i}{N},y'')+\beta_t[v](x+\frac{j}{N},y'')$. And Assumption 5 implies

$$\begin{aligned}
y''-y' &< \frac{N}{j-i}\frac{1}{\triangle}\Big(\beta_t[v](x+\frac{i}{N},y')+\beta_t[v](x+\frac{j}{N},y')+\beta_t[v](x+\frac{i}{N},y'')+\beta_t[v](x+\frac{j}{N},y'')\Big)\\
&< \frac{2N}{\triangle}\big(\max_{j\in\{0,\ldots,N-1\}}\beta_t[v](x+\frac{j}{N},y')+\max_{j\in\{0,\ldots,N-1\}}\beta_t[v](x+\frac{j}{N},y'')\big).
\end{aligned}$$

Integrating over all $\mathcal{A}_t^k[v](x)$ where $k>1$ then implies that[33]

$$\sum_{k>1}\int_{\mathcal{A}_t^k[v](x)} dy < \frac{4N}{\triangle}\int_0^1 2\frac{3}{\sqrt{2}}\sqrt{\max\{\max_{j\in\{0,\ldots,N-1\}}\big|v_t^X(x+\frac{j}{N})-\bar{v}_t^X(x+\frac{j}{N})\big|,\big|v_t^Y(y)-\bar{v}_t^Y(y)\big|\}}dy.$$

Step 4: Due to the preceding it must then hold that

$$\begin{aligned}
&\int_0^\infty\int_0^1\int_0^{\frac{1}{N}} e^{-t}\sum_{i=0}^{N-2}\sum_{j=i+1}^{N-1} 1\{y\in J_t^X\big(\beta_t[v](x+\frac{i}{N},y);x+\frac{i}{N}\big)\bigcap J_t^X\big(\beta_t[v](x+\frac{j}{N},y);x+\frac{j}{N}\big)\}dx\,dy\,dt\\
&\le \sum_{k\ge 1}\int_0^\infty\int_0^{\frac{1}{N}}\int_{\mathcal{A}_t^k[v](x)} e^{-t}k\,dy\,dx\,dt \le \frac{1}{N}+\int_0^\infty e^{-t}\int_0^{\frac{1}{N}}\sum_{k>1}\int_{\mathcal{A}_t^k[v](x)} dy\,dx\,dt\,\frac{N^2-3N+2}{2}
\end{aligned}$$

[33] Formally, the implication is due to the following result: let $h:[0,1]\to[0,1]$ be a measurable function and $U\subset[0,1]$ be such that $x',x''\in U$ only if $|x'-x''|<h(x')+h(x'')$. Then $\int_U dx \le \alpha$ where $\alpha = \frac{3}{\sqrt{2}}\big(\int_0^1 h(x)dx\big)^{1/2}$. Proof: Denote $\delta=\int_0^1 h(x)dx$. If the claim did not hold, there exist $\underline{x},\overline{x}\in U$ so that $\int_{[0,\underline{x}]\cap U} dx = \int_{[\underline{x},\overline{x}]\cap U} dx = \int_{[\overline{x},1]\cap U} dx > \frac{\alpha}{3}$. It follows that $\overline{x}-\underline{x}>\frac{\alpha}{3}$. Then pick for each $x'\in[0,\underline{x}]\cap U$ a unique $x''\in[\overline{x},1]\cap U$. For any such pair it must hold that $\frac{\alpha}{3}<|x''-x'|\le h(x')+h(x'')$. It then follows that $\int_0^1 h(x)dx \ge \int_0^{\underline{x}} h(x)dx+\int_{\overline{x}}^1 h(x)dx > 2\big(\frac{\alpha}{3}\big)^2$. This poses the desired contradiction, for by construction $\delta = 2\big(\frac{\alpha}{3}\big)^2$.

$$
\leq \frac{1}{N}+\frac{4N}{\triangle}\int_0^\infty e^{-t}\int_0^{\frac{1}{N}}\int_0^1 2\frac{3}{\sqrt{2}}\sqrt{\max\{\max_j\big|v_t^X(x+\frac{j}{N})-\bar{v}_t^X(x+\frac{j}{N})\big|,\big|v_t^Y(y)-\bar{v}_t^Y(y)\big|\}}dy\,dx\,dt\;\frac{N^2-3N+2}{2}
$$

$$
\leq \frac{1}{N}+12\sqrt{2}\frac{N}{\triangle}\int_0^\infty e^{-t}\Big(\int_0^1\int_0^1 \max\{|v_t^X(x)-\bar{v}_t^X(x)|,|v_t^Y(y)-\bar{v}_t^Y(y)|\}dy\,dx\Big)^{\frac{1}{2}}dt\;\frac{N^2-3N+2}{2}
$$

$$
\leq \frac{1}{N}+12\sqrt{2}\frac{N}{\triangle}\int_0^\infty e^{-t/2}\Big(\int_0^1\int_0^1 \max\{|v_t^X(x)-\bar{v}_t^X(x)|,|v_t^Y(y)-\bar{v}_t^Y(y)|\}dy\,dx\Big)^{\frac{1}{2}}dt\;\frac{N^2-3N+2}{2}
$$

$$
\leq \frac{1}{N}+12\sqrt{2}\frac{N}{\triangle}\big|\big|v-\bar{v}\big|\big|^{\frac{1}{2}}\;\frac{N^2-3N+2}{2}
$$

due to Jensen's inequality. In conclusion, we have established that for arbitrary N

$$
\big|\big|\overset{TU}{m}[v]-\overset{TU}{m}[\bar{v}]\big|\big|\leq\frac{1}{N}+\frac{1}{N}+12\sqrt{2}\frac{N}{\triangle}\big|\big|v-\bar{v}\big|\big|^{\frac{1}{2}}\;\frac{N^2-3N+2}{2}.
$$

□

E.5 Proof of Lemma 10

Proof of Lemma 10. Step 1: Manipulating (1) gives

$$
\begin{aligned}
\boldsymbol{\mu}_t^X[v](x)-\boldsymbol{\mu}_t^X[\bar{v}](x) &= \int_0^t\Big\{\boldsymbol{\mu}_\tau^X[\bar{v}](x)\int_0^1\lambda^X(\tau,\boldsymbol{\mu}_\tau[\bar{v}])(y|x)m_\tau[\bar{v}](x,y)dy\\
&\quad-\boldsymbol{\mu}_\tau^X[v](x)\int_0^1\lambda^X(\tau,\boldsymbol{\mu}_\tau[v])(y|x)m_\tau[v](x,y)dy+\eta^X(\tau,\boldsymbol{\mu}_\tau[v])(x)-\eta^X(\tau,\boldsymbol{\mu}_\tau[\bar{v}])(x)\Big\}d\tau\\
&=\int_0^t\Big\{\big(\boldsymbol{\mu}_\tau^X[\bar{v}](x)-\boldsymbol{\mu}_\tau^X[v](x)\big)\int_0^1\lambda^X(\tau,\boldsymbol{\mu}_\tau[\bar{v}])(y|x)m_\tau[\bar{v}](x,y)dy\\
&\quad+\boldsymbol{\mu}_\tau^X[v](x)\int_0^1\big(\lambda^X(\tau,\boldsymbol{\mu}_\tau[\bar{v}])(y|x)-\lambda^X(\tau,\boldsymbol{\mu}_\tau[v])(y|x)\big)m_\tau[\bar{v}](x,y)dy\\
&\quad+\boldsymbol{\mu}_\tau^X[v](x)\int_0^1\lambda^X(\tau,\boldsymbol{\mu}_\tau[v])(y|x)\big(m_\tau[\bar{v}](x,y)-m_\tau[v](x,y)\big)dy+\eta^X(\tau,\boldsymbol{\mu}_\tau[v])(x)-\eta^X(\tau,\boldsymbol{\mu}_\tau[\bar{v}])(x)\Big\}d\tau.
\end{aligned}
$$

Using Assumptions 2 and 3 we obtain the following upper bound:

$$
\int_0^1\big|\boldsymbol{\mu}_t^X[v](x)-\boldsymbol{\mu}_t^X[\bar{v}](x)\big|dx\leq\big(1+\bar{\bar{\mu}}_t\big)L^\lambda\int_0^t\int_0^1\big|\boldsymbol{\mu}_\tau^X[\bar{v}](x)-\boldsymbol{\mu}_\tau^X[v](x)\big|dx\,d\tau
$$

$$+\overline{\overline{\mu}}_t L^\lambda \int_0^t N(\boldsymbol{\mu}_\tau[v], \boldsymbol{\mu}_\tau[\overline{v}])d\tau + \overline{\overline{\mu}}_t(1+\overline{\overline{\mu}}_t)L^\lambda \int_0^t \int_0^1 \int_0^1 \big|m_\tau[\overline{v}](x,y) - m_\tau[v](x,y)\big| dy\, dx\, d\tau$$

$$+ L^\eta \int_0^t N(\boldsymbol{\mu}_\tau[v], \boldsymbol{\mu}_\tau[\overline{v}])d\tau. \tag{$\star$}$$

Step 2: The preceding Lemmata 8 and 9 imply that in both paradigms for all $\xi > 0$ (to be determined) there exists $\delta > 0$ such that for all $v: ||v - \overline{v}|| < \delta$:

$$\int_0^t \int_0^1 \int_0^1 \big|m_\tau[\overline{v}](x,y) - m_\tau[v](x,y)\big| dy\, dx\, d\tau < \xi. \tag{$\star\star$}$$

Step 3: We show that $\forall \epsilon > 0\ \exists \delta > 0$ such that $N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}]) < \epsilon \quad \forall\, v: ||v - \overline{v}|| < \delta$.

Indeed, inequalities ($\star$) and ($\star\star$) jointly imply that

$$N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}]) \leq \underbrace{\left((1+\overline{\overline{\mu}}_t)L^\lambda + \overline{\overline{\mu}}_t L^\lambda + L^\eta\right)}_{\equiv K_1} \int_0^t N(\boldsymbol{\mu}_\tau[v], \boldsymbol{\mu}_\tau[\overline{v}])d\tau + \underbrace{\overline{\overline{\mu}}_t\left(1+\overline{\overline{\mu}}_t\right)L^\lambda}_{\equiv K_2}\, \xi$$

for all $v: ||v - \overline{v}|| < \delta$. And an application of Grönwall's inequality gives $N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}]) \leq K_1 \xi e^{K_2 t}$.

Then to satisfy the $\epsilon - \delta$ argument, choose $\xi \equiv \frac{\epsilon}{K_1} e^{-K_2 t}$. □

Proof of Proposition 7

Proof of Proposition 7. Pick h such that $1/h \in \mathbb{N}$. Then, due to Corollary 2,

$$\big|\boldsymbol{V}^X_{t_0}[v](x) - \boldsymbol{V}^X_{t_0}[\overline{v}](x)\big| \leq e^{\rho t_0}\left\{ e^{-\rho t_0}\big|\boldsymbol{V}^X_{t_0}[v](x) - \boldsymbol{V}^X_{t_0}[\overline{v}](x)\big| - e^{-\rho t_1}\big|\boldsymbol{V}^X_{t_1}[v](x) - \boldsymbol{V}^X_{t_1}[\overline{v}](x)\big| \right\} + e^{-\rho(t_1 - t_0)}$$

$$= e^{\rho t_0} h \sum_{n=0}^{\frac{1}{h}-1} \left\{ e^{-\rho(t_0+nh)} \frac{\big|\boldsymbol{V}^X_{t_0+nh}[v](x) - \boldsymbol{V}^X_{t_0+nh}[\overline{v}](x)\big|}{h} - e^{-\rho(t_0+(n+1)h)} \frac{\big|\boldsymbol{V}^X_{t_0+(n+1)h}[v](x) - \boldsymbol{V}^X_{t_0+(n+1)h}[\overline{v}](x)\big|}{h} \right\}$$
$$+ e^{-\rho(t_1 - t_0)}$$

$$= e^{\rho t_0} h \sum_{n=0}^{\frac{1}{h}-1} \left\{ \frac{\big|\boldsymbol{v}^X_{t_0+nh}[v](x) - \boldsymbol{v}^X_{t_0+nh}[\overline{v}](x)\big|}{h} - \frac{\big|\boldsymbol{v}^X_{t_0+(n+1)h}[v](x) - \boldsymbol{v}^X_{t_0+(n+1)h}[\overline{v}](x)\big|}{h} \right\} + e^{-\rho(t_1 - t_0)}$$

$$\leq e^{\rho t_0} h \sum_{n=0}^{\frac{1}{h}-1} \left| \frac{\boldsymbol{v}^X_{t_0+(n+1)h}[v](x) - \boldsymbol{v}^X_{t_0+nh}[v](x)}{h} - \frac{\boldsymbol{v}^X_{t_0+(n+1)h}[\overline{v}](x) - \boldsymbol{v}^X_{t_0+nh}[\overline{v}](x)}{h} \right| + e^{-\rho(t_1 - t_0)}$$

$$
\leq e^{\rho t_0} h \sum_{n=0}^{\frac{1}{h}-1} \left| \frac{1}{h} \int_{t_0+nh}^{t_0+(n+1)h} e^{-\rho t} \int_0^1 \left\{ \big(\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\big) \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) \boldsymbol{m}_t[\overline{v}](x,y) \right.\right.
$$

$$
\left.\left. - \big(\pi^X(y|x) - \boldsymbol{V}_t^X[v](x)\big) \lambda^X(t, \boldsymbol{\mu}_t[v])(y|x) \boldsymbol{m}_t[v](x,y) \right\} dy\, dt \right| + o(1) + e^{-\rho(t_1-t_0)}
$$

$$
= e^{\rho t_0} \left| \int_{t_0}^{t_1} e^{-\rho t} \int_0^1 \left\{ \big(\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\big) \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) \boldsymbol{m}_t[\overline{v}](x,y) \right.\right.
$$

$$
\left.\left. - \big(\pi^X(y|x) - \boldsymbol{V}_t^X[v](x)\big) \lambda^X(t, \boldsymbol{\mu}_t[v])(y|x) \boldsymbol{m}_t[v](x,y) \right\} dy\, dt \right| + e^{-\rho(t_1-t_0)}.
$$

Next, recall the definition of $\boldsymbol{m}_t[v](x,y)$. In the NTU paradigm the preceding term is

$$
= e^{\rho t_0} \left| \int_{t_0}^{t_1} e^{-\rho t} \int_0^1 \left\{ \big(\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\big) \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) \right.\right.
$$

$$
\Big(1\big\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\big\} - 1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\}\Big) 1\big\{\pi^X(y|x) \geq \overline{v}_t^X(x)\big\}
$$

$$
+ \Bigg[\big(\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\big) \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) 1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\} 1\big\{\pi^X(y|x) \geq \overline{v}_t^X(x)\big\}
$$

$$
\left.\left. - \big(\pi^X(y|x) - \boldsymbol{V}_t^X[v](x)\big) \lambda^X(t, \boldsymbol{\mu}_t[v])(y|x) 1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\} 1\big\{\pi^X(y|x) \geq v_t^X(x)\big\} \Bigg] \right\} dy\, dt \right|
$$

$$
+ e^{-\rho(t_1-t_0)}
$$

$$
\leq e^{-\rho(t_1-t_0)} + \int_{t_0}^{t_1} e^{-\rho(t-t_0)} \int_0^1 \left\{ (1+\overline{\overline{\mu}}_t) L^\lambda \Big| 1\big\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\big\} - 1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\} \Big| \right.
$$

$$
\left. + \left| \Big[\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\Big]_+ \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) - \Big[\pi^X(y|x) - \boldsymbol{V}_t^X[v](x)\Big]_+ \lambda^X(t, \boldsymbol{\mu}_t[v])(y|x) \right| \right\} dy\, dt
$$

$$
\leq e^{-\rho(t_1-t_0)} + (1+\overline{\overline{\mu}}_{t_1}) L^\lambda \int_{t_0}^{t_1} \int_0^1 \Big| 1\big\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\big\} - 1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\} \Big| dy\, dt
$$

$$
+ (1+\overline{\overline{\mu}}_{t_1}) L^\lambda \int_{t_0}^{t_1} \int_0^1 \left| \Big[\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\Big]_+ - \Big[\pi^X(y|x) - \boldsymbol{V}_t^X[v](x)\Big]_+ \right| dy\, dt
$$

$$
+ \int_{t_0}^{t_1} \int_0^1 \Big| \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) - \lambda^X(t, \boldsymbol{\mu}_t[v])(y|x) \Big| dy\, dt
$$

$$
\leq (1+\overline{\overline{\mu}}_{t_1}) L^\lambda \int_{t_0}^{t_1} \int_0^1 \Big| 1\big\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\big\} - 1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\} \Big| dy\, dt + e^{-\rho(t_1-t_0)}
$$

$$+ (1+\overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1} \left|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\right| dt + L^\lambda \int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}])dt$$

where we have made use of Assumption 2. By integrating over all $x \in [0,1]$, it follows that $\int_0^1 \left|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\right| dx$ is bounded by

$$(1+\overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1}\int_0^1\int_0^1 \left|1\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\} - 1\{\pi^Y(x|y) \geq v_t^Y(y)\}\right| dx\, dy\, dt + e^{-\rho(t_1-t_0)}$$
$$+ (1+\overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1}\int_0^1 \left|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\right| dx\, dt + L^\lambda \int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}])dt.$$

To conclude, fix some ξ (yet to be determined). Let t_1 be the smallest time such that $e^{-\rho(t_1-t_0)} < \xi$. The proof of Lemma 8 implies that there exists $\delta_1 > 0$ such that for all $v : ||v - \overline{v}|| < \delta_1$

$$\int_{t_0}^{t_1}\int_0^1\int_0^1 \left|1\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\} - 1\{\pi^Y(x|y) \geq v_t^Y(y)\}\right| dx\, dy\, dt < \xi.$$

And Lemma 10 implies that there exists $\delta_2 > 0$ such that $\int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}])dt < \xi$ for all $v : ||v - \overline{v}|| < \delta_2$ Then set $\delta = \min\{\delta_1, \delta_2\}$. It follows that for all $v : ||v - \overline{v}|| < \delta$

$$\int_0^1 \left|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\right| dx \leq \underbrace{(1+\overline{\overline{\mu}}_{t_1})L^\lambda}_{\equiv K_1} \int_{t_0}^{t_1}\int_0^1 \left|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\right| dx\, dt + \underbrace{\left((2+\overline{\overline{\mu}}_{t_1})L^\lambda + 1\right)}_{\equiv K_2}\xi.$$

And an application of Grönwall's inequality gives $\int_0^1 \left|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\right| dx \leq K_1 \xi e^{K_2(t_1-t_0)}$. Then to satisfy the $\epsilon - \delta$ argument, choose $\xi \equiv \frac{\epsilon}{K_1} e^{-K_2(t_1-t_0)}$. □

E.6 Proof of Proposition 8

Proof of Proposition 8. Pick h such that $1/h \in \mathbb{N}$. Then (details for the first inequality that is not specific to the TU paradigm are given in the proof of Proposition 7)

$$\left|\boldsymbol{V}_{t_0}^X[v](x) - \boldsymbol{V}_{t_0}^X[\overline{v}](x)\right| \leq e^{\rho t_0}\left| \int_{t_0}^{t_1} e^{-\rho t} \int_0^1 \Big\{ \left(\pi_t^X[\overline{v}](y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\right)\lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x)\boldsymbol{m}_t[\overline{v}](x,y)\right.$$
$$\left. - \left(\pi_t^X[v](y|x) - \boldsymbol{V}_t^X[v](x)\right)\lambda^X(t, \boldsymbol{\mu}_t[v])(y|x)\boldsymbol{m}_t[v](x,y) \Big\} dy\, dt \right| + e^{-\rho(t_1-t_0)}$$

$$
\begin{aligned}
&\leq \int_{t_0}^{t_1}\int_0^1 \Bigg| \Big[\pi_t^X[v](y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\Big]_+ \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) \\
&\qquad - \Big[\pi_t^X[v](y|x) - \boldsymbol{V}_t^X[v](x)\Big]_+ \lambda^X(t, \boldsymbol{\mu}_t[v])(y|x)\Bigg| dy\, dt + e^{-\rho(t_1-t_0)} \\
&= \alpha^X \int_{t_0}^{t_1}\int_0^1 \Bigg\{ \Big| \Big[f(x,y) - \overline{v}_t^Y(y) - \boldsymbol{V}_t^X[\overline{v}](x)\Big]_+ - \Big[f(x,y) - v_t^Y(y) - \boldsymbol{V}_t^X[v](x)\Big]_+ \Big| \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) \\
&\qquad + \Big|\lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) - \lambda^X(t, \boldsymbol{\mu}_t[v])(y|x)\Big| \Bigg\} dy\, dt + e^{-\rho(t_1-t_0)} \\
&\leq \alpha^X(1+\overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1}\int_0^1 \Big| \big(\overline{v}_t^Y(y) - \boldsymbol{V}_t^X[\overline{v}](x)\big) - \big(v_t^Y(y) - \boldsymbol{V}_t^X[v](x)\big)\Big| dy\, dt \\
&\qquad + \alpha^X L^\lambda \int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}]) dt + e^{-\rho(t_1-t_0)} \\
&\leq \alpha^X(1+\overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1}\int_0^1 \big|\overline{v}_t^Y(y) - v_t^Y(y)\big| dy\, dt + \alpha^X(1+\overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1} \big|\boldsymbol{V}_t^X[\overline{v}](x) - \boldsymbol{V}_t^X[v](x)\big| dt \\
&\qquad + \alpha^X L^\lambda \int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}]) dt + e^{-\rho(t_1-t_0)}.
\end{aligned}
$$

To conclude, fix some ξ (yet to be determined). Let t_1 be the smallest time such that $e^{-\rho(t_1-t_0)} < \xi$. To bound the first term, set $\delta_1 = e^{-t_1}\xi$. Lemma 10 implies that there exists $\delta_2 > 0$ such for all $v : ||v - \overline{v}|| < \delta_2$: $\int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}]) dt < \xi$.

Then set $\delta = \min\{\delta_1, \delta_2\}$. It follows that for all $v : ||v - \overline{v}|| < \delta$

$$
\big|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\big| \leq \underbrace{\alpha^X(1+\overline{\overline{\mu}}_{t_1})L^\lambda}_{\equiv K_1} \int_{t_0}^{t_1} \big|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\big| dt + \underbrace{\alpha^X\big((2+\overline{\overline{\mu}}_{t_1})L^\lambda + 1\big)}_{\equiv K_2}\xi.
$$

And an application of Grönwall's inequality gives $\big|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\big| \leq K_1 \xi e^{K_2(t_1-t_0)}$.

Then to satisfy the $\epsilon - \delta$ argument, choose $\xi \equiv \frac{\epsilon}{K_1} e^{-K_2(t_1-t_0)}$.

□

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