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Placement With Assignment Guarantees and Semi-Flexible Capacities

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Placement with Assignment Guarantees and Semi-Flexible Capacities

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Abstract

We analyze an extension of the many-to-one placement problem, where some doctors are exogenously guaranteed a seat at a program, which defines a lower bound on their assignment. Respecting assignment guarantees, combined with the limited capacities of programs often violates fairness and leaves more preferred doctors unemployed. In pursuance of restoring fairness, a designer often has to deviate from the target capacities of programs, and imposing the traditional notion of fairness results in excessive deviations from the target capacities. In order to prevent excessive deviations, we introduce two notions that are tailored to the environment: q-fairness and avoiding unnecessary slots. Furthermore, we introduce the Assignment-Guarantees-Adjusted Mechanism (AGAM) and show that it is the unique strategy-proof mechanism that satisfies q-fairness and avoids unnecessary slots whilst respecting assignment guarantees. Furthermore, among the mechanisms that satisfy q-fairness and respect guarantees, AGAM minimizes the deviation from the target capacities.

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1 Introduction

In most of the standard matching theory and its applications, the analysis often includes the preferences of both sides as the main component. In a residency matching environment, this component would consist of doctors' preferences over residency programs and programs' preferences over sets of doctors. In addition to baseline preferences as such, many real-life matching applications include ex-ante entitlements. The entitlements constitute assignment guarantees and define a lower bound on placements. In the case of overbooking by airline companies, all ticket owners are entitled to fly to their destination. Only occasionally is the demand to check-in higher than the capacity of the plane. Companies then compensate the ineligible passengers with highly attractive offers so that they voluntarily back down from their claim on the flight.

Nevertheless, in many other situations, such monetary transfers are not possible. Yet, it might be feasible to relax the capacities to some extent. In many countries where civil servants are appointed to locations, governments are responsible for protecting family integrity and can assign spouses to workplaces that are only up to a certain distance level from each other. Similarly, whenever there are second-round placements for empty seats in schools or colleges, the assignment in the first round defines a lower bound for candidates who participate in the second round. In both examples, candidates are exogenously guaranteed to have certain seats. The placement procedure can only send them to places they like better than their assignment guarantee, or else they will have to be assigned to their entitlement.

In this paper, we take an axiomatic approach to many-to-one matching environments with assignment guarantees and semi-flexible capacities. Adapting the jargon of the resident matching problem, doctors have preferences over residency programs and programs have preferences over doctors. Furthermore, some doctors have assignment guarantees. A mechanism respects assignment guarantees if it assigns doctors to programs that they like at least as much as their assignment guarantee.

We instantly observe that the hybrid placement problem with assignment guarantees reveals a many-to-one matching trilemma: *assignment guarantees* and *fixed capacities* are generically not compatible with *fairness*. This is because whenever the less preferred candidates have guarantees, the mechanism can not assign seats to more preferred candidates, hence is not "fair" to them. If a designer wants to respect the exogenously determined assignment guarantees as well as remain fair to doctors, she has to relax the capacity constraints of residency programs. Therefore, with such assignment guarantees, we often observe deviations from the initially determined capacities.

However, the capacities of residency programs are exogenously given in placement problems, and they reflect limited resources. Therefore, simply relaxing them as much as needed would probably be undesirable for the designer. With the aim of finding a balance between fairness and capacity concerns, we present two novel axioms, which are tailored for environments with assignment guarantees and semi-flexible capacities.

One of the axioms defines the eligible doctors for a seat in a program. This axiom is called *avoiding unnecessary slots* (shortly AUS) and ensures that a doctor can only earn a seat through her assignment guarantee or merit ranking for a program. By AUS, if a doctor is assigned a seat at a program despite having no guarantee, we can conclude that she is ranked within the target capacity among the doctors who are placed there. Similarly, if a doctor is not ranked within the target capacity of a program, the placement must be due to her placement guarantee.

As mentioned above, deviations from the target capacities are inevitable if the designer aims to be fair to more preferred candidates. However, only slightly relaxing the capacities might not be sufficient to ensure fairness. On the contrary, the fairness of any mechanism is endangered unless the capacity limits of programs are abolished completely. When the worst doctors have assignment guarantees, imposing the traditional fairness notion requires creating additional capacities, even for some doctors, who would not have received a seat if it were not for the assignment guarantees.

Observing that the traditional notion of fairness is too strict for environments with guarantees and requires major deviations from the target capacities, the other axiom we introduce is a relaxed notion of fairness, which we name as *capacity respecting fairness* (shortly q-fairness). Given an assignment, if a doctor does not receive a seat at a program that she likes better, she envies the candidates who are placed there. With the traditional notion of fairness, her envy is justified as soon as there's a candidate in that program that is less preferred than her. With q-fairness, on the other hand, there is another requirement to justify the envy: Among the doctor pool of the program, the doctor has to be ranked within the target capacity. In the absence of guarantees, q-fairness is equivalent to the traditional notion of fairness.

Crucially, q-fairness and AUS have different implications, such that q-fairness puts constraints on doctors who do not get into programs, whereas AUS constrains those who do get in. For instance, assigning every doctor to their favorite program would trivially satisfy q-fairness but fail AUS, thus heavily damaging the capacities. On the other hand, assigning everybody to their guaranteed seat would raise fairness concerns. Thus, whilst respecting the guarantees, a suitable mechanism needs to find the balance between capacity constraints and fairness.

After introducing the axioms, we present the Assignment-Guarantees-Adjusted Mechanism (AGAM), which is the deferred acceptance algorithm induced by the Assignment-Guarantees-Adjusted choice function. This special choice function is tailored for the student placement environments with assignment guarantees, aiming to create as least excessive capacities as possible. At each step of the algorithm, programs first admit the best candidates in their application pool as many as their target capacity. If there are remaining doctors in the pool that have assignment guarantees at that program, they are admitted additionally. We show that AGAM admits many favorable features. First, it satisfies q-fairness and AUS, respects the assignment guarantees, and is non-wasteful. Second, it produces a stable matching and is strategy-proof on the doctor side. In fact, it is the unique strategy-proof mechanism that is q-fair, avoids unnecessary slots, and respects assignment guarantees. Furthermore, among the mechanisms that are q-fair and respect the assignment guarantees, it minimizes the deviations from the target capacities with its AUS property.

The environment described in this paper has many applications. For instance, most firms consist of different types of professionals who have different skills. The firms frequently analyze their status and restructure the firm if needed. From time to time, efficient restructuring might involve replacing existing workers with new ones who have different skills. However, it is often too costly for the firm to lay off existing workers due to regulations. If hiring the new worker is essential for the firm, it might deviate from the target capacity and hire the new worker anyway, even though the firm did not intend to expand in the first place.

Similarly, in the European Union, any worker is entitled to the same job and wage when they go back to work after their parental leave. If the firm has employed someone else during the new parent's absence, they either must relocate the new parent to another position that the parent prefers to the older one, or let both workers work for the same position. Similarly, when civil servants apply for a change of location, they only agree to the locations they like more than their current place of duty.

In order to have a better understanding of all such environments with entitlements, this paper provides the suitable axioms that would match the policymaker's expectations. Moreover, it presents a plausible way to relax the capacities if it is allowed.

This paper is organized as follows: In Section 2, we present our model. In Section 3 we show the preliminary shortcomings of the existing notions and define the axioms that are tailored for the specific environment. We present the Assignment-Guarantees-Adjusted

Mechanism and discuss its properties in section 4. In Section 5, we present three real-life applications to this paper. We conclude in Section 6.

1.1 Related Literature

The paper connects to both many-to-one matching and many-to-one matching with contracts literature. On the one hand, programs have capacities and preferences over doctors, which reflects the key elements of a many-to-one environment such as in the seminal papers by Roth (1984), Balinski and Sönmez (1999), Abdulkadiroğlu and Sönmez (2003). As in those papers, the only strategic agents are the doctors, and the mechanism chosen by the designer, as well as the residential programs' preferences are commonly observed. In addition to these basic components, the existence of assignment guarantees complexifies the many-to-one environment towards a many-to-one matching with contracts framework described by Hatfield and Milgrom (2005), Hatfield and Kojima (2008) Westkamp (2013), and many others.

Intuitively, the placement guarantees reflect yet another form of affirmative action policy. Similar to the existing literature, some candidates are exogenously prioritized at some programs, such as in Abdulkadiroğlu and Sönmez (2003), Kojima (2012), Hafalir, Yenmez, and Yildirim (2013), and Doğan (2016). Similar to them, there might be various underlying reasons for such a claim at a seat, for example, the location of the spouse or acquired rights. However, because a doctor cannot be assigned to a least preferred alternative than their guarantee, the assignment guarantees are more strict compared to the priorities in those papers. A similar lower bound constraint can be found in Combe et al. (2022). In their setting, the existing teachers of schools cannot be sent anywhere that they like less than their current assignment.

Similar to Westkamp (2013), Kominers and Sönmez (2016), Aygün and Sönmez (2013), the mechanism we propose is essentially a Gale-Shapley deferred acceptance algorithm Gale and L. S. Shapley (1962) along with a choice rule to be implemented at each step. Furthermore, the mechanism involves a dynamic flavor as in Aygün and Turhan (2020) in the sense that the balance between the guarantee seats and regular seats evolves as the mechanism moves forward. Different than those papers, doctors do not have a preference about the type of seat they acquire. Therefore, even if they were asked to reveal their guarantees to the mechanism, no candidate would have incentives to hide their guarantee status strategically.

The environment resembles the benchmark housing market as individuals' guarantees define a lower bound for them in a Pareto sense, candidates with guarantees can only be better off L. Shapley and Scarf (1974). However, the complex preferences of programs prevent us from implementing Gale's TTC. Furthermore, our mechanism creates additional seats. In that sense, the problem also resembles the House Allocation with Existing Tenants in Abdulkadiroğlu and Sönmez (1999), with a substantial twist that creating new rooms is also possible. Nevertheless, the additional rooms, if created, can only be used by the candidates with a guarantee at the specific program. Increasing efficiency further by exchanging guarantees is restricted due to stability concerns.

To the best of our knowledge, this paper is the first to study how to relax capacities when the need arises, while respecting the exogenous lower bounds on the assignments as well as upholding some sort of fairness. Pursuing an axiomatic approach to characterize a suitable mechanism for such environments, we translate the semi-flexible capacities into the choice functions of programs, and the realized capacities then depend on the application pool. Similar to Westkamp (2013), Sönmez and Switzer (2013), and Dimakopoulos and Heller (2019), our analysis has a direct application in a current market that concerns tens of thousands of people every year.

2 Model

Same as in other many-to-one matching environments, we have a finite set of doctors and a finite set of residency programs denoted by $D = \{d_1, d_2, ..., d_n\}$ and $H = \{h_1, h_2, ..., h_m\}$, respectively. The generic doctor d has strict preferences, P_d (occasionally $\succ_d$ for convenience) over programs H along with an outside option $\emptyset$, where $P_{D'}$ is the collection of the preferences of doctors in set $D' \subset D$.

Similarly, the generic residency program is denoted by h and has an exogenously determined capacity q_h, where the collection of the capacities of all programs is denoted by $q_H = \{q_{h_1}, ..., q_{h_m}\}$.

While considering a student placement problem with assignment guarantees, a convenient approach is adapted from the matching with contracts framework: We define choice functions for both sides of the matching platform. From any set of residency programs $H' \subset H$, d chooses according to the choice function that is driven from her strict preference P_d over $H \cup \{\emptyset\}$, $C_d : 2^H \to H \cup \{\emptyset\}$, such that $C_d(H') = max_{P_d}(H' \cup \{\emptyset\})$. This implies that the doctors have unit demand. When $C_d(H') = \emptyset$, doctor d prefers the outside option among the choices, i.e. to remain unemployed.

Analogously, C_h is the choice function of program h. Similar to C_d, C_h allows the programs to choose no doctor from any application pool. However, C_h is different than C_d in two aspects: First, programs choose sets of doctors from application pools, thus $C_h : 2^D \to 2^D$, and for any $D' \subset D$, $C_h(D') \subset D'$. Second, C_h does not need to be driven from a preference relation. In fact, any choice process will involve two components:

First, each program h has an exogenously given strict preference over individual doctors that is denoted by P_h ($\succ_h$, and the collection of preferences P_H).[1] For notational convenience, we occasionally use the rankings of doctors in an application pool D' instead of the preferences. Reasonably, for program h, the ranking of doctor d in any application pool D' is defined as a function $z_h(d|P_h, D') : D' \to \mathbb{N}^+$ and $z_h(d)$ decreasing with P_h, such that the more preferred d is for h, the higher ranking she has in an application pool[2]. Without assignment guarantees, the components described so far constitute a student placement problem: (D, H, P_D, P_H, q_H).

Nevertheless, the contribution of this paper is to implement assignment guarantees into such a many-to-one placement problem, which is the second component of a program's choice process. Doctors can have assignment guarantees at different programs. In line with this purpose, let E_h denote the set of doctors, who are guaranteed a seat at program h. With a similar notation logic as above, E_H is the collection of doctors that have assignment guarantees at each program. The student placement problem with assignment guarantees therefore consists of the tuple: $(D, H, P_D, P_H, q_H, E_H)$.

Once doctors and programs are assigned to each other, a matching μ is a set of doctor-program (d, h) pairs such that each doctor d appears in at most one pair and $\mu(d) = h$ if and only if $d \in \mu(h)$, where $\mu(d)$ and $\mu(h)$ denote the match of the doctor d and program h under matching μ, respectively.

A direct mechanism is then a function ϕ and selects a matching for each preference profile, capacity vector and guarantee scheme of the doctors. In this paper, we denote the matching, which is the outcome of the direct mechanism ϕ as μ^ϕ.

3 Placement Problem with Assignment Guarantees

In this section, we formally define the concepts that are specific to our environment. Moreover, we discuss the peculiarities and challenges of the student placement problem with

[1] The existence of a centralized exam score is a special case of this framework such that the programs have the same preference over doctors.

[2] Formally, $z_h(d|P_h, D') = |d' \in D' : d' \succ_h d| + 1$

assignment guarantees that arise due to the characteristics of the problem.

3.1 Placement Problem with Assignment Guarantees when Capacities are Fixed

This subsection studies the incompatibility between assignment guarantees and fixed capacities in the placement problem. Namely, we show and discuss that it is not possible to respect assignment guarantees and satisfy fairness while keeping the capacities of the programs constant. Assignment guarantees of doctors put constraints on the outcome, such that a doctor may never be assigned to a program that she prefers less than the program at which she has an assignment guarantee. Formally:

Definition 1. *Mechanism ϕ* ***respects assignment guarantees*** *if for any fixed problem $(D, H, P_D, P_H, q_H, E_H)$, $\nexists d \in D$ such that $d \in E_h$ for some h and $h \succ_d \mu^\phi(d)$.*[3]

Next, we introduce the *fairness* criterion, which is adapted from the student placement literature. Fairness requires that the more preferred doctors are assigned to better alternatives. Formally:

Definition 2. *Mechanism ϕ satisfies* ***fairness*** *if for any fixed problem $(D, H, P_D, P_H, q_H, E_H)$, $\nexists\{d, d', h\}$ such that $d \succ_h d'$ and $h \succ_d \mu^\phi(d)$ whereas $d' \in \mu^\phi(h)$.*

The outcome of a mechanism is not fair, if there is an unmatched doctor-program pair (d, h), where doctor d prefers program h to her own assignment and the program h prefers her to another doctor d' who is assigned a seat there. Observe that both criteria are quite intuitive. The difference between the two is that fairness considers the preferences of both sides only. However, programs should take into account the assignment guarantees of doctors as well. Therefore, assignment guarantees might diversify programs' "preferences".

Example 1. Let us now consider the following simple example: There are two doctors, $D = \{a, b\}$ and one single program $H = \{h\}$ with a single capacity $q_h = 1$. Suppose a is guaranteed a seat at hospital h, meaning that if a mechanism is to respect assignment guarantees, she cannot be assigned to a worse alternative than h. Doctor b has no assignment guarantee, therefore $E_h = \{a\}$. If $a \succ_h b$, it is quite easy for any mechanism ϕ to satisfy fairness as well as respect the guarantees, that is to match a with h and leave b unemployed.

[3] Observe that this definition does not prevent a doctor from having assignment guarantees at multiple programs. If a doctor is guaranteed a seat at multiple programs, we can WLOG restrict attention to her most preferred alternative amongst her assignment guarantees.

However, it is not as simple once $b \succ_h a$. In that case, fairness requires $\mu^\phi(b) = h$ and assignment guarantees require $\mu^\phi(a) = h$. Thereupon, it is not possible to satisfy both without creating an additional capacity at program h.

Admitting the impossibility to respect assignment guarantees and satisfy fairness without creating additional capacities, deviation from the "target" capacities q_H is still undesired. The unintended acceptance of additional doctors results in inefficiency in many ways. First of all, it is harmful to the government budget to employ two doctors instead of one. Second, from program h's point of view, if the program was optimally designed for one resident only, then the additional resident may reduce the overall quality of the education.

Additionally, recall the Rural Hospitals Theorem by Roth (1986). A quite intuitive fact that names the theorem is that the young residential candidates usually prefer the programs in the urban areas rather than the rural ones. Thus, this unintended creation of additional capacities is likely to disturb the balance between hospitals in terms of the number of residents employed.

On the one hand, it is clear that additional capacities have to be created if the designer aims to implement a fair mechanism. However, how many additional capacities will be required still remains an unanswered question. We illustrate this extent with the following example:

Example 2. Suppose there are three doctors, $D = \{a, b, c\}$ and one single program $H = \{h\}$ with a single capacity $q_h = 1$. Same as the previous example, only a is guaranteed a seat at hospital h. Furthermore, h ranks the candidates as $b \succ c \succ a$.

Any mechanism that respects assignment guarantees has to satisfy $\mu^\phi(a) = h$.

Furthermore, once a is assigned to program h, fairness would require all three doctors to be assigned to h since b and c both are preferred over a. However, in the absence of assignment guarantees, c would not receive a seat in a fair mechanism. In other words, a's assignment guarantee at program h creates another assignment guarantee for c indirectly.

We could take this scenario to an even more extreme point: What if there are many doctors as c, whose ranking satisfy $b \succ c \succ ... \succ z \succ a$? Would all the candidates receive a seat at h only because of a's assignment guarantee? In fact, the first proposition of the paper gives answers this question and shows the extent of demandingness of fairness in environments with placement guarantees:

Proposition 1. *There does not exist a mechanism that satisfies fairness and respects assignment guarantees without creating capacities equal to the number of candidates for each* *program.*

If the designer wants to ensure fairness for all candidates, she has to abolish the capacities of all programs. The discussion above is only strengthened for no capacity regulations, and therefore is undesirable. Admitting that imposing the traditional fairness axiom is too strict in such environments, we propose an alternative, relaxed notion of fairness in the next section, which is more suitable to the placement environments with assignment guarantees.

3.2 Capacity Respecting Fairness (q-fairness)

The main reason for proposing a new notion of fairness is to avoid creating additional capacities for those, who would not acquire a seat in the absence of assignment guarantees of other candidates. The traditional notion of fairness is too strict in the sense that additional capacities are created unintendedly even to doctors such as c, who are ranked outside the target capacity and do not have assignment guarantees at programs. In other words, if it were not for the guarantees, doctors such as c would not have a claim on the seats based on fairness, because they are ranked outside the capacity of program h. For this very reason, we present *capacity respecting fairness*, which is a relaxed version of the traditional fairness axiom:

Definition 3. *Mechanism ϕ satisfies **capacity respecting fairness** (or is q-fair) if for any fixed problem $(D, H, P_D, P_H, q_H, E_H)$,*

$$\nexists (d, h) \text{ such that } h \succ_d \mu^\phi(d) \ \& \ z(d|P_h, \mu^\phi(h) \cup \{d\}) \leq q_h.$$

Intuitively, q-fairness suggests that a mechanism ϕ is *capacity respectingly fair* to doctors, as long as it doesn't assign a doctor d to a worse alternative $\mu^\phi(d)$, who would be ranked within the target capacity of a program h along with the to-h-matched doctors $\mu^\phi(h)$. Observe that q-fairness is clearly a weaker condition than fairness. Specifically, q-fairness allows violation of fairness for those, who are ranked outside the target capacity for an application pool (such as doctor c in Example 2 above, but not doctor b). Furthermore, in an environment where agents can earn seats by no means but simple preferences on the program side (such as their exam scores), q-fairness is the same as the traditional fairness axiom.

The problem of having to create additional capacities is still not beside the mark with q-fairness. There still does not exist a mechanism that satisfies q-fairness and respects assignment guarantees, without creating additional capacities for some program. This could again be observed in Example 2 with doctors a, b, and c. As a has a right at the only seat of h, b also has the right due to q-fairness. However, q-fairness now allows us to leave c out of the program. In that sense, q-fairness is a minimal "fairness" requirement on the way to decrease the number of additional capacities. It is the least we could expect from a mechanism that has some fairness concerns when assignment guarantees are present.

If a mechanism is q-fair, a side benefit is that no seat of a program is left empty as long as there is some doctor who prefers the seat to her current alternative. This notion is formally called *non-wastefulness*, which indeed means that no seat is wasted throughout the procedure. [4]

3.3 Avoiding Unnecessary Slots

Capacity respecting fairness is still not enough to prevent the creation of unnecessary capacities to the full extent. To see this, consider the following example:

Example 3. Suppose there are four doctors, $D = \{a, b, c, d\}$ and two programs $H = \{x, y\}$ such that $q_x = q_y = 1$. The program's preferences both satisfy $a \succ b \succ c \succ d$ and the least preferred candidates have assignment guarantees, $E_x = \{c\}$ and $E_y = \{d\}$. Suppose x is preferred over y by all doctors but c (c prefers y to x). If mechanism ϕ places candidates such that $\mu^\phi(x) = \{a, d\}$, $\mu^\phi(y) = \{b, c\}$, q-fairness is still not violated. However, it is neither clear nor natural that c and d receive an additional seat at a program at which they have no assignment guarantee.

Capacity respecting fairness constraints the envy on the side of those who did not get into the residential programs. However, as can be seen above, another notion is needed as well, which restricts the ones who actually receive a slot. Only that way we can make sure that the excess capacities are only created for those, who have assignment guarantees at programs. For this reason, we introduce the following notion:

Definition 4. *Mechanism ϕ* ***avoids unnecessary slots*** *(or satisfies AUS) if for any fixed problem $(D, H, P_D, P_H, q_H, E_H)$ and any pair (d, h),*

$$\mu^\phi(d) = h \text{ and } d \notin E_h \Rightarrow z(d|P_h, \mu^\phi(h)) \leq q_h$$

Verbally, a mechanism avoids unnecessary slots if a doctor is placed at a program only because of her assignment guarantee or if she is ranked within the target capacity among the assigned set of doctors to the program.

[4]The formal definition, as well as the proof for q-fairness implying non-wastefulness can be found in the appendix.

The two notions of q-fairness and AUS are introduced to find a balance between the preferences of doctors, preferences of programs, and assignment guarantees. Even though they might sound similar, their implications are quite different. For example, assigning every doctor to their favorite program would satisfy assignment guarantees and q-fairness trivially (because there is no envy at all) but would possibly create many unnecessary slots, failing AUS. On the other hand, allocating seats only to doctors with assignment guarantees would not create unnecessary seats but raise fairness concerns.

When there are assignment guarantees in addition to program preferences, the requirements we expect from a mechanism have to be relaxed relative to the benchmark student placement problem. For the two new notions of q-fairness and avoiding unnecessary slots, we can say that q-fairness takes into account of preferences of doctors in a relaxed way that is compatible with assignment guarantees, whereas AUS considers the preferences of programs in the same relaxed way. In the absence of assignment guarantees, Student Proposing Deferred Acceptance Algorithm is naturally q-fair (also fair) and avoids unnecessary slots.

4 The Assignment-Guarantees-Adjusted Mechanism

Having presented the appropriate notions, we now define a new choice function, with the intent of building towards a mechanism that creates excess additional capacities to the least, while respecting assignment guarantees. Because of the rather complex nature of the student placement problem with assignment guarantees, the choice function will resemble the choice functions in matching with contracts framework, especially the choice functions in the Cadet-Branch Matching problem by Sönmez and Switzer (2013). The Assignment-Guarantees-Adjusted Choice Function (shortly AGA Choice Function, denoted by C_h^A) defines a selection rule of program h from any application pool $D' \subset D$ and proceeds as follows:

1. Rank all the doctors in D' according to preferences of h.
2. Based on their rankings, add doctors one-by-one to $C_h^A(D')$ until q_h is full or all doctors are considered.
3. Add all the remaining doctors such that $d \in (E_h \cap D')$ to $C_h^A(D')$.
4. Terminate the procedure, reject all other doctors.

For any given application pool, the choice function first considers its own preferences and adds doctors one-by-one according to their ranking. In this step, there may or may not be doctors who have assignment guarantees among the chosen doctors. After the target capacity is full with the merit candidates, the program does not immediately reject all the remaining candidates. Instead, if there are doctors left in the application pool with assignment guarantee, it adds them to the chosen set and expands its capacity.

Almost trivially, one can show that the choice function satisfies certain well-behaving properties that a mechanism designer would expect from a choice function such as substitutes, law of aggregate demand (LAD), and irrelevance of rejected contracts (IRC), proofs of which can be found in the Appendix B. Intuitively, the substitutes condition ensures that there are no complementarities between the doctors for the programs. LAD guarantees the expansion of the rejection set as the choice set expands and with IRC, removing the rejected alternatives does not affect the choice set.

After the introduction of AGA Choice Function, we now shift our attention to mechanism design. The mechanism we introduce in order to minimize the number of additional seats is the doctor proposing deferred acceptance algorithm induced by the AGA Choice Function. Formally:

Step 1: Each doctor proposes to her first choice. Each program tentatively assigns its seats to the doctors in its application pool according to the AGA Choice Function.

$\vdots$

Step k: Each doctor who was rejected by any program in the previous step proposes to her next choice. Each program considers the doctors it has been holding so far, together with the new applicants and tentatively assigns its seats to the doctors in its new application pool according to the AGA Choice Function.

Because this mechanism uses the AGA Choice Function in every step of the deferred acceptance algorithm, we call this special mechanism "Assignment-Guarantees-Adjusted Mechanism", shortly AGAM.

In an environment where doctors have assignment guarantees at some programs in addition to the programs' preferences, creating additional capacities is inevitable. However, the adjusted choice function helps implement assignment guarantees into the benchmark placement problem such that assignment guarantees are respected and the deviation from the target capacities is only due to the guarantees.

It is almost trivial that for any application pool of any program, the choice function itself exhibits the plausible features of the environment with assignment guarantees. If there is a single program, the choice function admits candidates in a single step such that it is q-fair, avoids unnecessary slots, and respects assignment guarantees by construction.[5] Furthermore, the candidates then would not have any incentive to misreport their preferences. However, when it comes to the mechanism which includes more than one program and takes several steps to conclude, it is not that straightforward that the plausible properties are still satisfied. In the following section, we rigorously discuss the properties of AGAM. All omitted proofs can be found in Appendix C.

4.1 Fairness, q-fairness, AUS, and Assignment Guarantees

At this point, it is already clear that AGAM violates traditional fairness[6]. It was the first acknowledgment of the paper that fairness is too strict in placement environments with assignment guarantees. On the other hand, the mechanism satisfies other properties that were defined in the previous chapters.

Proposition 2. *AGAM satisfies capacity respecting fairness, avoids unnecessary slots, and respects assignment guarantees.*

The fact that AGAM satisfies q-fairness, AUS, and respects assignment guarantees proves that the mechanism is a suitable candidate for a placement environment with assignment guarantees. The mechanism acknowledges the merit rankings of doctors and respects the exogenously given assignment guarantees. When some fairness concerns are present, recall that any mechanism has to deviate from the target capacities but AGAM does that in a minimally harmful way. This is because AGAM makes sure that the extra capacities belong to exogenously guaranteed candidates at each program. Furthermore, as mentioned before, it is non-wasteful which is implied by q-fairness.

4.2 Strategy-Proofness

While designing a mechanism for a placement problem with or without assignment guarantees, most of the properties we look for and expect from our mechanism depend on the preferences of the doctors (q-fairness of the mechanism, stability of the outcome, etc.) Therefore, if the doctors have incentives to misreport their preferences, it would be pointless to analyze those properties. Strategy-proofness is therefore an essential property of the mechanisms to eliminate such incentives of the doctors.

[5]When there is a single program, a mechanism being q-fair is equivalent to the choice function being q-responsive as in Aygün and Turhan (2020).

[6]Recall Example 2, the outcome of AGAM would be $\mu^A(h) = \{a, b\}$, leaving c unemployed, thus violating fairness.

Definition 5. *A mechanism ϕ is **strategy-proof** (for doctors) if for any doctor d and preference profile (P_d, P_{-d}), where P_{-d} is the collection of the preference profiles of all doctors but d, there is no preference $P'_d \in \mathcal{P}_d$ such that $\mu^{\phi(P'_d, P_{-d})}(d) \succ_d \mu^{\phi(P_d, P_{-d})}(d)$.*

Proposition 3. *AGAM is strategy-proof.*

Strategy-proofness ensures that the doctors reveal their true preferences to the mechanism, without which, all the other properties would trivially fail according to true preferences. Moreover, note that the assignment guarantees of doctors are automatically revealed to the mechanism. However, the guarantees can only improve the placement of a doctor and the doctors do not differentiate between different types of seats. Therefore, even if we allowed the doctors to report their assignment guarantees alongside their preferences, they would not have an incentive to hide their guarantee status either.

The fact that AGAM is also strategy-proof strengthens our claim that the mechanism is suitable to use in placement environments with assignment guarantees. Furthermore, the following theorem concludes that AGAM is in fact the unique mechanism if the designer has merit concerns, is constrained by the guarantees as well as aims to create as least additional seats as possible while eliciting the true preferences of doctors.

Theorem 1. *AGAM is the unique strategy-proof mechanism that respects assignment guarantees, is q-fair, and avoids unnecessary slots.*

Proof. The mechanism is essentially a deferred acceptance algorithm that uses the AGA choice function at each iterative step. In Appendix D, we show that the axioms of respecting assignment guarantees, q-fairness, and AUS are together equivalent to stability with respect to the AGA Choice Function. The AGA Choice Function satisfies substitutes and the law of aggregate demand conditions, the deferred acceptance induced by this function produces a stable outcome and is strategy-proof. By Hirata and Kasuya (2017), the doctor proposing deferred acceptance algorithm is the unique candidate for a strategy-proof mechanism that produces a stable outcome. □

4.3 Deviation

Until this point, we characterized and showed many favorable properties of AGAM. In this section, we analyze how the mechanism deviates from the target capacities, which are carefully designed by planners, and deviations from which are rather undesired.

Admitting that the additional capacities will have to be created in a student placement problem with assignment guarantees, we calculate the **deviation** of an outcome from the original target capacity of a program by taking the difference between the realized capacity and the target capacity only if the realized capacity exceeds the target capacity. Formally, the deviation of an outcome from the original target capacity of a program is $max\{\mu(h) - q_h, 0\}$. The main reasoning behind this absolute value approach is that it is the unintended and inevitable "excess" placements that cause the complications in the first place. By now, it is clear that the notion of avoiding unnecessary slots is required to control the complication dimension. The theorem below shows that, indeed, AGAM is one of the mechanisms that minimize the deviation from the target capacities whilst satisfying q-fairness and respecting the assignment guarantees of doctors.

Theorem 2. *Among the mechanisms that are q-fair and respect assignment guarantees of doctors, AGAM minimizes the deviation from the target capacities.*
Formally, for all mechanisms ϕ that are q-fair and respect assignment guarantees, and problems $(D, H, P_D, P_H, q_H, E_H)$, we have $\forall h \in H$, $max\{\mu^A(h) - q_h, 0\} \leq max\{\mu^\phi(h) - q_h, 0\}$.

A very short and verbal intuition for the proof would be: In order to obtain an outcome that deviates less than μ^A, a chain must be constructed over μ^A, which starts with a doctor who is placed to that program under μ^A and ends at a vacant capacity. We show in the appendix rigorously that such a chain conflicts with either q-fairness or assignment guarantees or both. In fact, the deviation result relies on the AUS property of AGAM, which is one way to prevent excessive deviations. After proving this theorem, we can now conclude that: In a student placement problem with assignment guarantees, if the deviation from the target capacities is undesirable, AGAM is one of the best mechanisms which can be implemented. The formal proof, as well as an illustrative example, is to be found in the appendix.[7]

5 Applications

In this section, we present three applications of the theoretical analysis in this paper. First, we introduce the Re-placement of Residents Matching Problem in Turkey, where some candidates have guaranteed seats at a program. Second, we discuss implications for couple matching where the authorities guarantee that spouses will be appointed to geographically

[7]Observe that there might be other mechanisms that result in the same deviation for each program. For example, one other mechanism that minimizes deviation applying TTC after AGAM. This mechanism fails AUS. An example of the outcome can be found here 3.

similar locations. Third, we consider an application in the job market: Workers can claim their previous positions after their parental leave.

5.1 Re-placement of Residents Matching Problem in Turkey

Medical students in Turkey, who want to continue their education with specialization take an exam called the Examination of Specialty in Medicine (ESM). The state agency Measuring, Selection, and Placement Center (MSPC) is responsible for conducting the ESM twice a year as well as the assignment procedure in the aftermath of the exam. In the ESM, residential candidates receive a score and they are ranked according to those scores. Thereafter, they are placed at residency programs according to the doctor proposing deferred acceptance algorithm (similar to many exams conducted by the MSPC). However, one of the two exams in the years 2010, 2013, 2014, and 2016 were special and the placement process was not that straightforward. After the exam, some questions were found flawed by authorities and were officially canceled. Scores of the doctors were calculated according to the remaining accurate questions. As usual, placements were done by the doctor proposing DA algorithm.

Nonetheless, after the placements, the State Council revoked the cancellation which re-established the accuracy of the canceled questions. This led to a change in the scores and hence the rankings of the doctors. However, the placements had already been done and the residential candidates had started working at their assigned programs. Hence, the original placement was obviously not *fair* to some residential candidates, especially to those, whose rankings have increased after the score recalculation.

As a compensation, it was announced that there was going to be a re-placement procedure, which aimed to provide fairness to the doctors with increased rankings. The re-placement procedure would preserve the acquired rights (which correspond to the assignment guarantees in our setting) of the existing doctors in the programs, i.e. who were assigned a seat during the original placement procedure [8]. In other words, any compensation mechanism should make the candidates who were placed in the initial placement at least as well of.

All in all, restoring fairness and respecting the acquired rights cannot take place unless the target capacities of the programs are relaxed. For some scores and preferences of doctors, some residential programs end up with more doctors than their target capacity. The

[8]The acquired rights are defined by law and prevent the existing residents of programs being assigned to other programs which they prefer less than their initial assignments (also called as vested interests in the literature)

unintended acceptance of these doctors results in inefficiency in many ways. First, from the governmental point of view, if they were not placed at the original placements, they are harming the government's budget. Second, from a program's point of view, if the program was optimally designed for one resident, then the additional resident may reduce the quality of the education for each resident. Third, if a doctor has already started with another residency program, her being accepted by another program means a loss for her original assignment. Depending on the presence and quality of the other doctors in its application pool, that program might face other complications and this situation will keep snowballing towards the less preferred programs. Additionally, young residential candidates usually prefer programs in urban areas rather than rural ones. Thus, this unintended creation of additional capacities is likely to disturb the balance between rural and urban hospitals in terms of the number of residents employed, who are an important chain ring in the middle of the health industry.

In another work in progress (Aygün and Bilgin (n.d.)), we analyze the mechanism used by MSPC and compare it to AGAM rigorously. Under the dominant strategy of doctors, the mechanism used by MSPC creates additional seats to doctors even without the top-q merit or an assignment guarantee. As shown in previous chapters, in the Re-Placement of Medical Residents Problem, AGAM is the only strategy-proof mechanism that respects the acquired rights, is q-fair, and avoids unnecessary slot creation. In fact, for this case with exam scores, the characterization is even stronger. The only stable matching in a re-placement of residents matching problem can be found via AGAM. Therefore, in a student placement environment with central exam scores and assignment guarantees, the designer can implement AGAM to find a q-fair matching that respects acquired rights and avoid excess capacity deviations simultaneously. Furthermore, among the q-fair mechanisms that respect the acquired rights, AGAM minimizes the deviation from the target capacities.

5.2 Spousal Matching

Matching markets with couples is an interesting theoretical problem. The problem of matching with couples relies on the fact that spouses have a preference to be appointed to geographically similar locations, and there is already a wide literature about whether and under what conditions stability can be achieved (Roth (1984), Kojima, Pathak, and Roth (2013)).

In order to protect family integrity, different central planners adopt different solutions. The famous National Resident Matching Program (NRMP) states that "When applicants participate in a Match as a couple, their rank order lists form pairs of program choices that are considered by the matching algorithm. A couple will match to the most preferred pair of programs on their rank order lists where each partner has been offered a position.[9]". In Turkey, civil servants are guaranteed to be appointed to the same location as their spouse, provided that either their spouse is also a civil servant, or the spouse has been working for the same private firm for a sufficiently long time.

For instance, after medical education, doctors have to complete a mandatory civil service at a place that is determined by a lottery to validate their diplomas. Married doctors who satisfy the above criteria can apply for a spouse-related appointment to be separated from the general lottery and are guaranteed to be appointed to a hospital in the same city as their spouse.[10]

In that case, the departments might have to create excess capacities even though they are not looking for additional workers. The excess capacity creation is only due to the marital status of doctors, which usually is exogenous to the appointment problem.

5.3 Return to Work After Parental Leave

Having and raising offspring is a basic instinct for human beings. Furthermore, for a functioning social security system and a balanced society, every country needs a sufficient amount of young population to join the labor force. Nevertheless, the fertile time window usually conflicts with the early career plans of young individuals. Hence, it is not always an easy decision to take a break from their career. Therefore, many countries work on regulations that will give young individuals incentives to childbearing to have a balanced population.

In the European Union, the law ensures that "working men and women are entitled to return to their jobs or to equivalent posts on terms and conditions which are no less favorable to them.[11]" With that regulation, the potential fear of losing their job is eliminated, so young individuals are incentivized towards childbearing.

However fair and reasonable that protection is, the firms might have nondeferrable needs. Suppose the newly parent's temporarily vacant position is crucial for the firm's structure. In that case, the firm might consider hiring an additional worker, even though it is aware that the parent has the right, and will return to the same position after their parental leave. In that case, the firm will have hired an additional worker even though it does not have a prospect of expanding the company.

[9]Source: The official website of NRMP https://www.nrmp.org/

[10]Source: The official website of Ministry of Health in Turkey`https://yhgm.saglik.gov.tr/`

[11]Source: Directive 2006/54/EC of the European Parliament and of the Council of 5 July 2006 on the implementation of the principle of equal opportunities and equal treatment of men and women in matters of employment and occupation `https://commission.europa.eu`

6 Conclusion

In this paper, we analyze student placement problems with assignment guarantees, where the designer aims to preserve assignment guarantees as well as has some fairness concerns. We show that conflict between fairness and assignment guarantees is unavoidable when the capacities of programs are fixed.

Nevertheless, since the capacities of the programs are already optimized, any deviation from the target capacities is costly and undesirable. Imposing the traditional notion of fairness, however, results in excessive deviations from the target capacities in such an environment. Thus, the traditional fairness notion is not suitable for student placement environments with assignment guarantees. In order to reduce excessive deviations and still redeem some form of fairness, we define the notions of capacity respecting fairness and avoiding unnecessary slots. Capacity respecting fairness relaxes fairness to the extent that a mechanism can be unfair to candidates that are ranked outside the target capacity but can still be q-fair. Avoiding unnecessary slots ensures that additional seats are used either by merit candidates or guaranteed candidates.

Moreover, we define a new selection rule for the programs: Assignment-Guarantees-Adjusted Choice Function and propose a new mechanism, Assignment-Guarantees-Adjusted Mechanism to be used in student placement procedures with assignment guarantees.

The Assignment-Guarantees-Adjusted Mechanism is the deferred acceptance algorithm induced by the Assignment-Guarantees-Adjusted Choice Function. It is the only strategy-proof mechanism that satisfies the q-fairness, and avoids unnecessary slots while respecting assignment guarantees. Moreover, the Assignment-Guarantees-Adjusted Mechanism minimizes the deviation from the target capacities, whilst respecting the assignment guarantees of doctors and satisfying q-fairness.

For future research, it might be useful to consider programs' preferences more elaborately. For instance, the central planner might commit to smaller target capacities if there are many candidates with assignment guarantees and the deviation is costly. On the other hand, if the emphasis is rather on fairness, we would observe larger target capacities. Quantifying the analysis might help us better understand environments with assignment guarantees.

Appendix

A Non-wastefulness

Definition 6. *Mechanism ϕ is* ***non-wasteful*** *if for any fixed problem $(D, H, P_D, q_H, E_H, s_D)$, $\forall(d, h) \in (D \times H)$, $h \succ_d \mu^\phi(d) \implies |\mu^\phi(h)| \geq q_h$.*

We can easily show that if a mechanism satisfies q-fairness, it is also non-wasteful. Suppose ϕ violates non-wastefulness. Then $\exists(d, h)$ such that $h \succ_d \mu^\phi(d)$ and $|\mu^\phi(h)| < q_h$. Then, $z(d|P_h, \mu^\phi(h) \cup \{d\}) \leq q_h$ which means ϕ also violates q-fairness.

B Properties of the AGA Choice Function (C_h^A)

1. Substitutes

   Definition 7. *Elements of Y are* ***substitutes*** *for program h if for all subsets $Y' \subset Y'' \subset D$ we have $Y' \setminus C_h(Y') \subset Y'' \setminus C_h(Y'')$. (Hatfield & Milgrom, 2005)*

   Substitutes condition requires that the rejection set expands (weakly) as the application pool expands. Intuitively, it implies that there are no complementarities between doctors.

   Lemma 1. *C_h^A satisfies substitutes.*

   Proof. Any violation of substitutes would require the existence of doctor d such that, $d \notin C_h^A(Y')$ but $d \in C_h^A(Y'')$ for some Y'' such that $Y' \subset Y''$. All by-h-prioritized doctors are chosen by the choice function, so our violation of substitutes, if any, must stem from the non-prioritized candidates in the application pool. Suppose d it not prioritized at h and $d \notin C_h^A(Y')$. Then d has not a high enough ranking in Y'. Clearly, doctor d's ranking in the set Y' weakly decreases while the set expands by the addition new doctors. As a consequence, d will still not be chosen from any set Y'' such that $Y' \subset Y''$ either. Thus, C_h^A satisfies substitutes. $\square$

2. Law of Aggregate Demand

   Definition 8. *The preferences of hospital $h \subset H$ satisfy the law of aggregate demand if for all $X' \subset X'' \subset D$, $C_h(X') \leq C_h(X'')$. (Hatfield & Milgrom, 2005)*

   The Law of Aggregate Demand is an intuitive condition, which implies that the chosen set does not get smaller as the application pool expands.

Lemma 2. *C_h^A satisfies LAD.*

Proof. Observe that for any $Y' \subset Y'' \subset D$, if $|C_h^A(Y')| \leq q_h$, at least $|C_h^A(Y')|$ candidates will be chosen from Y''. If $|C_h^A(Y')| > q_h$, it means there are prioritized candidates in (Y'), who are ranked outside the target capacity. As the set expands, those prioritized doctors will still be outside the capacity, thus again at least $|C_h^A(Y')|$ candidates will be chosen from Y'' as well. Thus, for all $Y' \subset Y'' \subset D$, $|C_h^A(Y')| \leq |C_h^A(Y'')|$. □

3. IRC

   Definition 9. *Given a set of doctors D, a choice function satisfies the* ***irrelevance of rejected contracts (IRC)*** *if and only if:*
   $\forall Y \subset D, \quad \forall z \in D \setminus Y \qquad z \notin C(Y \cup \{z\}) \implies C(Y) = C(Y \cup \{z\})$. *(Aygün & Sönmez, 2013)*

   The IRC condition requires that the removal of not chosen (rejected) contracts from the application pool do not affect the chosen set.

   Lemma 3. *C_h^A satisfies IRC.*

   Proof. Suppose the choice function chooses $C_h^A(Y)$ from the application pool Y, where $d \notin C_h^A(Y)$. As above, $d \in (Y \setminus E_h)$ and d's ranking in Y is lower than the target capacity. Then, removing d from Y would have no effect on the top q_h candidates of the prioritized candidates, and thus on the chosen set, namely $\forall d \in Y$ such that $d \notin C_h^A(Y)$, $C_h^A(Y) = C_h^A(Y \setminus \{d\})$. Thus, C_h^A satisfies IRC. □

C Properties of AGAM

Proposition 2:

Proof. Below we show that AGAM is q-fair, AUS, and respects assignment guarantees.

1. q-fairness:
   AGAM is the mechanism that uses the AGA Choice Function for programs at each iterative step of the deferred acceptance algorithm. Therefore, if $\exists(d, h)$ such that $h \succ_d \mu^A(d)$, d must have proposed to h at earlier steps before the algorithm concluded, and h can only reject d for the doctors it prefers over d to fill its seats, which implies $z(P_h, \mu^A(h) \cup \{d\}) > q_h$.

2. AUS:

   The AUS property of AGAM straightforwardly follows from the AGA Choice Function and the deferred acceptance procedure. At each step of the DA, the choice function selects applicants such that they are either in the top-q for the program, or they have a guaranteed seat. The top-q candidates admitted in the first step can only be replaced with better-ranked candidates in the following rounds, whereas the guarantees candidates never lose their additional seats. Therefore, the end allocation ensures the same for the selected doctors for all programs.

3. Assignment Guarantees:

   Similar to AUS, respecting assignment guarantees follows from the fact that the AGA Choice Function respects guarantees at each step. A candidate who has a guaranteed seat at h proposes to h only when she is rejected from all other programs that she prefers to h and she once she proposes to h she receives either a top-q seat or an additional seat created because of her guarantee. The status of the seat can only change from top-q to an additional seat but she is never rejected by h.

□

Proposition 3:

Proof. AGAM is strategy-proof because the choice function that induces the DA in this mechanism satisfies the substitutes and the Law of Aggregate Demand (LAD) conditions, which are sufficient properties for a deferred acceptance algorithm to be strategy-proof (Hatfield and Milgrom (2005)). □

D Stability

The minimal requirement one would expect from a mechanism is that it produces a stable matching, which is the most common equilibrium concept in matching theory. Among several different approaches to stability in the literature, we use pairwise stability, intuitively meaning that no parties can individually or mutually be better off by opting out of the mechanism.

Formally, a matching μ is stable if it is:

1. individually rational, $C_i(\mu(i)) = \mu(i)$ for all $i \in (D \cup H)$.

2. not blocked, $\nexists (d,h)$ pair such that $\mu(d) \neq h$, $C_d(\mu(d) \cup h) = h$ and $d \in C_h(\mu(h) \cup d)$.

Below, we show that AGAM always creates a stable outcome with respect to the AGA Choice Function. In fact, we prove this not only by relying on the properties of the choice function but also by rigorously showing that the outcome is always individually rational and there are no blocking pairs. Furthermore, we show the properties of q-fairness, AUS, and respecting guarantees together are equivalent to the stability with respect to the AGA choice function in our framework, which is yet another way of proving the stability of the outcome of AGAM.

Proposition 4. *AGAM produces a stable outcome with respect to the AGA Choice Function.*

Proof. As mentioned above, one proof would be relying on the properties of the choice function. Since C_h^A satisfies substitutes, LAD, and IRC conditions, the existence of a stable outcome is guaranteed. Furthermore, the deferred acceptance algorithm induced by this choice function creates a stable outcome.

Alternatively, we show that the mechanism is individually rational and there are no blocking pairs.

1. Individual Rationality: Along the DA, doctors only propose to acceptable programs and programs only accept doctors that either belong to top q or have an assignment guarantee, which corresponds to AUS for programs.
2. Blocking pairs: Suppose (d,h) constitute a blocking pair under μ^A. The nature of DA requires d having proposed to h. Since they are not matched under μ^A, h rejects d after the proposal because h has employed at least q_h candidates that are preferred over d and d has no guarantee at h.

□

Theorem 3. *Any mechanism satisfies q-fairness, avoids unnecessary slots, respects assignment guarantees, and is individually rational for doctors if and only if it is stable with respect to the AGA Choice Function.*

Proof. $\Rightarrow$ Suppose a mechanism ϕ is individually rational for the doctors, satisfies q-fairness, avoids unnecessary slots, respects assignment guarantees, and however, is not stable. Since we already assumed it is individually rational for the doctors, it can only fail IR for the programs. A doctor d is not acceptable for program h under μ^ϕ if $d \in \mu^\phi(h)$ but $d \notin C_h(\mu^\phi(h))$. Doctor d not being chosen implies d not being one of the top q_h candidates in

$\mu^\phi(h)$ and $d \notin E_h$, which conflicts with ϕ avoiding unnecessary slots. Therefore, ϕ has to be individually rational.

The only possibility of ϕ not being stable is then blocking pairs. Suppose (d, h) such that $d \in C_h(\mu^\phi(h) \cup \{d\})$ and $h = C_d(\mu^\phi(d) \cup h)$. Since ϕ respects assignment guarantees, $d \notin E_h$. Then $d \in C_h(\mu^\phi(h) \cup \{d\})$ implies $z(d|P_h, \mu^\phi(h)) \leq q_h$, which conflicts with ϕ satisfying q-fairness.

$\Leftarrow$ Suppose a mechanism ϕ is stable with respect to the AGA Choice Function. It is individually rational for the doctors since it is stable. Suppose it does not respect assignment guarantees. This means $\exists(d, h)$ such that $\mu^\phi(d) = h'$, $h \succ_d h'$, and $d \in E_h$. In that case, (d, h) would constitute a blocking pair with respect to the AGA Choice Function (because $C_d(\mu^\phi(d)\ cup\{h\}) = h$ and $C_h(\mu^\phi(h) \cup \{d\})$ is either $\mu^\phi(h) \cup \{d\}$ or $\mu^\phi(h) \setminus \{d'\} \cup \{d\}$ for some d'). With the same logic, ϕ has to satisfy q-fairness (or the candidate who is ranked within capacity would form a blocking pair with the respective program). Lastly, suppose ϕ does not avoid unnecessary slots, i.e. $\exists(d, h)$ such that $\mu^\phi(d) = h$ but $z(d|p_h, \mu^\phi(h)) > q_h$ and $d \notin E_h$, which means d received a seat at h despite her ranking and having no guarantee. The AGA Choice Function would then reject at least d, $d \in \mu^\phi(h) \setminus C_h(\mu^\phi(h))$, which would imply ϕ not being individually rational for the programs. $\square$

E Deviation

Theorem 2:

Proof. The theorem is equivalent to the following: $\nexists$ a mechanism ϕ which satisfies q-fairness and respects guarantees, along with a problem $(D, H, P_D, P_H, q_H, E_H)$ such that $\exists h$ for which $max\{\mu^\phi(h) - q'_h, 0\} < max\{\mu^A(h) - q'_h, 0\}$.

Suppose there exists such ϕ, which satisfies q-fairness, respects guarantees, and creates less deviation at program h from the target capacities than AGAM for a fixed problem $(D, H, P_D, P_H, q_H, E_H)$. The existence of a mechanism ϕ with less deviation implies that there is excess employment under AGAM. The only way there is excess employment under AGAM is that for some h the least preferred doctor $d \in \mu^A(h)$ is a doctor with a guarantee (because of AUS). Since ϕ results in less deviation, the excess capacity creation at h must be strictly reduced.

Step 0: Furthermore, we also know that if ϕ creates less deviation, it must place some doctor, who was placed elsewhere under μ^A to a vacant capacity. This can happen via cycles and chains, but there has to be at least 1 chain. Note that this vacant capacity can either be at another hospital, or it can be the unemployment scenario. Call this doctor, who is placed in a vacant capacity under ϕ as d_n. Recall that AGAM is individually rational and non-wasteful. Hence, we know that d_n prefers $h_n = \mu^A(d_n)$ to this vacant capacity. Because ϕ respects guarantees, it also follows that $d_n \notin E_{h_n}$. Hence, $d_n \neq d$.

Step 1: Since ϕ also satisfies q-fairness, there must be at least q_{h_n} doctors under μ^ϕ at h_n that h_n prefers to d_n. In words, this means all the seats in the target capacity of the program must have been filled with better candidates so that d_n can not reclaim her seat at h_n. This also means that there is at least one doctor who is preferred to d_n, **who was matched to somewhere else under AGAM**, but is assigned to h_n under ϕ. Call the by-h_n-least-preferred doctor among those as d_{n-1}. Let d_n hypothetically point to d_{n-1} and let $h_{n-1} = \mu^A(d_{n-1})$. If d_{n-1} prefers h_n over h_{n-1}, AGAM would fail q-fairness, which by definition is impossible. Hence d_{n-1} must prefer h_{n-1} over h_n. Because ϕ respects guarantees, it follows that $d_{n-1} \notin E_{h_{n-1}}$.

$\vdots$

Step k: Since ϕ also satisfies q-fairness, there must be at least $q_{h_{n-(k-1)}}$ doctors under μ^ϕ at $h_{n-(k-1)}$ that $h_{n-(k-1)}$ prefers to $d_{n-(k-1)}$. This also means that there is at least one doctor who is preferred to $d_{n-(k-1)}$, **who was matched to somewhere else under AGAM**, but is assigned to $h_{n-(k-1)}$ under ϕ and **who has not been pointed until this step**. Call the least preferred doctor among those as $d_{n-(k-1)-1}$. Let $d_{n-(k-1)}$ hypothetically point to $d_{n-(k-1)-1}$ and let $h_{n-(k-1)-1} = \mu^A(d_{n-(k-1)-1})$. If $d_{n-(k-1)-1}$ prefers $h_{n-(k-1)}$ to $h_{n-(k-1)-1}$, AGAM would fail q-fairness, which by definition is impossible. Hence $d_{n-(k-1)-1}$ must prefer $h_{n-(k-1)-1}$ to $h_{n-(k-1)}$. Because ϕ preserves assignment guarantees, it follows that $d_{n-(k-1)-1} \notin E_{h_{n-(k-1)-1}}$.

Observe that this induction can be traced back with finitely many steps until all the relocated candidates are pointed. Furthermore, since ϕ strictly reduces the deviation at h, the chain's last step is a doctor that was employed at h under AGAM. Then, there are two cases to consider:

1. If we encounter d in one of the steps:

   Recall that d was the least preferred doctor along the doctors in $\mu^A(h)$. We cleared before that $d \in E_h$. By construction of the pointing, $\mu^\phi(d) = h' \neq h$. Now, if d prefers h over h', ϕ does not respect her guarantee at h. If d prefers h' over h then AGAM fails q-fairness, contradiction.

2. If we don't encounter d in one of the steps:

This means that $d \in \mu^{\phi}(h)$ as well. To reduce the deviation, some other doctor $d' \in \mu^{A}(h)$, such that $d' \succ_h d$ must have been relocated to some other program, hence constitutes one end of the chain, $d' = d_1$. For notational convenience, rename $\mu^{A}(d_i) = h_i$ and $\mu^{\phi}(d_i) = h_{i+1}$ for all $i = 1, ..., n$. The chain takes d_1 from h_1 to h_2, takes d_2 from h_2 to h_3,...., until the under AGAM vacant capacity under h_{n+1} is reached. From the construction of the chain, no candidate has a guarantee at their assignment under μ^{A}. Since AGAM is AUS and d_1 has no priority at h_1, d_1 is ranked within top q_{h_1} for h_1. Therefore, it must be that $h_2 \succ_{d_1} h_1$ so that d_1 and h_1 do not block mu^{ϕ} (q-fairness). If $d_1 \succ_{h_1} d_2$ contradicts with q-fairness of AGAM, so $d_2 \succ_{h_1} d_1$. Again, $h_3 \succ_{d_2} h_2$ for q-fairness of ϕ. Following these steps, we have $h_{i+1} \succ_{d_i} h_i$ and $d_{i+1} \succ_{h_{i+1}} d_i$. However, for h_n, $d_n \succ_{h_n} d_{n-1}$ contradicts with ϕ's q-fairness (d_n is at a vacant seat) and $d_{n-1} \succ_{h_n} d_n$ contradicts with AGAM's q-fairness (d_{n-1} and h_n would block mu^{A}).

Another possibility is that the chain does not include the least preferred doctor in $\mu^{A}(h)$ but another one among the least preferred doctors who are ranked outside the target capacity. Such a doctor has also a guaranteed seat at h and the proof goes through. $\square$

Let us illustrate the arguments of the proof with the example below. Similar to the proof, we try to construct an alternative outcome with less deviation from the target capacities.

In order to consider less deviation from the target capacities, the outcome of AGAM must have placed excessive residents in at least one program. There must be an existing candidate of this program, who wanted to use her guarantee to be placed in the same program. In the example, let the set of programs and doctors be $H = \{h_1, h_2, h_3\}$ and $D = \{d_1, d_2, d_3\}$ respectively, with each program having a target capacity of 1. Suppose the outcome of AGAM is such that $\mu^{A}(h_1) = \{d_3, d_1\}$, $\mu^{A}(h_2) = \{d_2\}$ and $\mu^{A}(h_3) = \emptyset$. $h_3 = \emptyset$ is analogous to some vacant seat at a program or the unemployment case. Let us without loss of generality assume $d_3 \succ_{h_1} d_1$ and impose no further score relations on the candidates. For the other case, the flow will be analogous.

From the structure above, we already have some information:

- h_1 likes d_3 more than d_1 and d_1 creates excess capacity at $h_1 \implies d_1 \in E_{h_1}$.
- $\mu^{A}(h_3) = \emptyset \implies$ each candidate prefers own allocation under μ^{A} to h_3.
- d_1 used her guarantee to be placed at $h_1 \implies$
  she either prefers h_1 to h_2 or she prefers h_2 to h_1 but h_2 prefers d_2 over d_1.

Now consider the following alternative allocations μ^{ϕ} which deviate less than μ^{A}. In any alternative, we start the chain from the residential candidate placed at the vacant seat at h_3.

This doctor will (hypothetically) point to the lowest-scored doctor who claimed her seat at her previous assignment by outscoring her. If the chain includes d_1, ϕ violates guarantees or AGAM violates q-fairness. If the chain does not include d_1, ϕ violates q-fairness. In either case, we find a contradiction that ϕ creates less deviation while respecting guarantees and satisfying q-fairness.

- $\mu(h_1) = d_1$, $\mu(h_2) = d_2$, $\mu(h_3) = d_3$
  d_3 cannot point to anyone, the chain ends without starting
  Since $d_3 \succ_{h_1} d_1$, ϕ fails q-fairness.
- $\mu(h_1) = d_1$, $\mu(h_2) = d_3$, $\mu(h_3) = d_2$
  d_2 points to d_3, d_3 cannot point to anyone, the chain ends. $h_2 \succ_{d_3} h_1$ (or AGAM is not q-fair). If $d_3 \succ_{h_2} d_2$, AGAM fails q-fairness. If $d_2 \succ_{h_2} d_3$, ϕ fails q-fairness.
- $\mu(h_1) = d_2$, $\mu(h_2) = d_1$, $\mu(h_3) = d_3$
  d_3 points to d_2, d_2 points to d_1.
  We encounter d_1, a guaranteed candidate. $d_2 \succ_{h_1} d_3$ (or ϕ is not q-fair), $h_2 \succ_{d_2} h_1$ (or AGAM is not q-fair), $d_1 \succ_{h_2} d_2$ (or ϕ is not q-fair). If $h_2 \succ_{d_1} h_1$ AGAM is not q-fair, if $h_1 \succ_{d_1} h_2$ ϕ fails guarantees.
- $\mu(h_1) = d_2$, $\mu(h_2) = d_3$, $\mu(h_3) = d_1$
  d_1's guarantee at h_1 is not respected.
- $\mu(h_1) = d_3$, $\mu(h_2) = d_1$, $\mu(h_3) = d_2$
  d_2 points to d_1, the chain ends. $d_1 \succ_{h_2} d_2$ (or ϕ is not q-fair). If $h_2 \succ_{d_1} h_1$ AGAM is not q-fair, if $h_1 \succ_{d_1} h_2$ ϕ fails guarantees.
- $\mu(h_1) = d_3$, $\mu(h_2) = d_2$, $\mu(h_3) = d_1$
  d_1's guarantee h_1 is not respected.

Observe that leaving $\mu(h_3) = \emptyset$ as it was in the μ^A and sending the candidates to unemployment will not be possible due to similar arguments as above. So, we can conclude that it is not possible to create an allocation with less deviation than AGAM outcome, whilst preserving assignment guarantees and satisfying q-fairness.

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