Discussion Paper Series – CRC TR 224

Discussion Paper No. 413
Project B 01

Order Independence in Sequential, Issue-by-Issue Voting

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April 2023

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Support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 is gratefully acknowledged.

Collaborative Research Center Transregio 224 - www.crctr224.de
Rheinische Friedrich-Wilhelms-Universität Bonn - Universität Mannheim

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August 11, 2022

Abstract

We study when the voting outcome is independent of the order of issues put up for vote in a spacial multi-dimensional voting model. Agents equipped with norm-based preferences that use a norm to measure the distance from their ideal policy vote sequentially and issue-by-issue via simple majority. If the underlying norm is generated by an inner-product – such as the Euclidean norm – then the voting outcome is order independent if and only if the issues are orthogonal. If the underlying norm is a general one, then the outcome is order independent if the basis defining the issues to be voted upon satisfies the following property: for any vector in the basis, any linear combination of the other vectors is *Birkhoff-James orthogonal* to it. We prove a partial converse in the case of two dimensions: if the underlying basis fails the above property then the voting order matters. Finally, despite existence results for the two-dimensional case and for the general l_p case, we show that non-existence of bases with the above property is generic.

1 Introduction

Complex collective decision problems are often divided in simpler issues that are put to a vote. In this paper we ask the following questions: How can the simpler issues be structured such that the voting outcome is robust to manipulations of the voting order among the issues? Is it always possible to find such robust procedures? We assume that the space of alternatives is multi-dimensional and that voters have utility functions that measure the distance of an outcome from their ideal point (which is their private information) according to an arbitrary norm defined on the space of alternatives.

*Gershkov's research is supported by a grant from Israel Science Foundation, Moldovanu's research is supported by the German Science Foundation through the Hausdorff Center for Mathematics and CRC TR-224, and Shi's research is supported by a grant from the Social Sciences and Humanities Research Council of Canada. Gershkov: Department of Economics, Hebrew University of Jerusalem, Israel and School of Economics, University of Surrey, UK, alexg@huji.ac.il; Moldovanu: Department of Economics, University of Bonn, Germany, mold@uni-bonn.de; Shi: Department of Economics, University of Toronto, Canada, xianwen.shi@utoronto.ca.

Following Davis, DeGroot and Hinich [1972], "spatial" models have become ubiquitous in Political Science, Economics and Operations Research: faced with policy bundles (or locations, or attributes) voters are assumed to have a most preferred bundle – an ideal point or "peak" – and utilities that decrease as one "moves away" from the peak. In one dimension, the spatial approach reduces to the classical domain of single-peaked preferences a la Black [1948] and choosing the median peak is a strategy proof mechanism, i.e., revealing the true preferences is a dominant strategy for each agent.[1] Moreover, the peak of the median voter is then a Condorcet winner.

Unfortunately, in multi-dimensional collective decision problems a Condorcet winner exists only under very restrictive symmetry conditions. As a result, voting cycles – where the preference of the majority changes from one alternative to another and cycles back – lead to equilibrium non-existence in spatial models (Plott [1967], Kramer [1973] and McKelvey [1979]).

1.1 Tullock's Puzzle: Institutional Stability

As Tullock [1984] famously noted, in real-life we observe considerable stability of complex legislative outcomes, contrasting the inherent instability described above. How can this puzzle be explained?

One avenue is the use of super-majorities to stabilize the status quo. Caplin and Nalebuff [1988] famously showed that, for a large number of voters and for log-concave distributions of peaks, the average of all peaks – viewed as status quo – cannot be displaced by any another alternative if a reform requires a super-majority of at least 64% in favor. Super-majorities are not neutral, and are thus best suited for special situations, e.g., the 2/3 majority needed for a constitutional amendment in the US or in Germany. More importantly, choosing the average peak is not a strategy-proof mechanism for a finite number of voters: extremists – called "cranks" in Galton's [1907] pioneering article which proposed the median as a better opinion aggregator – can easily manipulate the outcome by significantly exaggerating their position in order to pull the collective choice closer to their desired position.

Another prominent resolution of Tullock's puzzle was offered by Shepsle [1979] who argued that the division of a complex decision into several different jurisdictions (which he called *germaneness*), creates equilibria that would not be stable in a general, unconstrained collective decision model.[2] In Shepsle's approach, the focus on the lack of equilibria in spatial models of unconstrained voting is replaced by the study of particular strategic situations induced by the institutions governing proposal making, agenda formation, coalition

[1] Generalized medians that allow for the presence of additional "phantom" voters with fixed, known peaks exhaust the set of DIC mechanisms in various settings where the preference domain is sufficiently rich. The first, fundamental result was obtained by Moulin [1980]. Gershkov, Moldovanu and Shi [2017] and Kleiner and Moldovanu [2017] analyze the implementation of generalized medians via sequentially binary procedures with varying majority requirements.

[2] See also Feld and Grofman [1988] and Kramer [1972], [1973].

formation and so on.

The best known example of such an institution is “issue-by-issue” voting: a complex policy problem is first divided into several one-dimensional issues, each possibly controlled by a different committee or jurisdiction; then voting by majority takes place on each issue; finally, the outcomes, one on each issue, are aggregated to yield a solution to the original multi-dimensional problem – the so-called issue-by-issue or “coordinate-wise” or “marginal” median.

Issue-by-issue voting guarantees equilibrium existence (Kramer [1972]): even if a Condorcet winner does not exist, the stability of the issue-by-issue median is ensured by the rigid institutional constraints put on the voting procedure.

Since the one-dimensional median is not a linear function of its inputs (thus contrasting the mean), the issue-by-issue median varies with the underlying system of coordinates along which voting takes place (see Haldane [1948]). Thus, the simpler policy issues into which a complex, multidimensional decision can be possibly carved are endogenous, and their choice matters! This simple insight is not too prominent in the spatial voting literature where it is implicitly assumed that the issues on which voting proceeds are exogenously fixed (e.g., the standard Cartesian basis).

Consider, for example, a legislature that has to decide how much money to allocate to two programs in a given fiscal year. One intuitive budgeting procedure, called “bottom-up”, is to vote (say by simple majority) on each program separately, in which case the total budget will be the sum of the individual budgets. Another intuitive alternative, called “top-down”, is to vote on the total budget first, and then vote on how to divide the total budget among the two items.[3,4] It is clear that the “top-down” procedure determines the expenditures on each program in an indirect way. The outcomes of these two procedures are generically different, and thus conflicts may arise about which one to employ in particular circumstances.

Assume that an agreement on what to put on the ballot – the issues – has been obtained. Does the order in which issue-by-issue sequential voting proceeds matter? When is the outcome independent of the order in which the policy issues are put on the ballot? In this paper we show that this order-independent property is intimately connected to the following one: when is it the case that truthful voting in a direct (and static) marginal median mechanism forms an equilibrium outcome in dominant strategies? We also show that the answer to both questions is determined by the geometry induced on the space of multi-dimensional alternatives by the assumed utility functions (recall that here these are norm-generated distances from given individual peaks).

Depending on the chosen coordinates (e.g., issues that are put to vote) and on the

[3]The U.S. Congress switched from bottom-up to top-down following the 1974 Congressional Budget and Impoundments Control Act. See also Poterba and von Hagen [1999].

[4]Note that in this example there is no exogenous budget constraint: both the size and the allocation of the budget are equilibrium outcomes of legislative voting.

utility functions, the same voting procedure may, or may not be, strategy proof and order-independent. Viewed in light of Shepsle's "structure-induced" equilibrium theory, our goal here is to endogenize the choice of issues (or jurisdictions) in order to induce strategically stable and robust outcomes that are not subject to voters' manipulations nor to manipulations of an agenda setter. We consider spacial preferences based on norms, and show that the results obtained for general norms are related, but significantly different from the relatively simple results obtained for inner product norms (such as the Euclidean norm)

We note that questions about the influence of order in sequential voting have been posed before, e.g., by De Donder et al. [2012a] [2012b], in a model where the decision is multi-dimensional while types are one-dimensional and preferences satisfy some single-crossing conditions (these preferences are not norm-based). Their model is specially constructed in order to incorporate famous examples such as the public good model in Alesina et al. [1999] where voters have preferences over level and type. Alesina et al. only considered the order of votes where size is determined first, while alluding to the top-down budgetary procedure.

1.2 The Euclidean Norm

The ubiquitous way of measuring distance from a peak in a multi-dimensional framework is the standard Euclidean norm.[5] The main reason behind this choice is technical: it allows the use of basic geometric intuitions.

Euclidean preferences are relatively well-understood: for example, an issue-by-issue median procedure is strategy-proof if and only if the axes (i.e., the issues) along which it is computed are orthogonal to each other (see Kim and Roush [1984] and Peters, van der Stel, and Storcken [1992]).[6] Intuitively, this says that a change in the voting outcome on one issue does not influence the outcome on the other issues (because, by orthogonality, its projections on the other axes does not change).

Our first result (Proposition 1) connects to the above insight and shows that, for any normed space induced by an inner-product – such as the Euclidean norm – the outcome of sequential, issue-by-issue voting is order independent if and only if the issues are orthogonal to each other. We also explain how the outcome comes about if sequential voting is not along orthogonal axes (Proposition 2).

[5] See, for example, the textbook by Austen-Smith and Banks [2005].For empirical methodologies see, for example, Clinton, Jackman and Rivers [2004]).

[6] Gershkov, Moldovanu and Shi [2019] showed how to maximize utilitarian welfare over the class of issue-by-issue medians (each corresponding to an orthogonal rotation of the axes) and illustrated when the top-down budgeting procedure dominates the bottom-up one.

1.3 More General Norms

The Euclidean norm is not suitable for most applications because its complete spherical symmetry implies equal losses induced by deviations in a multitude of directions.[7] A smaller literature, both theoretical and empirical, has considered distances based on the "city block" (or "Manhattan") norm, or on the general class of l_p norms, $p \geq 1$ (where the city-block norm is l_1). Eguia [2013] offers a critical discussion of the Euclidean norm, and references papers that empirically test alternative distance functions.

Since issue orthogonality – the main attribute behind of robust sequential voting – is defined via the notion of an inner-product, it is not at all clear how to extend the above insights to general distance functions that are not generated by inner-products (recall that even within the standard l_p class, only the Euclidean norm l_2, admits an inner-product). Does this mean that such spaces never admit strategy-proof or order independent issue-by-issue medians? This would be a bit surprising since, for example, in two dimensions, the issue-by-issue median according to the Cartesian coordinates is actually a Condorcet winner if voters use the city-block l_1 norm according to the Cartesian basis (see Wendell and Thorson [1974])!

A first step towards obtaining strategy-proofness for more general two-dimensional policy spaces has been made by Peters, van der Stel, and Storcken [1993]: they showed that marginal medians are strategy-proof if and only if majority voting takes place along two coordinates that satisfy an extended notion of orthogonality, due to Birkhoff [1935], and analysed by James [1947]. Peters, van der Stel, and Storcken also completely characterized the class of strategy-proof mechanisms that satisfy anonymity and Pareto-optimality: these must be issue-by-issue medians.

Unfortunately, *Birkhoff-James (BJ) orthogonality* lacks the symmetry and additivity properties of the orthogonality defined by a zero inner product. As a consequence, the results of Peters, van der Stel, and Storcken [1993] cannot hold in that form for higher-dimensional spaces.

For general normed spaces with any number of policy dimensions, Gershkov, Moldovanu and Shi [2020] showed that an issue-by-issue median is strategy-proof if and only if it is computed with respect to a basis that satisfies a property called "left-additive mutual orthogonality" (LAMO) (see Theorem 2 below). This property is stronger than requiring the basis vectors to be BJ-mutually orthogonal. In fact, with three dimensions or more, only a small, possibly empty subset of bases with BJ-mutually orthogonal basis vectors satisfy LAMO.

This paper shows that, if the sequential, issue-by-issue voting is conducted according to a basis that satisfies LAMO, then the voting outcome is order-independent and coincides with the outcome of the corresponding static, issue-by-issue median (Proposition 3). If

[7] Some variants allow for different weights on different coordinates, but still embody a symmetry, for example, among increases or decreases in the same direction.

the associated basis fails to satisfy LAMO, then we show in the two-dimension case that different voting orders may lead to different voting outcomes (Proposition 4).

1.4 The Existence Question

Given the above characterization, the following question becomes pertinent: Does there always exist a system of one-dimensional policy issues such that issue-by-issue voting by simple majority on these issues is strategy-proof or order-independent?

For two-dimensional spaces, the answer is positive by a classical theorem due to Auerbach (see Theorem 1 below). An affirmative answer is also provided for the class of l_p norm (in any finite dimension) by Gershkov, Moldovanu and Shi [2020] who fully characterized all LAMO bases for all $p \neq 2$. The number of such bases is small and they do not depend at all on p (unless $p = 2$). Thus, the outcome of sequential, issue-by-issue voting with respect to these bases would remain order-independent even for situations where the norm is allowed to vary across agents within the l_p class, and is their private information.

Despite the partial existence results mentioned above, our final main result in this paper is a generic non-existence result: for any arbitrary distance (utility) function derived from a norm there exists a nearby distance defined from another norm such that the set of LAMO bases for the latter norm is empty (Theorem 3).

2 Norm-Based Preferences

Throughout of the paper, the **bold** font is used to denote vectors in $\mathbb{R}^d$. An odd number n of agents collectively choose a decision $\mathbf{v} \in \mathbb{R}^d$ where d is a positive integer. Agent i's ideal position is given by a "peak" $\mathbf{t}_i \in \mathbb{R}^d$. The peak $\mathbf{t}_i$ is agent i's private information.

The utility of agent i with peak $\mathbf{t}_i$ from decision $\mathbf{v}$ is given by

$$- \|\mathbf{t}_i - \mathbf{v}\|$$

where $\|\cdot\|$ is a *norm* on $\mathbb{R}^d$.[8] Recall that a norm $\|\cdot\|$ is a real-valued function on $\mathbb{R}^d$ that satisfies four basic properties associated with "length" of vectors:

1. $\|\mathbf{x}\| \geq 0$;
2. $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$, $\forall \mathbf{x} \in \mathbb{R}^d$;
3. $\|a\mathbf{x}\| = |a| \|\mathbf{x}\|$, $\forall \mathbf{x} \in \mathbb{R}^d,\ a \in \mathbb{R}$; and
4. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$, $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

[8] Since our analysis and results are purely ordinal, they immediately apply to all utility functions of the form $-\Delta(\|\mathbf{t}_i - \mathbf{v}\|)$ where Δ is a strictly increasing function: all these cardinal utilities represent the same ordinal preferences as the basic norm $\|\cdot\|$.

We denote by $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ a generic algebraic basis for $\mathbb{R}^d$, where $\mathbf{x}^1, ..., \mathbf{x}^d$ are linearly independent, and by $\{\mathbf{e}^1, ..., \mathbf{e}^d\}$ the standard Cartesian basis where, for vector $\mathbf{e}^j$, only the j-th coordinate is different from zero, and equals one:

$$\mathbf{e}^j = (\underbrace{0, 0, ..., 1}_{j}, 0, ..., 0).$$

Fix a basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ in $\mathbb{R}^d$. Recall that we can represent each $\mathbf{x} \in \mathbb{R}^d$ as

$$\mathbf{x} = \sum_{j=1}^{d} \alpha^j(\mathbf{x})\mathbf{x}^j,$$

where $\alpha^j(\mathbf{x})$ is the j-th coordinate of $\mathbf{x}$ according to this basis. To simplify notation when confusion cannot arise, we write

$$(x_1, ..., x_d) = \left(\alpha^1(\mathbf{x}), ..., \alpha^d(\mathbf{x})\right)$$

and identify $\mathbf{x}$ with the vector of coordinates $(x_1, ..., x_d)$.

2.1 The Endogeneity of Issues

As we discussed above, equilibrium may not exist if voters vote on all issues simultaneously. Equilibrium existence, however, is guaranteed under issue-by-issue voting. The approach of modeling legislative procedures via issue-by-issue voting is justified by the structure of committees and jurisdictions. Hence, we will focus on it and, for simplicity, we assume that a simple majority is applied in determining the voting outcome for each dimension.

Due to the multi-dimensionality of the decision space, the dimensions on which voting takes place are not uniquely defined. In other words, there are many different ways of framing the issues that are put on the ballot, and each feasible set of issues yields a potentially different multi-dimensional median. We interpret different algebraic bases $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ as different ways of structuring the issues that are put to vote.

Example 1 (Issues and Bases I) *Consider a legislature that decides how much money to allocate to defence (D) and to everything else (E). The decision space can be represented by $\mathbb{R}^2_+ = \{(D, E) : D \geq 0, E \geq 0\}$. One budgeting procedure, called "bottom-up", is to vote on D and E sequentially, in which case the total budget will be the sum of the individual budgets, $D+E$. An alternative, called "top-down", is to first vote on the total budget $D+E$, and then vote on how to divide the total budget between D and E. Let $\mathbf{e}^1$ and $\mathbf{e}^2$ denote the unit vector along the dimensions of D and E, respectively. Then the bottom-up budgeting procedure can be represented by voting along the coordinates of the standard Cartesian basis $\{\mathbf{e}^1, \mathbf{e}^2\}$ with either $\mathbf{e}^1$ or $\mathbf{e}^2$ being voted first, while the top-down budgeting procedure can be modelled by voting along $\{\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1 - \mathbf{e}^2\}$ with $\mathbf{e}^1 + \mathbf{e}^2$ being voted first. Both bases consist of orthogonal vectors, but their outcomes are generically different.*

Example 2 (Issues and Bases II) *A community decides on the size (or level, or quantity) of a public good Q and on the total charge to each member T. Members of the community have preferences on bundles (Q,T) – these are derived from considerations of this versus alternative public goods, budget constraints, costs, etc.. The decision space can be represented by $\mathbb{R}^2_+ = \{(Q,T) : Q \geq 0, T \geq 0\}$. Suppose that instead of voting on quantity Q and on total charge T, voting is on Q and on a "flat fee" F, with the understanding that $T = F + pQ$ where p is an exogenously given marginal price (this may reflect marginal cost, etc.).*

In a sequential vote first on coordinate $\mathbf{Q}$ and then on $\mathbf{F}$, then, naturally, the first vote determines Q and the second fixes F, and thus T.[9] *This is equivalent to a vote on $\mathbf{Q}$ and then on $\mathbf{T}$. But, consider now the sequential voting in the opposite order, $\mathbf{F}$ and then $\mathbf{Q}$. This is equivalent to voting according to the "rotated issue" $\mathbf{T}-p\mathbf{Q}$ and then according to $\mathbf{Q}$. Now the first vote does not fully determine T – it will be completely determined only after the second vote on $\mathbf{Q}$.*

Note also each fixed marginal price p determines a different voting mechanism, where $p = 0$ corresponds to a vote on $\mathbf{T}$ and then on $\mathbf{Q}$. Even with standard Euclidean preferences, if $p \neq 0$, the outcome in the order $\mathbf{F}$ and then $\mathbf{Q}$ need not be equivalent to the outcome in the order $\mathbf{Q}$ and then $\mathbf{F}$ (or $\mathbf{T}$), nor to the outcome of simultaneous voting.

Example 2 is illustrative for a common situation where voters have preferences in terms of basic issues (say efficiency, distribution, nature preservation, immigration, etc.) but actual voting takes place partly on proxy instruments such as taxes and subsidies.

3 Sequential Issue-by-Issue Voting

Fix now the issues to be voted upon, i.e. an algebraic basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$. Voters sequentially report preferred points (for example, these could be the projections of their ideal peak) on dimensions $\mathbf{x}^1, ..., \mathbf{x}^d$, in that order. When voters decide on dimension $\mathbf{x}^k$, they observe all previous voting outcomes along dimensions $(\mathbf{x}^1, ..., \mathbf{x}^{k-1})$. Taking the medians of the voters' reports on each dimension, and combining the obtained medians into a d-dimensional vector, we obtain a *sequential issue-by-issue median* voting rule ψ with respect to basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ and order $\left(\mathbf{x}^1, ..., \mathbf{x}^d\right)$.

For the analysis of sequential voting procedures, we use as solution concept the *ex-post perfect equilibrium*. An n-tuple strategy profile constitutes an ex-post perfect equilibrium if at every stage, and for every possible realization of private information, the n-tuple of continuation strategies constitutes a Nash equilibrium of the subgame in which the realization of the agents' types is common knowledge. This is a robust equilibrium concept since it does not depend on the voters' beliefs about each other.

For each fixed basis $\left\{\mathbf{x}^1, ..., \mathbf{x}^d\right\}$ let Σ_d be the set of permutations on d elements. Each permutation $\sigma = (\sigma_1, ..., \sigma_d) \in \Sigma_d$ of the elements in the set $\left\{\mathbf{x}^1, ..., \mathbf{x}^d\right\}$ defines a poten-

[9] Here, as above, variables in bold are vectors.

tially different sequential issue-by-issue voting procedure ψ_σ as above, where agents report points in each coordinate, and where the median is computed along the respective coordinates, one after the other, given the chosen permutation.

Definition 1 *The set* $\{\psi_\sigma\}_{\sigma\in\Sigma_d}$ *of sequential issue-by-issue voting procedures with respect to a basis* $\left\{\mathbf{x}^1,...,\mathbf{x}^d\right\}$ *is order independent if the voting equilibrium outcome is invariant over all permutations of the basis' elements.*

The goal of the paper is to characterize the bases – the issues – with respect to which the equilibrium outcome sequential issue-by-issue median is order-independent, and to study whether and when such bases exist. In other words, we seek to structure the endogenous one-dimensional issues into which a complex multi-dimensional decision problem can be divided so that incentives to manipulate the voting order are avoided. We also link the property of order-independence to the incentive properties of sequential voting rules. In particular, is sincere voting necessarily an equilibrium in sequential voting if the voting rule is order-independent? Conversely, what incentive properties of a voting rule can guarantee order independence?

To study the incentive properties of voting rules, we also introduce a simultaneous version of the sequential issue-by-issue voting rule. A *simultaneous issue-by-issue median* $\psi\left(\mathbf{t}_1,...,\mathbf{t}_n\right)$ with respect to basis $\{\mathbf{x}^1,...,\mathbf{x}^d\}$ is defined as

$$\psi\left(\mathbf{t}_1,...,\mathbf{t}_n\right)=\sum_{j=1}^{d} med(\alpha^j(\mathbf{t}_1),...,\alpha^j(\mathbf{t}_n))\mathbf{x}^j,$$

where $med(\alpha^j(\mathbf{t}_1),...,\alpha^j(\mathbf{t}_n))$ is the (one-dimensional) median of the j-th coordinates of the agents' peaks. Under a simultaneous issue-by-issue median, voters are asked to report the coordinates of their peaks on dimensions $\mathbf{x}^1,...,\mathbf{x}^d$ all at once. Alternatively, we can imagine that the issue-by-issue voting still proceeds sequentially, but earlier voting outcomes are not revealed.

The simultaneous issue-by-issue median is a (static) *direct revelation mechanism* $\psi:(\mathbb{R}^d)^n\to\mathbb{R}^d$. That is, all agents report about their peaks, and an outcome is computed based on the reports.

Definition 2 *A direct revelation mechanism* ψ *is strategy-proof or dominant-strategy incentive compatible (DIC) if for any voter* i *and for any* $\mathbf{t}_i$, $\hat{\mathbf{t}}_i$ *and* $\mathbf{t}_{-i}$, *it holds that*

$$\left\|\mathbf{t}_i-\psi\left(\mathbf{t}_i,\mathbf{t}_{-i}\right)\right\|\leq\left\|\mathbf{t}_i-\psi\left(\hat{\mathbf{t}}_i,\mathbf{t}_{-i}\right)\right\|.$$

In other words, if ψ is strategy-proof, any individual manipulation $\hat{\mathbf{t}}_i$ forces the social choice to move weakly away from the true peak $\mathbf{t}_i$, as measured by the distance function used by the manipulating agent.[10]

[10]The constraint for agent i only uses the norm considered by agent i, and hence the above definition easily generalizes to situations where different agents use different norms.

4 Order Independence under Inner-Product Norms

We start with preferences that are defined by the ubiquitous Euclidean norm, also known as the l_2 norm. This is the standard inner-product norm on the Euclidean space $\mathbb{R}^d$. As mentioned in the Introduction, almost all applied spatial voting models use this and other related inner-product norms to measure the perceived distance between alternatives.

An inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d \times \mathbb{R}^d$ is a real-valued function that satisfies:

1. additivity $\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$;

2. scalar multiplication $\langle a\mathbf{x}, \mathbf{y} \rangle = a \langle \mathbf{x}, \mathbf{y} \rangle$ for all $a \in \mathbb{R}$;

3. symmetry $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$; and

4. non-negativity $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Definition 3 *In an inner-product space, two vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *are orthogonal (denoted by* $\mathbf{x} \perp \mathbf{y}$*) if and only if* $\langle \mathbf{x}, \mathbf{y} \rangle = 0$*. In an inner-product space, an algebraic basis* $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ *of* $\mathbb{R}^d$ *is an orthogonal basis if and only if all vectors in it are mutually orthogonal.*

The above orthogonality relation is, by definition, symmetric: that is, $\mathbf{x} \perp \mathbf{y} \Leftrightarrow \mathbf{y} \perp \mathbf{x}$. Moreover, again by definition, the orthogonality relation is additive from both sides: if $\mathbf{x} \perp \mathbf{y}$ and $\mathbf{z} \perp \mathbf{y}$, then $(\mathbf{x} + \mathbf{z}) \perp \mathbf{y}$; if $\mathbf{x} \perp \mathbf{y}$ and $\mathbf{x} \perp \mathbf{z}$, then $\mathbf{x} \perp (\mathbf{y} + \mathbf{z})$.

The standard Euclidean norm on $\mathbb{R}^d$ is defined via the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^d x_i y_i$ as

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \left(\sum\nolimits_{i=1}^d x_i^2 \right)^{1/2}.$$

An arbitrary inner-product norm on $\mathbb{R}^d$ is generated by the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{M} \mathbf{y}$ where $\mathbf{M}$ is a positive semi-definite matrix. This yields

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\mathbf{x}^T \mathbf{M} \mathbf{x}}$$

For example, if $d = 2$ and if

$$\mathbf{M} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix},$$

with $\beta_1 > 0$ and $\beta_2 > 0$, then

$$\|\mathbf{x}\| = \sqrt{\beta_1 x_1^2 + \beta_2 x_2^2}$$

is a weighted Euclidean norm.

It is relatively straightforward to see that, if the basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ is orthogonal, then the voting outcome of a sequential issue-by-issue median ψ is order independent. To illustrate, let us return to the budgeting problem in Example 1.

Example 3 (Euclidean Norm and Orthogonal Basis) *Recall that the bottom-up budgeting procedure can be represented by voting along the standard and orthogonal Cartesian basis* $\{\mathbf{e}^1, \mathbf{e}^2\}$. *There are two possible voting orders,* $\sigma = (\mathbf{e}^1, \mathbf{e}^2)$ *and* $\pi = (\mathbf{e}^2, \mathbf{e}^1)$. *Consider first voting order* σ *and suppose that the first-stage voting outcome is* y_1. *In the second stage, conditional on being pivotal, voter* i *with ideal point* (x_1^i, x_2^i) *chooses a report* $\hat{x}_2^i$ *to minimize*

$$\left(x_1^i - y_1\right)^2 + \left(x_2^i - \hat{x}_2^i\right)^2.$$

Hence, it is optimal for voter i *to report truthfully in the second stage. It follows that the second stage outcome* y_2 *is independent of the first stage outcome. In the first stage, anticipating that everyone reports truthfully in the second stage, voter* i *chooses report* $\hat{x}_1^i$, *conditional on being pivotal, to minimize*

$$\left(x_1^i - \hat{x}_1^i\right)^2 + \mathbb{E}\left[x_2^i - med\left(x_2^1, ..., x_2^n\right)\right]^2.$$

Hence, it is optimal for voter i *to vote sincerely in the first stage. The same logic applies to voting order* π. *Therefore, the voting outcome of a sequential issue-by-issue voting rule with respect to this orthogonal basis is order independent.*

The top-down budgeting procedure can be represented by voting with respect to basis $\{\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1 - \mathbf{e}^2\}$ *with dimension* $\mathbf{e}^1 + \mathbf{e}^2$ *being voted first. The basis* $\{\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1 - \mathbf{e}^2\}$ *is also orthogonal, and the ideal point* (x_1^i, x_2^i) *of voter* i *can be represented as*

$$x_1^i \mathbf{e}^1 + x_2^i \mathbf{e}^2 = \frac{1}{2}\left(x_1^i + x_2^i\right)\left(\mathbf{e}^1 + \mathbf{e}^2\right) + \frac{1}{2}\left(x_1^i - x_2^i\right)\left(\mathbf{e}^1 - \mathbf{e}^2\right).$$

As above, it is optimal for all voters to sincerely report the projections of their idea points on $\mathbf{e}^1 + \mathbf{e}^2$ *and* $\mathbf{e}^1 - \mathbf{e}^2$. *The same is true when dimension* $\mathbf{e}^1 - \mathbf{e}^2$ *is voted first.*

The above example illustrates that, if the associated basis is orthogonal, independently of the voting order voters always vote sincerely in the sequential issue-by-issue median voting rule. Therefore, for all possible voting orders, the voting outcome of the sequential issue-by-issue median coincides with the outcome of the simultaneous issue-by-issue median according to the respective basis.

We next present an example where the basis underlying sequential issue-by-issue voting is not orthogonal.

Example 4 (Non-Orthogonal Basis) *Consider the basis* $\{\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1\}$ *and standard Euclidean preferences. The basis is not orthogonal because* $\langle \mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1 \rangle = 1$. *Let* $\sigma = (\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1)$ *and* $\pi = (\mathbf{e}^1, \mathbf{e}^1 + \mathbf{e}^2)$ *denote the two possible voting orders. We will show that these voting orders lead to different voting outcomes.*

Fix first voting order σ *and suppose that the first-stage voting outcome on dimension* $\mathbf{e}^1 + \mathbf{e}^2$ *is* y_1. *Consider voter* i *with ideal point* $x_1^i \mathbf{e}^1 + x_2^i \mathbf{e}^2$. *In the second stage of voting (on dimension* $\mathbf{e}^1$*) conditional on being pivotal, voter* i *chooses* $\hat{r}_2$ *to minimize*

$$\left\|x_1^i \mathbf{e}^1 + x_2^i \mathbf{e}^2 - \left(y_1\left(\mathbf{e}^1 + \mathbf{e}^2\right) + \hat{r}_2 \mathbf{e}^1\right)\right\|^2 = \left(x_1^i - y_1 - \hat{r}_2\right)^2 + \left(x_2^i - y_1\right)^2$$

It follows that, in the second stage, it is optimal for voter i to report $\hat{r}_2^* = x_1^i - y_1$. Hence, given the first stage outcome y_1, the second stage outcome is $med(x_1^1, ..., x_1^n) - y_1$. Now consider the first stage voting on dimension $\mathbf{e}^1 + \mathbf{e}^2$. Conditional on being pivotal, voter i chooses report $\hat{r}_1$ to minimize

$$\begin{aligned}
& \mathbb{E}\left\|x_1^i\mathbf{e}^1 + x_2^i\mathbf{e}^2 - \left(\hat{r}_1\left(\mathbf{e}^1+\mathbf{e}^2\right) + \left(med(x_1^1,...,x_1^n) - \hat{r}_1\right)\mathbf{e}^1\right)\right\|^2 \\
= & \mathbb{E}\left\|x_1^i\mathbf{e}^1 + x_2^i\mathbf{e}^2 - \left(med(x_1^1,...,x_1^n)\mathbf{e}^1 + \hat{r}_1\mathbf{e}^2\right)\right\|^2 \\
= & \mathbb{E}\left[x_1^i - med(x_1^1,...,x_1^n)\right]^2 + \left(x_2^i - \hat{r}_1\right)^2
\end{aligned}$$

Hence voter i will report $\hat{r}_1^* = x_2^i$ in the first stage. Therefore, the voting outcome under order σ is

$$\begin{aligned}
& med(x_2^1,...,x_2^n)\left(\mathbf{e}^1+\mathbf{e}^2\right) + \left(med(x_1^1,...,x_1^n) - med(x_2^1,...,x_2^n)\right)\mathbf{e}^1 \\
= & med(x_1^1,...,x_1^n)\mathbf{e}^1 + med(x_2^1,...,x_2^n)\mathbf{e}^2,
\end{aligned}$$

which is the same as the one in sequential voting with respect to orthogonal basis $\left\{\mathbf{e}^1, \mathbf{e}^2\right\}$!

Next, consider the other voting order π and let y_1 be the first-stage voting outcome on dimension $\mathbf{e}^1$. It follows from a logic similar to the above that voter i will report $\hat{r}_2^* = \frac{1}{2}\left(x_1^i + x_2^i - y_1\right)$ on dimension $\mathbf{e}^1 + \mathbf{e}^2$ in the second stage, and report $\hat{r}_1^* = x_1^i - x_2^i$ on dimension $\mathbf{e}^1$ in the first stage. Therefore, the voting outcome under order π is

$$\begin{aligned}
& med(x_1^1 - x_2^1,...,x_1^n - x_2^n)\mathbf{e}^1 \\
& + \frac{1}{2}\left(med(x_1^1 + x_2^1,...,x_1^n + x_2^n) - med(x_1^1 - x_2^1,...,x_1^n - x_2^n)\right)\left(\mathbf{e}^1+\mathbf{e}^2\right) \\
= & \frac{1}{2}med\left(x_1^1 + x_2^1,...,x_1^n + x_2^n\right)\left(\mathbf{e}^1+\mathbf{e}^2\right) + \frac{1}{2}med\left(x_1^1 - x_2^1,...,x_1^n - x_2^n\right)\left(\mathbf{e}^1-\mathbf{e}^2\right),
\end{aligned}$$

which is the same as the one in sequential voting with respect to orthogonal basis $\left\{\mathbf{e}^1+\mathbf{e}^2, \mathbf{e}^1-\mathbf{e}^2\right\}$!

In Example 3 – equipped with an orthogonal basis and with the standard Euclidean norm – sequential issue-by-issue voting was order independent. In contrast, in Example 4, sequential issue-by-issue voting with respect to a non-orthogonal basis is order dependent. The following proposition generalizes these insights to higher dimensions and to any inner-product norm.

Proposition 1 *Let $\left\{\mathbf{x}^1, ..., \mathbf{x}^d\right\}$ be a basis and let $\|\cdot\|$ be an inner-product norm on $\mathbb{R}^d$. The set $\{\psi_\sigma\}_{\sigma\in\Sigma_d}$ of sequential issue-by-issue voting procedures with respect to the basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ is order-independent if and only if $\left\{\mathbf{x}^1, ..., \mathbf{x}^d\right\}$ is an orthogonal basis.*

Proof. That orthogonality yields order-independence follows from Proposition 3 for general norms in Section 5. For the other direction, we prove the contrapositive: that is, if orthogonality fails, then the outcome of sequential issue-by-issue voting varies according to the order in which issues are put to vote. Consider an inner-product norm $\|\mathbf{x}\| = \langle\mathbf{x}, \mathbf{x}\rangle^{1/2}$ and a basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$. The basis vectors can always be normalized so that $\|\mathbf{x}^i\| = 1$. Let $\gamma_{j,k}$ denote the "correlation coefficient" of basis vectors of $\mathbf{x}^j$ and $\mathbf{x}^k$, that is, $\gamma_{j,k} = \langle \mathbf{x}^j, \mathbf{x}^k \rangle$.

If orthogonality fails, there must exist two basis vectors that are not orthogonal. Without loss of generality, denote these two non-orthogonal vectors as $\mathbf{x}^{d-1}$ and $\mathbf{x}^d$. Hence $\gamma_{d,d-1} = \langle \mathbf{x}^d, \mathbf{x}^{d-1} \rangle \neq 0$.

Fix the profile of ideal points for the n voters, and consider the voting orders given by the permutations $\sigma = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^{d-1}, \mathbf{x}^d)$ and $\pi = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^{d-2}, \mathbf{x}^d, \mathbf{x}^{d-1})$.

Let $(y_1, ..., y_{d-1}, y_d)$ and $(z_1, ..., z_{d-1}, z_d)$ be the voting outcomes under these voting orders, respectively. If $y_j \neq z_j$ for some $j \in \{1, ..., d-2\}$, then we are done. Hence, we assume below that

$$(y_1, ..., y_{d-2}) = (z_1, ..., z_{d-2}). \tag{1}$$

We need to argue that either $y_{d-1} \neq z_{d-1}$ holds or $y_d \neq z_d$ holds (or both hold).

Consider first the voting order $\sigma = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^{d-1}, \mathbf{x}^d)$ and the last stage voting in the $\mathbf{x}^d$ dimension. Look at a voter a with ideal point

$$\mathbf{t}_a = (x_1^a, ..., x_d^a) = x_1^a \mathbf{x}^1 + ... + x_d^a \mathbf{x}^d.$$

Given the voting outcomes $(y_1, ..., y_{d-1})$ that were determined in the previous $d-1$ stages, let $\hat{x}_d^a(y_1, ..., y_{d-1})$ denote voter a's optimal report in the $\mathbf{x}^d$ dimension, i.e., the one that maximizes this voter's payoff conditional on being pivotal at this stage with this report. That is, $\hat{x}_d^a(y_1, ..., y_{d-1})$ must solve

$$\begin{aligned}
& \min_{\hat{x}_d} \left\| y_1 \mathbf{x}^1 + ... + y_{d-1} \mathbf{x}^{d-1} + \hat{x}_d \mathbf{x}^d - \mathbf{t}_a \right\|^2 \\
= & \min_{\hat{x}_d} \left\| \sum\nolimits_{j=1}^{d-1} \left(y_j - x_j^a\right) \mathbf{x}^j + \left(\hat{x}_d - x_d^a\right) \mathbf{x}^d \right\|^2 \\
= & \min_{\hat{x}_d} \left\langle \sum\nolimits_{j=1}^{d-1} \left(y_j - x_j^a\right) \mathbf{x}^j + \left(\hat{x}_d - x_d^a\right) \mathbf{x}^d, \; \sum\nolimits_{j=1}^{d-1} \left(y_j - x_j^a\right) \mathbf{x}^j + \left(\hat{x}_d - x_d^a\right) \mathbf{x}^d \right\rangle
\end{aligned}$$

For the derivations below we use the inner-product derivative formula:

$$\frac{d}{dx} \langle f(x), \; g(x) \rangle = \left\langle f'(x), \; g(x) \right\rangle + \left\langle f(x), \; g'(x) \right\rangle,$$

where f and g are real-valued and differentiable functions defined on $\mathbb{R}$.

The first-order condition for this maximization problem is

$$\begin{aligned}
0 &= 2 \left\langle \mathbf{x}^d, \; \sum\nolimits_{j=1}^{d-1} \left(y_j - x_j^a\right) \mathbf{x}^j + \left(\hat{x}_d - x_d^a\right) \mathbf{x}^d \right\rangle \\
&= 2 \sum\nolimits_{j=1}^{d-1} \left(y_j - x_j^a\right) \left\langle \mathbf{x}^d, \mathbf{x}^j \right\rangle + 2\left(\hat{x}_d - x_d^a\right) \left\langle \mathbf{x}^d, \mathbf{x}^d \right\rangle \\
&= 2 \sum\nolimits_{j=1}^{d-1} \left(y_j - x_j^a\right) \gamma_{d,j} + 2\left(\hat{x}_d - x_d^a\right)
\end{aligned}$$

which implies that

$$\hat{x}_d^a(y_1, ..., y_{d-1}) = x_d^a - \sum\nolimits_{j=1}^{d-1} \left(y_j - x_j^a\right) \gamma_{d,j}. \tag{2}$$

For the second-to-the-last stage vote (in the $\mathbf{x}^{d-1}$ dimension), consider a voter b with ideal point

$$\mathbf{t}_b = \left(x_1^b, ..., x_d^b\right) = x_1^b \mathbf{x}^1 + ... + x_d^b \mathbf{x}^d.$$

Anticipating how his choice would affect voting in the $\mathbf{x}^d$ dimension, if voter b is pivotal with report $\hat{x}_{d-1}$, then this voter chooses $\hat{x}_{d-1}$ that solves

$$\begin{aligned} & \min_{\hat{x}_{d-1}} \mathbb{E} \left\| y_1 \mathbf{x}^1 + ... + \hat{x}_{d-1} \mathbf{x}^{d-1} + x_d^a \left(y_1, ..., \hat{x}_{d-1}\right) \mathbf{x}^d - \mathbf{t}_b \right\|^2 \\ = & \min_{\hat{x}_{d-1}} \mathbb{E} \left\| \sum\nolimits_{j=1}^{d-2} \left(y_j - x_j^b\right) \mathbf{x}^j + \left(\hat{x}_{d-1} - x_{d-1}^b\right) \mathbf{x}^{d-1} + \left(x_d^a \left(y_1, ..., \hat{x}_{d-1}\right) - x_d^a\right) \mathbf{x}^d \right\|^2 \end{aligned}$$

The associated first-order condition is

$$\begin{aligned} 0 &= 2\mathbb{E} \left\langle \mathbf{x}^{d-1} - \gamma_{d,d-1} \mathbf{x}^d, \sum\nolimits_{j=1}^{d-2} \left(y_j - x_j^b\right) \mathbf{x}^j + \left(\hat{x}_{d-1} - x_{d-1}^b\right) \mathbf{x}^{d-1} + \left(x_d^a \left(y_1, ..., \hat{x}_{d-1}\right) - x_d^b\right) \mathbf{x}^d \right\rangle \\ &= 2 \sum\nolimits_{j=1}^{d-2} \left(y_j - x_j^b\right) \gamma_{d-1,j} - 2 \sum\nolimits_{j=1}^{d-2} \left(y_j - x_j^b\right) \gamma_{d,d-1} \gamma_{d,j} + 2 \left(1 - \left(\gamma_{d,d-1}\right)^2\right) \left(\hat{x}_{d-1} - x_{d-1}^b\right) \end{aligned}$$

which implies that

$$\hat{x}_{d-1}^b \left(y_1, ..., y_{d-2}\right) = x_{d-1}^b - \frac{\sum_{j=1}^{d-2} \left(y_j - x_j^b\right) \left(\gamma_{d,d-1} \gamma_{d,j} - \gamma_{d-1,j}\right)}{1 - \left(\gamma_{d,d-1}\right)^2}. \tag{3}$$

The analysis for voting order $\pi = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^{d-2}, \mathbf{x}^d, \mathbf{x}^{d-1})$ is similar. Let $\hat{x}_{d-1}^b \left(z_1, ..., z_{d-2}, z_d\right)$ denote voter's b optimal report in the $\mathbf{x}^{d-1}$ dimension (i.e., conditional on being pivotal and given voting outcome $\left(z_1, ..., z_{d-2}, z_d\right)$ determined at the previous $d-1$ stages). Then $\hat{x}_{d-1}^b \left(z_1, ..., z_{d-2}, z_d\right)$ must solve

$$\begin{aligned} & \min_{\hat{x}_{d-1}} \left\| z_1 \mathbf{x}^1 + ... + \hat{x}_{d-1} \mathbf{x}^{d-1} + z_d \mathbf{x}^d - \mathbf{t}_b \right\|^2 \\ = & \min_{\hat{x}_{d-1}} \left\| \sum\nolimits_{j \in \{1,...,d\}/\{d-1\}} \left(z_j - x_j^b\right) \mathbf{x}^j + \left(\hat{x}_{d-1} - x_{d-1}^b\right) \mathbf{x}^{d-1} \right\|^2 \end{aligned}$$

The first-order condition is

$$\begin{aligned} 0 &= 2 \left\langle \mathbf{x}^{d-1}, \ \sum\nolimits_{j \in \{1,...,d\}/\{d-1\}} \left(z_j - x_j^b\right) \mathbf{x}^j + \left(\hat{x}_{d-1} - x_{d-1}^b\right) \mathbf{x}^{d-1} \right\rangle \\ &= 2 \sum\nolimits_{j \in \{1,...,d\}/\{d-1\}} \left(z_j - x_j^b\right) \gamma_{d-1,j} + 2 \left(\hat{x}_{d-1} - x_{d-1}^b\right) \end{aligned}$$

which implies that

$$\hat{x}_{d-1}^b \left(z_1, ..., z_{d-2}, z_d\right) = x_d^b - \sum\nolimits_{j \in \{1,...,d\}/\{d-1\}} \left(z_j - x_j^b\right) \gamma_{d-1,j}. \tag{4}$$

Next consider voter's a optimal report at the the second-to-the-last stage voting on the $\mathbf{x}^d$ dimension. Conditional on being pivotal, given the voting outcome $\left(z_1, ..., z_{d-2}\right)$ that was decided upon in the earlier $d-2$ stages, and anticipating how his choice would affect voting in the last dimension $\mathbf{x}^{d-1}$, voter a chooses $\hat{x}_d$ to solve

$$\begin{aligned}
& \min_{\hat{x}_d} \mathbb{E} \left\| z_1 \mathbf{x}^1 + \ldots + x^b_{d-1}\left(z_1, \ldots, z_{d-2}, \hat{x}_d\right) \mathbf{x}^{d-1} + \hat{x}_d \mathbf{x}^d - \mathbf{t}_a \right\|^2 \\
= \; & \min_{\hat{x}_d} \mathbb{E} \left\| \sum\nolimits_{j=1}^{d-2} \left(z_j - x^a_j\right) \mathbf{x}^j + \left(x^b_{d-1}\left(z_1, \ldots, z_{d-2}, \hat{x}_d\right) - x^a_{d-1}\right) \mathbf{x}^{d-1} + \left(\hat{x}_d - x^a_d\right) \mathbf{x}^d \right\|^2
\end{aligned}$$

The first-order condition is

$$\begin{aligned}
0 \; &= \; 2\mathbb{E} \left\langle \mathbf{x}^d - \gamma_{d,d-1} \mathbf{x}^{d-1}, \sum\nolimits_{j=1}^{d-2} \left(z_j - x^a_j\right) \mathbf{x}^j + \left(x^b_{d-1}\left(z_1, \ldots, z_{d-2}, \hat{x}_d\right) - x^a_{d-1}\right) \mathbf{x}^{d-1} + \left(\hat{x}_d - x^a_d\right) \mathbf{x}^d \right\rangle \\
&= \; 2 \sum\nolimits_{j=1}^{d-2} \left(z_j - x^a_j\right) \gamma_{d,j} - 2 \sum\nolimits_{j=1}^{d-2} \left(z_j - x^b_j\right) \gamma_{d,d-1} \gamma_{d-1,j} + 2\left(1 - \left(\gamma_{d,d-1}\right)^2\right) \left(\hat{x}_d - x^a_d\right)
\end{aligned}$$

which implies that

$$\hat{x}^a_d\left(z_1, \ldots, z_{d-2}\right) = x^a_d - \frac{\sum_{j=1}^{d-2} \left(z_j - x^a_j\right) \left(\gamma_{d,d-1} \gamma_{d-1,j} - \gamma_{d,j}\right)}{1 - \left(\gamma_{d,d-1}\right)^2}. \tag{5}$$

From (5) we obtain that voter's a equilibrium vote on the $\mathbf{x}^d$ dimension under order π is independent of the outcome z_{d-1} in the $\mathbf{x}^{d-1}$ dimension. In contrast, we obtain from (2) that, in the equilibrium under the order σ, voter a's vote on the $\mathbf{x}^d$ dimension varies with the outcome y_{d-1} in the $\mathbf{x}^{d-1}$ dimension if $\gamma_{d,d-1} \neq 0$. Therefore, the two voting orders generate different voting outcomes if $\mathbf{x}^d$ and $\mathbf{x}^{d-1}$ are not orthogonal. In other words, y_d depends on y_{d-1} but z_d does not depend on z_{d-1}. On the other hand, y_{d-1} does not depend on y_d but z_{d-1} depends on z_d. Therefore, generically, the voting outcomes will be different under the two voting orders. ■

Example 4 has shown that the voting outcome under the non-orthogonal basis $\left\{\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1\right\}$ with voting order $\left(\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1\right)$ is the same as the one under the orthogonal basis $\left\{\mathbf{e}^2, \mathbf{e}^1\right\}$, and that the voting outcome under the non-orthogonal basis $\left\{\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1\right\}$ with alternative voting order $\left(\mathbf{e}^1, \mathbf{e}^1 + \mathbf{e}^2\right)$ is the same as the one under the orthogonal basis $\left\{\mathbf{e}^1 - \mathbf{e}^2, \mathbf{e}^1 + \mathbf{e}^2\right\}$. The following proposition generalizes this insight, which will be useful also for general norms:[11]

Proposition 2 *Consider the standard Euclidean norm and let $d = 2$. The voting outcome of sequential issue-by-issue voting under any basis $\left\{\mathbf{x}^1, \mathbf{x}^2\right\}$ with voting order $\left(\mathbf{x}^1, \mathbf{x}^2\right)$ is the same as the one under the orthogonal basis $\left\{\hat{\mathbf{x}}^1, \mathbf{x}^2\right\}$.*

Proof. Let $\mathbf{x}^1 = \alpha \hat{\mathbf{x}}^1 + \beta \mathbf{x}^2$ and consider the second stage voting given voting outcome y_1 in the first stage. The set of feasible outcomes in stage two can be rewritten as

$$y_1 \mathbf{x}^1 + \hat{r}_2 \mathbf{x}^2 = \alpha y_1 \hat{\mathbf{x}}^1 + \left(y_1 \beta + \hat{r}_2\right) \mathbf{x}^2.$$

[11] It is straightforward to generalize this insight to higher dimensions.

Therefore, from the perspective of basis $\{\hat{\mathbf{x}}^1, \mathbf{x}^2\}$, the coordinate in dimension $\hat{\mathbf{x}}^1$ is fixed at αy_1 by stage-one voting, and stage-two voting is to determine the coordinate in the other dimension $\mathbf{x}^2$. Therefore, voters' strategic consideration in stage-two voting under $\{\mathbf{x}^1, \mathbf{x}^2\}$ is the same as in stage-two voting under $\{\hat{\mathbf{x}}^1, \mathbf{x}^2\}$. Voter i with ideal point

$$x_1^i \mathbf{x}^1 + x_2^i \mathbf{x}^2 = \alpha x_1^i \hat{\mathbf{x}}^1 + \left(x_1^i \beta + x_2^i\right) \mathbf{x}^2 \tag{6}$$

will choose report $\hat{r}_2^*$ optimally to match the projection on $\mathbf{x}^2$ in the representation (6):

$$\hat{r}_2^* = x_1^i \beta + x_2^i - y_1 \beta.$$

Now consider voter i in stage-one voting. If voter i is pivotal with report $\hat{r}_1$, then i chooses $\hat{r}_1$ to minimize

$$\mathbb{E}\left\|\alpha x_1^i \hat{\mathbf{x}}^1 + \left(x_1^i \beta + x_2^i\right) \mathbf{x}^2 - \left(\alpha \hat{r}_1 \hat{\mathbf{x}}^1 + med\left(x_1^1 \beta + x_2^1, ..., x_1^n \beta + x_2^n\right) \mathbf{x}^2\right)\right\|^2$$

which implies that

$$\hat{r}_1^* = x_1^i.$$

Hence, the voting outcome under $\{\mathbf{x}^1, \mathbf{x}^2\}$ with voting order $(\mathbf{x}^1, \mathbf{x}^2)$ is

$$\begin{aligned}
& med\left(x_1^1, ..., x_1^n\right) \mathbf{x}^1 + \left[med\left(x_1^1 \beta + x_2^1, ..., x_1^n \beta + x_2^n\right) - \beta med\left(x_1^1, ..., x_1^n\right)\right] \mathbf{x}^2 \\
= \; & med\left(x_1^1, ..., x_1^n\right) \left(\alpha \hat{\mathbf{x}}^1 + \beta \mathbf{x}^2\right) + \left[med\left(x_1^1 \beta + x_2^1, ..., x_1^n \beta + x_2^n\right) - \beta med\left(x_1^1, ..., x_1^n\right)\right] \mathbf{x}^2 \\
= \; & med\left(\alpha x_1^1, ..., \alpha x_1^n\right) \hat{\mathbf{x}}^1 + med\left(x_1^1 \beta + x_2^1, ..., x_1^n \beta + x_2^n\right) \mathbf{x}^2
\end{aligned}$$

which is the same as if the voting is cast with respect to $\{\hat{\mathbf{x}}^1, \mathbf{x}^2\}$. ■

The intuition is as follows: since $\{\mathbf{x}^1, \mathbf{x}^2\}$ are not orthogonal, voters know that the outcome on the first issue that is put to vote, $\mathbf{x}^1$, is not final: the second stage vote on issue $\mathbf{x}^2$ will have a non-zero effect also on issue $\mathbf{x}^1$. The only thing that can be determined at the first stage is the outcome in dimension $\hat{\mathbf{x}}^1$ that is orthogonal to $\mathbf{x}^2$ – this part cannot be affected anymore by the second stage vote, and hence the projection of the ideal point along the $\hat{\mathbf{x}}^1$ dimension should be truthfully revealed.

5 Order Independence under General Norms

As mentioned in the Introduction, some applications are better captured by preferences based on non-Euclidean norms. For example, an important class of norms that are not generated by inner products is the l_p class, $p \neq 2$. Fix any basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$. Let $\mathbf{x} = (x_1, ..., x_d)$ and $p \geq 1$ and define

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^d |x_j|^p\right)^{1/p}.$$

This is the class of l_p^d norms with respect to the given basis. The limit norm when $p \to \infty$ is given by $\|\mathbf{x}\|_\infty = \max_j |x_j|$.

5.1 Birkhoff-James Orthogonality

The standard definition of orthogonality (i.e., zero inner-product) only applies if the norm is generated by an inner product. We need here a more general notion, due to Birkhoff [1935] and James [1947]: A vector $\mathbf{x}$ is *Birkhoff-James (BJ) orthogonal* to another vector $\mathbf{y}$ if $\mathbf{x}$ has the smallest norm among all vectors on the line through $\mathbf{x}$ that is parallel to $\mathbf{y}$. Equivalently, the line through $\mathbf{x}$ that is parallel to $\mathbf{y}$ is tangent to the "ball" with radius $\|\mathbf{x}\|$.

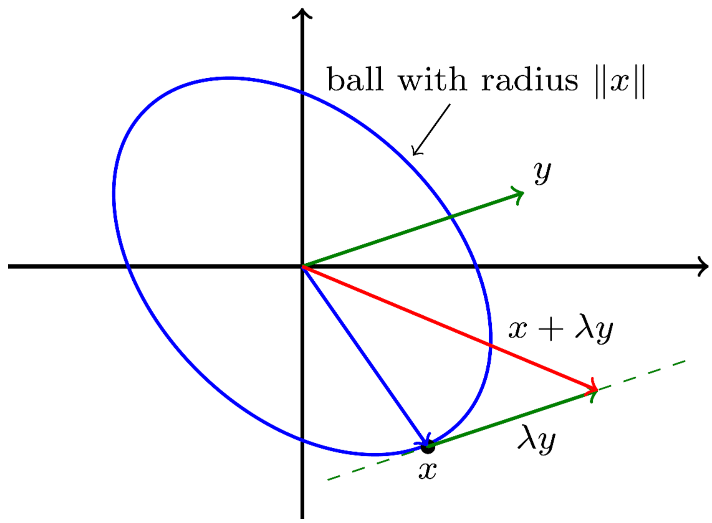

Figure 1: Vector $\mathbf{x}$ is BJ-orthogonal to vector $\mathbf{y}$

Definition 4 (BJ-Orthogonality: Birkhoff [1935], James [1947]) *Consider a general norm $\|\cdot\|$.*

1. *A vector $\mathbf{x}$ is BJ-orthogonal to another vector $\mathbf{y}$, denoted $\mathbf{x} \dashv \mathbf{y}$, if $\|\mathbf{x} + \lambda \mathbf{y}\| \geq \|\mathbf{x}\|$ for all real λ.*

2. *Vectors $\mathbf{x}$ and $\mathbf{y}$ are BJ-mutually orthogonal if $\mathbf{x} \dashv \mathbf{y}$ and $\mathbf{y} \dashv \mathbf{x}$.*

3. *A vector $\mathbf{x}$ is BJ-orthogonal to a subspace M, denoted $\mathbf{x} \dashv M$, if $\mathbf{x} \dashv \mathbf{y}$ for all $\mathbf{y} \in M$. A subspace M is BJ-orthogonal to a vector $\mathbf{x}$, denoted $M \dashv \mathbf{x}$, if $\mathbf{y} \dashv \mathbf{x}$ for all $\mathbf{y} \in M$.*

4. *An* Auerbach basis *is an algebraic basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ such that, for each $j = 1, ..., d$, $\mathbf{x}^j \dashv X^{-j}$ where X^{-j} is the subspace spanned by all basis vectors except $\mathbf{x}^j$.*

BJ-orthogonality reduces to the standard (symmetric and additive) definition of orthogonality if the norm is generated by an inner-product. But, the BJ-orthogonality relation is generally **not** symmetric: $\mathbf{x}$ can be orthogonal to $\mathbf{y}$ but not vice-versa. Moreover, the BJ-orthogonality relation is generally **not** additive, neither on the left, nor on the right: $\mathbf{y} \dashv \mathbf{x}$ and $\mathbf{z} \dashv \mathbf{x}$ need not imply $(\mathbf{y} + \mathbf{z}) \dashv \mathbf{x}$, and also $\mathbf{x} \dashv \mathbf{y}$ and $\mathbf{x} \dashv \mathbf{z}$ need not imply $\mathbf{x} \dashv (\mathbf{y} + \mathbf{z})$.

Bases that consist of mutually BJ-orthogonal vectors always exist. In fact, a stronger statement holds:

Theorem 1 (Day [1947], Taylor [1947]) *In any normed space there exist at least two distinct Auerbach bases.*

5.2 Left-Additive Mutual Orthogonality (LAMO)

While Auerbach bases - that always consist of mutually orthogonal vectors - generalize the orthogonal bases of inner-product spaces, they are not too useful for our purposes. Instead, another concept takes center-stage. The main result in Gershkov, Moldovanu and Shi [2020] connects another special subset of mutually BJ-orthogonal bases to strategy-proofness of static marginal median mechanisms.

Theorem 2 (Gershkov, Moldovanu and Shi [2020]) *Let ψ be a (static) issue-by-issue median with respect to a basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$, and denote by $X^{-j} \subset \mathbb{R}^d$ the subspace spanned by all vectors in the basis except $\mathbf{x}^j$. Let $||\cdot||$ be a norm on $\mathbb{R}^d$. Then the following properties are equivalent:*

1. *ψ is strategy-proof with respect to the utility function induced by $||\cdot||$;*
2. *The vectors in the basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ are BJ-mutually orthogonal and the orthogonality is additive on the left: $X^{-j} \dashv \mathbf{x}^j$ for all $j = 1, ..., d$.*

We shall call the property in the above theorem *left-additive mutual orthogonality*, abbreviated LAMO. A necessary condition for basis $\left\{\mathbf{x}^1, ..., \mathbf{x}^d\right\}$ to be either Auerbach or LAMO is that all vectors in the basis are BJ-mutually orthogonal. Auerbach bases correct for symmetry of the BJ-orthogonally condition and for its additivity on the right, so they are mirror images of bases that satisfy LAMO – the latter correct for symmetry and for additivity on the left.

For the special case $d = 2$, additivity is mute, and LAMO is equivalent to mutual orthogonality and the same is true for the property defining Auerbach bases. Therefore, Theorem 1 implies that, in any normed, two-dimensional space, there exist at least two distinct LAMO bases. The following example illustrates how LAMO can be verified and how it is linked to the strategy-proofness of issue-by-issue median mechanisms.

Example 5 (LAMO Property and Strategy-Proofness) *Let $d = 2$ and fix the Cartesian basis $\left\{\mathbf{e}^1, \mathbf{e}^2\right\}$. Consider the norm with unit ball defined as the parallelogram with* *vertices at* $\pm(2,2)$ *and* $\pm(1,-1)$.

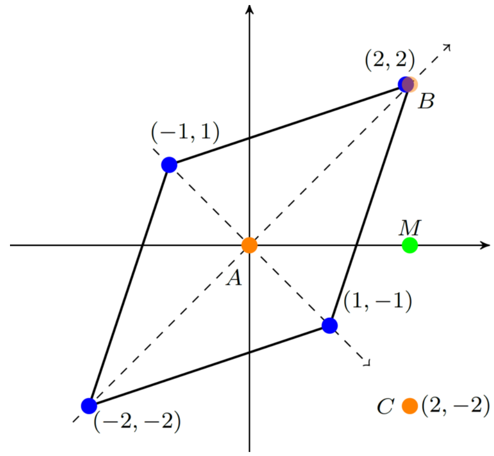

Figure 2: LAMO bases with respect to the norm defined by the parallelogram.

Vectors $\mathbf{e}^1$ *and* $\mathbf{e}^2$ *are not mutually BJ-orthogonal with respect to this norm, and hence basis* $\{\mathbf{e}^1, \mathbf{e}^2\}$ *is not LAMO. To see this, note that*

$$2\left\|\mathbf{e}^1 + \mathbf{e}^2\right\| = ||(2,2)|| = 1 < ||(2,0)|| = 2\left\|\mathbf{e}^1\right\|,$$

which implies for $\lambda = 1$ *that*

$$\left\|\mathbf{e}^1 + \lambda\mathbf{e}^2\right\| < \left\|\mathbf{e}^1\right\|,$$

and hence $\mathbf{e}^1 \not\dashv \mathbf{e}^2$ *by Definition 4. Similarly, we can argue* $\mathbf{e}^2 \not\dashv \mathbf{e}^1$ *by noting that*

$$2\left\|\mathbf{e}^2 + \mathbf{e}^1\right\| = ||(2,2)|| = 1 < ||(0,2)|| = 2\left\|\mathbf{e}^2\right\|.$$

In contrast, the two vectors, $\mathbf{z}^1 = (-1,1)$ *and* $\mathbf{z}^2 = (1,1)$, *are mutually BJ-orthogonal, and hence the basis* $\{\mathbf{z}^1, \mathbf{z}^2\}$ *is LAMO. This holds because, for all* $\lambda \in \mathbb{R}$,

$$\left\|\mathbf{z}^1 + \lambda\mathbf{z}^2\right\| = \|(-1+\lambda, 1+\lambda)\| \geq 1 = \left\|\mathbf{z}^1\right\|,$$

and

$$2\left\|\mathbf{z}^2 + \lambda\mathbf{z}^1\right\| = \|(2-2\lambda, 2+2\lambda)\| \geq 1 = 2\left\|\mathbf{z}^2\right\|,$$

where the two inequalities follow from the fact that both points $(-1+\lambda, 1+\lambda)$ *and* $(2-2\lambda, 2+2\lambda)$ *lie outside of the parallelogram. One can similarly verify that the basis* $\{(1,3),(1,1/3)\}$ *is also LAMO.*

Now suppose that three agents (A, B, C), with peaks at $(0,0)$, $(2,2)$ *and* $(2,-2)$ *respectively, are asked to vote according to the Cartesian basis* $\{\mathbf{e}^1, \mathbf{e}^2\}$. *If all agents report truthfully, the issue-by-issue median is* $M = (2,0)$. *If agent A deviates and reports instead* $(2,2)$, *the median becomes* $(2,2)$ *and this deviation is clearly profitable (since* $||(2,2)|| = 1 < \|(2,0)\|$*). In contrast, if a marginal median is computed with respect to the dashed coordinates defined by the basis* $\{(-1,1),(1,1)\}$ *or by the basis* $\{(1,3),(1,1/3)\}$, *then by Theorem 2 this mechanism is strategy-proof.*

We are now ready to present the connection between LAMO and order independence in sequential voting.

Proposition 3 *Let* $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ *be a basis and let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^d$. *If the basis* $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ *satisfies LAMO with respect to this norm, then the set* $\{\psi_\sigma\}_{\sigma\ \in\Sigma_d}$ *of sequential issue-by-issue voting procedures with respect to this basis is order-independent, and all their outcomes coincide with the outcome of the static and strategy proof issue-by-issue median* ψ*.*

Proof. Consider a sequential issue-by-issue voting procedure ψ_σ. We prove by backward induction that all voters vote sincerely in all stages. Hence, by Theorem 2, the outcome in any such mechanism coincides with the outcome in the static, strategy-proof marginal median mechanism

Consider the decision of voter i at the last stage where the outcome in the $\mathbf{x}^{\sigma_d}$ dimension is determined. Let voter i's ideal point be

$$\mathbf{t}_i = \left(x^i_{\sigma_1}, ..., x^i_{\sigma_d}\right) = x^i_{\sigma_1}\mathbf{x}^{\sigma_1} + ... + x^i_{\sigma_d}\mathbf{x}^{\sigma_d},$$

and let $\left(y_1, ...y_{\sigma_{d-1}}\right)$ denote the voting outcome in earlier stages. This voter chooses a reported peak $\hat{x}^i_{\sigma_d}$ in order to maximize his payoff, conditional on being pivotal. The optimization problem is equivalent to

$$\begin{aligned} & \min_{\hat{x}^i_{\sigma_d}} \left\| \sum\nolimits_{j\in\{\sigma_1,....,\sigma_{d-1}\}} y_j\mathbf{x}^j + \hat{x}^i_{\sigma_d}\mathbf{x}^{\sigma_d} - \sum\nolimits_{j\in\{\sigma_1,....,\sigma_d\}} x^i_j\mathbf{x}^j \right\| \\ = & \min_{\hat{x}^i_{\sigma_d}} \left\| \sum\nolimits_{j\in\{\sigma_1,....,\sigma_{d-1}\}} \left(y_j - x^i_j\right)\mathbf{x}^j + \left(\hat{x}^i_{\sigma_d} - x^i_{\sigma_d}\right)\mathbf{x}^{\sigma_d} \right\| \end{aligned}$$

Note that

$$\sum\nolimits_{j\in\{\sigma_1,....,\sigma_{d-1}\}} \left(y_j - x^i_j\right)\mathbf{x}^j \in X^{-\sigma_d}.$$

If $X^{-\sigma_d} \dashv \mathbf{x}^{\sigma_d}$, then the definition of BJ-orthogonality yields that

$$\left\| \sum\nolimits_{j\in\{\sigma_1,....,\sigma_{d-1}\}} \left(y_j - x^i_j\right)\mathbf{x}^j \right\| \leq \left\| \sum\nolimits_{j\in\{\sigma_1,....,\sigma_{d-1}\}} \left(y_j - x^i_j\right)\mathbf{x}^j + \left(\hat{x}^i_{\sigma_d} - x^i_{\sigma_d}\right)\mathbf{x}^{\sigma_d} \right\|.$$

This means that truthful reporting $\hat{x}^i_{\sigma_d} = x^i_{\sigma_d}$ maximizes voter i's payoff. Therefore, the voting outcome in the last voting stage will be the median of all voters' projections on the $\mathbf{x}^{\sigma_j}$ dimension. In particular, this outcome does not depend on the outcome of earlier stages, and we denote it as $x^*_{\sigma_d}$. Consider next voter i's decision at the second to the last stage, where the $\mathbf{x}^{\sigma_{d-1}}$ dimension outcome is determined. This voter chooses a reported peak $\hat{x}^i_{\sigma_{d-1}}$ in order to maximize his payoff, if the voter is pivotal with this report. Let $\left(y_1, ...y_{\sigma_{d-2}}\right)$ denote the voting outcome in earlier stages. Voter i chooses $\hat{x}^i_{\sigma_{d-1}}$ to solve

$$\begin{aligned} & \min_{\hat{x}^i_{\sigma_{d-1}}} \mathbb{E} \left\| \sum\nolimits_{j\in\{\sigma_1,....,\sigma_{d-2}\}} y_j\mathbf{x}^j + \hat{x}^i_{\sigma_{d-1}}\mathbf{x}^{\sigma_{d-1}} + x^*_{\sigma_d}\mathbf{x}^{\sigma_d} - \sum\nolimits_{j\in\{\sigma_1,....,\sigma_d\}} x^i_j\mathbf{x}^j \right\| \\ = & \min_{\hat{x}^i_{\sigma_{d-1}}} \mathbb{E} \left\| \sum\nolimits_{j\in\{\sigma_1,....,\sigma_{d-2}\}} \left(y_j - x^i_j\right)\mathbf{x}^j + \left(x^*_{\sigma_d} - x^i_{\sigma_d}\right)\mathbf{x}^{\sigma_d} + \left(\hat{x}^i_{\sigma_{d-1}} - x^i_{\sigma_{d-1}}\right)\mathbf{x}^{\sigma_{d-1}} \right\| \end{aligned}$$

Note that

$$\sum\nolimits_{j\in\{\sigma_1,\ldots,\sigma_{d-2}\}}\left(y_j - x_j^i\right)\mathbf{x}^j + \left(x_{\sigma_d}^* - x_{\sigma_d}^i\right)\mathbf{x}^{\sigma_d} \in X^{-\sigma_{d-1}}.$$

If $X^{-\sigma_{d-1}} \dashv \mathbf{x}^{\sigma_{d-1}}$, then the definition of BJ-orthogonality yields

$$\begin{aligned} &\left\|\sum\nolimits_{j\in\{\sigma_1,\ldots,\sigma_{d-2}\}}\left(y_j - x_j^i\right)\mathbf{x}^j + \left(x_{\sigma_d}^* - x_{\sigma_d}^i\right)\mathbf{x}^{\sigma_d}\right\| \\ \leq\ &\left\|\sum\nolimits_{j\in\{\sigma_1,\ldots,\sigma_{d-2}\}}\left(y_j - x_j^i\right)\mathbf{x}^j + \left(x_{\sigma_d}^* - x_{\sigma_d}^i\right)\mathbf{x}^{\sigma_d} + \left(\hat{x}_{\sigma_{d-1}}^i - x_{\sigma_{d-1}}^i\right)\mathbf{x}^{\sigma_{d-1}}\right\|. \end{aligned}$$

This means that voting sincerely maximizes voter i's payoff. Since voter i was chosen arbitrarily, all voters vote sincerely in the second to the last stage voting. By successive backward induction, we conclude that, as long as LAMO holds, all voters vote sincerely under any order. ■

What is the equilibrium voting outcome if the chosen basis does not satisfy LAMO? We conjecture that the outcome of sequential issue-by-issue voting is then necessarily order dependent. To prove that, we would need to derive voters' optimal reporting strategy which is rather difficult with general norms. Hence, the method we previously used to prove the necessity of orthogonality in Proposition 1 is less feasible at this level of generality.

An alternative approach is to follow Proposition 2 and to show that the voting outcomes in sequential voting with respect to a non-orthogonal basis under different voting orders are equivalent to the voting outcomes under different LAMO bases. Since the marginal medians associated with different LAMO bases are generically different (note that this is true even for inner-product norms), we can conclude that the voting outcomes of sequential voting with respect to a non-orthogonal basis is order dependent. The problem of this approach is that the existence of a LAMO basis is not always guaranteed, as shown in the next section. For the case of $d = 2$, we have, however, the following equivalence result that parallels the insight of Proposition 2 for the Euclidean norm.

Proposition 4 *Consider a two-dimensional normed space, and two distinct mutually BJ-orthogonal bases* $\left\{\mathbf{x}^1, \mathbf{x}^2\right\}$ *and* $\left\{\mathbf{z}^1, \mathbf{z}^2\right\}$*, whose existence is assured by Auerbach's Theorem. Suppose that the sequential issue-by-issue voting is carried out with respect to the non-orthogonal basis* $\left\{\mathbf{z}^1, \mathbf{x}^2\right\}$ *and according to voting order* $\sigma = \{1, 2\}$*. Then the voting outcome is the same as that the outcome of issue by issue voting with respect to the orthogonal basis* $\left\{\mathbf{x}^1, \mathbf{x}^2\right\}$ *with the same voting order. Similarly, the voting outcome with respect to the non-orthogonal basis* $\left\{\mathbf{x}^2, \mathbf{z}^1\right\}$ *coincides with the one with respect to the BJ-orthogonal basis* $\left\{\mathbf{z}^2, \mathbf{z}^1\right\}$*.*

Proof. Let us write

$$\mathbf{z}^1 = \alpha\mathbf{x}^1 + \beta\mathbf{x}^2.$$

Consider voter $i's$ decision at stage two, where the $\mathbf{x}^2$ dimension outcome is determined. Let voter i's ideal point be

$$\mathbf{t}_i = x_1^i\mathbf{z}^1 + x_2^i\mathbf{x}^2,$$

and let y_1 denote the voting outcome in stage one. This voter chooses a reported peak $\hat{x}_2^i$ in order to maximize his payoff, conditional on being pivotal. The optimization problem is equivalent to

$$\begin{aligned}
& \min_{\hat{x}_2^i} \left\| y_1 \mathbf{z}^1 + \hat{x}_2^i \mathbf{x}^2 - \left(x_1^i \mathbf{z}^1 + x_2^i \mathbf{x}^2 \right) \right\| \\
= & \min_{\hat{x}_2^i} \left\| \left(y_1 - x_1^i \right) \left(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2 \right) + \left(\hat{x}_2^i - x_2^i \right) \mathbf{x}^2 \right\| \\
= & \min_{\hat{x}_2^i} \left\| \left(y_1 - x_1^i \right) \alpha \mathbf{x}^1 + \left(\hat{x}_2^i - x_2^i + \left(y_1 - x_1^i \right) \beta \right) \mathbf{x}^2 \right\|
\end{aligned}$$

By assumption $\left\{ \mathbf{x}^1, \mathbf{x}^2 \right\}$ is LAMO„ so $\mathbf{x}^1 \dashv \mathbf{x}^2$. The definition of BJ-orthogonality yields

$$\left\| \left(y_1 - x_1^i \right) \alpha \mathbf{x}^1 \right\| \leq \left\| \left(y_1 - x_1^i \right) \alpha \mathbf{x}^1 + \left(\hat{x}_2^i - x_2^i + \left(y_1 - x_1^i \right) \beta \right) \mathbf{x}^2 \right\|$$

This means that, when voting is with respect to basis $\left\{ \mathbf{z}^1, \mathbf{x}^2 \right\}$ under order $\sigma = \{1, 2\}$, agent i would report at the last stage:

$$\hat{x}_2^i = \beta x_1^i + x_2^i - \beta y_1.$$

The voting outcome in stage two is then

$$x_2^* \left(y_1 \right) = med \left(\beta x_1^1 + x_2^1, ..., \beta x_1^n + x_2^n \right) - \beta y_1.$$

Consider next voter $i's$ decision at stage one (this voter is not necessarily the pivotal voter in stage two), where the $\mathbf{z}^1$ dimension outcome is determined. Voter i chooses to report a peak $\hat{x}_1^i$ in order to maximize his payoff, if i is pivotal with this report. If voter i is pivotal in stage one, we have

$$y_1 = \hat{x}_1^i.$$

The optimization problem of voter i is equivalent to

$$\begin{aligned}
& \min_{\hat{x}_1^i} \mathbb{E} \left\| \hat{x}_1^i \mathbf{z}^1 + x_2^* \left(\hat{x}_1^i \right) \mathbf{x}^2 - \left(x_1^i \mathbf{z}^1 + x_2^i \mathbf{x}^2 \right) \right\| \\
= & \min_{\hat{x}_1^i} \mathbb{E} \left\| \left(\hat{x}_1^i - x_1^i \right) \left(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2 \right) + \left[x_2^* \left(\hat{x}_1^i \right) - x_2^i \right] \mathbf{x}^2 \right\| \\
= & \min_{\hat{x}_1^i} \mathbb{E} \left\| \left(\hat{x}_1^i - x_1^i \right) \left(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2 \right) + \left[med \left(\beta x_1^1 + x_2^1, ..., \beta x_1^n + x_2^n \right) - \beta \hat{x}_1^i - x_2^i \right] \mathbf{x}^2 \right\| \\
= & \min_{\hat{x}_1^i} \mathbb{E} \left\| \left(\hat{x}_1^i - x_1^i \right) \alpha \mathbf{x}^1 + \left[\beta \left(\hat{x}_1^i - x_1^i \right) + med \left(\beta x_1^1 + x_2^1, ..., \beta x_1^n + x_2^n \right) - \beta \hat{x}_1^i - x_2^i \right] \mathbf{x}^2 \right\| \\
= & \min_{\hat{x}_1^i} \mathbb{E} \left\| \left(\hat{x}_1^i - x_1^i \right) \alpha \mathbf{x}^1 + \left[med \left(\beta x_1^1 + x_2^1, ..., \beta x_1^n + x_2^n \right) - \left(\beta x_1^i + x_2^i \right) \right] \mathbf{x}^2 \right\|
\end{aligned}$$

By assumption $\left\{ \mathbf{x}^1, \mathbf{x}^2 \right\}$ is LAMO, so we also have $\mathbf{x}^2 \dashv \mathbf{x}^1$. The definition of BJ-orthogonality yields

$$\begin{aligned}
& \left\| \left[med \left(\beta x_1^1 + x_2^1, ..., \beta x_1^n + x_2^n \right) - \left(\beta x_1^i + x_2^i \right) \right] \mathbf{x}^2 \right\| \\
\leq & \left\| \left(\hat{x}_1^i - x_1^i \right) \alpha \mathbf{x}^1 + \left[med \left(\beta x_1^1 + x_2^1, ..., \beta x_1^n + x_2^n \right) - \left(\beta x_1^i + x_2^i \right) \right] \mathbf{x}^2 \right\|.
\end{aligned}$$

In other words, it is optimal for voter i to report

$$\hat{x}_1^i = x_1^i.$$

Therefore, the voting outcome under order $\sigma = (1,2)$ with respect to basis $\{\mathbf{z}^1, \mathbf{x}^2\}$ is

$$\begin{aligned}
& med(x_2^1,...,x_2^n)\mathbf{z}^1 + \left[med\left(\beta x_1^1 + x_2^1,...,\beta x_1^n + x_2^n\right) - \beta med(x_2^1,...,x_2^n)\right]\mathbf{x}^2 \\
= \; & med(x_2^1,...,x_2^n)\left(\alpha\mathbf{x}^1 + \beta\mathbf{x}^2\right) + \left[med\left(\beta x_1^1 + x_2^1,...,\beta x_1^n + x_2^n\right) - \beta med(x_2^1,...,x_2^n)\right]\mathbf{x}^2 \\
= \; & med(x_2^1,...,x_2^n)\alpha\mathbf{x}^1 + med\left(\beta x_1^1 + x_2^1,...,\beta x_1^n + x_2^n\right)\mathbf{x}^2
\end{aligned}$$

If the sequential voting is carried out with respect to $\{\mathbf{x}^1, \mathbf{x}^2\}$ in the order $\sigma = \{1,2\}$, then voters report truthfully since

$$\mathbf{t}_i = x_1^i\mathbf{z}^1 + x_2^i\mathbf{x}^2 = \mathbf{t}_i = x_1^i\left(\alpha\mathbf{x}^1 + \beta\mathbf{x}^2\right) + x_2^i\mathbf{x}^2 = x_1^i\alpha\mathbf{x}^1 + \left(\beta x_1^i + x_2^i\right)\mathbf{x}^2$$

The voting outcome is

$$med(x_2^1,...,x_2^n)\alpha\mathbf{x}^1 + med\left(\beta x_1^1 + x_2^1,...,\beta x_1^n + x_2^n\right)\mathbf{x}^2.$$

In other words, the voting outcome under order $\sigma = (1,2)$ with respect to the non-orthogonal basis $\{\mathbf{z}^1, \mathbf{x}^2\}$ coincides with the voting outcome under order $\sigma = (1,2)$ with respect to the orthogonal basis $\{\mathbf{x}^1, \mathbf{x}^2\}$. ■

6 When do LAMO Bases Exist?

We conclude the paper with a discussion of the general existence question of LAMO bases – the property that has proved crucial for both strategy-proofness and for order independence of issue-by-issue medians.

We show that, beyond the two-dimensional case for which existence is always assured by Auerbach's theorem, the non-existence of LAMO bases is generic in a sense to be made precise below. This means that for "almost all" norms on spaces with at least three dimensions, it is impossible to find policy issues along which issue-by-issue voting by majority has the desirable properties discussed above. Otherwise put, for generic preferences induced by norms, it is impossible to decompose a multi-dimensional problem into simpler one-dimensional issues such that the utility function becomes "separable" in those dimensions.

Theorem 3 *Let X be a finite d-dimensional space with $d > 2$, and assume that agents' utilities are induced by a norm $\|\cdot\|$. Then, for any $\varepsilon > 0$, there exists another norm $\|\cdot\|_\varepsilon$ on X with*

$$(1-\varepsilon)\|\mathbf{x}\| \leq \|\mathbf{x}\|_\varepsilon \leq (1+\varepsilon)\|\mathbf{x}\|$$

for every $\mathbf{x} \in \mathbf{X}$, such that $\|\cdot\|_\varepsilon$ admits no LAMO basis.

For the proof of Theorem 3, we first need several abstract definitions

Definition 5 (Projection and Operator Norm) *Consider a closed subspace $Y \subseteq \mathbb{R}^d$.*

1. *A linear operator P such that $P^2 = P$ and such that $P(\mathbb{R}^d) = Y$ is called a projection onto Y.*
2. *The norm of a linear operator P is defined by $\|P\| \equiv \sup_{\mathbf{x}\in\mathbb{R}^d} \frac{\|P(\mathbf{x})\|}{\|\mathbf{x}\|}$.*
3. *A subspace Y is 1-complemented if there exists a projection P onto Y such that $\|P\| = 1$.*

Note that all subspaces of a finite dimensional inner-product space are 1-complemented.[12]

Definition 6 (Orthant Monotonicity: Gries [1967]) *Consider an algebraic basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ for $\mathbb{R}^d$ and represent any $\mathbf{x} \in \mathbb{R}^d$ by its coordinates $(x_1, ..., x_d)$. A norm $\|\cdot\|$ on $\mathbb{R}^d$ is orthant-monotonic with respect to this basis if*

$$\|(x_1, ..., x_d)\| \leq \|(y_1, ..., y_d)\|$$

whenever

$$x_j y_j \geq 0 \text{ and } |x_j| \leq |y_j| \text{ for all } j = 1, ..., d.$$

Lemma 1 (Johnson and Nylen [1991]) *A norm is orthant-monotonic if and only if it satisfies*

$$\|(x_1, ..., x_{j-1}, 0, ..., x_d)\| \leq \|(x_1, ..., x_{j-1}, x_j, ..., x_d)\|$$

for all $\mathbf{x} \in \mathbb{R}^d$ and all j.

Consider then algebraic basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ and represent any $\mathbf{x} \in \mathbb{R}^d$ by its coordinates $(x_1, ..., x_d)$. Let the *natural projection* P_j on X^{-j}, the hyperplane spanned by the vectors $\{\mathbf{x}^1, ..., \mathbf{x}^{j-1}, \mathbf{x}^{j+1}, ..., \mathbf{x}^d\}$ be defined by:

$$P_j(\mathbf{x}) = (x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_d) := \left(0, \mathbf{x}^{-j}\right).$$

If the norm is orthant-monotonic with respect to this basis, then Lemma 1 says that

$$\|P_j(\mathbf{x})\| = \left\|\left(0, \mathbf{x}^{-j}\right)\right\| \leq \|\mathbf{x}\|$$

for all $\mathbf{x} \in \mathbb{R}^d$ and for all j. By the definition of the natural projection, we know that $\left\|\mathbf{P}_j\left(0, \mathbf{x}^{-j}\right)\right\| = \left\|\left(0, \mathbf{x}^{-j}\right)\right\|$. Hence, all the natural projections $\{\mathbf{P}_j\}_{j=1}^d$ satisfy:

$$\|\mathbf{P}_j\| = \sup_{\mathbf{x}} \frac{\|\mathbf{P}_j(\mathbf{x})\|}{\|\mathbf{x}\|} = 1 \text{ for all } j.$$

This yields:

[12] Kakutani [1940] and Lindenstrauss and Tzafriri [1971] related the existence of bounded-norm projections on rich families of subspaces to the existence of an inner-product. The study of 1-complemented spaces in infinite-dimensional spaces is deep and the 1988 Fields medal was awarded to Timothy Gowers for work in this area.

Proposition 5 *If the norm is orthant-monotonic with respect to the basis* $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ *then each hyperplane* X^{-j} , $j = 1, ..., d$ *is* 1-complemented.

Finally, Gershkov, Moldovanu and Shi [2020] have established that a norm $\|\cdot\|$ is orthant-monotonic with respect to a given basis if and only if the basis is LAMO. Thus, the non-existence Theorem 3 follows now by an application of the following result due to Bosznay and Garay [1986]:[13]

Theorem 4 (Bosznay and Garay [1986]) *Let* $\|\cdot\|$ *be a given norm on a finite d-dimensional space* X *where* $d > 2$. *Then for any* $\varepsilon > 0$, *there exists a norm* $\|\cdot\|_\varepsilon$ *on* X *such that*

$$(1-\varepsilon)\|\mathbf{x}\| \leq \|\mathbf{x}\|_\varepsilon \leq (1+\varepsilon)\|\mathbf{x}\|$$

and such that $(X, \|\cdot\|_\varepsilon)$ *does not have any 1-complemented subspaces besides the whole space itself and one-dimensional subspaces.*[14]

The above Theorem implies that, for any normed space, there is a nearby normed space without any LAMO basis. Hence, that latter space does not admit strategy proof issue-by-issue medians and sequential issue-by-issue voting will be generally order dependent.

We stress that the above generic impossibility result is of a different type from most classical results available in the literature, e.g., the celebrated Gibbard-Satterthwaite Theorem. Classical results rely on a richness condition on preferences in order to create a large set of possible strategical manipulations that eventually destroy strategy-proofness for any non-trivial mechanism. This is also the basic approach in Border and Jordan [1983] and Barbera, Gul and Stachetti [1993] who characterize the "small sets" of remaining strategy-proof mechanisms in their multi-dimensional frameworks (these authors did not consider norm-based utilities).[15]

It is relatively easy to also provide here counterparts to such "impossibility by richness" results: consider, for example, a two-dimensional setting where agents can have any individually weighted Euclidean norm – all these norms have unit balls that are ellipses with axes that are parallel to the Cartesian ones. Then, generically, there is exactly one strategy proof issue-by-issue median: only the standard Cartesian coordinates are jointly orthogonal ones for all the norms in this class. Introducing even the slightest degree of interaction among the issues – by allowing utility functions derived from other, more general, inner-product norms (where the units balls are tilted ellipses) yields an impossibility result.

[13] Bohnenblust [1941] already provided a counter-example to general existence. Theorem 4 shows that non-existence is "generic". See also Theorem 7.2 in the survey by Randrianantoanina [2001]).

[14] One-dimensional subspaces are always 1-complemented. This follows by the Hahn-Banach theorem.

[15] Barbera, Gul and Stacchetti [1993] assumed that the decision set is a product of lines and studied a rich class of preferences called *multidimensional single-peaked* (m.s.p.). They showed that, on the class of m.s.p. preferences, a mechanism is strategy-proof if and only if it is a generalized marginal median. Border and Jordan [1983] considered a different rich domain of preferences which they called *star-shaped and separable* and obtained results that generalize Moulin's one-dimensional finding.

What we did in Theorem 3 is different: the generic impossibility result is obtained despite the fact that all agents share the same underlying utility function and they only differ in the location of their peak.

We recall that existence is assured for any two-dimensional spaces, for any inner-product spaces and for any l_p space. As an application consider the two-dimensional budgeting problem described in the Introduction, and assume that agents may use any l_p norm (these can be different across agents!) and that at least one agent uses an l_p norm with $p \neq 2$. Our results then imply that the "bottom-up" procedures where the budget is sequentially determined on the two individual issues, and the "top-down" budgeting procedure where first a total budget is determined followed by a division among the issues are the **only** order-independent possibilities!

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