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Imitation Perfection - a Simple Rule to Prevent Discrimination in Procurement

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Imitation perfection – a simple rule to prevent discrimination in procurement*

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Abstract

Procurement regulation aimed at curbing discrimination requires equal treatment of sellers. However, Deb and Pai (2017) show that such regulation imposes virtually no restrictions on the ability to discriminate. We propose a simple rule — imitation perfection — that restricts discrimination significantly. It ensures that in every equilibrium bidders with the same valuation distribution and the same valuation earn the same expected utility. If all bidders are homogeneous, revenue and social surplus optimal auctions consistent with imitation perfection exist. For heterogeneous bidders, however, it is incompatible with revenue and social surplus optimization. Thus, a trade-off between non-discrimination and optimality exists.

JEL classification: D44, D73, D82, L13

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1 Introduction

Regulators go to great lengths to prevent discrimination in procurement. In its rules for public procurement, for example, the World Trade Organization (WTO) demands that governments comply with "non-discrimination, equality of treatment, transparency and mutual recognition". Furthermore, the WTO seeks "to avoid introducing or continuing discriminatory measures that distort open procurement."[1]

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[1]See the General Agreement on Tariffs and Trade (GATT) (Article 1), General Agreement on Trade in Services (GATS) (Article 2), and Agreement on Trade-Related Aspects of Intellectual Property Rights (TRIPS) (Article 4) and World Trade Organization (2012).

The European Commission requires public buyers to reach their decision "in full accordance with the principles of equal treatment, non-discrimination, and transparency."[2] These regulations imply that the *rules and procedures* of a procurement process should treat suppliers equally. However, Deb and Pai (2017) show that regulation requiring equal treatment of suppliers on its own imposes virtually no restrictions on the ability to discriminate. In particular, such *symmetric auctions* allow for *perfect discrimination*. That is, there exists a symmetric auction and an equilibrium of this auction, in which the project is always awarded to a particular bidder at the most favorable price. Hence, an auctioneer can favor a particular bidder in the most extreme way without violating existing legal hurdles. This in turn, indicates that existing legal hurdles are not sufficient to prevent discrimination and that regulators should not remain satisfied with *rules* that imply equal treatment but need to go further to guarantee discrimination-free *outcomes*.

This article is complementary to Deb and Pai (2017) and provides an answer to the question: what rules are sufficient in order to achieve discrimination-free outcomes? We propose a simple rule named *imitation perfection*. Imitation perfection requires that for any realization of bids and the resulting allocation and payments, every bidder had the opportunity to imitate the allocation and payment of any other bidder. We show that imposing imitation perfection rules out perfect discrimination. This is due to the fact that imitation perfection implies that every bidder could have won the auction at (almost) the same price as the winning bidder by slightly outbidding the winning bidder. More generally, in an imitation-perfect auction each bidder had the opportunity to come arbitrarily close to the ex-post allocation and payment of every other bidder.

We denote an equilibrium as *non-discriminatory* if among a group of (possibly heterogeneous) bidders a pair of homogeneous bidders with the same valuation expects the same utility. Furthermore, we denote a mechanism as *discrimination-free* if all of its equilibria are non-discriminatory. We show that each imitation-perfect auction is *discrimination-free*.

For a pair of ex-ante heterogeneous bidders there is no clear definition of a non-discriminatory equilibrium. We introduce a measure of how two ex-ante heterogeneous bidders' distributions differ. We show that in an imitation-perfect auction the difference in the expected utility of two ex-ante heterogeneous bidders with the same valuation is limited by the measure of their heterogeneity. Thus, we show that the auction designer's ability to discriminate between (heterogeneous) bidders in

[2]See Directive 2004/18/EC of the European Parliament and of the Council of 31 March 2004 on the coordination of procedures for the award of public works contracts, public supply contracts and public service contracts.

an imitation-perfect auction is limited by the heterogeneity between these bidders regardless of the other bidders' distributions. In particular, this implies that the result, that a pair of ex-ante homogeneous bidders expects the same utility given their valuation, is robust with respect to small perturbations of homogeneity, even if the heterogeneity among the other bidders is arbitrarily high. Moreover, we analyze whether an auctioneer can discriminate in favor of a bidder by choosing among different imitation-perfect auctions. We introduce a measure of the heterogeneity of all bidders and show that the difference in expected utilities of a bidder with a given valuation in two different imitation-perfect auctions is limited by the measure of the heterogeneity of all bidders.

Usually, the beneficiary of a procurement organization (the people of a country, the chief procurement officer of a company, or its shareholders) is responsible for thousands of different procurement projects with thousands of different bidders. According to the European Commission, there are over 250,000 public authorities involved in procurement in the EU. Delegating the specific procurement project to a (potentially large) group of agents is therefore unavoidable. Most of these agents will have the buyer's best interest in mind and will use the optimal procedures. There may, however, be some agents who are corrupt and/or favor certain bidders.[3] For the buyer, it is impossible to monitor each of the procurement transactions and to check whether the implemented procedures were optimal. Thus, there is a need to determine general procurement rules. The set of procurement regulations should have the following properties. Firstly, it should be easy to check whether these regulations have been followed. In particular, this should not require knowledge of unobservables such as subjective beliefs, or the use of complicated calculations such as equilibrium analyses. Secondly, the regulation should restrict corrupt agents in a meaningful way. Finally, honest agents should maintain enough freedom to enable them to implement optimal procedures. Imitation perfection has all of these desirable properties. Firstly, a quick look at the rules of the particular auction is sufficient to verify whether the procurement process satisfies imitation perfection. This is due to the fact that imitation perfection is a property of the payment rule. Hence, the verification does not require information on any details of the procurement project and can also be done ex-ante or ex-post without the calculation of equilibria. Secondly, imitation perfection prevents corrupt agents from implementing perfectly discriminatory outcomes and guarantees discrimination-free outcomes. Finally, imitation perfection gives honest agents the opportunity to implement an efficient auction as well as a revenue-optimal one if bidders are homogeneous. In

[3] See Mironov and Zhuravskaya (2016) for some recent empirical evidence.

this respect, ensuring that the procurement mechanism is imitation-perfect comes at no costs if all bidders are ex-ante homogeneous.

If bidders are ex-ante heterogeneous, imitation perfection is neither compatible with social surplus maximization nor with revenue maximization. Efficiency requires that bidders with the same valuation place the same bids. We will show that in imitation-perfect auctions the payment of a winning bidder depends only on her own bid. This, however, implies that if bidders with the same valuation have different beliefs about the bids they are competing against, it cannot be optimal for these bidders to place the same bid. Applying similar reasoning to virtual valuations indicates that imitation perfection is not compatible with revenue maximization in the case of ex-ante heterogeneous bidders. Thus, there is a trade-off between non-discrimination and optimality.

Common auction formats that are compatible with imitation perfection are first-price auctions and all-pay auctions with a reservation bid. A common auction format that is ruled out by imitation perfection is the second-price auction. It cannot be imitation-perfect since it has a perfect discrimination equilibrium where one bidder bids an arbitrarily high bid b and all other bidders bid zero. It is also easy to see that none of the bidders bidding zero can imitate the bidder bidding b since by bidding slightly above b, the imitating bidder would have to pay b and not zero.

Relation to the literature

Only a few papers deal with the question of how general procurement rules must be designed in order to achieve the goals of procurement organizations. Deb and Pai (2017) analyze the common desideratum of "non-discrimination". However, they show that even equal and anonymous treatment of all bidders does not prevent discrimination. Gretschko and Wambach (2016) analyze how far public scrutiny can help to prevent corruption and discrimination. They consider a setting in which the agent is privately informed about the preferences of the buyer regarding the specifications of the horizontally differentiated sellers. The agent colludes with one exogenously chosen seller. They show that in the optimal mechanism the agent should have no discretion with respect to the probability of the favorite seller winning, which in turn induces the agent to truthfully report the preference of the buyer whenever his favorite seller fails to win. Moreover, they demonstrate that intransparent negotiations have this feature of the optimal mechanism whenever the favorite bidder fails to win the project and thus may outperform transparent auctions. Even though we do not explicitly model an agent of the buyer, our model could easily be extended by the introduction of an agent who, in exchange for a bribe, would bend the rules of the mechanism in the most favorable way that is consistent with the procurement regulations. Contrary to Gretschko and Wambach (2016), we do not focus on the ability of the agent to manipulate the quality assessment of the buyer but rather on the ability of an agent to design procurement mechanisms. To the best of our knowledge, our article is the first to investigate the design of procurement regulations in the presence of corruption and manipulation of the rules of the mechanism.[4]

In the majority of work on corruption in auctions, the ability of the agent to manipulate is defined with respect to the particular mechanism. Either the agent is able to favor one of the sellers within the rules of a particular mechanism (typically, bid-rigging in first-price auctions) or the agent is able to manipulate the quality assessment of the sellers for a particular mechanism. Examples of the first strand of literature include Arozamena and Weinschelbaum (2009), Burguet and Perry (2007), Burguet and Perry (2009), Cai et al. (2013), Compte et al. (2005), Lengwiler and Wolfstetter (2010), and Menezes and Monteiro (2006). Examples of the second strand include Burguet and Che (2004), Koessler and Lambert-Mogiliansky (2013), and Laffont and Tirole (1991).

Finally, our article is related to the literature on mechanism design with fairness concerns. As pointed out by Bolton et al. (2005) and Saito (2013) (among others), market participants care about whether the rules governing a particular market are procedurally fair. Thus, imitation perfection can be seen not only as a device to prevent favoritism and corruption, but also as a possible way of ensuring that all equilibria of a particular mechanism yield fair (discrimination-free) outcomes. Previous approaches to mechanism design with fairness concerns in auctions and other settings include Bierbrauer et al. (2017), Bierbrauer and Netzer (2016), Budish (2011), Englmaier and Wambach (2010) and Rasch et al. (2012).

2 Model

Environment Let $\{1, \ldots, n\}$ denote a set of risk-neutral bidders that compete for one indivisible item. Bidder i's valuation v_i for the item is her private information and is drawn independently from the interval $[0, \overline{v}]$ according to a continuous (i.e. atomless) differentiable distribution function F_i with corresponding continuous density f_i which is strictly positive at zero. That is, there exists some interval $(0, v_i)$

[4] Previous work on mechanism design with corruption focused on the ability of the agent to manipulate the quality assessment and the principal's optimal reaction to this. In particular, the mechanism designed by the principal is tailored to the situation at hand and does not imply general procurement regulations. See Celentani and Ganuza (2002) and Burguet (2017) for details.

such that f_i is strictly positive on $(0, v_i)$.[5] The functions F_i are common knowledge among the bidders. Denote by $\boldsymbol{v_{-i}} \in [0, \overline{v}]^{n-1}$ the vector containing all the valuations of bidder i's competitors.[6]

Symmetric auctions We consider an auction mechanism in which all participants submit bids $b_i \in \mathbb{R}^+$ and the auction mechanism assigns the item based on these bids.[7] An *auction mechanism* is a double (x, p) of an allocation function x and a payment function p. For every number of bidders n, and for every vector of bids $\boldsymbol{b} = (b_1, \ldots, b_n) \in (\mathbb{R}^+)^n$, the allocation function

$$x^n : \boldsymbol{b} \to (x_1, \ldots, x_n) \quad \text{with } x_i \in [0,1],\ \sum x_i \leq 1$$

determines for each participant the probability of receiving the item. For every number of bidders n and for every vector of bids $\boldsymbol{b} = (b_1, \ldots, b_n) \in (\mathbb{R}^+)^n$, the payment function

$$p^n : \boldsymbol{b} \to (p_1, \ldots, p_n) \quad \text{with } p_i \in \mathbb{R}^+$$

determines each participant's payment.[8] We require that the payment function fulfills the following minimal consistency condition.

For all bidders i and for all bid vectors $(b_1, \ldots, b_i, \ldots b_n)$ it holds that

$$p_i^n (b_1, \ldots, b_i, \ldots b_n) = p_i^{n+1} (b_1, \ldots, b_i, \ldots b_n, 0).$$ [9]

In order to be able to properly account for ties throughout the paper, we introduce the term *winner with a tie*. For a given vector of bids $(b_i, \boldsymbol{b_{-i}})$ bidder i is a winner with a tie if it holds that $b_i = \max_{j \neq i} b_j$. That is, there exists a bidder $k \neq i$ such that $b_k = \max_{j \neq k} b_j = b_i$. If bidder i is not a winner with a tie, there are two

[5]We allow for the fact that the support of F_i is a strict subset of $[0, \overline{v}]$.

[6]For a vector $(v_1, \ldots, v_n)$ we denote by $\boldsymbol{v_{-i}}$ the vector $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$.

[7]The process of a procurement auction is mostly similar to the process of a sales auction, the only difference being that the lowest bid is awarded the contract. The bidders do not have valuations for the good but costs for fulfilling the contract. Due to the existence of the correspondence between selling auctions and procurement auctions, the formal framework will be set up for selling auctions and we will use the term auctions from now on. This has the advantage that most readers are more familiar with this notation.

[8]Since the *Directive 2004/18/EC of the European Parliament and of the Council of 31 March 2004 on the coordination of procedures for the award of public works contracts, public supply contracts and public service contracts* requires the auctioneers "to post in advance all decision criteria", we define the allocation and payment functions for all possible number of bidders. That is, the auctioneer has to commit to an auction mechanism *before* observing the number of bidders. In the following all results and definitions hold for all number of bidders.

[9]Whenever the number of bidders is not ambiguous, we will simplify notation and omit the superscript indicating the number of bidders.

different possibilities: either bidder i is a *unique winner*, i.e. $b_i > \max_{j \neq i} b_j$ or bidder i is a *losing bidder*, i.e. $b_i < \max_{j \neq i} b_j$. Whenever the difference between these two cases does not matter, we will use the term "not a winner with a tie".

A *pure strategy* of bidder i is a mapping

$$\beta_i : [0, \overline{v}] \to \mathbb{R}^+.$$

A *(mixed) strategy* of bidder i is a map from the set of valuations to the set of bid distributions on $\mathbb{R}^+$:

$$\beta_i : [0, \overline{v}] \to \mathcal{P}\left(\mathbb{R}^+\right).$$

That is, for all valuations v_i and all bids b, $\beta_i(v_i)(b)$ denotes the probability that bidder i places a bid lower or equal than b. Let $supp\left(\beta_i(v_i)\right)$ denote the support of the bid distribution $\beta_i(v_i)$ and $g_{v_i}^{\beta_i}$ the corresponding density.[10]

Our solution concept is Bayes-Nash equilibrium in (almost surely) continuous strategies.[11] We will define Bayes-Nash equilibria in pure and mixed strategies. A tuple $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ of pure strategies constitutes an *equilibrium* of a mechanism (x, p) if for all i and for all $v_i \in V$ the bid $\beta_i(v_i)$ maximizes over all bids b bidder i's expected utility

$$U_i^{\boldsymbol{\beta_{-i}}}(v_i, b) = \int_{[0,\overline{v}]^{n-1}} \left[v_i \cdot x_i\left(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})\right) - p_i\left(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})\right)\right] \boldsymbol{f_{-i}}(\boldsymbol{v_{-i}})\, d\boldsymbol{v_{-i}}.$$

A tuple $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ of (mixed) strategies constitutes an *equilibrium* of a mechanism if for all i, for all $v_i \in [0, \overline{v}]$ and for all $b_i \in supp\left(\beta_i(v_i)\right)$ the bid b_i maximizes over all bids b bidder i's expected utility

$$U_i^{\boldsymbol{\beta_{-i}}}(v_i, b) = \int_{[0,\overline{v}]^{n-1}} \int_{b_{-i} \in supp\left(\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})\right)} \left[v_i \cdot x_i\left(b, \boldsymbol{b_{-i}}\right) - p_i\left(b, \boldsymbol{b_{-i}}\right)\right] \prod_{j \neq i} g_{v_j}^{\beta_j}(b_j)\, \boldsymbol{f_{-i}}(\boldsymbol{v_{-i}})\, d\boldsymbol{v_{-i}}$$

where $supp\left(\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})\right) = \mathsf{X}_{j \neq i} supp\left(\beta_j(v_j)\right)$. The expected equilibrium utility of bidder i with valuation v_i, which is given by $U_i^{\boldsymbol{\beta_{-i}}}(v_i, b_i)$ for $b_i \in supp\left(\beta_i(v_i)\right)$, is denoted by $U_i^{\boldsymbol{\beta}}(v_i)$.[12] In the remainder of this paper we allow for mixed strategies if

[10] A pure strategy can be interpreted as distribution of bids which puts probability weight 1 on one bid. We abuse notation since in the case of a pure strategy, $\beta_i(v_i)$ denotes an element in $\mathbb{R}^+$ while in the case of a (mixed) strategy $\beta_i(v_i)$ denotes an element in $\mathcal{P}(\mathbb{R}^+)$. However, in the following it will be clear whether β_i is a pure or a mixed strategy.

[11] As usual, we say that an event happens almost surely (abbreviated as a.s.) if the set of possible exceptions may be non-empty, but it has probability zero.

[12] In the following we will use the notation $U^{\boldsymbol{\beta}}(v_i)$ or $U^{\boldsymbol{\beta}}(v_i, b_i)$ in order to denote bidder i's

we use the term strategy or equilibrium. In particular, all results hold for mixed strategies unless specified otherwise.

Current public procurement regulation aimed at preventing discrimination requires equal treatment of bidders. The restrictiveness of this requirement is analyzed by Deb and Pai (2017), who provide the following definition.

Definition 1 (Symmetric auction)**.** *A* symmetric auction *with reservation bid r is an auction mechanism which fulfills the following two conditions:*

(i) The highest bidder wins. That is, the allocation is given by

$$x_i\left(b_i, \boldsymbol{b_{-i}}\right) = \begin{cases} \frac{1}{\#\{j\in\{1,\dots,n\}:b_j=b_i\}} & \text{if } b_i \geq \max\limits_{j\neq i}\{b_j, r\} \\ 0 & \text{otherwise,} \end{cases}$$

where r is a reservation bid.

(ii) The payment does not depend on the identity of the bidder and every bidder is treated equally. Formally, let π_n be a permutation of the elements $1,\dots,n$. In a symmetric auction, it holds true for all $\boldsymbol{b} = (b_1,\dots,b_n)$ that

$$p_i\left(b_{\pi_n(1)},\dots,b_{\pi_n(i-1)}, b_{\pi_n(i)}, b_{\pi_n(i+1)},\dots,b_{\pi_n(n)}\right) = p_{\pi_n(i)}\left(b_i, \boldsymbol{b_{-i}}\right).$$

In a symmetric auction, the highest bidder wins and the payment function is anonymous. Hence, a bidder's payment depends only on the bids and not on her identity. Moreover, a permutation of all bids would lead to the same permutation of payments and allocations.

In addition to the requirements of a symmetric auction, we assume that an auction mechanism fulfills some monotonicity conditions. First, we require that the payment of a bidder is non-decreasing in her own bid. Second, we require that conditional on winning or losing the payment of a bidder is non-decreasing in the other bidders' bids. Third, we require that the payment of a unique winner is strictly increasing in at least one component of the bid vector.

Assumption 1. *We assume that the payment function of every auction mechanism is monotone. We call a payment function p* monotone *if for every bidder i and for each vector of bids $\left(b_i, \boldsymbol{b_{-i}}\right)$ the following holds:*[13]

equilibrium utility. We will use the notation $U^{\boldsymbol{\beta_{-i}}}\left(v_i, b\right)$ in order to indicate that bidder i deviates from equilibrium to bid b.

[13]From now on the first bid in a vector $\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right)$ will always denote bidder i's bid while the second bid will denote bidder j's bid.

(i) The payment of bidder i is non-decreasing in her bid, i.e. for all b_i' with $b_i \leq b_i'$ it holds that

$$p_i\left(b_i, \boldsymbol{b_{-i}}\right) \leq p_i\left(b_i', \boldsymbol{b_{-i}}\right).$$

(ii) Conditional on losing or winning, her payment is non-decreasing in the other bidders' bids. That is, if $b_i < \max_{j \neq i} b_j$, then for every bid b_j' with $b_j \leq b_j'$ it holds that

$$p_i\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) \leq p_i\left(b_i, b_j', \boldsymbol{b_{-(i,j)}}\right)$$

and if $b_i \geq \max_{j \neq i} b_j$, then for every bid b_j' with $b_j \leq b_j' < b_i$ it holds that

$$p_i\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) \leq p_i\left(b_i, b_j', \boldsymbol{b_{-(i,j)}}\right).$$

(iii) If i is a unique winner, her payment is strictly increasing either in the bid of another bidder or her own bid. That is, if $b_i > \max_{j \neq i} \{b_j, r\}$, then either there exists a bidder $j \neq i$ such that for all b_j' with $b_i > b_j' > \max\{b_j, r\}$ it holds that

$$p_i\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) < p_i\left(b_i, b_j', \boldsymbol{b_{-(i,j)}}\right)$$

or it holds for all $b_i' > b_i$ that

$$p_i\left(b_i, \boldsymbol{b_{-i}}\right) < p_i\left(b_i', \boldsymbol{b_{-i}}\right).$$

These conditions ensure equilibrium existence.[14] If the payment of a bidder was strictly decreasing in her own bid, she would place arbitrarily high bids. Similar reasoning applies to the second and third condition. Consider a symmetric auction with two bidders where the payment rule is given by:[15]

$$p_i\left(b_i, b_j\right) = \max\{b_i - b_j, 0\}.$$

In this auction, for any potential equilibrium strategy of her competitor, it is not optimal for a bidder to bid below the mean of the bid distribution of her competitor. Thus, an equilibrium does not exist. Finally, consider an auction in which a bidder pays a constant independent of her bid, which contradicts the third condition. Again this bidder has an incentive to place arbitrarily high bids and an equilibrium does not exist. Although requiring a monotone payment function is a technical assumption, it is not restrictive in the sense that it does not rule out any of the auction formats

[14] We use these conditions in the proof of Proposition 1.

[15] In the following we will use the terms auction and auction mechanism interchangeably.

that are popular in practice, like the first-price auction or the second-price auction.[16]

Moreover, we assume that every bidder has the possibility to achieve at least an expected utility of zero by bidding below the reservation bid or bidding zero.

Assumption 2. *We assume that for every bidder $i \in \{1, \dots, n\}$ and every bid vector $(b_i, \boldsymbol{b_{-i}})$ with $b_i < r$ or $b_i = 0$ it holds that*

$$p_i\left(b_i, \boldsymbol{b_{-i}}\right) = 0.$$

2.1 Discrimination-free auctions

The main insight of Deb and Pai (2017) is that even though the *rules* of a symmetric auction treat all bidders equally, mechanisms with discriminating *outcomes* can still be implemented. In particular, they demonstrate that almost every reasonable mechanism has an implementation as a symmetric auction. Thus, requiring a symmetric auction, i.e. equal treatment, is not an effective anti-discrimination measure. To get an idea of the discrimination that is possible in symmetric auctions, consider the following example.

Example 1. *An agency is in charge of running an auction among n bidders with valuations in $[0, 1]$. One of the bidders, say bidder 1, has close ties to the agency. Thus, the agency does not aim at maximizing revenue but instead seeks to maximize the utility of bidder 1. In this case, the agency can implement the following symmetric auction. If only one bidder bids a strictly positive amount, all payments are zero. If more than one bidder bids a strictly positive amount, all bidders who bid a strictly positive amount pay their own bid plus (a penalty of) one. This auction has a Bayes-Nash equilibrium in undominated strategies in which bidder 1, irrespective of her valuation, bids some strictly positive amount $b_1 > 0$. All other bidders bid zero, irrespective of their valuations. In this case, bidder 1 receives the object with probability one and pays nothing.*

We call an equilibrium a *perfect discrimination equilibrium* if one bidder wins the auction with probability one independent of her valuation and pays nothing.

Definition 2 (Perfect discrimination equilibrium). *An equilibrium $(\beta_1, \dots, \beta_n)$ of an auction mechanism (x, p) is called a* perfect discrimination equilibrium *if there exists a bidder i such that for any vector of valuations $(v_1, \dots v_n)$ and every vector*

[16] Note that Deb and Pai (2017) and Example 1 show that symmetric auctions with a monotone payment function do not prevent perfect discrimination as defined in Definition 2.

of bids $(b_1, \ldots, b_n)$ *such that for all* $j \in \{1, \ldots, n\}$, $b_j \in supp\,(\beta_j\,(v_j))$, *it holds that*

$$x_i\,(b_1, \ldots, b_n) = 1$$

and

$$p_i\,(b_1, \ldots, b_n) = 0.$$

Given that symmetric auctions do not prevent perfect discrimination, the aim of this article is to provide a simple extension to the existing rules that restricts discrimination in a meaningful way. A minimum requirement for the extension is that it rules out perfect discrimination equilibria.[17] In addition, we demand that in a non-discriminatory equilibrium ex-ante homogeneous bidders with the same valuation expect the same utility. We denote a symmetric auction as discrimination-free if all of its equilibria are non-discriminatory.

Definition 3 (Discrimination-free auction)**.** *An equilibrium* $(\beta_1, \ldots, \beta_n)$ *of a symmetric auction is called* non-discriminatory *if for all bidders* i, j *with* $F_i = F_j$ *it holds for all* $v \in [0, \overline{v}]$ *that*

$$U_i^{\boldsymbol{\beta}}\,(v) = U_j^{\boldsymbol{\beta}}\,(v)\,.$$

A symmetric auction is called discrimination-free *if all equilibria of this auction are non-discriminatory.*

3 Imitation perfection

In what follows we introduce a simple extension of the existing symmetric rules which require equal treatment. We call this extension *imitation perfection* and show that all imitation-perfect auctions are discrimination-free.

Imitation perfection requires that for any realization of bids each bidder could have achieved the same allocation and payment as any other bidder by bidding slightly higher than a bidder with a higher bid or bidding slightly lower than a bidder with a lower bid.

Definition 4 (Imitation perfection)**.** *A symmetric auction* (x, p) *is imitation-perfect if for all bidders* i, *all bids* b_i, *and all* $\epsilon > 0$

[17] Note that Deb and Pai (2017) propose adjustments of symmetric auctions that may restrict the class of implementable mechanisms. In particular, they consider auction mechanisms with inactive losers, continuous payment rules, monotonic payment rules, and ex-post individual rationality. However, it is easy to see that none of these adjustments prevents the existence of perfect discrimination equilibria. This is due to the fact that any of these adjustments allow for the implementation of the second-price auction. The second-price auction has perfectly-discriminating equilibria in which one of the bidders bids $b_i \geq \overline{v}$ and all other bidders bid zero.

(i) For all vectors of bids $\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right)$ *such that bidder* i *is not a winner with a tie and for all* $j \in \{1, \ldots, n\}$ *with* $b_i > b_j$ *there exists a bid* $\bar{b} > b_i$ *such that*

$$\left|p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right) - p_j\left(b_i, \bar{b}, \boldsymbol{b}_{-(i,j)}\right)\right| < \epsilon.$$

That is, all bidders can imitate the allocation and payment of a higher bidder who is not a winner with a tie by bidding slightly higher.

(ii) For all vectors of bids $\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right)$ *and for all* $j \in \{1, \ldots, n\}$ *with* $0 < b_i < b_j$ *there exists a bid* $\underline{b} < b_i$ *such that*

$$\left|p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right) - p_j\left(b_i, \underline{b}, \boldsymbol{b}_{-(i,j)}\right)\right| < \epsilon.$$

That is, all bidders can imitate the allocation and payment of a lower bidder by bidding slightly lower.[18]

In an imitation-perfect auction, every bidder could have imitated the (ex-post) allocation and payment of each bidder who is not a winner with a tie. The definition does not cover the imitation of winners with a tie. By bidding slightly above a winner with a tie a bidder would become the unique winner and therefore cannot imitate the allocation and payment of a bidder with a tie. However, payments in the case of ties can be extended in a natural way for imitation-perfect auctions. This will become clear after Proposition 1. Thus, we postpone the discussion to after Proposition 1.

A strength of our proposed rule is that the verification of whether an auction is imitation-perfect can be done without knowledge about the environment, such as the beliefs of the bidders or the selection of a particular equilibrium. A simple verification of the payment rule is sufficient.

In order to gain some intuition for the definition of imitation perfection, we consider the following examples.

Example 2. *Consider the mechanism proposed in Example 1. Recall that bidder 1 is the favorite bidder and if more than one bidder places a strictly positive bid, all bidders who placed a strictly positive bid pay their bid plus a penalty of one. This mechanism is not imitation-perfect. For* $b_1 > 0$ *it holds that*

$$p_1\left(b_1, 0, \ldots, 0\right) = 0.$$

[18] It is sufficient to consider only the payment function, because in a symmetric auction the allocation rule is fixed.

Bidder 1 wins the auction and pays nothing. For every $b_j > b_1$ it holds that

$$p_j\left(b_1, 0, \ldots, 0, b_j, 0, \ldots 0\right) - p_1\left(b_1, 0, \ldots, 0\right) > 1,$$

which implies that bidder 1 cannot be imitated.

Example 3. *Consider a second-price auction with two bidders. If bidder 1 is bidding $b_1 = 1$ and bidder 2 is bidding $b_2 = 0$, bidder 1 will receive the object and pay a price of zero. Bidder 2 cannot imitate this outcome. By bidding above* 1*, bidder 2 would win the object but her payment would be* 1*.*

Example 4. *Consider a first-price auction with two bidders. If bidder 1 is bidding $b_1 = 1$ and bidder 2 is bidding $b_2 = 2$, bidder 2 will receive the object and pay a price of* 2 *while bidder 1 pays zero. By placing a bid marginally higher than* 2 *bidder 1 can imitate bidder 2's allocation and payment. Bidder 2 can imitate bidder 1's allocation and payment by placing a bid marginally lower than 1.*

In the following, we will present the properties of an imitation-perfect auction and of its outcomes. We start with a property of imitation perfection which we need for subsequent proofs:

Proposition 1. *In an imitation-perfect auction the payment of a bidder depends only on her own bid conditional on winning or losing. That is, for all bidders i the following holds true:*

(i) For all bid vectors $\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right)$ such that $b_i > \max_{j \neq i} b_j$ it holds for all bids b'_j with $b_i > b'_j$ that

$$p_i\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) = p_i\left(b_i, b'_j, \boldsymbol{b_{-(i,j)}}\right).$$

(ii) For all bid vectors $\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right)$ such that $b_i < \max_{j} b_j$ it holds for all bids b'_j with $b_i < \max\{b'_j, \boldsymbol{b_{-(i,j)}}\}$ that

$$p_i\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) = p_i\left(b_i, b'_j, \boldsymbol{b_{-(i,j)}}\right).$$

To gain some intuition, consider an auction with two bidders, i and j, where the payment of a bidder depends also on the other bidder's bid. Then w.l.o.g. there exists some vector (b_i, b_j) with $b_i > b_j$ such that bidder i's payment depends also on b_j. Due to Assumption 1, bidder i's payment has to be strictly increasing in bidder j's bid at (b_i, b_j). Moreover, it follows from Assumption 1 that bidder i's payment is non-decreasing in her bid. Thus, we can conclude that for any bid $b'_j > b_i$ it holds that

$$p_j(b_i, b'_j) = p_i(b'_j, b_i) \geq p_i(b_i, b_i) > p_i(b_i, b_j).$$

where the equality follows from symmetry. Therefore, bidder j cannot imitate bidder i's allocation and payment. The formal proof is relegated to Appendix B.

We are now in the position to extend the definition of imitation perfection to winners with a tie. Proposition 1 implies that if there is a unique highest bidder i with bid b_i, then her payment depends only on her bid. That is, there exists a function p^{win} such that bidder i's payment is equal to $p^{win}(b_i)$. If bidder i is a losing bidder, then there exists a function p^{lose} such that her payment is equal to $p^{lose}(b_i)$.

Since the allocation rule breaks ties randomly, an analogous property should hold for the payment rule. We can use the functions p^{win} and p^{lose} in order to define such a payment rule. If a bidder is the highest bidder and ties with $k-1$ other bidders, she wins with probability $\frac{1}{k}$ and the payment of such a bidder depends on the outcome of a fair lottery. Thus, for a winner with a tie, we can formalize only the expected payment as it is done in the following definition.

Definition 5. *Define for every bidder i and for every vector of bids*

$$(b_i, \boldsymbol{b_{-i}})$$

such that $b_i = \max_{j \neq i} b_j$ and $k = \#\{j \in \{1, \ldots, n\} \mid b_j = b_i\}$, i.e., i is a winner with a tie bidding b_i, the expected payment of bidder i by

$$p_i(b_i, \boldsymbol{b_{-i}}) = \frac{1}{k} p^{win}(b_i) + \left(1 - \frac{1}{k}\right) p^{lose}(b_i)$$

where for every $b_i > 0$, $p^{win}(b_i)$ is defined by

$$p^{win}(b_i) = p_i(b_i, 0, \ldots, 0)$$

and for every b_i, $p^{lose}(b_i)$ is defined by

$$p^{lose}(b_i) = p_i(b_i, b_j, 0 \ldots, 0)$$

for any b_j with $b_j > b_i$.

It follows from Proposition 1 that the definition of the payment for a bidder who is a winner with a tie and the definitions of p^{win} and p^{lose} are well-defined.

We have shown that conditional on winning or losing the payment of a bidder depends only on her own bid. In addition, we state the following properties of imitation perfection that will be useful in the sections to follow. They also serve as necessary and sufficient conditions for imitation perfection. First, we need the following definition.

Definition 6 (Bid-determines-payment auction)**.** *A symmetric auction is a* bid-determines- payment auction *if the payment of every bidder depends only on whether or not she wins and on her bid. Formally, an auction satisfies the bid-determines-payment rule if there exist functions $p^{win}, p^{lose} : \mathbb{R}^{+} \to \mathbb{R}^{+}$ such that for bidder i her payment can be written as*[19]

$$p_i\left(b_i, \boldsymbol{b_{-i}}\right) = x_i\left(b_i, \boldsymbol{b_{-i}}\right) p^{win}\left(b_i\right) + \left[1 - x_i\left(b_i, \boldsymbol{b_{-i}}\right)\right]\ p^{lose}\left(b_i\right).$$

Proposition 2. *An auction is imitation-perfect if and only if the following holds true:*

(i) (Bid-determines-payment) The auction is a bid-determines-payment auction.

(ii) (Continuity) If bidder i is not a winner with a tie, her payment is right-continuous in her bid. That is, for every bidder i, for every bid vector $(b_i, \boldsymbol{b_{-i}})$ such that $b_i \neq \max\limits_{j \neq i} b_j$ and for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all b_i' with $b_i < b_i' < b_i + \delta$ it holds that

$$\left|p_i\left(b_i, \boldsymbol{b_{-i}}\right) - p_i\left(b_i', \boldsymbol{b_{-i}}\right)\right| < \epsilon.$$

Moreover, if a bidder does not place the highest bid, then her payment is left-continuous in her bid. That is, for every bidder i, for every bid vector such that $0 < b_i < \max\limits_{j \neq i} b_j$, and for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all b_i' with $b_i - \delta < b_i' < b_i$ it holds that

$$\left|p_i\left(b_i, \boldsymbol{b_{-i}}\right) - p_i\left(b_i', \boldsymbol{b_{-i}}\right)\right| < \epsilon.$$

The proof is relegated to Appendix C.

We continue with the results specifying the desirable properties imitation-perfect auctions have. We illustrated in Examples 2 and 3 how imitation perfection can prevent perfect discrimination equilibria. The following proposition states that imita-

[19]If bidder i is a winner with tie, her payment depends on the outcome of a lottery. Thus, as in Definition 5, in this case $p_i(b_i, \boldsymbol{b_{-i}})$ denotes bidder i's expected payment.

tion perfection prevents, in general, the existence of perfect discrimination equilibria in symmetric auctions.

Proposition 3. *An imitation-perfect symmetric auction does not have a perfect discrimination equilibrium.*

We sketch the proof for the case of pure strategies. The formal proof is relegated to Appendix E. Assume that there exists a perfect discrimination equilibrium in which bidder i wins the auction with probability 1 and pays zero. We show in Lemma 6 in Appendix D that every equilibrium bidding strategy is non-decreasing. Thus, the highest bid placed by bidder i is given by $\beta_i(\overline{v})$. Let j be a bidder with valuation $v_j > 0$. Fix an ϵ with $0 < \epsilon < v_j$. Due to imitation perfection, there exists a bid $\overline{b} > \beta_i(\overline{v})$ such that

$$|p_i\left(\beta_i(\overline{v}), \beta_j(v_j), \boldsymbol{\beta}_{-(i,j)}\left(\boldsymbol{v}_{-(i,j)}\right)\right) - p_j\left(\beta_i(\overline{v}), \overline{b}, \boldsymbol{\beta}_{-(i,j)}\left(\boldsymbol{v}_{-(i,j)}\right)\right)| < \epsilon$$

for every vector of valuations $\boldsymbol{v}_{-i}$. Since in equilibrium bidder i always pays zero, this implies that by bidding $\overline{b}$, bidder j would win the auction with probability 1 and pay an amount which is strictly lower than her valuation. Therefore, she has an incentive to deviate. Hence, a perfect discrimination equilibrium cannot exist in an imitation-perfect auction. Each bidder $j \neq i$ would have an incentive to deviate whenever she has a strictly positive valuation for the good.

We have established that imitation perfection fulfills the minimum requirement of preventing perfect discrimination. The following theorem states that imitation-perfect auctions are discrimination-free.

Theorem 1. *A symmetric and imitation-perfect auction is discrimination-free.*

Intuitively, Theorem 1 builds on the fundamental idea of imitation perfection that bidders can imitate the allocation and payment of the other bidders that have outbid them. Formally, we prove that homogeneous bidders follow identical strategies. This ensures that ex-ante homogeneous bidders with the same valuation have the same expected utility. In order to do so, we adapt a technique of Chawla and Hartline (2013). They show that for a given auction, if some interval $[\underline{z}, \overline{z}]$ satisfies *utility crossing*, that is, if for some bidders i and j it holds that $U_i^\beta(\overline{z}) \geq U_j^\beta(\overline{z})$ and $U_j^\beta(\underline{z}) \geq U_i^\beta(\underline{z})$ and $\beta_j(v) \geq \beta_i(v)$ for all $v \in [\underline{z}, \overline{z}]$, then the strategies of bidder i and bidder j must be identical on this interval. If there is an interval of valuations of positive measure such that the equilibrium prescribes that one bidder strictly outbids the other, we apply imitation perfection at the endpoints of this interval in order to demonstrate that this interval satisfies utility crossing. Due to imitation perfection, at the upper endpoint $\overline{z}$ a deviating bid for bidder i exists, such that bidder i can achieve the same expected utility as bidder j by bidding slightly higher than bidder j. Bidder i's utility in equilibrium cannot, therefore, be lower than bidder j's utility as bidder i would otherwise have an incentive to deviate. Similarly, bidder j can achieve the same expected utility as bidder i at the lower endpoint $\underline{z}$. The formal proof is relegated to Appendix F.

4 Imitation perfection with homogeneous bidders

In this section we present further results for the case that bidders are ex-ante homogeneous. Bidders are ex-ante homogeneous if all bidders draw their valuations from the same distribution. That is, it holds for all $i, j \in \{1, \dots n\}$ that $F_i = F_j$. We provide conditions for the existence and uniqueness of equilibria in imitation-perfect auctions. Furthermore, we show that imitation perfection is compatible with revenue and social surplus maximization.

Proposition 4. *Assume that the bid spaces of all bidders are compact intervals, i.e. every bidder is allowed to submit bids in some interval $[0, \overline{b}]$. Then the following holds true in an imitation-perfect auction:*

(i) There exists an equilibrium.

(ii) If bidders are ex-ante homogeneous, then there exists a unique non-decreasing equilibrium in pure strategies.

The proof is relegated to Appendix G.

If bidders are ex-ante homogeneous, a revenue-optimal auction can be implemented as a first-price auction, which is an imitation-perfect auction, with an appropriate reservation bid (see Krishna 2009). Similarly, an efficient auction can be implemented as a first-price auction without a reservation bid. Thus, it follows from Proposition 2 that there exist imitation-perfect auctions which are revenue and social surplus optimal, as stated in the following corollary.

Corollary 1. *If bidders are ex-ante homogeneous, the following holds true:*

(i) Let $v - \frac{1-F_i(v)}{f_i(v)}$ be the virtual valuation of bidder i with valuation v. If for every bidder i the virtual valuation is increasing in v, then there exists a symmetric and discrimination-free auction that is revenue-optimal among all incentive compatible mechanisms.

(ii) There exists a symmetric and discrimination-free auction that is social surplus maximizing among all incentive compatible mechanisms.

Thus, the implementation of a discrimination-free auction is not in conflict with the aims of revenue or social surplus maximization if all bidders are ex-ante homogeneous.

5 Imitation perfection with heterogeneous bidders

In this section, we analyze the extent to which imitation perfection limits discrimination between bidders that are ex-ante heterogeneous and examine whether imitation perfection is compatible with revenue and social surplus maximization.

If bidders are ex-ante heterogeneous, it is not reasonable to require that bidders with the same valuation earn the same expected utility in equilibrium. The heterogeneity implies that different bidders face different degrees of competition even if they have the same valuation.

Nevertheless, we will show that even in settings with ex-ante heterogeneous bidders imitation perfection effectively limits the possible extent of discrimination. In order to provide a precise and tractable measure of heterogeneity, we follow Fibich et al. (2004). They show that by defining

$$H = \frac{1}{n} \sum_{i=1}^{n} F_i$$

$$\Delta = \max_i \max_v |F_i(v) - H(v)|$$
$$H_i(v) = (F_i(v) - H(v)) / \Delta,$$

for any set of distribution functions $F_1, \ldots, F_n$ defined on some interval $[0, \overline{v}]$ and for every $i \in \{1, \ldots, n\}$ the distribution function F_i can be decomposed in the following way

$$F_i(v) = H(v) + \Delta H_i(v) \tag{1}$$

where $H(0) = 0$, $H(\overline{v}) = 1$, $H_i(0) = H_i(\overline{v}) = 0$, $|H_i| \leq 1$ on $[0, \overline{v}]$ and $\Delta \geq 0$. Among all H, $\{H_i\}_i$ and Δ which allow such a decomposition, Δ as defined above is minimal. The parameter Δ formalizes the degree of heterogeneity between all bidders.

In order to measure the heterogeneity between two bidders, we define

$$H_{i,j} = \frac{1}{2}(F_i + F_j)$$

$$\Delta_{i,j} = \max_{k\in\{i,j\}} \max_{v} |F_k(v) - H_{i,j}(v)|$$

$$H_k(v) = (F_k(v) - H_{i,j}(v))/\Delta_{i,j} \quad \text{for } k \in \{i,j\}$$

for every pair of bidders i and j it holds that

$$F_i(v) = H_{i,j}(v) + \Delta_{i,j} H_i(v), \quad F_j(v) = H_{i,j}(v) + \Delta_{i,j} H_j(v) \tag{2}$$

where $H(0) = 0$, $H(\overline{v}) = 1$, $H_k(0) = H_k(\overline{v}) = 0$, and $|H_k| \leq 1$ on $[0, \overline{v}]$ for $k \in \{i,j\}$. Analogously, among all $H_{i,j}$, H_i, H_j and $\Delta_{i,j}$ which allow such a decomposition, $\Delta_{i,j}$ as defined above is minimal. The parameter $\Delta_{i,j}$ formalizes the degree of heterogeneity between two specific bidders i and j.

The following proposition provides an upper bound on the difference in expected utilities of two bidders with the same valuation in an imitation-perfect auction.

Proposition 5. *In an imitation-perfect auction it holds for every equilibrium $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, for every pair of bidders i, j and for every valuation v that*

$$|U_i^{\boldsymbol{\beta}}(v) - U_j^{\boldsymbol{\beta}}(v)| \leq \Delta_{i,j} + \Delta_{i,j}(\overline{v} - v).$$

That is, the difference in the expected utilities of two bidders with the same valuation in the same imitation-perfect auction is given by at most $\Delta_{i,j} + \Delta_{i,j}(\overline{v} - v)$ independent of the degree of heterogeneity of the other $n-2$ bidders.

The proof is relegated to Appendix H.

Theorem 1 states that in an imitation-perfect auction two ex-ante homogeneous bidders with the same valuation expect the same utility even if the heterogeneity among the other bidders is arbitrarily strong. Proposition 5 implies that this finding is robust towards small perturbations of homogeneity.

So far, we have analyzed the auctioneer's possibility to discriminate between two heterogeneous bidders in the same imitation-perfect auction. Now we turn our attention to the auctioneer's possibility to increase a favorite bidder's expected utility by choosing among different imitation-perfect auctions. If bidders are ex-ante heterogeneous, the revenue equivalence theorem does not hold. Hence, the expected utility of a bidder with a given valuation can differ between different imitation-perfect auctions. Proposition 6 demonstrates that the possible extent of discrimination is limited by the degree of heterogeneity. If the heterogeneity between bidders is small, so is the extent to which the auctioneer can discriminate between them by choosing different auction formats.

Since we make use of the Revenue Equivalence Principle and Lemma 1 in Fibich et al. (2004) in the proof of the following Proposition, it holds true for all pure strictly increasing and differentiable equilibria in imitation-perfect auctions with reservation bid zero.

Proposition 6. *Let A and B be imitation-perfect auctions with reservation bid zero and $\boldsymbol{\beta}$ be an equilibrium of A and $\boldsymbol{\beta'}$ be an equilibrium of B. If the equilibria $\boldsymbol{\beta}$ and $\boldsymbol{\beta'}$ are in pure strictly increasing and differentiable strategies, then for every bidder i and every valuation v it holds that*

$$\left| U_i\left(v\right)^{\boldsymbol{\beta}} - U_i\left(v\right)^{\boldsymbol{\beta'}} \right| \leq 2\left(\Delta\overline{v} + \Delta\left(\overline{v} - v\right)\right) + P(\Delta^2)$$

where $P(\Delta^2) = \sum_{i=2}^{\infty} c_i \Delta^i$ with appropriately chosen $c_i \in \mathbb{R}$ for $i \geq 2$.

That is, for every bidder i with a given valuation v the difference in the expected utilities in any equilibrium of A and B is given by at most

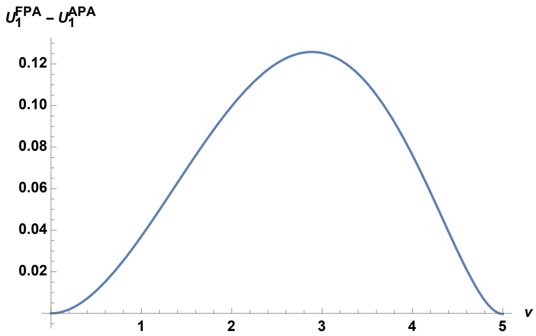
$$2\left(\Delta\overline{v} + \Delta\left(\overline{v} - v\right)\right) + P\left(\Delta^2\right).$$

The proof is relegated to Appendix I.

If the ex-ante heterogeneity among bidders is sufficiently pronounced, an auctioneer who knows the distributions of the bidders is able to substantially influence her favorite bidder's expected utility by choosing among imitation-perfect auctions. We illustrate the auctioneer's possibility to influence her favorite bidder's expected utility with the following example.

Example 5. *Consider an auctioneer who has to conduct an imitation-perfect auction with two bidders. The valuation of bidder 1 is uniformly distributed on the interval $[0, 5]$ and the valuation of bidder 2 is uniformly distributed on the interval $[0, 10]$. Assume that the auctioneer can either conduct a first-price auction or an all-pay auction. Following Krishna (2009) and Amann and Leininger (1996), we can compute the unique equilibrium bidding functions for both bidders in both auctions. If the auctioneer wants to favor bidder 1, he will conduct a first-price auction. Independent from her valuation, bidder 1 expects a weakly higher utility in a first-price auction than in an all-pay auction. Figure 1 illustrates the difference in the* *expected utility of bidder 1 in the first-price and the all-pay auction for all possible valuations.*

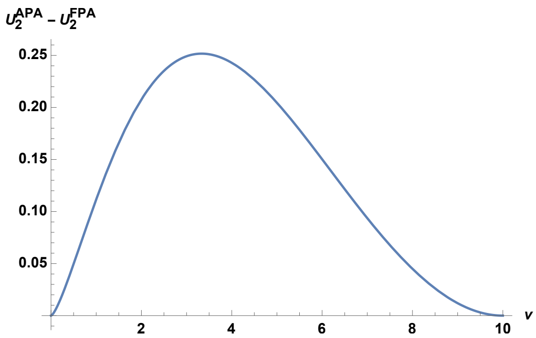
Figure 1: $U_1^{FPA}(v) - U_1^{APA}(v)$



Notes. The difference in the expected utilities $U_1^{FPA}(v) - U_1^{APA}(v)$ obtains its maximum value of 0.126 at $v = 2.9$. In this case, bidder 1's utility in a first-price auction is 39 percent larger than in an all-pay auction.

Vice versa, the auctioneer can favor bidder 2 by conducting an all-pay auction. Figure 2 illustrates that independent of her valuation, bidder 2 expects a (weakly) larger utility in an all-pay auction.

Figure 2: $U_2^{APA}(v) - U_2^{FPA}(v)$



Notes. The difference in the expected utilities $U_2^{APA}(v) - U_2^{FPA}(v)$ obtains its maximum value of 0.252 at $v = 3.4$. In this case, bidder 2's utility in an all-pay auction is 24 percent larger than in a first-price auction.

Finally, we will show that imitation perfection is not compatible with efficiency and revenue maximization if bidders are ex-ante heterogeneous.

Proposition 7. *Assume there exists at least one pair of bidders i, j such that $\int_0^{\overline{v}} F_i(z)\,dz \neq \int_0^{\overline{v}} F_j(z)\,dz$, then there does not exist an efficient equilibrium in any imitation-perfect auction.*

The proof is relegated to Appendix J.

In symmetric auctions efficiency requires that bidders with the same valuation place the same bid. As a consequence, ex-ante heterogeneous bidders face different bid distributions. The winner's payment in an imitation-perfect auction cannot depend on other bidders' bids. This implies that following the same bidding strategy cannot be optimal for ex-ante heterogeneous bidders. Applying similar reasoning to virtual valuations indicates that imitation perfection is not compatible with revenue maximization in the case of ex-ante heterogeneous bidders.

Proposition 8. *Denote by* $V_i(v) = v - \frac{1-F_i(v)}{f_i(v)}$ *the virtual valuation of bidder* i *with valuation* v*. Assume there exists at least one pair of bidders* i, j *such that* $\int_0^{\overline{v}} F_i\left(V_i^{-1}\left(V_j\left(z\right)\right)\right) dz \neq \int_0^{\overline{v}} F_j\left(V_j^{-1}\left(V_i\left(z\right)\right)\right) dz$*. In this case, all equilibria of an imitation-perfect auction are not revenue-maximizing. That is, the object is not always allocated to the bidder with the highest virtual valuation.*

The proof is relegated to Appendix J.

6 Conclusion

This article demonstrates that the existing rules imposed to prevent discrimination in procurement, which require equal treatment of bidders, are not sufficient to prevent perfect discrimination. We introduce a simple extension to the existing rules called imitation perfection. Imitation perfection requires that for any realization of bids and the resulting allocation and payments, every bidder had the opportunity to imitate the allocation and payment of every other bidder. Imitation perfection can be easily verified without specific knowledge of details of the environment and guarantees discrimination-free outcomes. If all bidders are ex-ante homogeneous, both an imitation-perfect revenue-optimal auction and an imitation-perfect social surplus optimal auction exist. If bidders are heterogeneous, imitation perfection still ensures that the difference in the expected utilities of two bidders with the same valuation is limited by the heterogeneity of their valuation distributions. Moreover, the difference in the expected utilities of a bidder with a given valuation in two different imitation-perfect auctions is limited by the heterogeneity of the valuation distributions of all bidders.

Appendices

A Definition of expected allocation and payment

If $(\beta_1, \ldots, \beta_n)$ is an equilibrium of an auction mechanism (x, p), then the expected (interim) allocation and payment of a bidder i who bids b_i are defined by

$$X_i^{\boldsymbol{\beta_{-i}}}(b_i) = \int_{[0,\overline{v}]^{n-1}} x_i\left(b_i, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})\right) \boldsymbol{f_{-i}}(\boldsymbol{v_{-i}})\, d(\boldsymbol{v_{-i}}) \tag{3}$$

$$P_i^{\boldsymbol{\beta_{-i}}}(b_i) = \int_{[0,\overline{v}]^{n-1}} p_i\left(b_i, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})\right) \boldsymbol{f_{-i}}(\boldsymbol{v_{-i}})\, d(\boldsymbol{v_{-i}}). \tag{4}$$

Similarly as in the notation for expected utility, we will use the notation $X_i^{\boldsymbol{\beta}}(v_i)$ or $X_i^{\boldsymbol{\beta}}(\beta_i(v_i))$ in order to denote the equilibrium allocation of bidder i with valuation v_i. We will use the notation $X_i^{\boldsymbol{\beta_{-i}}}(b)$ in order to indicate that bidder i deviated from equilibrium to bid b. The analogous notation holds for the expected payment.

B Proof of Proposition 1

Proof. The proof requires four lemmas. The statements in these lemmas can be (informally) summarized as follows:

- Lemma 1: The payment of a bidder does not depend on lower bids.
- Lemma 2: The payment of a bidder does not depend on higher bids.
- Lemma 3: The payment of a bidder who is not a winner with a tie is right-continuous in her bid.
- Lemma 4: The payment of a bidder who is not the highest bidder is left-continuous in her bid.

We will formally state and prove the four lemmas and then continue with the proof of Proposition 1.

Lemma 1. *In an imitation-perfect auction, for every bidder i and for every pair of vectors*

$$\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) \text{ and } \left(b_i, b_j', \boldsymbol{b_{-(i,j)}}\right)$$

where $b_i > b_j$, $b_i > b_j'$ and bidder i is not a winner with a tie, it holds that

$$p_i\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) = p_i\left(b_i, b_j', \boldsymbol{b_{-(i,j)}}\right).$$

That is, the payment of a bidder does not depend on the bids of competitors who placed lower bids.

Proof. Let $\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right)$ and $\left(b_i, b'_j, \boldsymbol{b}_{-(i,j)}\right)$ be bid vectors where bidder i is not a winner with a tie and it holds that $b_i > b_j$, $b_i > b'_j$. Imitation perfection implies that for every $\epsilon > 0$ there exist bids $\overline{b}$ and $\overline{b}'$ with $\overline{b} > b_i$, $\overline{b}' > b_i$ such that

$$\left|p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right) - p_j\left(b_i, \overline{b}, \boldsymbol{b}_{-(i,j)}\right)\right| < \frac{\epsilon}{2} \tag{5}$$

and

$$\left|p_i\left(b_i, b'_j, \boldsymbol{b}_{-(i,j)}\right) - p_j\left(b_i, \overline{b}', \boldsymbol{b}_{-(i,j)}\right)\right| < \frac{\epsilon}{2}.$$

W.l.o.g. it holds that $\overline{b} \leq \overline{b}'$. Since the payment function of a bidder is non-decreasing in her own bid and non-decreasing in the other bidders' bids, it holds that

$$p_j\left(b_i, \overline{b}, \boldsymbol{b}_{-(i,j)}\right) \geq p_j\left(b_j, \overline{b}, \boldsymbol{b}_{-(i,j)}\right) \geq p_j\left(b_j, b_i, \boldsymbol{b}_{-(i,j)}\right)$$

and

$$p_j\left(b_i, \overline{b}, \boldsymbol{b}_{-(i,j)}\right) \geq p_j\left(b'_j, \overline{b}, \boldsymbol{b}_{-(i,j)}\right) \geq p_j\left(b'_j, b_i, \boldsymbol{b}_{-(i,j)}\right).$$

Since the auction is symmetric, it holds that

$$p_j\left(b_j, b_i, \boldsymbol{b}_{-(i,j)}\right) = p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right)$$

and

$$p_j\left(b'_j, b_i, \boldsymbol{b}_{-(i,j)}\right) = p_i\left(b_i, b'_j, \boldsymbol{b}_{-(i,j)}\right).$$

It follows that

$$p_j\left(b_i, \overline{b}, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right) \geq 0$$

and

$$p_j\left(b_i, \overline{b}, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i, b'_j, \boldsymbol{b}_{-(i,j)}\right) \geq 0.$$

Since the payment function of a bidder is non-decreasing in her own bid and $\overline{b} \leq \overline{b}'$, it follows that

$$p_j\left(b_i, \overline{b}, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i, b'_j, \boldsymbol{b}_{-(i,j)}\right) \leq p_j\left(b_i, \overline{b}', \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i, b'_j, \boldsymbol{b}_{-(i,j)}\right) < \frac{\epsilon}{2}. \tag{6}$$

Due to the triangle inequality, it follows from (5) and (6) that

$$\left|p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i, b'_j, \boldsymbol{b}_{-(i,j)}\right)\right| < \epsilon.$$

Since ϵ can be chosen arbitrarily, it holds that

$$p_i\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) = p_i\left(b_i, b_j', \boldsymbol{b_{-(i,j)}}\right),$$

$\square$

Lemma 2. *In an imitation-perfect auction, for every bidder i and for every pair of vectors*

$$\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) \text{ and } \left(b_i, b_j', \boldsymbol{b_{-(i,j)}}\right)$$

where $b_i < b_j$ and $b_i < b_j'$, it holds that

$$p_i\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) = p_i\left(b_i, b_j', \boldsymbol{b_{-(i,j)}}\right).$$

That is, the payment of a bidder does not depend on the bids of competitors who placed higher bids.

The proof is omitted since it works analogously to the proof of Lemma 1.

Lemma 3. *In an imitation-perfect auction, for every bidder i, for all bid vectors $(b_i, \boldsymbol{b_{-i}})$ such that $b_i \neq \max_{j \neq i} b_j$ and for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all b_i' with $b_i < b_i' < b_i + \delta$ it holds that*

$$\left|p_i\left(b_i, \boldsymbol{b_{-i}}\right) - p_i\left(b_i', \boldsymbol{b_{-i}}\right)\right| < \epsilon.$$

That is, the payment of a bidder who is not a winner with a tie is right-continuous in her bid.

Proof. Let i be a bidder with bid b_i and $\epsilon > 0$. Let $(b_1, \ldots, b_i, \ldots, b_n)$ be a bid vector where bidder i is not a winner with a tie. It follows from imitation perfection that there exists a bid $\overline{b} > b_i$ such that

$$\left|p_i^{n+1}\left(b_1, \ldots, b_i, \ldots, b_n, 0\right) - p_{n+1}^{n+1}\left(b_1, \ldots, b_i, \ldots, b_n, \overline{b}\right)\right| < \epsilon.$$[20]

Since the auction is symmetric, it holds that

$$p_{n+1}^{n+1}\left(b_1, \ldots, b_i, \ldots, b_n, \overline{b}\right) = p_i^{n+1}\left(b_1, \ldots, \overline{b}, \ldots, b_n, b_i\right)$$

[20] We need the construction with the $(n+1)$th bidder merely to ensure that the lowest bidder in a bid vector can be also imitated by a higher bid.

and therefore

$$\left|p_i^{n+1}\left(b_1,\ldots,b_i,\ldots,b_n,0\right)-p_i^{n+1}\left(b_1,\ldots,\overline{b},\ldots,b_n,b_i\right)\right|<\epsilon.$$

Since $\overline{b}>b_i\geq 0$, it follows from Lemma 1 that

$$p_i^{n+1}\left(b_1,\ldots,\overline{b},\ldots,b_n,b_i\right)=p_i^{n+1}\left(b_1,\ldots,\overline{b},\ldots,b_n,0\right).$$

Therefore,

$$\left|p_i^{n+1}\left(b_1,\ldots,b_i,\ldots,b_n,0\right)-p_i^{n+1}\left(b_1,\ldots,\overline{b},\ldots,b_n,0\right)\right|<\epsilon$$

$$\Leftrightarrow\left|p_i^{n}\left(b_1,\ldots,b_i,\ldots,b_n\right)-p_i^{n}\left(b_1,\ldots,\overline{b},\ldots,b_n\right)\right|<\epsilon.$$

Define δ by $\delta:=\overline{b}-b_i$. Since the payment function of bidder i is non-decreasing in her own bid, it holds for every b with $b_i<b<b_i+\delta$ that

$$\left|p_i\left(b_i,\boldsymbol{b_{-i}}\right)-p_i\left(b,\boldsymbol{b_{-i}}\right)\right|<\epsilon$$

for all bid vectors where bidder i is not a winner with a tie. Hence, we have shown that the payment of bidder i is right-continuous if she is not a winner with a tie. □

Lemma 4. *In an imitation-perfect auction, for every bidder i, for all bid vectors* $(b_i,\boldsymbol{b_{-i}})$ *such that* $0<b_i<\max_{j\neq i}b_j$, *and for every* $\epsilon>0$ *there exists a* $\delta>0$ *such that for all* b *with* $b_i-\delta<b<b_i$ *it holds that*

$$\left|p_i\left(b_i,\boldsymbol{b_{-i}}\right)-p_i\left(b,\boldsymbol{b_{-i}}\right)\right|<\epsilon.$$

That is, the payment of a bidder who is not the highest bidder is left-continuous in her bid.

The proof is omitted since it works analogously to the proof of Lemma 3.

We continue with the proof of Proposition 1. The first part of Proposition 1 follows from Lemma 1. However, the second part does not directly follow from Lemma 2. We have not yet shown that the payment of a losing bidder with bid b_i does not change if another bidder changes her bid from b_j to b'_j (or from b'_j to b_j) with $b'_j>b_i>b_j$ given that bidder i remains a losing bidder. First, we will show the following claim.

Let $(b_i, \boldsymbol{b_{-i}})$ be a bid vector such that i is a losing bidder, i.e. it holds that $b_i < \max\limits_{j \neq i} b_j$. Then for every b_j with $j \in \{1, \ldots, n\}$ it holds that

$$p_i\left(b_i, b_i, \boldsymbol{b}_{-(i,j)}\right) = p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right).$$

First, we consider the case that $b_j < b_i$. Since bidder i's payment is right-continuous in her own bid if she is not a winner with a tie, for every $\epsilon > 0$ there exists a bid $b_i' > b_i$ such that

$$\left|p_i\left(b_i', b_i, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i, b_i, \boldsymbol{b}_{-(i,j)}\right)\right| < \frac{\epsilon}{2}$$

and

$$\left|p_i\left(b_i', b_j, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right)\right| < \frac{\epsilon}{2}.$$

It holds that

$$\begin{aligned} &\left|p_i\left(b_i, b_i, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right)\right| \\ \leq &\left|p_i\left(b_i, b_i, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i', b_i, \boldsymbol{b}_{-(i,j)}\right)\right| + \left|p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i', b_i, \boldsymbol{b}_{-(i,j)}\right)\right|. \end{aligned}$$

Since $b_i' > b_i > b_j$, it follows from Lemma 1 that this is equal to

$$\begin{aligned} &\left|p_i\left(b_i, b_i, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i', b_i, \boldsymbol{b}_{-(i,j)}\right)\right| + \left|p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i', b_j, \boldsymbol{b}_{-(i,j)}\right)\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Next, we consider the case that $b_j > b_i$. If $b_i = 0$, it follows from Assumption 2 that $p_i\left(b_i, b_i, \boldsymbol{b}_{-(i,j)}\right) = p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right) = 0$. If $b_i > 0$, then since bidder i's payment is left-continuous in her own bid if she is not the highest bidder, for every $\epsilon > 0$ there exists a bid $b_i' < b_i$ such that

$$\left|p_i\left(b_i', b_i, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i, b_i, \boldsymbol{b}_{-(i,j)}\right)\right| < \frac{\epsilon}{2}$$

and

$$\left|p_i\left(b_i', b_j, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right)\right| < \frac{\epsilon}{2}.$$

It holds that

$$\begin{aligned} &\left|p_i\left(b_i, b_i, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right)\right| \\ \leq &\left|p_i\left(b_i, b_i, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i', b_i, \boldsymbol{b}_{-(i,j)}\right)\right| + \left|p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i', b_i, \boldsymbol{b}_{-(i,j)}\right)\right|. \end{aligned}$$

Since $b_i' < b_i < b_j$, it follows from Lemma 2 that this is equal to

$$\left|p_i\left(b_i, b_i, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i', b_i, \boldsymbol{b}_{-(i,j)}\right)\right| + \left|p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(b_i', b_j, \boldsymbol{b}_{-(i,j)}\right)\right|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since in both cases ϵ can be chosen arbitrarily, it holds that

$$p_i\left(b_i, b_i, \boldsymbol{b_{-(i,j)}}\right) = p_i\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right)$$

for every vector of bids $(b_i, \boldsymbol{b_{-i}})$ where i is a losing bidder.

We can conclude that for two vectors of bids $\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right)$ and $\left(b_i, b'_j, \boldsymbol{b_{-(i,j)}}\right)$ where bidder i is a losing bidder that

$$p_i\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) = p_i\left(b_i, b_i, \boldsymbol{b_{-(i,j)}}\right) = p_i\left(b_i, b'_j, \boldsymbol{b_{-(i,j)}}\right). \tag{7}$$

This shows the second part of Proposition 1. □

C Proof of Proposition 2

Proof. We begin by proving that imitation perfection implies the first part of Proposition 2. We begin by stating the following two statements:

1. For each bidder i and for all vectors of bids $(b_i, \boldsymbol{b_{-i}})$ and $\left(b_i, \boldsymbol{b'_{-i}}\right)$ such that $b_i > \max_{j \neq i} b_j$ and $b_i > \max_{j \neq i} b'_j$ it holds that
$$p_i\left(b_i, \boldsymbol{b_{-i}}\right) = p_i\left(b_i, \boldsymbol{b'_{-i}}\right).$$

2. For each bidder i and for all vectors of bids $(b_i, \boldsymbol{b_{-i}})$ and $\left(b_i, \boldsymbol{b'_{-i}}\right)$ such that $b_i < \max_{j \neq i} b_j$ and $b_i < \max_{j \neq i} b'_j$ it holds that
$$p_i\left(b_i, \boldsymbol{b_{-i}}\right) = p_i\left(b_i, \boldsymbol{b'_{-i}}\right).$$

The first statement follows from the repeated application of the statement in Lemma 1 and the second statement follows from the repeated application of (7).

Given these two statements, we can define the function p^{win} by:

$$p^{win}(b_i) = p_i^2(b_i, 0)$$

and define p^{lose} by

$$p^{lose}(b_i) = p_i^2(b_i, b_j)$$

for $b_j > b_i$. In order to see that this definition is consistent for all numbers of bidders and all bid vectors, consider the vector $(b_i, \boldsymbol{b_{-i}}) \in \left(\mathbb{R}^+\right)^n$ such that $b_i > \max_{j \neq i} b_j$.

Then, due to statement 1, it holds that

$$p_i^n (b_i, \boldsymbol{b_{-i}}) = p_i^n (b_i, 0, \dots, 0) = p_i^2 (b_i, 0) \, .$$

Now consider the vector $(b_i, \boldsymbol{b_{-i}}) \in (\mathbb{R}^+)^n$ such that $b_i < \max_{j \neq i} b_j$. Then due to statement 2, it holds for every $b_j > b_i$ that

$$p_i^n (b_i, \boldsymbol{b_{-i}}) = p_i^n (b_i, b_j, 0, \dots, 0) = p_i^2 (b_i, b_j) \, .$$

It is left to show the following statement: For all vectors of bids $(b_i, \boldsymbol{b_{-i}})$ and $\left(b_i, \boldsymbol{b'_{-i}}\right)$ such that

$$b_i = \max_{j \neq i} b_j, \quad b_i = \max_{j \neq i} b'_j$$

and

$$\#\{j \in \{1, \dots, n\} \mid b_j = b_i\} = \#\{j \in \{1, \dots, n\} \mid b'_j = b_i\}\}$$

it holds that

$$p_i (b_i, \boldsymbol{b_{-i}}) = p_i \left(b_i, \boldsymbol{b'_{-i}}\right) .$$

Let

$$k = \#\{j \in \{1, \dots, n\} \mid b_j = b_i\} = \#\{j \in \{1, \dots, n\} \mid b'_j = b_i\}.$$

According to Definition 5, it holds that

$$p_i (b_i, \boldsymbol{b_{-i}}) = \frac{1}{k} p^{win} (b_i) + \left(1 - \frac{1}{k}\right) p^{lose} (b_i) = p_i \left(b_i, \boldsymbol{b'_{-i}}\right) .$$

The fact that imitation perfection implies the second part of Proposition 2, follows directly from Lemma 3 and Lemma 4.

It remains for us to show that the two conditions in Proposition 2 imply that an auction is imitation-perfect. That is, we have to show that

(i) For every vector of bids $\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right)$ such that bidder i is not a winner with a tie and for all $j \in \{1, \dots, n\}$ with $b_i > b_j$ there exists a bid $\bar{b} > b_i$ such that

$$\left| p_i \left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) - p_j \left(b_i, \bar{b}, \boldsymbol{b_{-(i,j)}}\right) \right| < \epsilon.$$

(ii) For every vector of bids $\left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right)$ and for all $j \in \{1, \dots, n\}$ with $0 < b_i < b_j$ there exists a bid $\underline{b} < b_i$ such that

$$\left| p_i \left(b_i, b_j, \boldsymbol{b_{-(i,j)}}\right) - p_j \left(b_i, \underline{b}, \boldsymbol{b_{-(i,j)}}\right) \right| < \epsilon.$$

Let $\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right)$ be a bid vector such that bidder i is not a winner with a tie and $b_i > b_j$. Since bidder i's payment is right-continuous in b_i if she is not a winner with a tie, there exists a $\bar{b} > b_i$ such that

$$\left|p_i\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right) - p_i\left(\bar{b}, b_j, \boldsymbol{b}_{-(i,j)}\right)\right| < \epsilon.$$

Since the payment of bidder i does not depend on lower bids and $\bar{b} > b_i > b_j$, it holds that

$$p_i\left(b_1, \ldots, \bar{b}, \ldots, b_j, \ldots, b_n\right) = p_i\left(b_1, \ldots, \bar{b}, \ldots, b_i, \ldots, b_n\right).$$

Due to the symmetry of the auction, it holds that

$$p_i\left(b_1, \ldots, \bar{b}, \ldots, b_i, \ldots, b_n\right) = p_j\left(b_1, \ldots, b_i, \ldots, \bar{b}, \ldots, b_n\right)$$

from which follows that

$$\left|p_i\left(b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n\right) - p_j\left(b_1, \ldots, b_i, \ldots, \bar{b}, \ldots, b_n\right)\right| < \epsilon.$$

This completes the proof of the statement in (i). For the proof of part (ii) let $\left(b_i, b_j, \boldsymbol{b}_{-(i,j)}\right)$ be a bid vector such that $0 < b_i < b_j$. Since bidder i's payment is left-continuous in b_i if she is not the highest bidder, there exists a $\underline{b} < b_i$ such that

$$\left|p_i\left(b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n\right) - p_i\left(b_1, \ldots, \underline{b}, \ldots, b_j, \ldots, b_n\right)\right| < \epsilon.$$

Since the payment of bidder i does not depend on higher bids and $b_j > b_i > \underline{b}$, it holds that

$$p_i\left(b_1, \ldots, \underline{b}, \ldots, b_j, \ldots, b_n\right) = p_i\left(b_1, \ldots, \underline{b}, \ldots, b_i, \ldots, b_n\right).$$

Due to the symmetry of the auction, it holds that

$$p_i\left(b_1, \ldots, \underline{b}, \ldots, b_i, \ldots, b_n\right) = p_j\left(b_1, \ldots, b_i, \ldots, \underline{b}, \ldots, b_n\right)$$

from which follows that

$$\left|p_i\left(b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n\right) - p_j\left(b_1, \ldots, b_i, \ldots, \underline{b}, \ldots, b_n\right)\right| < \epsilon.$$

This completes the proof of part (ii). $\square$

D Lemmas

After proving Propositions 1 and 2, which characterized the payment rule in imitation-perfect auctions, we use these results in order to prove the following lemmas. They provide statements about possible equilibria in imitation-perfect auctions and will be used throughout most of the proofs.

Lemma 5. *In an imitation-perfect auction, for every equilibrium $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ the expected payment as defined in (4) is strictly increasing above the reservation bid. That is, for all bids b_i, b_i' with $r \leq b_i < b_i'$ it holds that*

$$P_i^{\boldsymbol{\beta}}(b_i) < P_i^{\boldsymbol{\beta}}(b_i').$$

Proof. Due to Assumption 1, it holds that the payment of a unique winner is strictly increasing in at least one component of the bid vector. Since we have shown in Proposition 1 that the payment of a unique winner in an imitation-perfect auction does not depend on other bids, we conclude that the payment of a unique winner is strictly increasing in her own bid.

Assume that the lemma is not true and there exists an equilibrium $\boldsymbol{\beta}$, a bidder i and bids b_i, b_i' with $r \leq b_i < b_i'$ such that

$$P_i^{\boldsymbol{\beta}}(b_i) \geq P_i^{\boldsymbol{\beta}}(b_i').$$

This implies that bidder i with bid b_i' wins the auction with probability zero given the equilibrium strategies of the other bidders $\boldsymbol{\beta_{-i}}$. It follows that there exists an interval $[0, v]$ such that for all bidders $j \neq i$, for all $z \in [0, v]$, and for all $b \in supp(\beta_j(z))$ it holds that $b > b_i'$ (except a measure zero set of valuations in $[0, v]$). Therefore, there exists a valuation v_ϵ, a bid $b > b_i'$ and a bidder $j \neq i$ such that $v_\epsilon < p^{win}(b_i')$ and $b \in supp(\beta_j(v_\epsilon))$. Since this cannot be optimal, this leads to a contradiction to the assumption

$$P_i^{\boldsymbol{\beta}}(b_i) \geq P_i^{\boldsymbol{\beta}}(b_i').$$

$\square$

Lemma 6. *In an imitation-perfect auction, every equilibrium is non-decreasing above the reservation bid. That is, for every equilibrium $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, for every bidder $i \in \{1, \ldots, n\}$, for every pair of valuations $v', v \in [0, \overline{v}]$ such that $v' > v$, and for every pair of bids b', b with $b \geq r, b' \geq r$ such that $b' \in supp(\beta_i(v'))$ and $b \in supp(\beta_i(v))$ it holds that $b' \geq b$.*

Proof. The proof works analogously to the proof of Lemma 3.9 in Chawla and

Hartline (2013). Assume the lemma is not true and there exists an equilibrium $\boldsymbol{\beta}$ of an imitation-perfect auction which is decreasing. Then there exists a bidder i, valuations $v', v \in [0, \overline{v}]$ with $v' > v$ and bids b', b with $b' \in \mathit{supp}\left(\beta_i\left(v'\right)\right)$ and $b \in \mathit{supp}\left(\beta_i\left(v\right)\right)$ such that $b' < b$. It holds that

$$U_i^{\boldsymbol{\beta}}\left(v', b\right) = U_i^{\boldsymbol{\beta}}\left(v, b\right) + \left(v' - v\right) X_i^{\boldsymbol{\beta}}\left(b\right)$$
$$U_i^{\boldsymbol{\beta}}\left(v', b'\right) = U_i^{\boldsymbol{\beta}}\left(v, b'\right) + \left(v' - v\right) X_i^{\boldsymbol{\beta}}\left(b'\right).$$

Since $b' \in \mathit{supp}\left(\beta_i\left(v'\right)\right)$ and $b \in \mathit{supp}\left(\beta_i\left(v\right)\right)$, it holds that

$$U_i^{\boldsymbol{\beta}}\left(v', b\right) \leq U_i^{\boldsymbol{\beta}}\left(v', b'\right)$$

and

$$U_i^{\boldsymbol{\beta}}\left(v, b\right) \geq U_i^{\boldsymbol{\beta}}\left(v, b'\right).$$

Therefore, it must hold that

$$\left(v' - v\right) X_i^{\boldsymbol{\beta}}\left(b'\right) \geq \left(v' - v\right) X_i^{\boldsymbol{\beta}}\left(b\right).$$

It follows from $b' < b$ that $X_i^{\boldsymbol{\beta}}\left(b'\right) \leq X_i^{\boldsymbol{\beta}}\left(b\right)$. Hence, it holds that $X_i^{\boldsymbol{\beta}}\left(b'\right) = X_i^{\boldsymbol{\beta}}\left(b\right)$ and $U_i^{\boldsymbol{\beta}}\left(v', b'\right) = U_i^{\boldsymbol{\beta}}\left(v', b\right)$. This implies that

$$X_i^{\boldsymbol{\beta}}\left(b'\right) v' - P_i^{\boldsymbol{\beta}}\left(b'\right) = X_i^{\boldsymbol{\beta}}\left(b\right) v' - P_i^{\boldsymbol{\beta}}\left(b\right)$$

and

$$P_i^{\boldsymbol{\beta}}\left(b'\right) = P_i^{\boldsymbol{\beta}}\left(b\right)$$

which is a contradiction to Lemma 5. □

In order to state the next lemma, we need the following definition.

Definition 7. *For an equilibrium $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ and a bidder $i \in \{1, \ldots, n\}$ we denote the endpoints of an interval of valuations over which $\beta_i\left(v\right) = b$ by $\underline{v}_i\left(b\right)$ and $\overline{v}_i\left(b\right)$. Formally, we define*

$$\underline{v}_i\left(b\right) = \inf\{v \in [0, \overline{v}] \mid \beta_i\left(v\right) = b\}$$

and

$$\overline{v}_i\left(b\right) = \sup\{v \in [0, \overline{v}] \mid \beta_i\left(v\right) = b\}.$$

Lemma 7. *In an imitation-perfect auction, for every pure strategy equilibrium $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, for every bidder $i \in \{1, \ldots, n\}$, and for every pair of valuations v, v'*

such that

$$r \leq b = \beta_i(v) < b' = \beta_i(v')$$

it holds that

$$\overline{v}_i(b) \leq \underline{v}_i(b').$$

Proof. Assume there exists a pure strategy equilibrium $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, a bidder i, and valuations v, v' with $r \leq b = \beta_i(v) < b' = \beta_i(v')$ such that

$$\overline{v}_i(b) > \underline{v}_i(b')$$

Then there exist $\hat{v}$ and $\hat{v}'$ such that

$$\beta_i(\hat{v}) = b, \ \beta_i(\hat{v}') = b' \text{ and } \hat{v}' < \hat{v}.$$

This is a contradiction to Lemma 6. Thus, we conclude that it must hold

$$\overline{v}_i(b) \leq \underline{v}_i(b').$$

□

In several proofs we will first show the statement for pure strategy equilibria and then use the following lemma in order to derive the statement for general strategies.

Lemma 8. *Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ be an equilibrium of an imitation-perfect auction. Then there exists a pure strategy equilibrium $\boldsymbol{\beta}' = (\beta'_1, \ldots, \beta'_n)$ such that*

$$X_i^{\boldsymbol{\beta}}(v) = X_i^{\boldsymbol{\beta}'}(v)$$

holds for all $i \in \{1, \ldots, n\}$ and for all $v \in [0, \overline{v}]$ except a set of valuations with measure zero.

Proof. Since we have shown that an imitation-perfect auction is a bid-determines-payment auction, we can follow the same steps as the proof of Lemma 3.10 in Chawla and Hartline (2013). □

E Proof of Proposition 3

Proof. Assume there exists a perfect discrimination equilibrium and let bidder i be the bidder who wins the auction with probability 1 and pays zero. We will show that in a perfect discrimination equilibrium a bidder with a strictly positive valuation can deviate to a bid which is strictly higher than any bid placed in equilibrium and come arbitrarily close to bidder i's payment which is zero.

Let j be a bidder with valuation $v_j > 0$ and let ϵ be such that $0 < \epsilon < v_j$. For every vector of valuations $(v_i, \boldsymbol{v_{-i}})$ and for every vector of bids $(b_i(v_i), \boldsymbol{b_{-i}}(\boldsymbol{v_{-i}}))$, where $b_k(v_k) \in supp(\beta_k(v_k))$ for $k \in \{1, \ldots, n\}$, bidder i is the unique winner. Thus, due to imitation perfection, for every $b_i(v_i) \in supp(\beta_i(v_i))$ there exists a bid $\overline{b}_j(b_i(v_i))$ such that

$$|p_j\left(b_i(v_i), \overline{b}_j(b_i(v_i)), \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right) - p_i\left(b_i(v_i), b_j(v_j), \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right)| < \frac{\epsilon}{2} \quad (8)$$

for every vector of valuations $\boldsymbol{v_{-i}}$ and for every $\boldsymbol{b_{-i}} \in supp(\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}}))$. We want to construct a deviating bid $\hat{b}_j$ for bidder j such that bidder j wins the auction with probability one and her payment is close to zero.

A deviating bid for bidder j which guarantees a winning probability of one has to be higher than $s := sup\ supp(\beta_i(v_i))$. Since s is not necessarily played in equilibrium, we cannot conclude that $p_i(s, \boldsymbol{b_{-i}(v_{-i})}) = 0$. Thus, we construct the bid $\hat{b}_j$ as follows.

Fix for every $b_i(v_i) \in supp(\beta_i(v_i))$ a bid $\overline{b}_j(b_i(v_i))$ as in (8) and define

$$\hat{b}_j = \sup\left\{\overline{b}_j(b_i(v_i)) \mid b_i(v_i) \in supp(\beta_i(v_i)),\ v_i \in [0, \overline{v}]\right\}.$$

It is not immediately clear that for a given $\epsilon > 0$ there exists some $b_i(v_i^*) \in supp(\beta_i(v_i^*))$ such that

$$\left|p_j\left(b_i(v_i^*), \hat{b}_j, \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right) - p_i\left(b_i(v_i^*), b_j(v_j), \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right)\right| < \epsilon$$

which we will show in the remainder of the proof.

Since for every vector of valuations $\boldsymbol{v_{-j}}$ and for every $\boldsymbol{b_{-j}} \in supp(\boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}}))$ in the vector

$$\left(b_i(v_i), \overline{b}_j(b_i(v_i)), \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right)$$

bidder j is the unique winner, it follows from Proposition 2 that her payment function is right-continuous in her bid and there exists a $\delta > 0$ such that for all b_j with $\overline{b}_j(b_i(v_i)) < b_j < \overline{b}_j(b_i(v_i)) + \delta$ it holds that

$$\left|p_j\left(b_i(v_i), \overline{b}_j(b_i(v_i)), \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right) - p_j\left(b_i(v_i), b_j, \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right)\right| < \frac{\epsilon}{2}. \quad (9)$$

Fix a δ for which (9) holds. There exists a valuation $v_i^* \in [0, \overline{v}]$ and a bid $b_i(v_i^*) \in$

$supp\left(\beta_i\left(v_i^*\right)\right)$ such that

$$\overline{b}_j\left(b_i\left(v_i^*\right)\right) < \hat{b}_j < \overline{b}_j\left(b_i\left(v_i^*\right)\right) + \delta.$$

Otherwise, it would hold for all $v_i \in [0, \overline{v}]$ and for all $b_i\left(v_i\right) \in supp\left(\beta_i\left(v_i\right)\right)$ that $\hat{b}_j \geq \overline{b}_j\left(b_i\left(v_i\right)\right) + \delta$. This is a contradiction to the fact that $\hat{b}_j$ is the smallest upper bound of the set

$$\left\{\overline{b}_j\left(b_i\left(v_i\right)\right) \mid b_i\left(v_i\right) \in supp\left(\beta_i\left(v_i\right)\right),\ v_i \in [0, \overline{v}]\right\}.$$

Let $v_i^* \in [0, \overline{v}]$ and $b_i\left(v_i^*\right) \in supp\left(\beta_i\left(v_i^*\right)\right)$ be such that

$$\overline{b}_j\left(b_i\left(v_i^*\right)\right) < \hat{b}_j < \overline{b}_j\left(b_i\left(v_i^*\right)\right) + \delta.$$

Then for every vector of valuations $\boldsymbol{v_{-i}}$ and for every $\boldsymbol{b_{-i}} \in supp\left(\boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right)$ it follows from (8) and (9) that

$$\left|p_j\left(b_i\left(v_i^*\right), \hat{b}_j, \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right) - p_i\left(b_i\left(v_i^*\right), b_j\left(v_j\right), \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right)\right|$$

$$\begin{aligned}
&\leq \left|p_j\left(b_i\left(v_i^*\right), \hat{b}_j, \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right) - p_j\left(b_i\left(v_i^*\right), \overline{b}_j\left(b_i\left(v_i^*\right)\right), \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right)\right| \\
&+ \left|p_j\left(b_i\left(v_i^*\right), \overline{b}_j\left(b_i\left(v_i^*\right)\right), \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right) - p_i\left(b_i\left(v_i^*\right), b_j\left(v_j\right), \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right)\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

For every vector of valuations $\boldsymbol{v_{-i}}$ and for every $\boldsymbol{b_{-i}} \in supp\left(\boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right)$ it follows from

$$p_i\left(b_i\left(v_i^*\right), b_j\left(v_j\right), \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right) = 0,$$

that

$$p_j\left(b_i\left(v_i^*\right), \hat{b}_j, \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right) < \epsilon.$$

Since $\hat{b}_j$ is higher than any other bid placed by any bidder in equilibrium, bidder j would win the auction when bidding $\hat{b}_j$. According to Proposition 2, the payment of a bidder does not depend on lower bids from which follows for every vector of valuations $\boldsymbol{v_{-j}}$ and for every $\boldsymbol{b_{-j}} \in supp\left(\boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right)$ that

$$p_j\left(b_i\left(v_i\right), \hat{b}_j, \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right) = p_j\left(b_i\left(v_i^*\right), \hat{b}_j, \boldsymbol{b_{-(i,j)}}\left(\boldsymbol{v_{-(i,j)}}\right)\right) < \epsilon < v_j.$$

That is, by bidding $\hat{b}_j$, bidder j would pay an amount which is strictly smaller than her valuation. Hence, a perfect discrimination equilibrium cannot exist in an imitation-perfect symmetric auction, because each bidder $j \neq i$ would have an incentive to deviate whenever she has a strictly positive valuation for the good. $\square$

F Proof of Theorem 1

Proof. Although Theorem 1 directly follows from Proposition 5, we provide a separate proof for Theorem 1 since this proof is less technical and may help to understand the intuition behind the results in Theorem 1 and Proposition 5. We prove that the auction is discrimination-free by demonstrating that in every equilibrium two bidders with the same distribution function follow identical strategies above the reservation bid except a set of valuations with measure zero. In order to do so, we adapt a proof by Chawla and Hartline (2013).

First, we show the theorem for the case of equilibria in pure strategies and afterwards use Lemma 8 in order to derive the result for mixed strategy equilibria. We begin the proof by showing the following two lemmas, Lemma 9 and Lemma 10.

Lemma 9. *Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ be a pure strategy equilibrium of an imitation-perfect auction.*

(i) Let i and j be two bidders with the same distribution function and v a valuation such that

$$r \leq \beta_i(v) < \beta_j(v).$$

Then it holds that

$$X_j^{\boldsymbol{\beta}}(v) > X_i^{\boldsymbol{\beta}}(v)$$

where $X_j^{\boldsymbol{\beta}}(v)$ and $X_i^{\boldsymbol{\beta}}(v)$ are defined as in (3) in Appendix A.

(ii) Let i and j be two bidders with the same distribution function and v a valuation such that

$$\beta_i(v) = \beta_j(v) = b \text{ and } \underline{v}_j(b) \leq \underline{v}_i(b), \ \overline{v}_j(b) \leq \overline{v}_i(b)$$

where $\underline{v}_j(b), \underline{v}_i(b), \overline{v}_j(b), \overline{v}_i(b)$ are defined as in Definition 7 in Appendix D. Then it holds that

$$X_j^{\boldsymbol{\beta}}(v) \geq X_i^{\boldsymbol{\beta}}(v).$$

Proof. Part *(i)*: Let $v \in [0, \overline{v}]$ be a valuation such that $b_j > b_i \geq r$ where $b_j = \beta_j(v)$ and $b_i = \beta_i(v)$. The allocation probability of bidder i is equal to or lower than her allocation probability if she wins every tie with probability one. In this case her winning probability is determined by the case that she bids equal to or higher than any other bidder. Formally, we define a new allocation rule $\tilde{x}$ which is identical to the allocation rule x except all bid vectors where bidder i is a winner with a tie. For such a bid vector it holds that $\tilde{x}_i(b_i, b_{-i}) = 1$. It holds that

$$X_i^{\boldsymbol{\beta_{-i}}}(b_i) \le \tilde{X}_i^{\boldsymbol{\beta_{-i}}}(b_i)$$

$$= \prod_{k \neq i} Pr\left[\text{bidder } k \text{ bids lower than } b_i + \text{bidder } k \text{ bids } b_i\right]$$

$$= \prod_{k \neq i} \left[F\left(\underline{v}_k(b_i)\right) + \left(F\left(\overline{v}_k(b_i)\right) - F\left(\underline{v}_k(b_i)\right)\right)\right] = \prod_{k \neq i} F\left(\overline{v}_k(b_i)\right)$$

$$= F\left(\overline{v}_j(b_i)\right) \prod_{k \neq i,j} F\left(\overline{v}_k(b_i)\right)$$

where F is defined by $F := F_i = F_j$. Due to Lemma 7, it holds that $\overline{v}_k(b_i) \le \underline{v}_k(b_j)$ for all bidders $k \in \{1, \ldots, n\}$. Therefore, it holds that

$$\overline{v}_j(b_i) \le \underline{v}_j(b_j) \le v \le \overline{v}_i(b_i) \le \underline{v}_i(b_j).$$

It follows that

$$F\left(\overline{v}_j(b_i)\right) \prod_{k \neq i,j} F\left(\overline{v}_k(b_i)\right) \le F\left(\underline{v}_i(b_j)\right) \prod_{k \neq i,j} F\left(\underline{v}_k(b_j)\right) = \prod_{k \neq i} F\left(\underline{v}_k(b_j)\right)$$

$$\le \prod_{k \neq i} F\left(\underline{v}_k(b_j)\right) + \sum_{k=1}^{n-1} \frac{1}{k+1} Pr\left(\text{k other bidders bid } b_j \text{ and none higher}\right) = X_j^{\boldsymbol{\beta_{-j}}}(b_j).$$

According to Lemma 5, the expected payment of a bidder is strictly increasing in her own bid above the reservation bid. Thus, it cannot hold that $X_i^{\boldsymbol{\beta_{-i}}}(b_i) = X_j^{\boldsymbol{\beta_{-j}}}(b_j)$ in equilibrium. Otherwise, bidder j could deviate to a bid b'_j with $b_i < b'_j < b_j$. With an analogous reasoning, one can show that $X_j^{\boldsymbol{\beta_{-j}}}\left(b'_j\right) \ge X_i^{\boldsymbol{\beta_{-i}}}(b_i) = X_j^{\boldsymbol{\beta_{-j}}}(b_j)$. Due to Lemma 5, it holds that $P_j^{\boldsymbol{\beta_{-j}}}\left(b'_j\right) < P_j^{\boldsymbol{\beta_{-j}}}(b_j)$. Hence, deviating to b'_j would increase bidder j's expected utility. Therefore, $b_j > b_i$ implies

$$X_i^{\boldsymbol{\beta_{-i}}}(b_i) < X_j^{\boldsymbol{\beta_{-j}}}(b_j).$$

Proof of part *(ii)*: If $b < r$, then the allocation probability for both bidders is zero and therefore the same. If $b \geq r$, it holds that

$$\begin{aligned} X_i^{\boldsymbol{\beta}}(b) = F\left(\underline{v}_j(b)\right) E_{\boldsymbol{v}_{-\boldsymbol{i}}}[x_i(b, \boldsymbol{\beta}_{-\boldsymbol{i}}(\boldsymbol{v}_{-\boldsymbol{i}})) \mid b > \beta_j(v_j)] \\ + \left[F(\overline{v}_j(b)) - F\left(\underline{v}_j(b)\right)\right] E_{\boldsymbol{v}_{-\boldsymbol{i}}}[x_i(b, \boldsymbol{\beta}_{-\boldsymbol{i}}(\boldsymbol{v}_{-\boldsymbol{i}})) \mid b = \beta_j(v_j)]. \end{aligned}$$

Since $\underline{v}_j(b) \leq \underline{v}_i(b)$ and $\overline{v}_j(b) \leq \overline{v}_i(b)$, this is smaller or equal than

$$\begin{aligned} F(\underline{v}_i(b)) E_{\boldsymbol{v}_{-\boldsymbol{i}}}[x_i(b, \boldsymbol{\beta}_{-\boldsymbol{i}}(\boldsymbol{v}_{-\boldsymbol{i}})) \mid b > \beta_j(v_j)] \\ + [F(\overline{v}_i(b)) - F(\underline{v}_i(b))] E_{\boldsymbol{v}_{-\boldsymbol{i}}}[x_i(b, \boldsymbol{\beta}_{-\boldsymbol{i}}(\boldsymbol{v}_{-\boldsymbol{i}})) \mid b = \beta_j(v_j)]. \end{aligned}$$

The term $E_{\boldsymbol{v}_{-\boldsymbol{i}}}[x_i(b, \boldsymbol{\beta}_{-\boldsymbol{i}}(\boldsymbol{v}_{-\boldsymbol{i}})) \mid b > \beta_j(v_j)]$ denotes the probability that bidder i wins against all other bidders besides bidder j if she bids b and given that she wins against bidder j. This is equal to the probability that bidder j wins against all other bidders besides bidder i if she bids b given that she wins against bidder i, which is denoted by the term $E_{\boldsymbol{v}_{-\boldsymbol{j}}}[x_j(b, \boldsymbol{\beta}_{-\boldsymbol{j}}(\boldsymbol{v}_{-\boldsymbol{j}})) \mid b > \beta_i(v_i)]$. An analogous reasoning applies to the term $E_{\boldsymbol{v}_{-\boldsymbol{i}}}[x_i(b, \boldsymbol{\beta}_{-\boldsymbol{i}}(\boldsymbol{v}_{-\boldsymbol{i}})) \mid b = \beta_j(v_j)]$. Therefore, the expression above is equal to

$$\begin{aligned} = F(\underline{v}_i(b)) E_{\boldsymbol{v}_{-\boldsymbol{j}}}[x_j(b, \boldsymbol{\beta}_{-\boldsymbol{j}}(\boldsymbol{v}_{-\boldsymbol{j}})) \mid b > \beta_i(v_i)] \\ + [F(\overline{v}_i(b)) - F(\underline{v}_i(b))] E_{\boldsymbol{v}_{-\boldsymbol{j}}}[x_j(b, \boldsymbol{\beta}_{-\boldsymbol{j}}(\boldsymbol{v}_{-\boldsymbol{j}})) \mid b = \beta_i(v_i)] = X_j^{\boldsymbol{\beta}}(b). \end{aligned}$$

□

Lemma 10. *If there exists a pure strategy equilibrium $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ of an imitation-perfect auction, a bidder k and an interval $[\underline{v}_k, \overline{v}_k]$ with $F_k(\overline{v}_k) - F_k(\underline{v}_k) > 0$ such that $\beta_k(v) = b_k \geq r$ for all $v \in [\underline{v}_k, \overline{v}_k]$, then there does not exist another bidder j and a valuation v_j such that $\beta_j(v_j) = b_k$ and*

$$Pr(b_k > \beta_i(v_i) \text{ for all } i \neq k, j)\left(v_j - p^{win}(b_k)\right) > 0.$$

That is, if bidder k's bidding strategy is constant over some interval, due to the continuity of the payment function, every other bidder would never bid the same amount but has an incentive to slightly overbid bidder k.

Proof. Assume there exists a bidder j and a valuation v_j such that $\beta_j(v_j) = b_k$ and $Pr(b_k > \beta_i(v_i) \text{ for all } i \neq k, j)(v_j - p^{win}(b_k)) > 0$. First, we consider the case that bidder k is the only bidder such that there exists an interval of valuations with measure larger than zero over which the bidder bids b_k. Let $\left(\hat{\underline{v}}_k, \hat{\overline{v}}_k\right)$ be the maximal interval over which it holds that $\beta_k(v) = b_k$ for all $v \in \left(\underline{\hat{v}}_k, \hat{\overline{v}}_k\right)$. That is,

$$\underline{\hat{v}}_k = \inf_{v \in [0,\overline{v}]} \beta_k(v) = b$$

$$\hat{\overline{v}}_k = \sup_{v \in [0,\overline{v}]} \beta_k(v) = b.$$

To simplify notation, define P by

$$P = Pr\left(b_k > \beta_i(v_i) \text{ for all } i \neq k, j\right).$$

Let $\epsilon > 0$ be such that

$$\frac{1}{2}[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\underline{\hat{v}}_k\right)][v_j - p^{win}\left(b_k\right)]P - \epsilon > 0.$$

Due to the right-continuity of the payment function, for all $\epsilon > 0$ there exists a bid $b' > b_k$ such that

$$p_j\left(b', \boldsymbol{b_{-j}}\right) - p_j\left(b_k, \boldsymbol{b_{-j}}\right) < \frac{\epsilon}{2}$$

for all vectors $\left(b_k, \boldsymbol{b_{-j}}\right)$ where bidder j is not a winner with a tie.

If bidder j deviates to b', her winning probability increases from

$$E_{\boldsymbol{v_{-j}}}[x_j\left(b_k, \boldsymbol{\beta_{-j}}\left(\boldsymbol{v_{-j}}\right)\right)] = \frac{1}{2}[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\underline{\hat{v}}_k\right)]P$$

to

$$E_{\boldsymbol{v_{-j}}}[x_j\left(b', \boldsymbol{\beta_{-j}}\left(\boldsymbol{v_{-j}}\right)\right)] = [F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\underline{\hat{v}}_k\right)]Pr\left(b' > \beta_i(v_i) \text{ for all } i \neq k, j\right).$$

This implies that the winning probability increases by at least

$$\frac{1}{2}[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\underline{\hat{v}}_k\right)]P.$$

Her expected payment increases by at most

$$\frac{1}{2}\left(p^{win}\left(b'\right)\right)[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\underline{\hat{v}}_k\right)]P + \int_{[0,\overline{v}]^{n-1}} \frac{\epsilon}{2} f\left(\boldsymbol{v_{-j}}\right) d\boldsymbol{v_{-j}}$$

$$\leq \frac{1}{2}\left(p^{win}\left(\beta_j(v_j)\right) + \frac{\epsilon}{2}\right)[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\underline{\hat{v}}_k\right)]P + \frac{\epsilon}{2}$$

$$\leq \frac{1}{2}\left(p^{win}\left(\beta_j(v_j)\right)\right)[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\underline{\hat{v}}_k\right)]P + \frac{\epsilon}{4}[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\underline{\hat{v}}_k\right)]P + \frac{\epsilon}{2}$$

$$\leq \frac{1}{2}\left(p^{win}\left(\beta_j(v_j)\right)\right)[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\underline{\hat{v}}_k\right)]P + \epsilon.$$

Therefore, the expected utility gain for bidder j from deviating to b' is given by at least

$$\frac{1}{2}[F_k(\hat{\overline{v}}_k) - F_k(\hat{\underline{v}}_k)]Pv_j - \frac{1}{2}\left(p^{win}(\beta_j(v_j))\right)[F_k(\hat{\overline{v}}_k) - F_k(\hat{\underline{v}}_k)]P - \epsilon$$

$$= \frac{1}{2}[F_k(\hat{\overline{v}}_k) - F_k(\hat{\underline{v}}_k)][v_j - p^{win}(\beta_j(v_j))]P - \epsilon > 0.$$

We conclude that there does not exist a bidder j and a valuation v_j such that $\beta_j(v_j) = b_k$ because otherwise bidder j would have an incentive to deviate.

The case where more than one bidder bids b_k over an interval of valuations with measure greater than zero, can be treated analogously. □

We continue with the proof of Theorem 1. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ be an equilibrium in pure strategies. In order to show that bidders with the same distribution function follow identical strategies, we consider two arbitrary bidders i and j who draw their valuations from the same distribution. Assume that the strategies of bidder i and bidder j differ above the reservation bid over some set of valuations with strictly positive measure. Since strategies are continuous except on a set of valuations with measure zero, there exists an interval of valuations with strictly positive measure over which the strategies differ. Consider the lowest valuation at which the strategies differ above the reservation bid over an interval of valuations with strictly positive measure. Formally, let

$$\underline{z} = \inf\left\{v' \mid \exists\, z'' > v' \text{ s.t. } \beta_j(v) \neq \beta_i(v) \text{ and } \beta_i(v) \geq r, \beta_j(v) \geq r \text{ for all } v \in [v', z'']\right\}.$$

Since strategies are continuous besides a set of valuations with measure zero, there exists a valuation v'' such that without loss it holds for all $z \in (\underline{z}, v'')$ that $\beta_j(z) > \beta_i(z)$.

In order to show that this leads to a contradiction, we use the following definition.

Definition 8 (Utility crossing)**.** *For a given equilibrium $\boldsymbol{\beta}$, an interval $(\underline{z}, \overline{z})$ satisfies* utility crossing *if $X_j^{\boldsymbol{\beta}}(v) \geq X_i^{\boldsymbol{\beta}}(v)$ for all $v \in (\underline{z}, \overline{z})$, $U_j^{\boldsymbol{\beta}}(\underline{z}) \geq U_i^{\boldsymbol{\beta}}(\underline{z})$, and $U_i^{\boldsymbol{\beta}}(\overline{z}) \geq U_j^{\boldsymbol{\beta}}(\overline{z})$.*

We will show that strategies have to be equal on an interval which satisfies utility crossing.

Lemma 11. *Let $\boldsymbol{\beta}$ be an equilibrium of an imitation-perfect auction and $(\underline{z}, \overline{z})$ be an interval satisfying utility crossing. Then it holds that $\beta_i(v) = \beta_j(v)$ for all $v \in (\underline{z}, \overline{z})$*

except a set of measure zero.

Proof. Suppose that $\beta_j \neq \beta_i$ over some subset of $(\underline{z}, \overline{z})$ with strictly positive measure. Since strategies are discontinuous on a set of valuations with measure zero, there exists a set with strictly positive measure such that either $\beta_j > \beta_i$ or $\beta_i > \beta_j$ for all valuations in this set. Since $X_j^{\boldsymbol{\beta}}(v) \geq X_i^{\boldsymbol{\beta}}(v)$ for all $v \in (\underline{z}, \overline{z})$, it follows from Lemma 9 that $\beta_j > \beta_i$ over some subset of $(\underline{z}, \overline{z})$ with strictly positive measure. Due to Lemma 9, it holds that $X_j^{\boldsymbol{\beta}}(v) > X_i^{\boldsymbol{\beta}}(v)$ for all v with $\beta_j(v) > \beta_i(v)$. According to Myerson (1981), in every auction mechanism, the expected utility of a bidder can be written as a function of the winning probability. Formally, it holds for every k and every v_k that

$$U_k^{\boldsymbol{\beta}}(v_k) = U_k^{\boldsymbol{\beta}}(0) + \int_0^{v_k} X_k^{\boldsymbol{\beta}}(z)\, dz. \tag{10}$$

Since the payment function of a unique winner is strictly increasing in her bid,[21] a bidder with valuation zero does not win with positive probability. Because payments are non-negative and by Assumption 2 a bidder pays zero by bidding zero, the expected payment of a bidder with valuation zero is zero. Therefore, the expected utility of a bidder with valuation zero is zero.

Thus, applying equation (10) to $\overline{z}$ and $\underline{z}$ and rearranging it accordingly yields

$$U_i^{\boldsymbol{\beta}}(\overline{z}) - U_i^{\boldsymbol{\beta}}(\underline{z}) = \int_{\underline{z}}^{\overline{z}} X_i(z)\, dz$$

and

$$U_j^{\boldsymbol{\beta}}(\overline{z}) - U_j^{\boldsymbol{\beta}}(\underline{z}) = \int_{\underline{z}}^{\overline{z}} X_j^{\boldsymbol{\beta}}(z)\, dz.$$

Since $X_j^{\boldsymbol{\beta}} > X_i^{\boldsymbol{\beta}}$ over a subset of $(\underline{z}, \overline{z})$ with strictly positive measure, it holds that

$$U_j^{\boldsymbol{\beta}}(\overline{z}) - U_j^{\boldsymbol{\beta}}(\underline{z}) > U_i^{\boldsymbol{\beta}}(\overline{z}) - U_i^{\boldsymbol{\beta}}(\underline{z}),$$

which contradicts utility crossing. It therefore holds that $\beta_i(v) = \beta_j(v)$ for all v in $(\underline{z}, \overline{z})$ except a set of measure zero. □

We will show that $(\underline{z}, v'')$ is a subset of an interval satisfying utility crossing. Hence, the strategy of bidder i and j cannot differ on the interval $(\underline{z}, v'')$. We will show that the interval which satisfies utility crossing is given by $(\underline{z}, \overline{z})$ where $\overline{z}$ is

[21] Recall the argument provided in the proof of Lemma 5: Due to Assumption 1, the payment of a unique winner has to be strictly increasing in at least one component of the bid vector. We have shown in Proposition 1 that the payment of a unique winner depends only on her own bid. Therefore, the payment of a unique winner has to be strictly increasing in her own bid.

defined by

$$\overline{z} = \inf \left\{ v > \underline{z} | \beta_i (v) \geq \beta_j (v) \right\}.$$

If the infimum does not exist, we set $\overline{z} = \overline{v}$. It follows from Lemma 9 that $X_j^{\boldsymbol{\beta}} (v) > X_i^{\boldsymbol{\beta}} (v)$ for all $v \in (\underline{z}, \overline{z})$. Thus, it is left to show that

$$U_i (\overline{z}) \geq U_j (\overline{z}) \text{ and } U_j (\underline{z}) \geq U_i (\underline{z}).$$

Choose $b < \beta_j (\underline{z})$ such that there exists a valuation $z \in [0, \underline{z}]$ with $\beta_i (z) = b$. Since $z < \underline{z}$ and $\underline{z}$ is the infimum of valuations at which the strategies of bidder i and bidder j differ, it holds that

$$\beta_i (z) = \beta_j (z) = b$$

and

$$\underline{v}_i (b) = \underline{v}_j (b).$$

Moreover, it holds that $\overline{v}_j (b) \leq \overline{v}_i (b)$. Assume this were not true. Then it holds that $\overline{v}_j (b) > \overline{v}_i (b)$. Since the equilibrium is non-decreasing, this implies that there exists an interval $(\overline{v}_i (b), \hat{v})$ such that $\beta_i (v) > \beta_j (v)$ for all $v \in (\overline{v}_i (b), \hat{v})$. This is a contradiction to the assumption that $\underline{z}$ is the infimum of valuations at which the strategies of bidder i and j differ and that bidder j bids higher than bidder i on some interval $(\underline{z}, v'')$.

Thus, it holds that $\overline{v}_j (b) \leq \overline{v}_i (b)$ and it follows from Lemma 9 that $X_i^{\boldsymbol{\beta}} (b) = X_j^{\boldsymbol{\beta}} (b)$.[22] If $\beta_j (\underline{z}) > \beta_i (\underline{z})$, it follows from part (ii) of Lemma 9 that $X_i^{\boldsymbol{\beta}} (b) < X_j^{\boldsymbol{\beta}} (b)$.

If $\beta_j (\underline{z}) = \beta_i (\underline{z})$, it holds that $\underline{v}_i (\beta_i (\underline{z})) = \underline{v}_j (\beta_i (\underline{z}))$ because strategies are equal below $\underline{z}$. Moreover, it holds that $\overline{v}_i (\beta_i (\underline{z})) \geq \overline{v}_j (\beta_i (\underline{z}))$ because otherwise bidder i's strategy would be decreasing. Thus, it follows from Lemma 9 that $X_i^{\boldsymbol{\beta}} (z) \leq X_j^{\boldsymbol{\beta}} (z)$ for all $z \in [\underline{v}_i (\beta_i (\underline{z})), \underline{z}]$.

We conclude that for all $z \in [0, \underline{z}]$, it holds that $X_j^{\boldsymbol{\beta}} (z) \geq X_i^{\boldsymbol{\beta}} (z)$. Hence, it follows from Myerson (1981) that

$$U_j^{\boldsymbol{\beta}} (\underline{z}) = \int_0^{\underline{z}} X_j^{\boldsymbol{\beta}} (z) \, dz \geq \int_0^{\underline{z}} X_i^{\boldsymbol{\beta}} (z) \, dz = U_i^{\boldsymbol{\beta}} (\underline{z}).$$

It is left to show that

$$U_i (\overline{z}) \geq U_j (\overline{z}).$$

[22] Recall that as stated in (3) in Appendix A, by $X_i^{\boldsymbol{\beta}} (b)$ we denote the allocation probability of a bidder i who submits bid b given the equilibrium $\boldsymbol{\beta}$.

In order to do so, we show that for every $\epsilon > 0$ there exists a bid for bidder i with which she could achieve an expected utility of at least $U_j^{\boldsymbol{\beta}}(\overline{z}) - \epsilon$ if she has valuation $\overline{z}$. Therefore, the expected utility of bidder i's equilibrium bid has to induce at least an expected utility of $U_j^{\boldsymbol{\beta}}(\overline{z})$ and it holds that

$$U_i^{\boldsymbol{\beta}}(\overline{z}) \geq U_j^{\boldsymbol{\beta}}(\overline{z}).$$

It follows from Proposition 2 that the expected payment in equilibrium of bidder j at $\overline{z}$ is given by

$$P_j^{\boldsymbol{\beta}}(\beta_j(\overline{z})) = X_j^{\boldsymbol{\beta}}(\beta_j(\overline{z}))\, p^{win}(\beta_j(\overline{z})) + \left(1 - X_j^{\boldsymbol{\beta}}(\beta_j(\overline{z}))\right)\, p^{lose}(\beta_j(\overline{z})).$$

Let ϵ be greater than zero. Due to the right-continuity of the functions p^{win} and p^{lose}, there exists a bid $b > \beta_j(\overline{z})$ such that

$$p^{win}(b) - p^{win}(\beta_j(\overline{z})) < \epsilon$$

and

$$p^{lose}(b) - p^{lose}(\beta_j(\overline{z})) < \epsilon.$$

We can assume that $Pr(\beta_j(\overline{z}) > \beta_k(v_k) \text{ for all } k \neq i, j)(\overline{z} - \beta_j(\overline{z})) > 0$. Otherwise, it immediately follows that $U_i^{\boldsymbol{\beta}}(\overline{z}) \geq U_j^{\boldsymbol{\beta}}(\overline{z}) = 0$. Thus, by Lemma 10, the event that bidder j is a winner with a tie bidding $\beta_j(\overline{z})$ has probability zero. In particular, the interval $[\underline{v}_i(\beta_j(\overline{z})), \overline{v}_i(\beta_j(\overline{z}))]$ has measure zero.

In equilibrium, the expected utility of bidder j bidding $\beta_j(\overline{z})$ is given by

$$\begin{aligned}
&U_j^{\boldsymbol{\beta}}(\overline{z}, \beta_j(\overline{z})) \\
&\quad = F(\underline{v}_i(\beta_j(\overline{z})))\, E_{\boldsymbol{v_{-j}}}[x_j(\beta_j(\overline{z}), \boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}})) \mid \beta_j(\overline{z}) > \beta_i(v_i)]\left(\overline{z} - p^{win}(\beta_j(\overline{z}))\right) \\
&\quad + \left(1 - F(\underline{v}_i(\beta_j(\overline{z})))\, E_{\boldsymbol{v_{-j}}}[x_j(\beta_j(\overline{z}), \boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}})) \mid \beta_j(\overline{z}) > \beta_i(v_i)]\right) p^{lose}(\beta_j(\overline{z})).
\end{aligned}$$

The term

$$F(\underline{v}_i(\beta_j(\overline{z})))\, E_{\boldsymbol{v_{-j}}}[x_j(\beta_j(\overline{z}), \boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}})) \mid \beta_j(\overline{z}) > \beta_i(v_i)]$$

denotes the probability of winning. Since the event that bidder j is a winner with a tie has measure zero, we can denote the probability of losing simply by one minus this term. That is, the probability of losing can be denoted by

$$1 - F(\underline{v}_i(\beta_j(\overline{z})))\, E_{\boldsymbol{v_{-j}}}[x_j(\beta_j(\overline{z}), \boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}})) \mid \beta_j(\overline{z}) > \beta_i(v_i)].$$

The expected utility of bidder i deviating to bid b at $\overline{z}$ is given by

$$
\begin{aligned}
U_i^{\boldsymbol{\beta_{-i}}}(\overline{z}, b) = {} & F\left(\underline{v}_j(b)\right) E_{\boldsymbol{v_{-i}}}[x_i(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid b > \beta_j(v_j)] \left(\overline{z} - p^{win}(b)\right) \\
& + [F(\overline{v}_j(b)) - F\left(\underline{v}_j(b)\right)] E_{\boldsymbol{v_{-i}}}[x_i(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid b = \beta_j(v_j)] \left(\overline{z} - p^{win}(b)\right) \\
& - \Big(1 - F\left(\underline{v}_j(b)\right) E_{\boldsymbol{v_{-i}}}[x_i(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid b > \beta_j(v_j)] \\
& - [F(\overline{v}_j(b)) - F\left(\underline{v}_j(b)\right)] E_{\boldsymbol{v_{-i}}}[x_i(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid b = \beta_j(v_j)]\Big) p^{lose}(b)
\end{aligned}
$$

$$
\begin{aligned}
\geq {} & F\left(\underline{v}_j(b)\right) E_{\boldsymbol{v_{-i}}}[x_i(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid b > \beta_j(v_j)] \left(\overline{z} - p^{win}(b)\right) \\
& - \Big(1 - F\left(\underline{v}_j(b)\right) E_{\boldsymbol{v_{-i}}}[x_i(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid b > \beta_j(v_j)]\Big) p^{lose}(b). \qquad (11)
\end{aligned}
$$

Since $b > \beta_j(\overline{z})$, it follows from Lemma 7 that $\overline{v}_j(b) \geq \underline{v}_j(b) \geq \overline{v}_j(\beta_j(\overline{z})) \geq \overline{z}$. Since the interval $[\underline{v}_i(\beta_j(\overline{z})), \overline{v}_i(\beta_j(\overline{z}))]$ has measure zero, it holds that $\beta_i(v) > \beta_i(\overline{z})$ for all $v > \overline{z}$ except on a set of valuations with measure zero. It follows that $\overline{v}_i(\beta_j(\overline{z})) \leq \overline{z}$. Hence, it holds that

$$
\underline{v}_i(\beta_j(\overline{z})) \leq \overline{v}_i(\beta_j(\overline{z})) \leq \overline{z} \leq \overline{v}_j(\beta_j(\overline{z})) \leq \underline{v}_j(b). \qquad (12)
$$

It follows from (12) that the term in (11) is greater or equal than

$$
\begin{aligned}
& F(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)] \left(\overline{z} - p^{win}(b)\right) \\
& - \Big(1 - F(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\Big) p^{lose}(b)
\end{aligned}
$$

$$
\begin{aligned}
> {} & F(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)] \left(\overline{z} - \left(p^{win}(\beta_j(\overline{z})) + \epsilon\right)\right) \\
& - \Big(1 - F(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\Big) \left(p^{lose}(\beta_j(\overline{z})) + \epsilon\right)
\end{aligned}
$$

$$
\begin{aligned}
= {} & F(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-j}}}[x_j(\beta_j(\overline{z}), \boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}})) \mid \beta_j(\overline{z}) > \beta_i(v_i)] \left(\overline{z} - \left(p^{win}(\beta_j(\overline{z})) + \epsilon\right)\right) \\
& - \Big(1 - F(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-j}}}[x_j(\beta_j(\overline{z}), \boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}})) \mid \beta_j(\overline{z}) > \beta_i(v_i)]\Big) \left(p^{lose}(\beta_j(\overline{z})) + \epsilon\right)
\end{aligned}
$$

$$
\begin{aligned}
= {} & F(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-j}}}[x_j(\beta_j(\overline{z}), \boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}})) \mid \beta_j(\overline{z}) > \beta_i(v_i)] \left(\overline{z} - p^{win}(\beta_j(\overline{z}))\right) \\
& - \Big(1 - F(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-j}}}[x_j(\beta_j(\overline{z}), \boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}})) \mid \beta_j(\overline{z}) > \beta_i(v_i)]\Big) p^{lose}(\beta_j(\overline{z})) - \epsilon.
\end{aligned}
$$

It follows that

$$U_j^{\boldsymbol{\beta}}(\overline{z}, \beta_j(\overline{z})) - U_i^{\boldsymbol{\beta}_{-i}}(\overline{z}, b) < \epsilon.$$

Hence, we have shown that for every $\epsilon > 0$ there exists a deviating bid b such that bidder i can achieve an expected utility of at least $U_j^{\boldsymbol{\beta}}(\overline{z}) - \epsilon$ from which follows that

$$U_i^{\boldsymbol{\beta}}(\overline{z}) \geq U_j^{\boldsymbol{\beta}}(\overline{z}).$$

We conclude that the interval $(\underline{z}, \overline{z})$ satisfies utility crossing. Therefore, it holds that $\beta_i(v) = \beta_j(v)$ for all $v \in (\underline{z}, \overline{z})$ except a set of valuations with measure zero. Thus, the assumption that there exists a measurable interval over which the bidding strategies of two bidders differ above the reservation bid, leads to a contradiction.

It is left to consider the case of mixed equilibria. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ be a (possibly mixed) equilibrium. According to Lemma 8, there exists a pure strategy equilibrium $\boldsymbol{\beta}' = (\beta_1', \ldots, \beta_n')$ such that it holds for all $i \in \{1, \ldots, n\}$ and for all $v \in \{1, \ldots, n\}$ that

$$X_i^{\boldsymbol{\beta}}(v) = X_i^{\boldsymbol{\beta}'}(v)$$

except a set of valuations with measure zero. Since we have shown that in a pure strategy equilibrium all bidders adopt identical strategies above the reservation bid, for every pair of bidders i and j and for every $v \in [0, \overline{v}]$, it holds that

$$X_i^{\boldsymbol{\beta}'}(v) = X_j^{\boldsymbol{\beta}'}(v).$$

Thus, for every $v \in [0, \overline{v}]$ it holds that

$$\begin{aligned}\left|U_i^{\boldsymbol{\beta}}(v) - U_j^{\boldsymbol{\beta}}(v)\right| &= \left|\int_0^v X_j^{\boldsymbol{\beta}}(z)\,dz - \int_0^v X_i^{\boldsymbol{\beta}}(z)\,dz\right| \\ &= \left|\int_0^v X_j^{\boldsymbol{\beta}'}(z)\,dz - \int_0^v X_i^{\boldsymbol{\beta}'}(z)\,dz\right| = 0.\end{aligned}$$

Thus, it hods that $U_i^{\boldsymbol{\beta}}(v) = U_j^{\boldsymbol{\beta}}(v)$ which completes the proof.

□

G Proof of Proposition 4

Proof of part (i). To prove that every imitation-perfect auction has an equilibrium, we will directly apply Corollary 5.2 from Reny (1999) that provides sufficient conditions for the existence of an equilibrium. We start by laying out the definitions that are necessary to understand the approach. We then show that imitation-perfect auctions fulfill those sufficient conditions.

Definitions. Let

$$X_i = \{\beta_i : \Theta_i \to [0, \bar{b}]\}$$

denote the strategy spaces of the bidders and

$$u_i(x_1, \dots, x_n) = \int_{[0,\overline{v}]} U_i^{\boldsymbol{x}}(v_i)\, dF_i(v_i)$$

denote the utility function as a function of the strategies x_i in X_i where U_i is defined as in Section 2.

A game $(X_i, u_i)_{i=1}^n$ is a *compact Hausdorff game* if for every $i \in \{1, \dots, n\}$ it holds that X_i is a compact Hausdorff space and u_i is bounded.

Let A, B denote topological spaces. A tuple (a, b) with $a \in A$ and $b \in B$ is in the *closure of the graph* of a function $f : A \to B$ if there exists a sequence $\{a^l\}_{l=1}^\infty$ converging to a such that $b = \lim_{l\to\infty} f\left(a^l\right)$.

Let M_i denote the set of (regular, countable, additive) probability measures on the Borel subsets of X_i. Extend each u_i to $M = \times_{i=1}^n M_i$ by defining $u_i(m) = \int_X u_i(x)\mathrm{d}m$ for all $m \in M$ and call $(M_i, u_i)_{i=1}^n$ the *mixed extension* of a game.

Player i can *secure* a utility $p \in \mathbb{R}$ at $m \in M$ if there exists $\hat{x}_i \in X_i$ such that $u_i\left(\hat{x}_i, \boldsymbol{m'_{-i}}\right) \geq p$ for all $\boldsymbol{m'_{-i}}$ in some open neighborhood of $\boldsymbol{m_{-i}}$.

A game $(M_i, u_i)_{i=1}^n$ is *better-reply secure* if whenever $(\boldsymbol{m}^*, \boldsymbol{u}^*)$ is in the closure of the graph of $\boldsymbol{u}$ and $\boldsymbol{m}^*$ is not an equilibrium, there exists a player i who can secure a utility strictly above $\boldsymbol{u_i^*}$ at $\boldsymbol{m}^*$.

Lemma 12 (Corollary 5.2 in Reny (1999))**.** *If the mixed extension of a compact Hausdorff game is better-reply secure, the game possesses a Nash-equilibrium in mixed strategies.*

Application to imitation-perfect auctions. We apply Lemma 12 to our setting by adopting the proof of Example 5.2 in Reny (1999) which applies Corollary 5.2 to first-price auctions.

As shown in Reny (1999), the space of pure bidding strategies for a bidder is a compact Hausdorff space with respect to the topology of pointwise convergence.[23]

[23]Recall that for a set A and a topological space $(B, \mathcal{T})$ the topology of pointwise convergence on the set of functions $f : A \to B$ is generated by the subbase $U_{a,O}$ for $a \in A$ and $O \in \mathcal{T}$ with $U_{a,O} = \{f : A \to B \mid f(a) \in O\}$. A sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges to some f in this topology if and only if for every $a \in A$ the sequence $f_n(a)$ converges to $f(a)$.

In particular, for a space of functions from $[0, \overline{v}]$ to $[0, \overline{b}]$ the topology of pointwise convergence is generated by the subbase $U_{v,a,b}$ for $v \in [0, \overline{v}], 0 < a < b < \overline{b}$ with $U_{v,a,b} = \{f : [0, \overline{v}] \to \mathbb{R} \mid a < f(v) < b\}$.

Thus, it is left to show that an imitation-perfect auction is better-reply secure.

Assume that $\boldsymbol{m}^* \in M$ is not an equilibrium and let $(\boldsymbol{m}^*, \boldsymbol{u}^*)$ be in the closure of the graph of $\boldsymbol{u}$.

Case 1: First, we consider the case that ties occur with probability zero at $\boldsymbol{m}^*$. In this case, $\boldsymbol{u}$ is continuous at $\boldsymbol{m}^*$ and thus $\boldsymbol{u}^* = u(\boldsymbol{m}^*)$.

Let $u_i^s\left(\boldsymbol{m}^*_{-\boldsymbol{i}}\right)$ denote bidder i's supremum utility at $\boldsymbol{m}^*_{-\boldsymbol{i}}$, i.e. for all $i \in \{1, \ldots, n\}$ it holds that

$$u_i^s\left(\boldsymbol{m}^*_{-i}\right) = \sup_{x_i \in X_i} u_i\left(x_i, \boldsymbol{m}^*_{-i}\right).$$

Since $\boldsymbol{m}^*$ is not an equilibrium by assumption, there exists a bidder i such that

$$u_i^s\left(\boldsymbol{m}^*_{-\boldsymbol{i}}\right) > u_i\left(\boldsymbol{m}^*\right).$$

Therefore, for every $\epsilon > 0$ there exists a strategy for bidder i, x_i^ϵ, such that

$$\left|u_i\left(x_i^\epsilon, \boldsymbol{m}^*_{-\boldsymbol{i}}\right) - u_i^s\left(\boldsymbol{m}^*_{-\boldsymbol{i}}\right)\right| < \epsilon.$$

For a sufficiently small ϵ it holds that

$$u_i\left(x_i^\epsilon, \boldsymbol{m}^*_{-\boldsymbol{i}}\right) > u_i\left(\boldsymbol{m}^*\right).$$

Since by assumption ties occur with probability zero at $\boldsymbol{m}^*$, it holds that $\boldsymbol{u}(x_i^\epsilon, \cdot)$ is continuous at $\boldsymbol{m}^*{}_{-i}$ and therefore there exists an open neighborhood of $\boldsymbol{m}_{-\boldsymbol{i}}{}^*$ such that

$$u_i\left(x_i^\epsilon, \boldsymbol{m}'_{-\boldsymbol{i}}\right) > u_i\left(\boldsymbol{m}^*\right) = \boldsymbol{u}_{\boldsymbol{i}}^*$$

for all $\boldsymbol{m}'_{-\boldsymbol{i}}$ in this neighborhood. Thus, the game is better-reply secure.

Case 2: Now we consider the case that ties occur with positive probability at $\boldsymbol{m}^*$. In this case $\boldsymbol{u}^*$ is not necessarily equal to $u(\boldsymbol{m}^*)$ and a little more work is necessary to prove better-reply security.

By definition, there exists a sequence of strategies $\{\boldsymbol{m}^{\boldsymbol{l}}\}_{l=1}^\infty$ converging to $\boldsymbol{m}^*$ such that $\boldsymbol{u}^* = \lim_{l \to \infty} \boldsymbol{u}\left(\boldsymbol{m}^{\boldsymbol{l}}\right)$. Let i and j be two bidders who tie with positive probability at $\boldsymbol{m}^*$. That is, there exists an interval $[\underline{v}_t, \overline{v}_t]$ on which they submit the same bid b_t.

For every strategy profile $\boldsymbol{m}^{\boldsymbol{l}}$ in $\{\boldsymbol{m}^{\boldsymbol{l}}\}_{l=1}^\infty$ either bidder i or bidder j loses with positive (ex-ante) probability over the interval $[\underline{v}_t, \overline{v}_t]$. This implies that there exists a subsequence $\{\boldsymbol{m}^{\boldsymbol{l_k}}\}$ such that one of these two bidders, say bidder i, loses with positive (ex-ante) probability on the interval $[\underline{v}_t, \overline{v}_t]$ for every $\boldsymbol{m}^{\boldsymbol{l_k}}$. As $\{\boldsymbol{m}^{\boldsymbol{l}}\}_{l=1}^\infty$

converges to $\boldsymbol{m}^*$, $\{\boldsymbol{m}^{\boldsymbol{l_k}}\}$ also converges to $\boldsymbol{m}^*$.

It must hold that $p^{win}(b_t) \leq \underline{v}_t < \overline{v}_t$. Therefore, bidder i would strictly increase her ex-ante utility if her ex-ante winning probability increases on a subset of $[\underline{v}_t, \overline{v}_t]$ with positive measure. For every $k > 0$ it holds that at $\boldsymbol{m}^{\boldsymbol{l_k}}$ bidder i can strictly increase her ex-ante winning probability on a subset of $[\underline{v}_t, \overline{v}_t]$ with positive measure by slightly increasing her bid. Or put differently, since the functions p^{win} and p^{lose} are right-continuous, slightly increasing the bid, to avoid the tie, increases the winning probability discontinuously while the winning price merely increases continuously. Overall, the ex-ante expected utility increases discontinuously.

Let

$$u_i^s\left(\boldsymbol{m}_{-i}^{l_k}\right) = \sup_{x_i \in X_i} u_i\left(x_i, \boldsymbol{m}_{-i}^{l_k}\right).$$

denote bidder i's supremum utility at $\boldsymbol{m}^{\boldsymbol{l_k}}_{\boldsymbol{-i}}$. By the argument above, there exists $K > 0$ such that for all $k > K$ there exists a constant $c > 0$ such that

$$u_i^s\left(\boldsymbol{m}^{\boldsymbol{l_k}}_{\boldsymbol{-i}}\right) \geq u_i\left(\boldsymbol{m}^{\boldsymbol{l_k}}\right) + c.$$

Since this holds for all $k > K$, this also holds at the limit. That is,

$$u_i^s\left(\boldsymbol{m}^*_{\boldsymbol{-i}}\right) > \boldsymbol{u}_{\boldsymbol{i}}^*.$$

Hence, for every $\epsilon > 0$ there exists a strategy x_i^ϵ such that

$$\left|u_i\left(x_i^\epsilon, \boldsymbol{m}^*_{\boldsymbol{-i}}\right) - u_i^s\left(\boldsymbol{m}^*_{\boldsymbol{-i}}\right)\right| < \epsilon.$$

Again, for a sufficiently small ϵ it holds that

$$u_i\left(x_i^\epsilon, \boldsymbol{m}^*_{\boldsymbol{-i}}\right) > \boldsymbol{u}_{\boldsymbol{i}}^*.$$

Since at $\left(x_i^\epsilon, \boldsymbol{m}^*_{\boldsymbol{-i}}\right)$ bidder i does not tie with another bidder, $u_i(x_i^\epsilon, \cdot)$ is continuous at $\boldsymbol{m}^*_{\boldsymbol{-i}}$. Therefore, there exists an open neighborhood of $\boldsymbol{m}^*_{\boldsymbol{-i}}$ such that

$$u_i\left(x_i^\epsilon, \boldsymbol{m'}_{\boldsymbol{-i}}\right) > \boldsymbol{u}_{\boldsymbol{i}}^*$$

for all $\boldsymbol{m'}_{\boldsymbol{-i}}$ in this neighborhood.

We conclude that an imitation-perfect auction with compact intervals as bid spaces is better-reply secure and therefore an equilibrium exists. □

Proof of part (ii). It follows from Theorem 4.5 in Chawla and Hartline (2013) that in a bid-determines-payment auction with homogeneous bidders there exists only one symmetric equilibrium, that is, an equilibrium where all bidders adopt identical strategies. Due to Proposition 2, all imitation-perfect auctions are bid-determines-payment auctions. As shown in the proof of Theorem 1, in an imitation-perfect auction all equilibria are symmetric. Therefore, an imitation-perfect auction with homogeneous bidders has a unique equilibrium. It follows from Lemma 8 that if a mixed strategy equilibrium exists, a pure strategy equilibrium also exists. Since the equilibrium is unique, it has to be a pure strategy equilibrium. It follows from Lemma 6 that the equilibrium is non-decreasing.

□

H Proof of Proposition 5

Proof. As in the proof of Theorem 1, we will first show the statement in Proposition 5 for pure strategy equilibria and then apply Lemma 8 in order to show the statement for mixed strategy equilibria. We will prove Proposition 5 for pure strategy equilibria by showing the following claim: For every equilibrium $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, every valuation v and every pair of bidders i and j it holds that

$$\left| \int_0^v X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz \right| \leq \Delta_{i,j}\overline{v} + \Delta_{i,j}(\overline{v} - v). \tag{13}$$

Given this claim, Proposition 5 directly follows from Myerson (1981) since

$$\left| U_j^{\boldsymbol{\beta}}(v) - U_i^{\boldsymbol{\beta}}(v) \right| = \left| \int_0^v X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz \right|.$$

The proof of the claim works similarly to the proof of Theorem 1. We start by proving the following lemma which provides a result analogous to Lemma 9.

Lemma 13. *Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ be a pure strategy equilibrium equilibrium of an imitation- perfect auction.*

(i) Let i and j be two bidders and v a valuation such that

$$\beta_i(v) < \beta_j(v).$$

Then it holds that

$$X_j^{\boldsymbol{\beta}}(v) + \Delta_{i,j} \geq X_i^{\boldsymbol{\beta}}(v)$$

with $\Delta_{i,j}$ defined as in (2).

(ii) Let i and j be two bidders and v a valuation such that

$$\beta_i(v) = \beta_j(v) = b \text{ and } \underline{v}_j(b) \le \underline{v}_i(b), \ \overline{v}_j(b) \le \overline{v}_i(b).$$

Then it holds that

$$X_j^{\boldsymbol{\beta}}(v) + \Delta_{i,j} \ge X_i^{\boldsymbol{\beta}}(v)$$

with $\Delta_{i,j}$ defined as in (2).

Proof. Part *(i)*: Let v be a valuation and define b_i, b_j by $\beta_i(v) = b_i$ and $\beta_j(v) = b_j$. If $b_i < r$, it holds that $X_i^{\boldsymbol{\beta}}(b_i) = 0$ and the statement follows directly. Otherwise, similarly to the proof of Lemma 9 we have:

$$X_i^{\boldsymbol{\beta}}(b_i) \le \tilde{X}_i^{\boldsymbol{\beta}}(b_i), \text{ with}$$

$$\tilde{X}_i^{\boldsymbol{\beta}}(b_i) = \prod_{k \neq i} Pr\left[\text{bidder } k \text{ bids lower than } b_i + \text{bidder } k \text{ bids } b_i\right]$$

$$= \prod_{k \neq i} \left[F_k(\underline{v}_k(b_i)) + (F_k(\overline{v}_k(b_i)) - F_k(\underline{v}_k(b_i)))\right] = \prod_{k \neq i} F_k(\overline{v}_k(b_i))$$

$$= F_j(\overline{v}_j(b_i)) \prod_{k \neq i,j} F_k(\overline{v}_k(b_i)) \le F_j(\underline{v}_i(b_j)) \prod_{k \neq i,j} F(\underline{v}_k(b_j)).$$

Since for every $v \in [0, \overline{v}]$ it holds that $F_j(v) \le F_i(v) + \Delta_{i,j}$, this is smaller or equal than

$$(F_i(\underline{v}_i(b_j)) + \Delta_{i,j}) \prod_{k \neq i,j} F(\underline{v}_k(b_j)) = \prod_{k \neq i} F(\underline{v}_k(b_j)) + \Delta_{i,j} \prod_{k \neq i,j} F(\underline{v}_k(b_j))$$

$$\le \prod_{k \neq i} F(\underline{v}_k(b_j)) + \Delta_{i,j}$$

$$\le \prod_{k \neq i} F(\underline{v}_k(b_j)) + \Delta_{i,j} + \sum_{k=1}^{n-1} \frac{1}{k+1} Pr\left(\text{k other bidders bid } b_j \text{ and none higher}\right)$$

$$= X_j^{\boldsymbol{\beta}}(b_j) + \Delta_{i,j}.$$

Proof of part *(ii)*: If $b < r$, the allocation probability for both bidders is zero and therefore the same. If $b \ge r$, it holds that

$$X_i^{\boldsymbol{\beta}}(b) = F_j\left(\underline{v}_j(b)\right) E_{\boldsymbol{v_{-i}}}[x_i(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid b > \beta_j(v_j)]$$
$$+ \left[F_j(\overline{v}_j(b)) - F_j\left(\underline{v}_j(b)\right)\right] E_{\boldsymbol{v_{-i}}}[x_i(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid b = \beta_j(v_j)]$$

$$= F_j\left(\underline{v}_j(b)\right)\left(E_{\boldsymbol{v_{-i}}}[x_i(b,\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}}))\mid b>\beta_j(v_j)]-E_{\boldsymbol{v_{-i}}}[x_i(b,\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}}))\mid b=\beta_j(v_j)]\right) \\ + F_j(\overline{v}_j(b))\,E_{\boldsymbol{v_{-i}}}[x_i(b,\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}}))\mid b=\beta_j(v_j)]$$

$$\le (F_i(\underline{v}_i(b))+\Delta_{i,j})\left(E_{\boldsymbol{v_{-i}}}[x_i(b,\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}}))\mid b>\beta_j(v_j)] \right. \\ \left. -E_{\boldsymbol{v_{-i}}}[x_i(b,\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}}))\mid b=\beta_j(v_j)]\right) \\ +(F_i(\overline{v}_i(b))+\Delta_{i,j})\,E_{\boldsymbol{v_{-i}}}[x_i(b,\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}}))\mid b=\beta_j(v_j)]$$

$$\le F_i(\underline{v}_i(b))\left(E_{\boldsymbol{v_{-i}}}[x_i(b,\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}}))\mid b>\beta_j(v_j)]-E_{\boldsymbol{v_{-i}}}[x_i(b,\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}}))\mid b=\beta_j(v_j)]\right) \\ F_i(\overline{v}_i(b))\,E_{\boldsymbol{v_{-i}}}[x_i(b,\boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}}))\mid b=\beta_j(v_j)]+\Delta_{i,j}$$

$$= F_i(\underline{v}_i(b))\,E_{\boldsymbol{v_{-j}}}[x_j(b,\boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}}))\mid b>\beta_i(v_i)] \\ +[F_i(\overline{v}_i(b))-F_i(\underline{v}_i(b))]\,E_{\boldsymbol{v_{-j}}}[x_j(b,\boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}}))\mid b=\beta_i(v_i)]+\Delta_{i,j}=X_j^{\boldsymbol{\beta}}(b)+\Delta_{i,j}$$

where the first inequality follows from the assumption that $\underline{v}_j(b)\le\underline{v}_i(b)$ and $\overline{v}_j(b)\le\overline{v}_i(b)$. ☐

Note that Lemma 10 holds independently of the bidders' homogeneity. We will show the claim in (13). Assume that there exist bidders i and j and a valuation v such that

$$\left|\int_0^v X_j^{\boldsymbol{\beta}}(z)-X_i^{\boldsymbol{\beta}}(z)\,dz\right|>\Delta_{i,j}\overline{v}+\Delta_{i,j}(\overline{v}-v).$$

Without loss we can assume that it holds

$$\int_0^v X_j^{\boldsymbol{\beta}}(z)-X_i^{\boldsymbol{\beta}}(z)\,dz>\Delta_{i,j}\overline{v}+\Delta_{i,j}(\overline{v}-v).$$

In order to complete the proof, we will need the following lemma:

Lemma 14. *Let $\boldsymbol{\beta}$ be an equilibrium of an imitation-perfect auction and $\overline{z}$ a valuation. Then the following holds true:*

(i) If $\beta_i(\overline{z})\ge\beta_j(\overline{z})$, it holds that

$$U_i^{\boldsymbol{\beta}}(\overline{z})+\Delta_{i,j}\overline{v}\ge U_j^{\boldsymbol{\beta}}(\overline{z}).$$

(ii) If there exists a valuation $\hat{v}>\overline{z}$ such that for all $z\in(\overline{z},\hat{v})$ it holds that

$\beta_i(z) \geq \beta_j(z)$, *it holds that*

$$U_i^{\boldsymbol{\beta}}(\overline{z}) + \Delta_{i,j}\overline{v} \geq U_j^{\boldsymbol{\beta}}(\overline{z}).$$

(iii) If $\overline{z} = \overline{v}$, *it holds that*

$$U_i^{\boldsymbol{\beta}}(\overline{z}) = U_j^{\boldsymbol{\beta}}(\overline{z}).$$

.

Proof. As a preparation for the proof we show the following lemma:

Lemma 15. *In an imitation-perfect auction it holds for all* $b \geq 0$ *that*

$$p^{win}(b) \geq p^{lose}(b).$$

Proof. Assume the statement is not true and there exists a bid $b \geq 0$ such that

$$p^{win}(b) < p^{lose}(b).$$

Let $\alpha > 0$ be defined by $\alpha = p^{lose}(b) - p^{win}(b)$. Since the function p^{win} is right-continuous, there exists a bid $b' > b$ such that $p^{win}(b') - p^{win}(b) < \frac{\alpha}{2}$. Thus, it holds that

$$\frac{1}{2}p^{lose}(b) = \frac{1}{2}p^{win}(b) + \frac{\alpha}{2} > \frac{1}{2}p^{win}(b) + p^{win}(b') - p^{win}(b) = p^{win}(b') - \frac{1}{2}p^{win}(b). \quad (14)$$

Since the payment of a bidder is non-decreasing in her own bid, it holds that

$$p^{win}(b') = p_1(b', b) \geq p_1(b, b) = \frac{1}{2}p^{win}(b) + \frac{1}{2}p^{lose}(b).$$

It follows that

$$\frac{1}{2}p^{lose}(b) \leq p^{win}(b') - \frac{1}{2}p^{win}(b)$$

which is a contradiction to 14. □

We continue with the proof of Lemma 14. The following two statements hold for (i)-(iii).

1. If $\beta_j(\overline{z}) < r$, it holds that $U_j^{\boldsymbol{\beta}}(\overline{z}) = 0$ and the lemma follows directly.

2. If $\beta_j(\overline{z}) \geq r$, let ϵ be greater than zero.

Due to Proposition 2, it holds that the expected payment of bidder j at $\overline{z}$ is given by

$$P_j^{\boldsymbol{\beta}}\left(\beta_j\left(\overline{z}\right)\right) = X_j^{\boldsymbol{\beta}}\left(\beta_j\left(\overline{z}\right)\right) p^{win}\left(\beta_j\left(\overline{z}\right)\right) + \left(1 - X_j^{\boldsymbol{\beta}}\left(\beta_j\left(\overline{z}\right)\right)\right)\, p^{lose}\left(\beta_j\left(\overline{z}\right)\right).$$

We continue with the proof for part *(i)* and *(ii)*. Analogously as in the proof of Theorem 1, we will prove that

$$U_i^{\boldsymbol{\beta}}\left(\overline{z}\right) + \Delta_{i,j}\overline{v} \geq U_j^{\boldsymbol{\beta}}\left(\overline{z}\right)$$

by showing that for every $\epsilon > 0$ there exists a deviating bid b for bidder i at valuation $\overline{z}$ with which she could achieve at least a utility of $U_j^{\boldsymbol{\beta}_i}\left(\overline{z}\right) - \Delta_{i,j}\overline{v} - \epsilon$.

In equilibrium, the expected utility of bidder j bidding $\beta_j\left(\overline{z}\right)$ is given by

$$\begin{aligned}
&U_j^{\boldsymbol{\beta}}\left(\overline{z}, \beta_j\left(\overline{z}\right)\right) \\
&= F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-j}}}[x_j\left(\beta_j\left(\overline{z}\right), \boldsymbol{\beta_{-j}}\left(\boldsymbol{v_{-j}}\right)\right) \mid \beta_j\left(\overline{z}\right) > \beta_i\left(v_i\right)] \left(\overline{z} - p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right) \\
&- \Big(1 - F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-j}}}[x_j\left(\beta_j\left(\overline{z}\right), \boldsymbol{\beta_{-j}}\left(\boldsymbol{v_{-j}}\right)\right) \mid \beta_j\left(\overline{z}\right) > \beta_i\left(v_i\right)]\Big) p^{lose}\left(\beta_j\left(\overline{z}\right)\right).
\end{aligned} \tag{15}$$

We can assume that $Pr\left(\beta_j\left(\overline{z}\right) > \beta_k\left(v_k\right) \text{ for all } k \neq i, j\right) > 0$ because otherwise it directly follows that $U_i^{\boldsymbol{\beta}}\left(\overline{z}\right) \geq U_j^{\boldsymbol{\beta}}\left(\overline{z}\right) = 0$. Thus, according to Lemma 10, the event that bidder j is a winner with a tie bidding $\beta_j\left(\overline{z}\right)$, has probability zero. In particular, the interval $[\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right), \overline{v}_i\left(\beta_j\left(\overline{z}\right)\right)]$ has measure zero. Thus, equation (15) does not account for the possibility of ties. Due to the right-continuity of the functions p^{win} and p^{lose}, there exists a bid $b > \beta_j\left(\overline{z}\right)$ such that

$$p^{win}\left(b\right) - p^{win}\left(\beta_j\left(\overline{z}\right)\right) < \epsilon$$

and

$$p^{lose}\left(b\right) - p^{lose}\left(\beta_j\left(\overline{z}\right)\right) < \epsilon.$$

The expected utility of bidder i deviating to bid b at $\overline{z}$ is given by

$$\begin{aligned}
U_i^{\boldsymbol{\beta_{-i}}}\left(\overline{z}, b\right) &= F_j\left(\underline{v}_j\left(b\right)\right) E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b > \beta_j\left(v_j\right)] \left(\overline{z} - p^{win}\left(b\right)\right) \\
&+ [F_j\left(\overline{v}_j\left(b\right)\right) - F_j\left(\underline{v}_j\left(b\right)\right)] E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b = \beta_j\left(v_j\right)] \left(\overline{z} - p^{win}\left(b\right)\right) \\
&- \Big(1 - F_j\left(\underline{v}_j\left(b\right)\right) E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b > \beta_j\left(v_j\right)] \\
&- [F_j\left(\overline{v}_j\left(b\right)\right) - F_j\left(\underline{v}_j\left(b\right)\right)] E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b = \beta_j\left(v_j\right)]\Big) p^{lose}\left(b\right)
\end{aligned}$$

$$\geq F_j\left(\underline{v}_j(b)\right) E_{\boldsymbol{v_{-i}}}[x_i(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid b > \beta_j(v_j)] \left(\overline{z} - p^{win}(b)\right)$$
$$- \left(1 - F_j\left(\underline{v}_j(b)\right) E_{\boldsymbol{v_{-i}}}[x_i(b, \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid b > \beta_j(v_j)]\right) p^{lose}(b). \quad (16)$$

Since $b > \beta_j(\overline{z})$, it follows from Lemma 7 that $\overline{v}_j(b) \geq \underline{v}_j(b) \geq \overline{v}_j(\beta_j(\overline{z})) \geq \overline{z}$. Since the interval $[\underline{v}_i(\beta_j(\overline{z})), \overline{v}_i(\beta_j(\overline{z}))]$ has measure zero, it holds that $\beta_i(v) > \beta_i(\overline{z})$ for all $v > \overline{z}$ except a set of valuations with measure zero. It follows that $\overline{v}_i(\beta_j(\overline{z})) \leq \overline{z}$. Hence, it holds that

$$\underline{v}_i(\beta_j(\overline{z})) \leq \overline{v}_i(\beta_j(\overline{z})) \leq \overline{z} \leq \overline{v}_j(\beta_j(\overline{z})) \leq \underline{v}_j(b). \quad (17)$$

It follows from (17) that the expression in (16) is greater or equal than

$$F_j(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)] \left(\overline{z} - p^{win}(b)\right)$$
$$- \left(1 - F_j(\underline{v}_i(\beta_j)(\overline{z})) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\right) p^{lose}(b)$$

$$> F_j(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)] \left(\overline{z} - \left(p^{win}(\beta_j(\overline{z})) + \epsilon\right)\right)$$
$$- \left(1 - F_j(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\right) \left(p^{lose}(\beta_j(\overline{z})) + \epsilon\right)$$

$$\geq (F_i(\underline{v}_i(\beta_j)(\overline{z})) - \Delta_{i,j}) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)] \left(\overline{z} - p^{win}(\beta_j(\overline{z}))\right)$$
$$- \left(1 - (F_i(\underline{v}_i(\beta_j(\overline{z})) - \Delta_{i,j})) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\right) \left(p^{lose}(\beta_j(\overline{z}))\right) - \epsilon$$

$$\geq F_i(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)] \left(\overline{z} - p^{win}(\beta_j(\overline{z}))\right)$$
$$- \left(1 - F_i(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\right) p^{lose}(\beta_j(\overline{z}))$$
$$- \Delta_{i,j}\left(\overline{z} - p^{win}(\beta_j(\overline{z})) + p^{lose}(\beta_j(\overline{z}))\right) - \epsilon$$

$$\geq F_i(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)] \left(\overline{z} - p^{win}(\beta_j(\overline{z}))\right)$$
$$- \left(1 - F_i(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-i}}}[x_i(\beta_j(\overline{z}), \boldsymbol{\beta_{-i}}(\boldsymbol{v_{-i}})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\right) p^{lose}(\beta_j(\overline{z}))$$
$$- \Delta_{i,j}\overline{z} - \epsilon.$$

$$= F_i(\underline{v}_i(\beta_j(\overline{z}))) E_{\boldsymbol{v_{-j}}}[x_j(\beta_j(\overline{z}), \boldsymbol{\beta_{-j}}(\boldsymbol{v_{-j}})) \mid \beta_j(\overline{z}) > \beta_i(v_i)] \left(\overline{z} - p^{win}(\beta_j(\overline{z}))\right)$$

$$+\Big(1 - F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-j}}}[x_j\left(\beta_j\left(\overline{z}\right), \boldsymbol{\beta_{-j}}\left(\boldsymbol{v_{-j}}\right)\right) \mid \beta_j\left(\overline{z}\right) > \beta_i\left(v_i\right)]\Big) p^{lose}\left(\beta_j\left(\overline{z}\right)\right)$$
$$-\Delta_{i,j}\overline{z} - \epsilon.$$

It follows that

$$U_j^{\boldsymbol{\beta}}\left(\overline{z}, \beta_j\left(\overline{z}\right)\right) - U_i^{\boldsymbol{\beta_{-i}}}\left(\overline{z}, b\right) < \Delta_{i,j}\overline{z} + \epsilon.$$

Hence, we have shown that for every $\epsilon > 0$ there exists a deviating bid b such that bidder i can achieve an expected utility of at least $U_j^{\boldsymbol{\beta}}\left(\overline{z}\right) - \Delta_{i,j}\overline{z} - \epsilon$ from which follows that

$$U_i^{\boldsymbol{\beta}}\left(\overline{z}\right) + \Delta_{i,j}\left(\overline{z}\right) \geq U_j^{\boldsymbol{\beta}}\left(\overline{z}\right).$$

Now we provide a proof for part *(iii)*. It is sufficient to show that $U_i^{\boldsymbol{\beta}}\left(\overline{z}\right) \geq U_j^{\boldsymbol{\beta}}\left(\overline{z}\right)$, since then $U_j^{\boldsymbol{\beta}}\left(\overline{z}\right) \geq U_i^{\boldsymbol{\beta}}\left(\overline{z}\right)$ follows from symmetry. As in part *(i)* and *(ii)*, the expected utility of bidder j bidding $\beta_j\left(\overline{z}\right)$ is given by

$$\begin{aligned}
&U_j^{\boldsymbol{\beta}}\left(\overline{z}, \beta_j\left(\overline{z}\right)\right) \\
&= F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-j}}}[x_j\left(\beta_j\left(\overline{z}\right), \boldsymbol{\beta_{-j}}\left(\boldsymbol{v_{-j}}\right)\right) \mid \beta_j\left(\overline{z}\right) > \beta_i\left(v_i\right)] \left(\overline{z} - p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right) \\
&- \Big(1 - F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-j}}}[x_j\left(\beta_j\left(\overline{z}\right), \boldsymbol{\beta_{-j}}\left(\boldsymbol{v_{-j}}\right)\right) \mid \beta_j\left(\overline{z}\right) > \beta_i\left(v_i\right)]\Big) p^{lose}\left(\beta_j\left(\overline{z}\right)\right).
\end{aligned}$$

Since $\overline{z} = \overline{v}$, by bidding b bidder i will bid higher than bidder j with probability one. Thus, the expected utility of bidder i bidding b at $\overline{z}$ is given by

$$\begin{aligned}
U_i^{\boldsymbol{\beta_{-i}}}\left(\overline{z}, b\right) &= E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b > \beta_j\left(v_j\right)] \left(\overline{z} - p^{win}\left(b\right)\right) \\
&\quad - \left(1 - E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b > \beta_j\left(v_j\right)]\right) p^{lose}\left(b\right)
\end{aligned}$$

$$\begin{aligned}
&\geq F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b > \beta_j\left(v_j\right)] \left(\overline{z} - p^{win}\left(b\right)\right) \\
&\quad - \Big(1 - F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b > \beta_j\left(v_j\right)]\Big) p^{lose}\left(b\right)
\end{aligned}$$

$$\begin{aligned}
&> F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b > \beta_j\left(v_j\right)] \left(\overline{z} - \left(p^{win}\left(\beta_j\left(\overline{z}\right)\right) + \epsilon\right)\right) \\
&\quad - \Big(1 - F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b > \beta_j\left(v_j\right)]\Big) \left(p^{lose}\left(\beta_j\left(\overline{z}\right)\right) + \epsilon\right)
\end{aligned}$$

$$\begin{aligned}
&\geq F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b > \beta_j\left(v_j\right)] \left(\overline{z} - p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right) \\
&\quad - \Big(1 - F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-i}}}[x_i\left(b, \boldsymbol{\beta_{-i}}\left(\boldsymbol{v_{-i}}\right)\right) \mid b > \beta_j\left(v_j\right)]\Big) p^{lose} \beta_j\left(\overline{z}\right) - \epsilon
\end{aligned}$$

$$= F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-j}}}[x_j\left(\beta_j\left(\overline{z}\right), \boldsymbol{\beta_{-j}}\left(\boldsymbol{v_{-j}}\right)\right) \mid \beta_j\left(\overline{z}\right) > \beta_i\left(v_i\right)]\left(\overline{z} - p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right)$$
$$- \Big(1 - F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{\boldsymbol{v_{-j}}}[x_j\left(\beta_j\left(\overline{z}\right), \boldsymbol{\beta_{-j}}\left(\boldsymbol{v_{-j}}\right)\right) \mid \beta_j\left(\overline{z}\right) > \beta_i\left(v_i\right)]\Big) p^{lose}\beta_j\left(\overline{z}\right) - \epsilon.$$

Hence, we have shown that for every $\epsilon > 0$ there exists a deviating bid b such that bidder i can achieve an expected utility of at least $U_j^{\boldsymbol{\beta}}\left(\overline{z}\right) - \epsilon$ from which follows that

$$U_i^{\boldsymbol{\beta}}\left(\overline{z}\right) \geq U_j^{\boldsymbol{\beta}}\left(\overline{z}\right).$$

□

We continue with the proof of Proposition 5 by considering the following three cases.

Case 1: There exists an interval (v', v) such that $\beta_j(z) > \beta_i(z)$ for all $z \in (v', v)$. In this case, let

$$\overline{z} = \inf\{z > v \mid \beta_i(z) \geq \beta_j(v)\}.$$

If the infimum does not exist, we redefine $\overline{z} = \overline{v}$.

It follows from Lemma 14 that

$$\int_0^{\overline{z}} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz = U_j^{\boldsymbol{\beta}}\left(\overline{z}\right) - U_i^{\boldsymbol{\beta}}\left(\overline{z}\right) \leq \Delta_{i,j}\overline{z}.$$

It holds that

$$\int_0^{v} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz = \int_0^{\overline{z}} X_j^{\boldsymbol{\beta}}(z) - X_j^{\boldsymbol{\beta}}(z)\, dz - \int_v^{\overline{z}} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz$$
$$= \int_0^{\overline{z}} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz + \int_v^{\overline{z}} X_i^{\boldsymbol{\beta}}(z) - X_j^{\boldsymbol{\beta}}(z)\, dz$$
$$\leq \Delta_{i,j}\overline{z} + \int_v^{\overline{z}} X_i^{\boldsymbol{\beta}}(z) - X_j^{\boldsymbol{\beta}}(z)\, dz$$
$$\leq \Delta_{i,j}\overline{z} + \Delta_{i,j}\left(\overline{z} - v\right).$$

Due to $\beta_j(z) > \beta_i(z)$, for all $z \in (v, \overline{z})$, the last inequality follows from Lemma 13.

We conclude that the assumption that

$$\int_0^{v} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz > \Delta_{i,j}\overline{v} + \Delta_{i,j}\left(\overline{v} - v\right)$$

leads to a contradiction.

Case 2: There exists an interval (v', v) such that $\beta_i(z) > \beta_j(z)$ for all $z \in (v', v)$.

It follows from Lemma 14 that

$$U_i^{\boldsymbol{\beta}}(v') + \Delta_{i,j} v' \geq U_j^{\boldsymbol{\beta}}(v'). \tag{18}$$

Therefore, it holds that

$$\int_0^{v'} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz \leq \Delta_{i,j} v'.$$

It follows from the fact that $\beta_i(z) > \beta_j(z)$ for all $z \in (v', v)$ and from Lemma 13 that

$$\int_{v'}^{v} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz \leq \Delta_{i,j}(v - v').$$

Therefore, we can conclude that

$$\int_0^{v} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz = \int_0^{v'} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz + \int_{v'}^{v} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz$$
$$\leq \Delta_{i,j} v' + \Delta_{i,j}(v - v') \leq \Delta_{i,j}\overline{v} + \Delta_{i,j}(\overline{v} - v)$$

which leads to a contradiction.

Case 3: There exists an interval (v', v) such that $\beta_i(z) = \beta_j(z)$ for all $z \in (v', v)$. Since the bidding functions of bidders i and j are continuous except on a set of valuations with measure zero, cases 1, 2, and 3 constitute all possible cases. It follows from Lemma 14 that

$$U_i^{\boldsymbol{\beta}}(v) + \Delta_{i,j} v \geq U_j^{\boldsymbol{\beta}}(v). \tag{19}$$

Thus, it holds that

$$\int_0^{v} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz \leq \Delta_{i,j} v \leq \Delta_{i,j}\overline{v} + \Delta_{i,j}(\overline{v} - v)$$

which leads to a contradiction.

We conclude that the assumption that

$$\int_0^{v} X_j^{\boldsymbol{\beta}}(z) - X_i^{\boldsymbol{\beta}}(z)\, dz > \Delta_{i,j}\overline{v} + \Delta_{i,j}(\overline{v} - v)$$

leads to a contradiction which completes the proof of the claim in (13) and hence the proof of Proposition 5 for pure strategy equilibria.

It is left to consider the case of mixed equilibria. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ be a (possibly mixed) equilibrium. According to Lemma 8, there exists a pure strategy equilibrium $\boldsymbol{\beta}' = (\beta'_1, \ldots, \beta'_n)$ such that it holds for all $i \in \{1, \ldots, n\}$ and for all $v \in \{1, \ldots, n\}$ that

$$X_i^{\boldsymbol{\beta}}(v) = X_i^{\boldsymbol{\beta}'}(v)$$

except on a set of valuations with measure zero. Since we have shown Proposition 5 for pure strategy equilibria, it holds that

$$\left| \int_0^v X_j^{\boldsymbol{\beta}'}(z)\, dz - \int_0^v X_i^{\boldsymbol{\beta}'}(z)\, dz \right| \leq \Delta_{i,j}\overline{v} + \Delta_{i,j}(\overline{v} - v).$$

Thus, for every $v \in [0, \overline{v}]$ it holds that

$$|U_i^{\boldsymbol{\beta}}(v) - U_j^{\boldsymbol{\beta}}(v)| = \left| \int_0^v X_j^{\boldsymbol{\beta}}(z)\, dz - \int_0^v X_i^{\boldsymbol{\beta}}(z)\, dz \right|$$

$$= \left| \int_0^v X_j^{\boldsymbol{\beta}'}(z)\, dz - \int_0^v X_i^{\boldsymbol{\beta}'}(z)\, dz \right| \leq \Delta_{i,j}\overline{v} + \Delta_{i,j}(\overline{v} - v).$$

This completes the proof.

□

I Proof of Proposition 6

Proof. Let A be a mechanism and i a bidder with valuation v. Let $\boldsymbol{\beta}^{\Delta}$ denote an equilibrium of mechanism A for a given Δ and let $U_i\left(v, \boldsymbol{\beta}^{\Delta}\right)$ denote the expected utility of bidder i with valuation v in the equilibrium $\boldsymbol{\beta}^{\Delta}$. If Δ equals to zero, then we can deduce from the Revenue Equivalence Theorem (e.g. as stated in Krishna (2009)) that for every mechanism A and every strictly increasing equilibrium $\boldsymbol{\beta}$ of A the expected utility of bidder i with valuation v is given by

$$U_i^{\boldsymbol{\beta}}(v) = vX_i^{\boldsymbol{\beta}}(v) - P_i^{\boldsymbol{\beta}}(v) = vG(v) - \int_0^v zg(z)\, dz$$

where $G(v) = H^{n-1}(v)$ and $g(v)$ denotes the corresponding density. That is, for $\Delta = 0$ the expression $U_i\left(v, \boldsymbol{\beta}^{\Delta}\right)$ neither depends on the mechanism nor on the equilibrium and therefore can be denoted by $U_i(v, 0)$.[24]

It holds that

$$\sum_{i=1}^{n} \frac{d}{d\Delta} U_i\left(v, \boldsymbol{\beta}^{\Delta}\right)\Big|_{\Delta=0} = \sum_{i=1}^{n} \frac{d}{d\Delta} \int_0^v X_i\left(z, \boldsymbol{\beta}^{\Delta}\right) dz\Big|_{\Delta=0}.$$

[24]Recall that we assume strictly increasing equilibria.

According to Lemma 1 in Fibich et al. (2004), this is equal to

$$\int_0^v (n-1) H^{n-2}(z) \sum_{i=1}^n H_i(z)\, dz.$$

Let $\mathscr{P}(\Delta^2) = \{\sum_{i=2}^{\infty} c_i \Delta^i \mid c_i \in \mathbb{R}\}$. As in the proof of Theorem 1 in Fibich et al. (2004), we use the Taylor series in order to conclude that for a given equilibrium $\boldsymbol{\beta}$ and a given Δ it holds that

$$\sum_{i=1}^n U_i^{\boldsymbol{\beta}}(v) = \sum_{i=1}^n U_i\left(v, \boldsymbol{\beta}^{\Delta}\right) = \sum_{i=1}^n U_i\left(v, \boldsymbol{\beta}^{0}\right) + \Delta \frac{d}{d\Delta} \sum_{i=1}^n U_i\left(v, \boldsymbol{\beta}^{\Delta}\right) \Big|_{\Delta=0} + P\left(\Delta^2\right)$$

$$= \sum_{i=1}^n U_i(v, 0) + \Delta \int_0^v (n-1) H^{n-2}(z) \sum_{i=1}^n H_i(z)\, dz + P\left(\Delta^2\right)$$

for some $P(\Delta^2) \in \mathscr{P}(\Delta^2)$. Here the term $P(\Delta^2)$ may depend on the particular mechanism while

$$\mathcal{U}(v) := \sum_{i=1}^n U_i(v, 0) + \Delta \int_0^v (n-1) H^{n-2}(z) \sum_{i=1}^n H_i(z)\, dz$$

does not. It follows from Proposition 5 that for every $j \neq i$ it holds that

$$\left| U_j^{\boldsymbol{\beta}}(v) - U_i^{\boldsymbol{\beta}}(v) \right| \leq \Delta_{i,j} + \Delta_{i,j}(\overline{v} - v) \leq \Delta + \Delta(\overline{v} - v).$$

It follows for every bidder i that

$$\left| \sum_{i=1}^n U_i^{\boldsymbol{\beta}}(v) - n U_i^{\boldsymbol{\beta}}(v) \right| \leq n\Delta + n\Delta(\overline{v} - v).$$

Hence it holds that

$$\left| \mathcal{U}(v) + P\left(\Delta^2\right) - n U_i^{\boldsymbol{\beta}}(v) \right| \leq n\Delta + n\Delta(\overline{v} - v).$$

$$\Leftrightarrow n U_i^{\boldsymbol{\beta}}(v) - n\left(\Delta\overline{v} + \Delta(\overline{v} - v)\right) \leq \mathcal{U}(v) + P\left(\Delta^2\right) \leq n U_i^{\boldsymbol{\beta}}(v) + n\left(\Delta\overline{v} + \Delta(\overline{v} - v)\right)$$

$$\Leftrightarrow \frac{1}{n}\left(\mathcal{U}(v) + P\left(\Delta^2\right) - n\left(\Delta\overline{v} + \Delta(\overline{v} - v)\right)\right) \leq U_i^{\boldsymbol{\beta}}(v) \leq \frac{1}{n}\left(\mathcal{U}(v) + P\left(\Delta^2\right) + n\left(\Delta\overline{v} + \Delta(\overline{v} - v)\right)\right).$$

Let B be a mechanism with equilibrium $\boldsymbol{\beta'}$. Since the same statement holds for equilibrium $\boldsymbol{\beta'}$, it follows that

$$\left|U_i(v)^{\boldsymbol{\beta}} - U_i(v)^{\boldsymbol{\beta'}}\right| \leq 2\left(\Delta\overline{v} + \Delta\left(\overline{v} - v\right)\right) + P\left(\Delta^2\right).$$

□

J Proof of Propositions 7 and 8

Proof. First, we will show Proposition 7 for pure strategy equilibria and afterwards apply Lemma 8 in order to derive the result for mixed strategy equilibria. We will prove Proposition 7 for pure strategy equilibria by contradiction. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ be an efficient equilibrium of an imitation-perfect auction. Let bidders i and j be such that

$$\int_{[0,\overline{v}]^{n-1}} \boldsymbol{F_{-j}}(z)\,dz > \int_{[0,\overline{v}]^{n-1}} \boldsymbol{F_{-i}}(z)\,dz.$$

Since the equilibrium is efficient, it follows from Myerson (1981) that

$$\begin{aligned} U_j^{\boldsymbol{\beta}}(\overline{v}) = \int_0^{\overline{v}} X_j(z)\,dz = \int_0^{\overline{v}} \boldsymbol{F_{-j}}(z)\,dz \\ > \int_0^{\overline{v}} \boldsymbol{F_{-i}}(z)\,dz = \int_0^{\overline{v}} X_i(z)\,dz = U_i^{\boldsymbol{\beta}}(\overline{v}). \quad (20) \end{aligned}$$

According to Lemma 14, it holds that $U_j^{\boldsymbol{\beta}}(\overline{v}) = U_i^{\boldsymbol{\beta}}(\overline{v})$ which leads to a contradiction. This completes the proof for pure strategy equilibria. It is left to consider the case of mixed strategy equilibria. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ be a (possibly mixed) equilibrium. According to Lemma 8, there exists a pure strategy equilibrium $\boldsymbol{\beta'} = (\beta'_1, \ldots, \beta'_n)$ such that it holds for all $i \in \{1, \ldots, n\}$ and for all $v \in \{1, \ldots, n\}$ that

$$X_i^{\boldsymbol{\beta}}(v) = X_i^{\boldsymbol{\beta'}}(v)$$

except a set of valuations with measure zero. Since the equilibrium $\boldsymbol{\beta'}$ is efficient, for every pair of bidders i and j and for every pair of valuations v_i and v_j such that $v_j > v_i$ it holds that $X_i^{\boldsymbol{\beta'}}(v_i) < X_j^{\boldsymbol{\beta'}}(v_j)$. Therefore, it holds that $X_i^{\boldsymbol{\beta}}(v_i) < X_j^{\boldsymbol{\beta}}(v_j)$ except a measure zero set of valuations. Conclusively, given equilibrium $\boldsymbol{\beta}$, the bidder with the highest valuation wins with probability one and the same reasoning as above applies.

The proof of Proposition 8 works in the same way with the only difference being that valuations are replaced with the corresponding virtual valuations. Assume there exists a pure strategy equilibrium $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ of an imitation-perfect auction such that the bidder with the highest virtual valuation wins with probability 1. Let bidders i and j be such that

$$\int_0^{\overline{v}} F_i\left(V_i^{-1}\left(V_j\left(z\right)\right)\right) dz > \int_0^{\overline{v}} F_j\left(V_j^{-1}\left(V_i\left(z\right)\right)\right) dz.$$

Since the bidder with the highest virtual valuation wins with probability 1, it follows from Myerson (1981) that

$$\begin{aligned} U_j^{\boldsymbol{\beta}}\left(\overline{v}\right) = \int_0^{\overline{v}} X_j\left(z\right) dz = \int_0^{\overline{v}} \prod_{k\neq j} F_k\left(V_k^{-1}\left(V_j\left(z\right)\right)\right)\left(z\right) dz \\ > \int_0^{\overline{v}} \prod_{k\neq i} F_k\left(V_k^{-1}\left(V_i\left(z\right)\right)\right) dz = \int_0^{\overline{v}} X_i\left(z\right) dz = U_i^{\boldsymbol{\beta}}\left(\overline{v}\right). \end{aligned}$$

As before, one can show that this leads to a contradiction since the expected utilities of bidders i and j at $\overline{v}$ are equal. The same reasoning as above applies to mixed strategy equilibria.

$\square$

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