Mechanism Design with Financially Constrained Agents and Costly Verification*

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December 31, 2017

Abstract

A principal wishes to distribute an indivisible good to a population of budget-constrained agents. Both valuation and budget are an agent's private information. The principal can inspect an agent's budget through a costly verification process and punish an agent who makes a false statement. I characterize the direct surplus-maximizing mechanism. This direct mechanism can be implemented by a two-stage mechanism in which agents only report their budgets. Specifically, all agents report their budgets in the first stage. The principal then provides budget-dependent cash subsidies to agents and assigns the goods randomly (with uniform probability) at budget-dependent prices. In the second stage, a resale market opens, but is regulated with budget-dependent sales taxes. Agents who report low budgets receive more subsidies in their initial purchases (the first stage), face higher taxes in the resale market (the second stage) and are inspected randomly. This implementation exhibits some of the features of some welfare programs, such as the affordable housing program in Singapore.

Keywords: Mechanism Design, Budget Constraints, Efficiency, Costly Verification

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*This paper is based on a chapter of my dissertation at University of Pennsylvania. I would like to express my gratitude to Rakesh Vohra for his encouraging support from the very beginning of the project. I am also grateful to Arjada Bardhi, Aislinn Bohren, Hanming Fang, Qing Gong, Daniel Hauser, Ilwoo Hwang, Nick Janetos, Shinsuke Kambe, George Mailath, Steven Matthews, Mariann Ollar, Mallesh Pai, Andrew Postlewaite, Debraj Ray, Nishant Ravi, Michael Richter, Phillip Strack and the participants in seminars at University of Pennsylvania, City University of Hong Kong, Tsinghua University, Hong Kong Baptist University, The University of Hong Kong, the 11th Japan-Taiwan-Hong Kong Contract Theory Conference, the 18th ACM Conference on Economics and Computation, 2016 NASM of the Econometric Society, the 27th International Conference on Game Theory and EconCon 2016 for valuable discussions. All remaining errors are my sole responsibility.

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1 Introduction

Governments around the world allocate a variety of valuable resources to agents who are financially constrained. In Singapore, for example, 80% of the population's housing needs are met by the Housing and Development Board (HDB), a government agency founded in 1960 to provide affordable housing.[1] In the United States, Medicaid has provided health care to individuals and families with low income and limited resources since 1965. Medicaid currently accounts for 16.1% of the state general funds[2] and provides health coverage to 80 million low-income people.[3] Similar public housing and social health care programs prevail in many other countries. In China, several cities limit the supply of vehicle licenses to curb the growth in private vehicles, and different cities have implemented different mechanisms. For example, Shanghai allocates vehicle licenses through an auction-like mechanism, while Beijing uses a vehicle license lottery (see Rong, Sun, and Wang 2015). The evaluation of existing mechanisms has attracted attention from researchers and policymakers.

One justification for this role of governments is that a competitive market outcome will not maximize social surplus if households are financially constrained. Financial constraints mean that in a competitive market some households with high valuations will not obtain resources, while households with low valuations but access to cash will. The natural question arises as to what the surplus-maximizing (or optimal) mechanism is in these circumstances when both valuations and financial constraints are households' private information.

The mechanism design literature concerning this question has focused on mechanisms with only monetary transfers and has ignored the possibility of governments (or the principal) verifying the private information supplied by households (or agents). Indeed, in many instances, governments rely on households' reports of their abilities to pay when deciding allocations, and governments can verify this information and punish an household who makes a false statement. For example, applicants for HDB flats in Singapore and Medicaid in the United States are subject to a set of eligibility conditions on age, family nucleus, monthly income, and so on. Verification presumably allows governments to more accurately target the individuals in need for help. However, verification process can be costly. First, in some developing countries, verifiable records on household income or wealth are rarely available,

[1] http://www.hdb.gov.sg/cs/infoweb/about-us

[2] http://ccf.georgetown.edu/wp-content/uploads/2012/03/Medicaid-state-budgets-2005.pdf

[3] http://www.cbpp.org/research/health/policy-basics-introduction-to-medicaid?fa=view&id=2223

and governments lack the administrative capacity to process this information. In such cases, alternative verification methods such as a visit to a household to inspect the visible living conditions are not uncommon but are often costly (see Coady, Grosh, and Hoddinott 2004). Second, certain types of income such as tips, side-jobs and cash receipts are costly to verify. Similarly, governments have few ways to verify the income reports by individuals who are self-employed or run small business without performing a costly investigation. Third, households may be financially constrained due to limited access to financial markets or high expenditures, such as medical expenses or education costs. This information is often costly for governments to verify. Last but not least, even if the verification cost for one individual is low, the total cost can be substantial for a large population.

Hence, it is important to explore how the option of costly verification affects the optimal mechanism. Verification allows the principal to better target financially constrained agents and potentially improve their welfare. However, verification is costly and reduces the amount of money available for subsidies. The principal must now trade allocative efficiency for verification cost. The cost of verification also influences whether the principal chooses to use *cash subsidies* or *in-kind subsidies* (the provision of goods at discounted prices). The latter is less efficient because it often involves rationing, but saves verification cost because it only benefits low-budget agents with high valuations.

To study these questions, I consider a mechanism design problem in which there is a unit mass of a continuum of agents and a limited supply of indivisible goods. Each agent has two-dimensional private information — his valuation $v \in [\underline{v}, \overline{v}]$ and his exogenous budget constraint b. The budget constraint is a hard one in the sense that agents cannot be compelled to pay more than their budgets. For simplicity, I assume that there are only two possible types of budgets, $b_2 > b_1$. The principal can inspect an agent at a cost, perfectly revealing his budget, and impose a penalty on detected misreporting. The principal is also subject to a budget balance constraint which requires that the revenue from selling goods must exceed the total inspection cost. This constraint rules out the possibility that the principal can simply inject money and relieve all budget constraints.

I first focus on direct mechanisms in which each agent reports his private information directly and is punished if and only if he is found to have lied about his budget. Given their reports, a direct mechanism specifies for each agent his probability of receiving one unit good, his payment and his probability of being inspected. I characterize the optimal direct mechanism which maximizes utilitarian efficiency among all mechanisms that are incentive compatible and individually rational, and that satisfy the resource constraint, agents' budget constraints and the principal's budget balance constraint.

The unique optimal direct mechanism can be described as follows. Let $u(\underline{v}, b)$ denote the utility of an agent with the lowest valuation $\underline{v}$ and budget b, which is also the amount of cash subsidies received by an agent with budget b. There exist three cutoffs $v_1^* \leq v_2^* \leq v_2^{**}$. Firstly, low-budget agents whose valuations are below v_1^* and high-budget agents whose valuations are below v_2^* receive only cash subsidies. Not surprisingly, these low-budget agents receive higher cash subsidies ($u(\underline{v}, b_1) \geq u(\underline{v}, b_2)$) and are inspected with a probability proportional to the difference in cash subsidies between the two budget types ($u(\underline{v}, b_1) - u(\underline{v}, b_2)$). Secondly, a low-budget agent whose valuations exceed v_1^* receives one unit good with probability $a^* \leq 1$ and makes a payment of $a^* v_1^* - u(\underline{v}, b_1)$. High-budget agents whose valuations lie in $[v_2^*, v_2^{**}]$ are pooled with low-budget agents whose valuations are above v_1^*. They also receive goods with probability a^*, but pay more: $a^* v_2^* - u(\underline{v}, b_2)$. These low-budget agents are inspected with a probability proportional to the difference in payments between the two budget types ($u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^*(v_2^* - v_1^*)$). Finally, high-budget agents receive goods for sure and pay $v_2^{**} - u(\underline{v}, b_2)$ if their valuations exceed v_2^{**}. Compared with these high-budget agents who receive goods for sure, low-budget agents whose valuations are above v_1^* and high-budget agents whose valuations lie in $[v_2^*, v_2^{**}]$ pay a lower per-unit price, which is a form of in-kind subsidies.

In a direct mechanism, agents report their budgets as well as valuations. But in practice households are rarely asked to report their valuations. As the second main result of the paper, I show that the optimal direct mechanism can be implemented by a simple two-stage mechanism in which agents report only their budgets. Specifically, all agents report their budgets in the first stage. The principal then provides budget-dependent cash subsidies to agents and the opportunities to participate in a lottery at budget-dependent prices. the principal then assigns the goods randomly (with uniform probability) among all lottery participants. Agents who report low budgets receive higher cash subsidies and face lower prices. In the second stage, a resale market opens, but is regulated with budget-dependent sales taxes. Agents who report low budgets in the first stage are subject to higher sales taxes. Only agents who report low budgets are inspected randomly. Unlike the case without inspection, in which all agents are subsidized and regulated equally regardless of their budgets, the principal now provides more subsidies to low-budget agents in their initial purchases (the first stage) and imposes more restrictions on them in the resale market (the second stage). Although in my analysis the principal's objective is to maximize social surplus, I conjecture that these features would continue to apply when the principal wants to

Table 1: Minimum occupation periods (MOP) of housing and development board (HDB) flats

Types of HDB flats	MOP	
	Sell	Sublet
Resale flats w/ Grants	5–7 years	5–7 years
Resale flats w/o Grants	0–5 years	3 years

Sources. — Sell: http://www.hdb.gov.sg/cs/infoweb/residential/selling-a-flat/eligibility; and Sublet: http://www.hdb.gov.sg/cs/infoweb/residential/renting-out-a-flat-bedroom/renting-out-your-flat/eligibility.

benefit only low-budget agents.

If we interpret the sales tax as a form of restriction in the resale market, this implementation exhibits some features of the affordable housing program in Singapore, as shown in Table 1. In Singapore, HDB flats are sold on a 99-year lease agreement and buyers of resale HDB flats can apply for additional housing grants. If these flats are purchased with housing grants, these buyers are required to reside in their flats for at least 5 years before they could resell or sublet. In contrast, flats purchased without housing grants are subject to no requirement or a shorter one. Although the more familiar form of affordable housing programs in many other countries is public rental housing, it is not uncommon for governments to help households gain home ownership (see, for example, Andersson, Ehlers, and Svensson (2016) and Marcantel (2014) for similar programs in Europe and the U.S.). The result of this paper can help us evaluate these existing programs as well as design similar programs in the future.

Having solved the optimal mechanism, it is interesting to see how verification cost, the supply of goods and other parameters affect the optimal mechanism and agents' welfare. In the paper, I provide analytic results of comparative statics for some extreme cases, such as when verification cost is sufficiently large and the supply of goods is sufficiently large or small, and explore the intermediate cases numerically.

Verification allows the principal to better target low-budget agents and improves their welfare. Intuitively, as verification becomes costly, the principal tends to provide relatively smaller subsidies to low-budget agents and inspect them less frequently. More interestingly, the change in verification cost affects that mechanism's reliance on cash and in-kind subsidies. If verification is cheap, then the principal achieves efficiency mainly by offering more cash subsidies to low-budget agents. As verification becomes costly, the difference in cash subsidies between the two budget types declines, but the difference in in-kind subsidies increases. This is because in-kind subsidies are attractive only to high-valuation agents, which is cheaper in terms of verification cost. Eventually, the difference in in-kind subsidies also declines as verification becomes sufficiently costly.

Another interesting observation is that although an increase in the supply of goods improves the total welfare, its impact on the welfare of each budget type is not monotonic. This is because an increase in the supply has two opposite effects. On the one hand, the principal becomes less budget constrained, and can direct more subsidies to low-budget agents and inspect them more frequently. On the other hand, low-budget agents also become less budget constrained as "price" declines, which reduces the needs to subsidize and inspect them. As a result, the differences in cash and in-kind subsidies and the inspection probability are hump-shaped. Initially, the welfare of both budget types increases as the supply increases. When the supply is large enough that the principal can afford to provide more subsidies to low-budget agents, the welfare of high-budget agents begins to decrease. Eventually, the need to subsidize low-budget agents decreases as the supply increases. As a result, the welfare of low-budget agents begins to decrease and that of high-budget agents begins to increase, until they coincide.

Introducing costly verification is technically challenging because it is no longer sufficient to consider "local" incentive compatibility (IC) constraints. Because the IC constraints between distant types can also bind, one cannot anticipate a priori the set of binding IC constraints. More specifically, if each agent has only one-dimensional private information (i.e., valuation), then it is sufficient to consider adjacent IC constraints; if each agent has two-dimensional private information but the principal cannot inspect budgets, then it is sufficient to consider two one-dimensional deviations. These, however, no longer apply in the case that each agent has two dimensional private information and the principal can inspect budget at a cost. In this case, in addition to downward adjacent IC constraints of misreporting values, one must consider deviations in which an agent can misreport both dimensions of his private information. As a result, the local approach commonly used does not work here.

In this paper, I develop a novel method that can potentially be used in solving other mechanism design problems with multidimensional types. First, I restrict attention to a class of allocation rules that have enough structures to help me keep track of binding IC constraints, and that are also rich enough to approximate any general allocation rule well. Specifically, I approximate the allocation rule of each budget type using step functions.

When restricting attention to step functions, binding IC constraints corresponding to the under-reporting of budgets are between different budget types whose valuations are the jump discontinuity points of their allocation rules. This structure allows me to write the optimal inspection rule as a function of the possible values and the jump discontinuity points of the allocation rule. I then solve a modified problem of the principal in which the allocation rule of low-budget types are restricted to take at most M distinct values. Finally, because for M sufficiently large step-functions can approximate the optimal allocation rule arbitrarily well, I can obtain a characterization of the optimal mechanism in the limit.

The rest of the paper is organized as follows. Section 1.1 discusses related work. Section 2 presents the model. Section 3 characterizes the direct optimal mechanism when all agents' budget constraints are common knowledge. Section 4 characterizes the direct optimal mechanism when an agent's budget is his private information. Section 5 provides a simple implementation. Section 6 studies the properties of the optimal mechanism. Section 7 considers various extensions of the model. Section 8 concludes. All the proofs are relegated to the appendix.

1.1 Related literature

This paper is mainly related to two branches of literature. First, it contributes to the literature studying mechanism design problems when agents are financially constrained by incorporating costly verification. Prior work analyzes the revenue or efficiency of a given mechanism or the design of an optimal mechanism when either budgets are common knowledge, or budgets are agents' private information but cannot be verified. See Che and Gale (1998, 2006, 2000), Laffont and Robert (1996), Maskin (2000), Benoit and Krishna (2001), Brusco and Lopomo (2008), Malakhov and Vohra (2008) and Pai and Vohra (2014).

In this first branch of literature, the two closest papers to the current paper are Che, Gale, and Kim (2013) and Richter (2015). In Che, Gale, and Kim (2013) and Richter (2015), like in this paper, there is a unit mass of a continuum of agents and a limited supply of goods. In Richter (2015) agents have linear preferences for an unlimited supply of goods. He finds that both the revenue-maximizing mechanism and surplus-maximizing mechanism feature a linear price for goods. In addition, the surplus-maximizing mechanism has a uniform cash subsidy. In both Che, Gale, and Kim (2013) and this paper, each agent has a unit demand for an indivisible good, and the surplus-maximizing mechanism can be implemented via a random assignment with a regulated resale and cash subsidy scheme. However, Che, Gale, and Kim (2013) does not consider the possibility that the principal can verify an agent's budget at a cost. This feature also distinguishes the current paper from all the other papers on mechanism design with financially constrained agents. Che, Gale, and Kim (2013) first compare three different methods of assigning goods when agents have a continuum of possible valuations and a continuum of possible budgets, and then characterize the optimal mechanism in a simple 2×2 model, in which each agent has two possible valuations and two possible budgets. In the presence of costly verification, unlike Che, Gale, and Kim (2013), in which all agents are subsidized and regulated equally regardless of their budgets in an optimal mechanism, I show that an optimal mechanism provides more subsidies to low-budget agents in their initial purchases and imposes more restrictions on them in the resale market.

Second, this paper is related to the costly state verification literature. The first significant contribution to this series is from Townsend (1979), who studies a model of a principal and a single agent. In Townsend (1979) verification is deterministic. Border and Sobel (1987) and Mookherjee and Png (1989) generalize it by allowing random inspection. Gale and Hellwig (1985) consider the effects of costly verification in the context of credit markets. Recently, Ben-Porath, Dekel, and Lipman (2014) study the allocation problem in the costly state verification framework when there are multiple agents and monetary transfer is not possible. Li (2016) extends Ben-Porath, Dekel, and Lipman (2014) to environments in which the principal's ability to punish an agent is sufficiently limited. These models differ from what I consider here in that in their models each agent has only one-dimensional private information.

This paper is also somewhat related to the matching literature which studies the allocation of public housing. In Andersson, Ehlers, and Svensson (2016), the number of houses and agents coincide, but agents have different preferences over houses. In the current paper, all goods are homogeneous but the supply of goods is limited. This paper also differs from the above literature by considering agents' financial constraints.

2 Model

There is a unit mass of a continuum of agents. There is a mass $S \in (0, 1)$ of indivisible goods.[4] Each agent has a private valuation of the good $v \in V := [\underline{v}, \overline{v}] \subset \mathbb{R}_+$, and a

[4]The model is also applicable to divisible goods when an agent's per-unit value for the good is constant up to an upper bound.

privately known budget $b \in B := \{b_1, b_2\}$. I assume that $b_1 > \underline{v}$ and $b_2 > \overline{v}$.[5] Thus, a high-budget agent is never budget constrained in an individually rational mechanism. The type of an agent is a pair consisting of his valuation and his budget: $t := (v, b)$; and the type space is $T := V \times B$.

I assume v and b are independent. Each agent has a high budget with probability π and a low budget with probability $1 - \pi$. The valuation v is distributed with cumulative distribution function F and strictly positive density f.

The principal can inspect an agent's budget at a cost $k \geq 0$, and can impose a penalty $c > 0$. Inspection perfectly reveals an agent's budget.[6] I assume that the penalty c is large enough that an agent never find it optimal to misreport his budget if he is certain that he will be inspected. For the main body of the paper, I assume that the penalty is not transferable. In Section 7.2, I study the case in which penalty is transferable and show that all results hold in that case. For later use, let $\rho := k/c$. As it will become clear, ρ measures the "effective" inspection cost to the principal. The cost to an agent to have his report verified is zero. This assumption is reasonable if the goods are valuable to agents and disclosure costs are negligible. In Section 7.3, I discuss what happens if it is also costly for an agent to have his report verified.

The usual version of the revelation principle (see, e.g., Myerson 1979 and Harris and Townsend 1981) does not apply to models with verification. However, it is not hard to extend the argument to this type of environment.[7] Specifically, I show in Appendix A that it is without loss of generality to restrict attention to direct mechanisms. Furthermore, I assume that the principal can only punish an agent who is inspected and found to have lied about his budget. This assumption, however, is not without loss of generality. Roughly speaking, if we relax this assumption, in an optimal mechanism the principal will sometimes punish an low-budget agent without verifying his budget. In this case, punishment plays a similar role as "red tape" in Banerjee (1997) and is used to screen agents with different valuations when their valuations exceed their abilities to pay.[8] In this paper I want to first abstract way from this role of punishment.

A direct mechanism is a triple (a, p, q), where $a : T \to [0, 1]$ is the allocation rule,

[5]All the results can be easily extended to any $b_1 \geq 0$. In the paper, I assume $b_1 > \underline{v}$ to make the statement more concise.

[6]The paper's results will not change if the principal cannot detect a lie with some probability.

[7]See Townsend (1988) and Ben-Porath, Dekel, and Lipman (2014) for more discussion and extension of the revelation principle to various verification models, not including the environment considered in this paper.

[8]Note that this argument is valid only if penalty is not transferable. Indeed, if penalty is transferable, this assumption is without loss of generality as demonstrated in Section 7.2.

$p : T \to \mathbb{R}$ is the payment rule and $q : T \to [0,1]$ is the inspection rule. Specifically, for each reported type $t \in T$, $a(t)$ denotes the probability an agent obtains the good, $p(t)$ denotes the payment an agent must make and $q(t)$ denotes the probability of inspection. In this definition, I implicitly assume that payment rules are deterministic. I discuss random payment rules in Appendix B and show that it is without loss of generality to focus on deterministic payment rules.

The utility of an agent who has type $t := (v, b)$ and reports $\hat{t}$ is

$$u(\hat{t}, t) = \begin{cases} a(\hat{t})v - p(\hat{t}) & \text{if } \hat{b} = b \text{ and } p(\hat{t}) \leq b, \\ a(\hat{t})v - q(\hat{t})c - p(\hat{t}) & \text{if } \hat{b} \neq b \text{ and } p(\hat{t}) \leq b, \\ -\infty & \text{if } p(\hat{t}) > b. \end{cases}$$

That is, an agent has a standard quasi-linear utility up to his budget constraint, and cannot pay more than his budget.

With transferable utilities, the welfare criterion I use is simply utilitarian efficiency. For why utilitarian efficiency is a reasonable welfare criterion, see Vickrey (1945) and Harsanyi (1955). Given quasi-linear preferences, the total value realized minus total inspection cost is an equivalent criterion.[9] The principal's problem is[10]

$$\max_{a,p,q} \mathbb{E}_t \left[a(t)v - q(t)k \right], \tag{$\mathcal{P}$}$$

subject to

$$u(t) := u(t,t) \geq 0, \ \forall t \in T, \tag{IR}$$

$$u(t) \geq u(\hat{t}, t), \ \forall t \in T, \hat{t} \in \left\{ \hat{t} \in T \,\middle|\, p(\hat{t}) \leq b \right\}, \tag{IC}$$

$$p(t) \leq b, \ \forall t \in T, \tag{BC}$$

$$\mathbb{E}_t \left[p(t) - q(t)k \right] \geq 0, \tag{BB}$$

$$\mathbb{E}_t \left[a(t) \right] \leq S. \tag{S}$$

The individual rationality (IR) constraint requires that each agent gets a non-negative ex-

[9] To see this, consider a feasible mechanism (a, p, q). Note that if (a, p, q) maximizes welfare, then (BB) must hold with equality. Otherwise the principal can improve welfare through lump-sum transfers. Then the principal's objective function becomes $\mathbb{E}[u(t)] = \mathbb{E}[va(t) - p(t)] = \mathbb{E}[va(t) - q(t)k]$, where the last equality holds since (BB) holds with equality.

[10] There are some subtle issues regarding a continuum of random variables (see Judd (1985)). However, if we interpret the continuum model as an approximation of a large economy, then Al-Najjar (2004) makes the limiting argument rigorous.

pected payoff from participating in the mechanism. The incentive compatibility (IC) constraint requires that it is weakly better for an agent to report his true type than any other type whose transfers he can afford. The budget constraint (BC) states that an agent cannot be ask to make a payment larger than his budget b. To be clear, note that (BC) follows from (IR). This budget constraint is the same as that found in Che and Gale (2000) and Pai and Vohra (2014), but different from Che, Gale, and Kim (2013), who use a per unit price constraint.[11] I discuss the differences of the two frameworks in Section 7.1. The principal's budget balance (BB) constraint requires that the revenue raised from selling the goods must exceed the inspection cost. (BB) rules out the possibility that the principal can inject money and relieve all budget constraints. Finally, the limited supply (S) constraint requires that the amount of good assigned cannot exceed the supply. We say a mechanism (a, p, q) is *feasible* if it satisfies constraints (IR), (IC), (BC), (BB) and (S).

Throughout the paper, I assume that $S < 1 - F(b_1)$ since otherwise the first-best can be achieved via a competitive market. I also impose the following two assumptions throughout the paper.

Assumption 1 $\frac{1-F}{f}$ *is non-increasing.*

Assumption 2 f *is non-increasing.*

Assumption 1 is the standard monotone hazard rate condition, which is often adopted in the mechanism design literature. This assumption ensures that allocating more goods to agents with higher valuations rather than to those with lower valuations yeilds higher revenues for the principal. Assumption 2 says that agents are less likely to have higher valuations than to have lower valuations. These two assumptions are also imposed in Richter (2015) and Pai and Vohra (2014). These two assumptions are satisfied by some commonly used distributions such as uniform distributions, exponential distributions and left truncation of a normal distribution.

3 Common knowledge budgets

As a benchmark, I first analyze the case in which all agents' budgets are common knowledge. This case can be viewed as situations in which the principal can inspect agents' budgets for free or the penalty is infinitely large so that agents never lie about their budgets (i.e., $\rho = k/c = 0$).

[11]This constraint is called ex-post budget constraint in Che, Gale, and Kim (2013).

Since budgets are common knowledge, IC constraints hold as long as no agent has incentive to misreport his value:

$$a(v,b)v - p(v,b) \leq a(\hat{v},b)v - p(\hat{v},b), \ \forall v,\hat{v},b. \tag{IC-v}$$

The principal's problem becomes

$$\max_{a,p,q} \mathbb{E}_t \left[a(t)v\right], \tag{$\mathcal{P}_{CB}$}$$

subject to (IR), (IC-v), (BC), (S) and

$$\mathbb{E}_t[p(t)] \geq 0, \forall t \in T. \tag{BB$_{CB}$}$$

By the standard argument, (IC-v) holds if and only if, for all $b \in B$, $a(v,b)$ is non-decreasing in v and the envelope condition holds: $p(v,b) = a(v,b)v - \int_{\underline{v}}^{v} a(\nu,b)\mathrm{d}\nu - u(\underline{v},b)$ for all v. Since $a(v,b)$ is non-decreasing in v, the payment $p(v,b)$ is also non-decreasing in v. Hence, (BC) holds if and only if $p(\overline{v},b) \leq b$ for all b.

Let χ denote the characteristic function. The following theorem characterizes the optimal mechanism.

Theorem 1 *Suppose Assumption 2 holds, and budgets are common knowledge. There exist $v_1^*(0)$, $v_2^*(0)$, $u_1^*(0)$ and $u_2^*(0)$ such that an optimal mechanism of $\mathcal{P}_{CB}$ is given by*

$$\begin{aligned} a(v,b_1) &= \chi_{\{v\geq v_1^*(0)\}} a^*(0), \quad & p(v,b_1) &= \chi_{\{v\geq v_1^*(0)\}}(u_1^*(0)+b_1) - u_1^*(0), \\ a(v,b_2) &= \chi_{\{v\geq v_2^*(0)\}} 1, & p(v,b_2) &= \chi_{\{v\geq v_2^*\}} v_2^*(0), \end{aligned}$$

where $a^*(0) = \left[u_1^*(0)+b_1\right]/v_1^*(0)$, $b_1 < v_1^*(0) \leq v_2^*(0) < \overline{v}$ and $0 = u_2^*(0) < u_1^*(0) \leq v_1^*(0) - b_1$.

In notations $a^*(0)$, $v_i^*(0)$ and $u_i^*(0)$ $(i = 1,2)$, subscript i indicates the corresponding budget b_i and argument 0 indicates that this can be viewed as an optimal mechanism when $\rho = 0$.

As expected, when budgets are common knowledge, the two budget groups can be treated separately. Only low-budget agents receive positive cash subsidies aiming to relax their budget constraints: $u(\underline{v},b_1) = u_1^*(0) > 0 = u_2^*(0) = u(\underline{v},b_2)$. There are two cutoffs: $v_1^*(0) \leq v_2^*(0)$. All high-budget agents whose valuations are above $v_2^*(0)$ receive the good with probability one. This allocation can be implemented by posting a price $v_2^*(0)$ for high-budget agents. All low-budget agents whose valuations are above $v_1^*(0)$ receive the good with positive probability but are possibly rationed. The intuition for rationing is familiar from the literature. Increasing allocations to low value agents reduces the payment of high value agents by increasing their information rents and therefore "relaxes" their budget constraints.

Clearly, a high-budget agent whose value is below $v_1^*(0)$ has a strict incentive to misreport as a low-budget agent to receive higher cash transfers since $u(\underline{v}, b_1) > 0 = u(\underline{v}, b_2)$. A high-budget agent whose value is slightly above $v_1^*(0)$ also has strict incentives to misreport as a low-budget agent:

$$\frac{v\left(u(\underline{v}, b_1) + b_1\right)}{v_1^*(0)} - b_1 > \frac{\left(v - v_1^*(0)\right) b_1}{v_1^*(0)} \geq \max\left\{v - v_2^*(0), 0\right\}.$$

The last inequality holds for $v > v_1^*(0)$ sufficiently close to $v_1^*(0)$. As it will become clear in Section 4.1, when budgets are agents' private information and cannot be inspected by the principal, to discourage agents from under reporting their budgets, it must be that $u(\underline{v}, b_1) = u(\underline{v}, b_2)$ and a high-budget agent must receive the good with a probability no less than that of a low-budget agent who has the same valuation.

4 Privately known budgets

In this section, I analyze the case in which an agent's budget is his private information. In this case, IC constraints can be separated into two categories:

$$\text{Misreport value: } a(v,b)v - p(v,b) \geq a(\hat{v},b)v - p(\hat{v},b),\ \forall v,\hat{v},b, \tag{IC-v}$$

$$\text{Misreport both: } a(v,b)v - p(v,b) \geq \chi_{\{p(\hat{v},\hat{b})\leq b\}}\left(a(\hat{v},\hat{b})v - q(\hat{v},\hat{b})c - p(\hat{v},\hat{b})\right),\ \forall v,\hat{v},b,\hat{b}. \tag{1}$$

As I stated in the previous section, (IC-v) holds if and only if, for all $b \in \boldsymbol{B}$, $a(v,b)$ is non-decreasing in v and the envelope condition holds: $p(v,b) = va(v,b) - \int_{\underline{v}}^{v} a(\nu,b)\mathrm{d}\nu - u(\underline{v},b)$ for all v. The difficulty arises from (1). In what follows, I first consider a relaxed problem by replacing (1) with the following constraint:

$$a(v,b_2)v - p(v,b_2) \geq a(\hat{v},b_1)v - q(\hat{v},b_1)c - p(\hat{v},b_1),\ \forall v,\hat{v}. \tag{IC-b}$$

This relaxation formalizes the intuition that the principal's main concern is to prevent high-budget agents from falsely claiming to be low-budget agents. Later, I verify that an optimal mechanism of the relaxed problem automatically satisfies IC constraints corresponding to over-reporting of budgets. In other words, a solution to the relaxed problem also solves the original problem.

To summarize, the principal's relaxed problem is

$$\max_{a,p,a} \mathbb{E}_t[a(t)v - q(t)k], \tag{$\mathcal{P}'$}$$

subject to (IR), (IC-v), (IC-b), (BC), (BB) and (S).

4.1 No verification

Before solving the general model, I first consider the special case in which the principal does not inspect agents, i.e., $q \equiv \mathbf{0}$. In this case, as will become clear in the discussion below, it is sufficient to consider two *one-dimensional deviations*, which greatly simplifies the analysis. Although some of the results may be familiar, it helps to highlight the technical challenge arises from introducing costly verification. Denote the principal's problem in this case by $\mathcal{P}_{NI}$ and the corresponding relaxed problem by $\mathcal{P}'_{NI}$. As will become clear in Section 6, if the inspection cost, k, is sufficiently high relative to the punishment, c, it is optimal for the principal not to use inspection. In particular, this is the case when the principal's effective inspection cost is infinity (i.e., $\rho = k/c = \infty$).

Observe first that in this case (IC-b) holds if and only if (IC-v) holds and a high-budget agent does not have any incentives to misreport only his budget:

$$a(v, b_2)v - p(v, b_2) \geq a(v, b_1)v - p(v, b_1),\ \forall v. \tag{2}$$

To see this, note that if both (IC-v) and (2) hold, then

$$\begin{aligned} a(v, b_2)v - p(v, b_2) &\geq a(v, b_1)v - p(v, b_1) \\ &\geq a(\hat{v}, b_1)v - p(\hat{v}, b_1). \end{aligned}$$

Thus, it is sufficient to consider the two one-dimensional deviations: misreport only value and misreport only budget. The above inequality says that if a type (v, b_2) agent has no incentive to misreport (v, b_1), then he has no incentive to misreport $(\hat{v}, b_1)$. This argument is not true when there is verification because it is possible that types (v, b_1) and $(\hat{v}, b_1)$ are inspected with different probabilities. Instead, one must identify for each high-budget type $(\hat{v}, b_1)$ the high-budget type who benefits most from misreporting $(\hat{v}, b_1)$ in the absence of inspection, which determines the set of binding (IC-b) constraints.

Using the envelope condition, (2) can be rewritten as

$$u(\underline{v}, b_2) + \int_{\underline{v}}^{v} a(\nu, b_2)\mathrm{d}\nu \geq u(\underline{v}, b_1) + \int_{\underline{v}}^{v} a(\nu, b_1)\mathrm{d}\nu, \ \forall v. \tag{3}$$

If $v = \underline{v}$, then (3) implies that $u(\underline{v}, b_2) \geq u(\underline{v}, b_1)$. If $u(\underline{v}, b_2) > u(\underline{v}, b_1)$, then one can construct another feasible mechanism by reducing cash subsidies to high-budget agents while increasing their probabilities of receiving goods, which strictly improves welfare. Hence, in an optimal mechanism, agents receive the same amount of cash subsidies regardless of their budgets. This result is summarized in Lemma 1, and a complete proof can be found in the appendix.[12]

Lemma 1 *Suppose Assumption 2 holds, and the principal does not inspect agents. In an optimal mechanism of* $\mathcal{P}'_{NI}$, $u(\underline{v}, b_1) = u(\underline{v}, b_2)$.

One implication of Lemma 1 is that in an optimal mechanism agents receive positive cash subsidies regardless of their budgets. This result contrasts the case of common knowledge budgets in which only low-budget agents receive positive cash subsidies.

Next, I show that, for any given v, an optimal mechanism on average allocates weakly more resources to high-budget agents whose valuations are below v than to low-budget agents whose valuations are below v.

Lemma 2 *Suppose Assumptions 1 and 2 hold, and the principal does not inspect agents. In an optimal mechanism of* $\mathcal{P}'_{NI}$*, the allocation rule satisfies*

$$\int_{\underline{v}}^{v} a(\nu, b_2) f(\nu)\mathrm{d}\nu \geq \int_{\underline{v}}^{v} a(\nu, b_1) f(\nu)\mathrm{d}\nu, \ \forall v. \tag{4}$$

Given Lemma 1, (4) follows immediately from (3) if v is uniformly distributed. Lemma 2 shows that the result holds more generally for any distribution with non-increasing density. Using Lemmas 1 and 2, one can prove the following theorem, which characterizes the optimal direct mechanism.

[12] It is immediate that $u(\underline{v}, b_1) = u(\underline{v}, b_2)$ if one also requires that a low-budget agent has no incentive to misreport as a high-budget agent.

Theorem 2 *Suppose Assumptions 1 and 2 hold, and the principal does not inspect agents. There exist $v_1^*(\infty)$, $v_2^*(\infty)$, $v_2^{**}(\infty)$, $u_1^*(\infty)$ and $u_2^*(\infty)$ such that an optimal mechanism of $\mathcal{P}_{NI}$ satisfies*

$$a(v, b_1) = \chi_{\{v \geq v^*(\infty)\}} a^*(\infty), \ p(v, b_1) = \chi_{\{v \geq v_1^*(\infty)\}} (u_1^*(\infty) + b_1) - u_1^*(\infty),$$
$$a(v, b_2) = \chi_{\{v \geq v_2^*(\infty)\}} a^*(\infty) + \chi_{\{v \geq v_2^{**}(\infty)\}} (1 - a^*(\infty)),$$
$$p(v, b_2) = \chi_{\{v \geq v_2^*(\infty)\}} (u_2^*(\infty) + b_1) + \chi_{\{v \geq v_2^{**}(\infty)\}} (1 - a^*(\infty)) v_2^{**}(\infty) - u_2^*(\infty),$$

where
$$a^*(\infty) = \min \left\{ \frac{u_1^*(\infty) + b_1}{v_1^*(\infty)}, 1 \right\}.$$
$b_1 < v_1^*(\infty) = v_2^*(\infty) \leq v_2^{**}(\infty) \leq \overline{v}$ *and* $u_1^*(\infty) = u_2^*(\infty) > 0$.

In notations $a^*(\infty)$, $v_i^*(\infty)$, $v_2^{**}(\infty)$ and $u_i^*(\infty)$ $(i = 1, 2)$, subscript i indicates the corresponding budget b_i and argument ∞ indicates that this can be viewed as an optimal mechanism when $\rho = \infty$.

Not surprisingly the optimal allocation rule obtained here shares similar features with the one found in Pai and Vohra (2014). There are three cutoffs: $v_1^*(\infty) = v_2^*(\infty) < v_2^{**}(\infty)$. All high-budget agents whose valuations are above $v_2^{**}(\infty)$ receive one unit good with probability one. All low-budget agents whose valuations are above $v_1^*(\infty)$ receive one unit good with positive probability but may be rationed. In addition, high-budget agents whose valuations are in $[v_2^*(\infty), v_2^{**}(\infty)]$ are pooled with low-budget agents whose valuations are at least $v_1^*(\infty)(= v_2^*(\infty))$. To understand this pooling result, consider two agents with the same valuation v, but different budgets $b_2 > b_1$. Then (IC-b) implies that as long as agent (v, b_2)'s payment is less than b_1, he must receive one unit good with the same probability as (v, b_1) does.

The proof of Theorem 2 follows a weight-shifting argument similar to that of Lemma 1 in Richter (2015). Consider a feasible mechanism $(a, p, \mathbf{0})$ in which a high-budget agent's allocation rule is indicated by the dotted blue curve and a low-budget agent's allocation rule is indicated by the dash-dotted red curve in Figure 1. One can construct another feasible mechanism $(a^*, p^*, \mathbf{0})$, in which a high-budget agent's allocation rule is indicated by the solid blue line and a low-budget agent's allocation rule is indicated by the dash-two-dotted red line, in the following way. Find a v_1^* and shift the allocation mass of low-budget agents from the region to the left of v_1^* to the region to the right of v_1^*. The choice of v_1^* is uniquely determined so that the supply to low-budget agents remains unchanged. Let $\hat{v}$ denote the

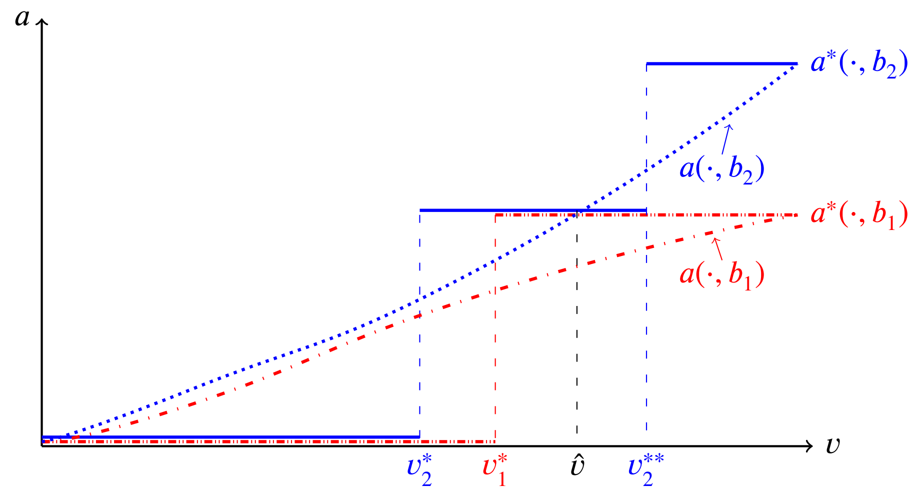


Figure 1: Proof sketch of Theorem 2

minimum valuation of high-budget agents who receive one unit good with a probability of at least $a(\overline{v}, b_1)\ \big(= a^*(\overline{v}, b_1)\big)$. Find v_2^* and v_2^{**} such that $v_2^* \leq \hat{v} \leq v_2^{**}$. Shift the allocation mass of high-budget agents from the region to the left of v_2^* to $[v_2^*, \hat{v}]$ and from $[\hat{v}, v_2^{**}]$ to the region to the right of v_2^{**}. The choice of v_2^* (and v_2^{**}, respectively) is uniquely determined so that the supply to high-budget agents whose valuations are in $[\underline{v}, \hat{v}]$ (and $[\hat{v}, \overline{v}]$, respectively) remains unchanged. Finally, define the new payment rule using the envelope condition. If f is "regular", i.e., satisfies Assumptions 1 and 2, then the new mechanism improves welfare while remaining affordable. Lemma 2 guarantees that $v_2^* \leq v_1^*$. Thus, no high-budget agent has incentive to misreport his budget. It is easy to see that one can further improve welfare by increasing v_2^* and reducing v_1^*. Hence, in an optimal mechanism $v_1^*(\infty) = v_2^*(\infty)$.

4.2 The general case

I now turn to the general problem of the principal. Using the envelope condition, (IC-b) becomes the following: For all v and $\hat{v}$,

$$u(\underline{v}, b_2) + \int_{\underline{v}}^{v} a(\nu, b_2)\mathrm{d}\nu \geq u(\underline{v}, b_1) + a(\hat{v}, b_1)(v - \hat{v}) - q(\hat{v}, b_1)c + \int_{\underline{v}}^{\hat{v}} a(\nu, b_1)\mathrm{d}\nu. \quad \text{(IC-b)}$$

First, for each $\hat{v}$, I identify the type of high-budget agents whose gains from falsely claiming to be a type $(\hat{v}, b_1)$ agent are the largest. (IC-b) holds if and only if for each $\hat{v} \in V$,

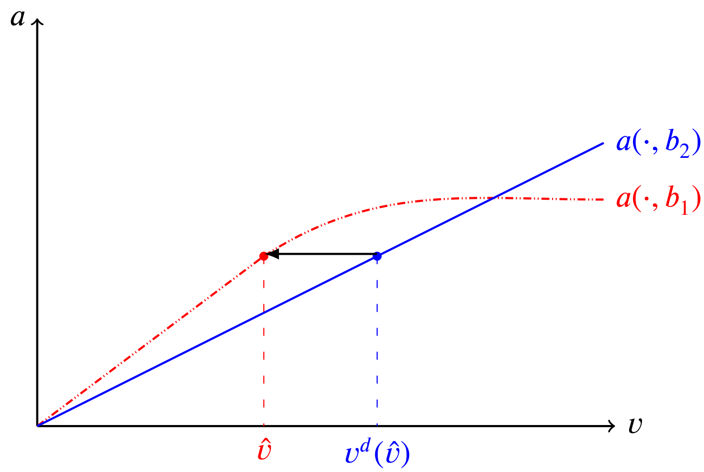


Figure 2: The set of binding (IC-b) constraints

$q(\hat{v}, b_1)c \geq \sup_v \Delta(v, \hat{v})$, where

$$\Delta(v, \hat{v}) := u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v} a(\nu, b_2)\mathrm{d}\nu + a(\hat{v}, b_1)(v - \hat{v}) + \int_{\underline{v}}^{\hat{v}} a(\nu, b_1)\mathrm{d}\nu.$$

Since $\partial\Delta(v, \hat{v})/\partial v = -a(v, b_2) + a(\hat{v}, b_1)$ is non-increasing in v, $\Delta(v, \hat{v})$ is concave in v and achieves its maximum at $v = v^d(\hat{v})$, where

$$v^d(\hat{v}) := \inf \left\{ v | a(v, b_2) \geq a(\hat{v}, b_1) \right\}. \tag{5}$$

If the allocation rules for both budget types are continuous in value v, the high-budget agents who benefit most from falsely claiming to be $(\hat{v}, b_1)$ are those who get the goods with the same probability as type $(\hat{v}, b_1)$ agents do. This point is illustrated by Figure 2, which plots an allocation rule for high-budget agents, $a(\cdot, b_2)$, and an allocation rule for low-budget agents, $a(\cdot, b_1)$, as a function of their valuations v.

Since the principal's objective function is strictly decreasing in q, the optimal inspection rule satisfies

$$q(\hat{v}, b_1) = \frac{1}{c} \max \left\{ 0, \Delta(v^d(\hat{v})) \right\}. \tag{6}$$

Note that $v^d(\cdot)$ is defined using the allocation rule. As a result, one cannot anticipate, a priori, which (IC-b) constraint binds. Furthermore, (IC-b) constraints are frequently binding not only among local types. These difficulties are inherent in all multidimensional problems, and as a result the existing approaches in the mechanism design literature do not apply to this problem. [13]

In order to keep track of the binding (IC-b) constraints, we solve the principal's problem by first approximating the allocation rule using step functions. Fix an integer $M \geq 2$. Let $\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}$ and $0 = a^0 \leq a^1 < a^2 < \cdots < a^M \leq a^{M+1} = 1$. Suppose that the allocation rule for type b_1 agents takes M distinct values: $a(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$. Lemma 3 below shows that the optimal allocation rule for type b_2 agents can take at most $M + 2$ distinct values: $a^0, a^1, \dots, a^{M+1}$.

Lemma 3 *Suppose Assumptions 1 and 2 hold. Suppose $a(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$. Then there exists $\underline{v} \leq v_2^0 \leq v_2^1 \leq \cdots \leq v_2^M \leq \overline{v}$ such that an optimal allocation rule for b_2 satisfies $a(v, b_2) = a^m$ if $v \in (v_2^{m-1}, v_2^m)$ for $m = 1, \dots, M$, $a(v, b_2) = 0$ if $v < v_2^0$ and $a(v, b_2) = 1$ if $v > v_2^M$.*

The proof of Lemma 3 is similar to that of Theorem 2 and illustrated by Figure 3, where the allocation rule for low-budget agents (the dash-dotted red line) takes three distinctive values: $a^1 < a^2 < a^3$. Consider a feasible allocation rule for high-budget agents indicated by the dotted blue curve. Suppose that there exist a payment rule and an inspection rule to be used in conjunction with the allocation rule so that the resulting mechanism is feasible. For ease of exposition, assume that $a(\cdot, b_2)$ is continuous and let $\hat{v}_2^m$ be such that $a(\hat{v}_2^m, b_2) = a^m$ for $m = 1, 2, 3$. For each $m = 1, 2, 3$, find v_2^m and move the allocation mass of high-budget agents from $[\hat{v}_2^m, v_2^m]$ to $[v_2^m, \hat{v}_2^{m+1}]$, where $\hat{v}_2^4 = \overline{v}$. The choice of v_2^m is uniquely determined so that the supply to high-budget agents whose value is in $[\hat{v}_2^m, \hat{v}_2^{m+1}]$ remains unchanged. Redefine the payment rule using the envelope condition and let the inspection rule remain the same. One can verify that the new mechanism is feasible and strictly improves welfare.

We say an allocation rule a is an *M-step allocation rule* if there exist $\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}$, $\underline{v} \leq v_2^0 \leq v_2^1 \leq \cdots \leq v_2^M \leq \overline{v}$ and $0 = a^0 \leq a^1 < a^2 < \cdots < a^M \leq a^{M+1} = 1$ for some integer $M \geq 2$ such that $a(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$ and $a(v, b_2) = a^m$ if $v \in (v_2^{m-1}, v_2^m)$ for $m = 0, 1, \dots, M + 1$. Lemma 3 shows that it is without loss of generality to focus on M-step-allocation rules among all step allocation rules.

Consider a mechanism using a M-step allocation rule. It is easy to see that for $v \in (v_1^{m-1}, v_1^m)$, the high-budget agents who benefit most from falsely claiming to be type (v, b_1) have valuations $v^d(v) = v_2^{m-1}$. Hence, we can keep track of the binding (IC-b) constraints by keeping track of the jump discontinuity points of the allocation rule. In this case, the

[13] See Rochet and Stole (2003) for a survey on multidimensional mechanism design problem.

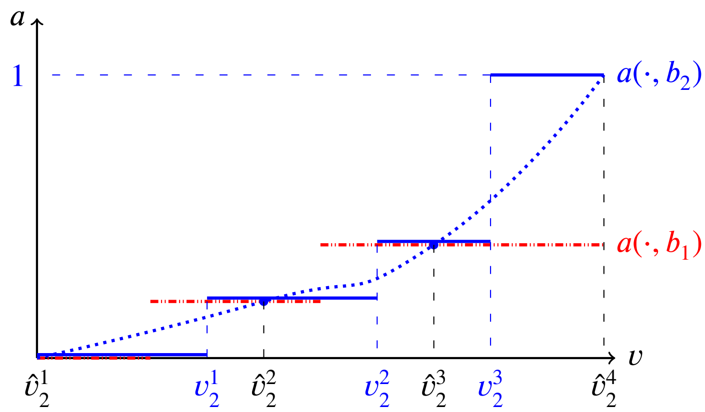


Figure 3: Proof Sketch of Lemma 3

optimal inspection rule satisfies $q(v, b_1) = q^m$ for all $v \in (v_1^{m-1}, v_1^m)$ and

$$q^m = \frac{1}{c} \max \left\{ 0, u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} \left(a^j - a^{j-1}\right) \left(v_2^{j-1} - v_1^{j-1}\right) \right\} \tag{7}$$

for $m = 1, \dots, M$.

Consider the principal's problem $\mathcal{P}'$ with two modifications:

$$\max_{a,p,q} \mathbb{E}_t[a(t)v - q(t)k], \tag{$\mathcal{P}'(M,d)$}$$

subject to (IR), (IC-v), (IC-b), (BC), (S),

$$a \text{ is a } M'\text{-step allocation rule for some } M' \leq M,$$

$$\mathbb{E}[p(t) - q(t)k] \geq -d. \tag{BB-d}$$

The first modification restricts our attention to step allocation rules whose number of steps is bounded from above. The second modification is to relax the government's budget balance constraint by $d \geq 0$. As it will become clear later, any feasible mechanism of $\mathcal{P}'$ can be approximated arbitrarily well by a feasible mechanism of $\mathcal{P}'(M, d)$ for M sufficiently large and d sufficiently small. In what follows, I first solve $\mathcal{P}'(M, d)$ for all $M \geq 2$ and $d > 0$, and then take $M \to \infty$ and $d \to 0$.

4.2.1 Solve $\mathcal{P}'(M, d)$

In this section, I solve $\mathcal{P}'(M, d)$. Let $V(M, d)$ denote the value of $\mathcal{P}'(M, d)$. The main result of this section is to show that $V(M, d) = V(2, d)$ for all $M \geq 2$ and $d \geq 0$. In other words, for all $M \geq 2$, in an optimal mechanism of $\mathcal{P}'(M, d)$ the allocation rule is a 2-step allocation rule. Readers who are not interested in the technical details can omit this section with little loss in continuity.

To solve this problem, I begin by proving that in an optimal mechanism of $\mathcal{P}'(M, d)$, in the absence of verification, either no high-budget agent has incentives to misreport as low budget agents, or all high-budget agents weakly prefer to misreport as low budget agents.

Lemma 4 *Suppose Assumptions 1 and 2 hold. An optimal mechanism of $\mathcal{P}'(M, d)$ satisfies one of the following two conditions:*

(C1) For all $m = 1, \ldots, M$,

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \geq 0. \tag{8}$$

(C2) For all $m = 1, \ldots, M$,

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \leq 0. \tag{9}$$

The basic intuition underlying Lemma 4 is as follows: As long as a mechanism satisfies neither (C1) nor (C2), one can strictly improve welfare by adjusting the allocation rule in regions in which high-budget agents find it strictly optimal to report their budgets truthfully.

Clearly, if (C2) holds, the optimal inspection rule is $q = \mathbf{0}$. In this case, the optimal mechanism of $\mathcal{P}'$, which is characterized in Section 4.1, is a feasible mechanism of $\mathcal{P}'(M, d)$ and satisfies (C1) with equality. Thus, I can conclude that an optimal mechanism of $\mathcal{P}'(M, d)$ satisfies (C1).

Corollary 1 *Suppose Assumptions 1 and 2 hold. An optimal mechanism of $\mathcal{P}'(M, d)$ satisfies (C1).*

Hence, an optimal inspection rule satisfies $q(v, b_1) = q^m$ for all $v \in (v_1^{m-1}, v_1^m)$, where

$$q^m = \frac{1}{c}\left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1})\right] \tag{10}$$

for $m = 1, \dots, M$. Now the principal's problem $\mathcal{P}'(M, d)$ can be written as follows, where the Greek letters in parentheses denote the corresponding Lagrangian multipliers.

$$\max_{\substack{u(\underline{v},b_1),u(\underline{v},b_2),\\ \{a^m\}_{m=1}^M,\{v_1^m\}_{m=1}^{M-1},\{v_2^m\}_{m=0}^M}} \pi\sum_{m=1}^{M+1}\int_{v_2^{m-1}}^{v_2^m} a^m v f(v)\mathrm{d}v + (1-\pi)\sum_{m=1}^{M}\int_{v_1^{m-1}}^{v_1^m} a^m v f(v)\mathrm{d}v$$
$$-(1-\pi)\frac{k}{c}\sum_{m=1}^{M}\int_{v_1^{m-1}}^{v_1^m}\left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1})\right] f(v)\mathrm{d}v,$$

subject to

$$\pi\sum_{m=1}^{M+1} a^m[F(v_2^m) - F(v_2^{m-1})] + (1-\pi)\sum_{m=1}^{M} a^m[F(v_1^m) - F(v_1^{m-1})] \le S, \tag{β}$$

$$a^M v_1^{M-1} - \sum_{j=1}^{M-1} a^j(v_1^j - v_1^{j-1}) - u(\underline{v}, b_1) \le b_1, \tag{η}$$

$$\begin{aligned}
&-(1-\pi)u(\underline{v}, b_1) + (1-\pi)\sum_{m=1}^{M}\int_{v_1^{m-1}}^{v_1^m} a^m\left[v - \frac{1-F(v)}{f(v)}\right] f(v)\mathrm{d}v\\
&-(1-\pi)\frac{k}{c}\sum_{m=1}^{M}\int_{v_1^{m-1}}^{v_1^m}\left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1})\right] f(v)\mathrm{d}v\\
&-\pi u(\underline{v}, b_2) + \pi\sum_{m=1}^{M+1}\int_{v_2^{m-1}}^{v_2^m} a^m\left[v - \frac{1-F(v)}{f(v)}\right] f(v)\mathrm{d}v \ge -d,
\end{aligned} \tag{λ}$$

$$u(\underline{v}, b_1) \ge 0, u(\underline{v}, b_2) \ge 0, \tag{ξ_1, ξ_2}$$

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \ge 0,\ m = 1, \dots, M, \tag{μ^m}$$

$$0 = a^0 \le a^1 \le a^2 \le \cdots \le a^M \le a^{M+1} = 1, \tag{$\alpha^1, \dots, \alpha^{M+1}$}$$

$$\underline{v} = v_1^0 \le v_1^1 \le \cdots \le v_1^M = \overline{v}, \tag{$\gamma_1^1, \dots, \gamma_1^M$}$$

$$\underline{v} \le v_2^0 \le v_2^1 \le \cdots \le v_2^M \le \overline{v}. \tag{$\gamma_2^0, \dots, \gamma_2^{M+1}$}$$

Next, I show that in an optimal mechanism of $\mathcal{P}'(M, d)$, the inspection probability is non-decreasing in a low-budget agent's reported value:

Lemma 5 *Suppose Assumptions 1 and 2 hold. In an optimal mechanism of $\mathcal{P}'(M,d)$, $v_2^1 - v_1^1 \geq 0$. Suppose in addition that $V(M,d) > V(M-1,d)$ for $M \geq 3$, then*

$$v_2^{M-1} - v_1^{M-1} > \cdots > v_2^1 - v_1^1 \geq 0.$$

As a result, the inspection probability in an optimal mechanism of $\mathcal{P}'(M,d)$ is non-decreasing in reported value, i.e., $q^M \geq \cdots \geq q^1 \geq 0$.

To understand the intuition behind this monotonicity result, consider a low-budget agent and a high-budget agent both receiving one unit good with probability a^m. Let p_1^m and p_2^m denote their payments, respectively. The difference in their payments, to which the inspection probability is proportional, is

$$p_2^m - p_1^m = u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}).$$

Clearly, this difference is non-decreasing in m since $v_2^{m-1} - v_1^{m-1} \geq 0$. In other words, low-budget agents who have higher valuations receive larger price discounts. This is not surprising as this is the group of agents that the social planner desires to help. Lemma 5 proves that an even stronger result holds in an optimal mechanism: the incremental per-unit price discount (i.e., $v_2^{m-1} - v_1^{m-1}$) is non-decreasing.

Using the above result, we can further simplify the principal's problem. Note that the inequality constraints corresponding to μ^m's in $\mathcal{P}'(M,d)$ are non-negativity constraints on inspection probabilities. As shown in Lemma 5, in an optimal mechanism of $\mathcal{P}'(M,d)$, the inspection probability is non-decreasing in a low-budget agent's reported value. As a result, it is sufficient to consider the inequality constraint corresponding to μ^1:

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) \geq 0.$$

Note that for fixed jump discontinuity points v_i^m's, the principal's problem $\mathcal{P}'(M,d)$ is linear in $u(\underline{v}, b_1)$, $u(\underline{v}, b_2)$ and a^m's. Hence, an optimal solution can be obtained at an extreme point of the feasible region. The monotonicity of inspection probability implies that in addition to the monotonicity constraints on a^m's there are only finitely many other constraints binding. As a result, for an M sufficiently large, there are finitely many distinct a^m's in an optimal mechanism. More formally, $V(M, d) = V(M-1, d)$ for M sufficiently large.[14] This result still holds even if (BC) is replaced by a per-unit price constraint, as shown in Section 7.1. If only (BC) is imposed, then I can further prove that in an optimal mechanism of $\mathcal{P}'(M, d)$ the allocation rule is a 2-step allocation rule, i.e., $V(M, d) = V(M-1, d)$ for $M \geq 3$.

Lemma 6 *Suppose Assumptions 1 and 2 hold. Then $V(M, d) = V(2, d)$ for all $M \geq 2$ and $d \geq 0$. Furthermore, for all $M \geq 2$, in an optimal mechanism of $\mathcal{P}'(M, d)$ the allocation rule is a 2-step allocation rule.*

4.2.2 Optimal mechanism

In this section, we characterize the optimal mechanism of the original problem $\mathcal{P}$. In particular, we show that an optimal mechanism of $\mathcal{P}'(2, 0)$ is also an optimal mechanism of $\mathcal{P}$. In other words, in an optimal mechanism of $\mathcal{P}$, the allocation rule is a 2-step allocation rule.

Let V denote the value of $\mathcal{P}'$. We prove the above claim by first showing that for any $d > 0$ there exists $\overline{M}(d) > 0$ such that for all $M > \overline{M}(d)$

$$V - V(M, d) \leq (1-\pi)(1+\rho)\frac{\mathbb{E}[v]}{M}.$$

The proof is by construction. Fix $d > 0$ and an integer $M > \overline{M}(d)$. We can construct a feasible mechanism of $\mathcal{P}'(M, d)$ that possibly violates (BB) by at most d and generates welfare which is at least $V - (1-\pi)(1+\rho)\mathbb{E}[v]/M$.[15] Lemma 6 in Section 4.2.1 proves that $V(M, d) = V(2, d)$ for all $M \geq 2$ and $d \geq 0$. Hence, $V - V(2, d) = V - V(M, d) \leq (1-\pi)(1+k/c)\mathbb{E}[v]/M$ for all $d > 0$ and $M > \overline{M}(d)$. Fixing $d > 0$ and taking M to infinity yields $V \leq V(2, d)$ for all $d > 0$. By definition, $V \geq V(2, 0)$. Hence, $V = V(2, 0)$ by the continuity of $V(2, \cdot)$. Thus, an optimal mechanism of $\mathcal{P}'(2, 0)$ also solves $\mathcal{P}'$. It is easy to verify that an optimal solution to $\mathcal{P}'(2, 0)$ satisfies (IC) constraints corresponding to agents over reporting their budgets and therefore solves $\mathcal{P}$. Finally, I show that $v_2^0 = \underline{v}$ and $a^2 = 0$ in an optimal mechanism. Let $a^*(\rho) = a^2$, $v_1^*(\rho) = v_1^1$, $v_2^*(\rho) = v_2^1$, $v_2^{**}(\rho) = v_2^2$, $u_1^*(\rho) = u(\underline{v}, b_1)$ and $u_2^*(\rho) = u(\underline{v}, b_2)$, then an optimal mechanism is characterized by the following Theorem 3.

[14] This result is formally proved in Lemma 10 in the appendix.
[15] This is formally proved in Lemma 11 in the appendix.

Theorem 3 *Suppose Assumptions 1 and 2 hold. There exist* $a^*(\rho)$, $v_1^*(\rho)$, $v_2^*(\rho)$, $v_2^{**}(\rho)$, $u_1^*(\rho)$ *and* $u_2^*(\rho)$ *such that an optimal mechanism of* $\mathcal{P}$ *is given by*

$$
\begin{aligned}
& a(v, b_1) = \chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho), \ p(v, b_1) = \chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho) v_1^*(\rho) - u_1^*(\rho), \\
& q(v, b_1) = \frac{1}{c} \left[\chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho) \left(v_2^*(\rho) - v_1^*(\rho) \right) + u_1^*(\rho) - u_2^*(\rho) \right], \\
& a(v, b_2) = \chi_{\{v \geq v_2^*(\rho)\}} a^*(\rho) + \chi_{\{v \geq v_2^{**}(\rho)\}} (1 - a^*(\rho)), \\
& p(v, b_2) = \chi_{\{v \geq v_2^*(\rho)\}} a^*(\rho) v_2^*(\rho) + \chi_{\{v \geq v_2^{**}(\rho)\}} (1 - a^*(\rho)) v_2^{**}(\rho) - u_2^*(\rho), \\
& q(v, b_2) = 0,
\end{aligned}
$$

where $a^*(\rho) = \left[u_1^*(\rho) + b_1 \right] / v_1^*(\rho)$, $\underline{v} < v_1^*(\rho) \leq v_2^*(\rho) \leq v_2^{**}(\rho) \leq \overline{v}$, $0 < a^*(\rho) \leq 1$ *and* $u_1^*(\rho) \geq u_2^*(\rho)$.

In notations $a^*(\rho)$, $v_i^*(\rho)$, $v_2^{**}(\rho)$ and $u_i^*(\rho)$ $(i = 1, 2)$, subscript i indicates the corresponding budget b_i and argument ρ indicates their dependence on ρ. As before, $u_i^*(\rho)$ measures the amount of cash subsidies received by an agent with type b_i. In an optimal mechanism, low-budget agents receive more cash subsidies (as in the case of common knowledge budgets), but high-budget agents may also receive strictly positive cash subsidies (as in the case of private budgets without inspection). There are three cutoffs: $v_1^*(\rho) \leq v_2^*(\rho) \leq v_2^{**}(\rho)$. All high-budget agents whose valuations are above $v_2^{**}(\rho)$ receive goods with probability 1. All low-budget agents whose valuations are above $v_1^*(\rho)$ receive goods with positive probability but may be rationed. Similar to the case of private budgets without inspection, some high-budget agents (whose valuations are in $[v_2^*(\rho), v_2^{**}(\rho)]$) are pooled with low-budget agents. However, $v_1^*(\rho) \leq v_2^*(\rho)$. This difference between $v_1^*(\rho)$ and $v_2^*(\rho)$, together with budget dependent cash subsidies, creates incentives for high-budget agents to under report their budgets. All agents who report low budgets are inspected with non-negative probability and those who receive goods are more likely to be inspected.

I note here that if $\rho = 0$, then $u_2^*(0) = 0$ and $v_2^*(0) = v_2^{**}(0)$, which is the case in Theorem 1. If $\rho = \infty$, then $u_1^*(\infty) = u_2^*(\infty)$ and $v_1^*(\infty) = v_2^*(\infty)$, which is the case in Theorem 2. To simplify notation, in what follows, I suppress the dependence of u_1^*, u_2^*, v_1^*, v_2^*, v_2^{**} and a^* on ρ whenever it is clear.

Theorem 3 also greatly simplifies the analysis. Now the principal's problem can be reduced to the following, where the Greek letters in parentheses denote the corresponding

Lagrangian multipliers.

$$\max_{\substack{u(\underline{v},b_1),u(\underline{v},b_2),\\ a^2,v_1^1,v_2^1,v_2^2}} \pi \left[\int_{v_2^1}^{v_2^2} a^2 v f(v)\mathrm{d}v + \int_{v_2^2}^{\overline{v}} v f(v)\mathrm{d}v\right] + (1-\pi)\int_{v_1^1}^{\overline{v}} a^2 v f(v)\mathrm{d}v$$

$$-(1-\pi)\frac{k}{c}\left[u(\underline{v},b_1)-u(\underline{v},b_2)\right]F(v_1^1) - (1-\pi)\frac{k}{c}\int_{v_1^1}^{\overline{v}}\left[u(\underline{v},b_1)-u(\underline{v},b_2)+a^2(v_2^1-v_1^1)\right]f(v)\mathrm{d}v,$$

subject to

$$\pi a^2\left[F(v_2^2)-F(v_2^1)\right] + \pi\left[1-F(v_2^2)\right] + (1-\pi)a^2[1-F(v_1^1)] \le S, \tag{β}$$

$$a^2 v_1^1 - u(\underline{v},b_1) \le b_1, \tag{η}$$

$$\begin{aligned}
&-(1-\pi)u(\underline{v},b_1) + (1-\pi)\int_{v_1^1}^{v_1^2} a^2\left[v-\frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v \\
&-(1-\pi)\frac{k}{c}\left[u(\underline{v},b_1)-u(\underline{v},b_2)\right]F(v_1^1) - (1-\pi)\frac{k}{c}\int_{v_1^1}^{\overline{v}}\left[u(\underline{v},b_1)-u(\underline{v},b_2)+a^2(v_2^1-v_1^1)\right]f(v)\mathrm{d}v \\
&-\pi u(\underline{v},b_2) + \pi\int_{v_2^1}^{v_2^2} a^2\left[v-\frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v + \pi\int_{v_2^2}^{\overline{v}}\left[v-\frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v \ge 0,
\end{aligned} \tag{λ}$$

$$u(\underline{v},b_1)\ge 0, u(\underline{v},b_2)\ge 0, \tag{ξ_1,ξ_2}$$

$$u(\underline{v},b_1)-u(\underline{v},b_2)\ge 0, \tag{μ^1}$$

$$u(\underline{v},b_1)-u(\underline{v},b_2)+a^2(v_2^1-v_1^1)\ge 0, \tag{μ^2}$$

$$0\le a^2\le a^3=1, \tag{α^2,α^3}$$

$$\underline{v}\le v_1^1\le\overline{v}, \tag{γ_1^1,γ_1^2}$$

$$\underline{v}\le v_2^1\le v_2^2\le\overline{v}. \tag{$\gamma_2^1,\gamma_2^2,\gamma_2^3$}$$

Furthermore, the optimal mechanism is *unique*. Suppose, on the contrary, that there are two optimal mechanism. Since $\mathcal{P}'$ is linear in (a,p,q),[16] the convex combination of these two optimal mechanisms is also optimal. However, the convex combination of two 2-step allocation rules is not a 2-step allocation rule in general, which cannot be optimal. Hence, there exists a unique optimal mechanism.

[16] $\mathcal{P}'(2,0)$ is not linear in $u(\underline{v},b_1)$, $u(\underline{v},b_2)$, a^2, v_1^1, v_2^1 and v_2^2.

Corollary 2 *Suppose Assumptions 1 and 2 hold. There exists a unique optimal mechanism of $\mathcal{P}$. Furthermore, u_1^*, u_2^*, v_1^*, v_2^*, v_2^{**} and a^* are continuous in k, c, π, b_1 and S.*

4.2.3 Subsidies in kind

I complete this section by discussing subsidies in kind. In the optimal mechanism, compared with high-budget agents who do not receive goods, high-budget agents whose valuations exceed v_2^{**} receive the good with probability one by making an additional payment $a^*v_2^* + (1 - a^*)v_2^{**}$. All high-budget agents whose valuations lie in $[v_2^*, v_2^{**}]$ receive goods with probability a^* by making an additional payment $a^*v_2^*$. This is a form *in-kind subsidy*. In the literature, the value of an in-kind subsidy is often measured by its market value. In this paper, I do not model private market explicitly, so I use the additional payment, $a^*v_2^* + (1 - a^*)v_2^{**}$, made by high-budget high-value agents as a measure of “price". Then the amount of in-kind subsidies received by a high-budget agent whose valuation lies in $[v_2^*, v_2^{**}]$ is $a^* \left[a^*v_2^* + (1 - a^*)v_2^{**}\right] - a^*v_2^*$. Note that high-budget agents do not receive any in-kind subsidies if $v_2^* = v_2^{**}$, as in the case when budgets are common knowledge. Similarly, the amount of in-kind subsidies received by a low-budget agent whose valuation is above v_1^* is $a^* \left[a^*v_2^* + (1 - a^*)v_2^{**}\right] - a^*v_1^*$. The difference in in-kind subsidies offered to the two budget types is $a^*(v_2^* - v_1^*)$.

In-kind subsidies are widespread around the world. The conventional wisdom rationalizing the prevalence of in-kind subsidies is paternalism. A more recent justification is based on the idea that agents have private information about their financial constraints, and governments cannot accurately identify low-budget agents in need of help. As a result, in-kind subsidies will be part of a surplus-maximizing mechanism as it is less susceptible to mimicking by high-budget agents. One difficulty with this justification is that, in many transfer programs, governments first “verify income, and then give benefits in kind, which would seem to rule out self-targeting as the primary reason for supplying benefits in-kind”.[17] Moreover, governments “generally expend considerable resources determining eligibility”.[18] In this paper, I formalize the idea that governments can verify agents’ private information about their financial constraints via a costly procedure, and show that in such an environment the optimal mechanism still makes use of in-kind subsidies.

[17] Currie and Gahvari (2008)
[18] Currie and Gahvari (2008)

5 Implementation

In this section, I provide one simple implementation of the direct optimal mechanism characterized in Section 4. This implementation exhibits some of the features of the affordable housing program in Singapore.

Consider the following *random assignment with regulated resale and cash subsidy* (RwRRC) scheme, which is based on the RwRRC scheme in Che, Gale, and Kim (2013). The scheme consists of two stages.

1. In the first stage, agents report their budgets. Agents who report low budget are inspected with a probability of $(u_1^* - u_2^*)/c$. The principal offers cash subsidies u_1^* to low-budget agents and u_2^* to high-budget agents. The principal also offers low-budget agents the choice of participating in a lottery at price $p_1^* := a^* v_1^*$ and high budget agents the choice of participating in the same lottery at price $p_2^* := a^* v_2^*$. The principal distributes goods randomly with uniform probability among all participants. Each participant receives one unit good with a probability no more than a^*.

2. In the second stage, the resale market opens, in which agents can purchase goods from each other and the principal if not all goods are distributed in the first stage. The per-unit sales taxes are $\tau_1^* := v_2^{**} - v_1^*$ for low-budget sellers and $\tau_2^* := v_2^{**} - v_2^*$ for high-budget sellers. Agents who report low budget in the first stage and choose not to sell goods in the second stage are inspected with probability $(v_2^* - v_1^*)/c$.

Let a denote a lottery participant's expected probability of receiving one unit good in the first stage, and p^s denote the expected price a buyer pays in the second stage. Assume without loss of generality that $p^s > b_1$ so that a low-budget agent cannot afford it. Consider a low-budget agent whose type is (v, b_1) and who reports his budget truthfully. Then his payoff is u_1^* if he does not enter the lottery. If he buys the lottery, there are two possibilities. If he keeps the good when he receives it in the first stage, then his payoff is $u_1^* + av - a^* v_1^*$; otherwise his payoff is $u_1^* - a^* v_1^* + a(p^s - v_2^{**} + v_1^*)$. Clearly, in the second stage, it is optimal for him to keep the good if and only if $v \geq p^s - v_2^{**} + v_1^*$. In the first stage, it is optimal for him to participate in the lottery if and only if $a \max\left\{v, p^s - v_2^{**} + v_1^*\right\} \geq a^* v_1^*$.

Similarly, consider a high-budget agent whose type is (v, b_2) and who reports his budget truthfully. It is easy to see that if it is optimal for an agent not to participate in the lottery in the first stage, then it is also optimal for him not to buy the good in the second stage. If it is optimal for an agent to sell the good he receives in the first stage, then it is optimal for him not to buy the good in the second stage when he does not receive it in the first stage. Then his payoff is u_2^* if he does not participate in the lottery. If he participates in the lottery, there are three possibilities. If he participates in the lottery, keeps the good when he receives it and buys it when he does not receive it, his payoff is $u_2^* - a^* v_2^* + av + (1-a)(v - p^s)$; if he participates the lottery, keeps the good when he receives it and does not buy when he does not receive it, his payoff is $u_1^* + a^*(v - v_1^*)$; if he participates in the lottery and sells the good when he receives it, then his payoff is $u_2^* - a^* v_2^* + a(p^s - v_2^{**} + v_2^*)$. Clearly, in the second stage, it is optimal for him to keep the good if and only if $v \geq p^s - v_2^{**} + v_2^*$ and buy the good if and only if $v \geq p^s$. In the first stage, it is optimal for him to participate in the lottery if and only if $a \max \left\{ v, p^s - v_2^{**} + v_2^* \right\} \geq a^* v_2^*$.

Hence, in the second stage, the demand of the goods is $\pi(1-a)\left[1 - F(p^s)\right]$ and the supply of the goods is

$$S - a(1-\pi)\left[1 - F\left(\max\left\{p^s - v_2^{**} + v_1^*, \frac{a^* v_1^*}{a}\right\}\right)\right] - a\pi\left[1 - F\left(\max\left\{p^s - v_2^{**} + v_2^*, \frac{a^* v_2^*}{a}\right\}\right)\right].$$

It is not hard to verify that $a = a^*$ and $p^s = v_2^{**}$ is the unique equilibrium.[19] Note that in this equilibrium, an agent is indifferent between not participating in the lottery, and participating the lottery but selling the good when he receives it. All low-budget agents whose valuations are above v_1^* strictly prefer to participate in the lottery and keep the goods they receive. All high-budget agents whose valuations are above v_2^* strictly prefer to participate in the lottery and keep the goods they receive. In addition, all high-budget agents whose valuations are above v_2^{**} will buy the goods in the second stage if they do not receive any in the first stage. These arguments prove the following result.

Proposition 1 *Suppose Assumptions 1 and 2 hold. The optimal mechanism is implemented by RwRRC with* $\underline{v} \leq v_1^* \leq v_2^* \leq v_2^{**} \leq \overline{v}$, $u_1^* \geq u_2^*$ *and* $0 \leq a^* \leq 1$ *given by Theorem 3.*

If inspection is sufficiently costly or the principal cannot inspect agents, then in the RwRRC scheme, regardless of their budgets, agents receive the same amount of cash subsidies ($u_1^* = u_2^*$) and the same price ($p_1^* = p_2^*$) in the first stage and face the same sales taxes ($\tau_1^* = \tau_2^*$) in the second stage. If inspection is not too costly, then the principal provides

[19] Clearly, for each a, there is a unique p^s such that demand is equal to supply in the second stage. By construction, $a \leq a^*$. Suppose $a < a^*$, then the market clearing condition in the second stage implies that $p^s < v_2^{**}$. This implies that a low-budget agent buys the lottery only if $v > v_1^*$ and a high-budget agent buys the lottery only if $v > v_2^*$, which in turn implies that $a = a^*$, a contradiction. Hence, $a = a^*$ and $p^s = v_2^{**}$.

financial aids to low-budget agents ($u_1^* \leq u_2^*, p_1^* \geq p_2^*$) in the first stage and discourages them from reselling by imposing a higher sales tax in the second stage.

If we interpret the sales tax as a form of restriction in the resale market, this implementation exhibits some of the features of Singapore's affordable housing program. In Singapore, the affordable housing program is administered by housing and development board (HDB). HDB develops new flats and sells them to eligible buyers.[20] Buyers can purchase new flats directly from HDB or resale flats from existing owners in the open market. Owners of HDB flats must have resided in their flats for a period of time, referred to as the minimum occupation period (MOP), before they are eligible to resell or sublet their flats, and the lengths of MOP positively depend on the financial aid they received. Specifically, buyers of resale HDB flats can apply for a CPF housing grant, which is a housing subsidy to help eligible households. HDB flats purchased with CPF housing grants are subject to longer MOPs as illustrated by Table 1.

6 Properties of the optimal mechanism

Having derived the optimal mechanism, I would like to investigate the following questions. When can the first-best outcome be achieved? What is the effect of a decrease in verification cost as, for example, a government's bureaucratic efficiency improves? What is the effect of an increase in the supply as, for example, a government builds more houses? In what follows, I investigate each of these questions in turn.

First of all, I give a necessary and sufficient condition under which the first-best is achieved.

Proposition 2 *Suppose Assumptions 1 and 2 hold. The first-best is achieved if and only if* $S \geq \hat{S}(b_1)$*, where* $\hat{S}(b_1)$ *is the solution to* $b_1 - v^* F(v^*) = 0$ *with* $v^* = F^{-1}(1 - S)$*. Furthermore,* $\hat{S}(b_1)$ *is strictly decreasing in* b_1.

Intuitively, the first-best is achieved if the supply of goods is abundant or agents have ample budgets. Note that the condition given in Proposition 2 is independent of inspection cost k, punishment c and the percentage of high-budget agents π. This is because when the first-best is achieved, agents of both budget types receive the same amount of cash subsidies

[20] 90% of HDB flats are owned by their residents. The remainder are rental flats for people who cannot afford to purchase the cheapest form of HDB flats despite financial aid.

and the same allocation rule, and the inspection probability is zero. For the rest of this section, I assume that the first-best cannot be achieved, i.e., $S < \hat{S}(b_1)$.

Next, I study the impact of changes in effective inspection cost ($\rho = k/c$), supply (S), budget (b_1) and the percentage of high-budget agents (π) on the optimal mechanism as well as welfare. The optimal mechanism is characterized by u_1^*, u_2^*, v_1^*, v_2^*, v_2^{**} and a^*, which (together with the corresponding Lagrangian multipliers) are solutions to a system of non-linear equations. As a result, it is hard to perform all comparative statics analysis analytically. In what follows, I give some analytic results for extreme cases such as when effective inspection cost is sufficiently large and explore the intermediate case numerically.

Effective Verification Cost (ρ). It is straightforward that an increase in ρ reduces the total welfare, as show in 3 below. How does this change affect the optimal mechanism? Intuitively, as verification becomes more costly (ρ increases), the principal tends to inspect agents less frequently in the optimal mechanism. To maintain incentive compatibility, the principal needs to reduce the differences in cash and in-kind subsidies offered to agents with different budgets. Proposition 3 shows that, for large ρ, agents of both budget types receive the same amount of cash subsidies. Eventually, for ρ sufficiently large, verification is never used.[21] If v is uniformly distributed, then one can further prove that, for fixed punishment c, the verification probability is non-increasing in verification cost k. However, the change in total verification cost may not be monotonic as illustrated by Figure 4c.

Proposition 3 *Suppose Assumptions 1 and 2 hold.*

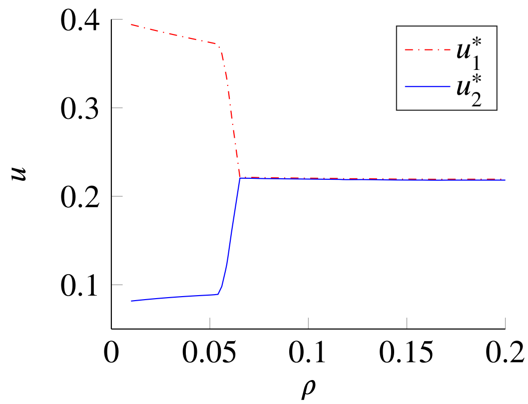
1. *If $\rho \geq \frac{\pi}{1-\pi}$, then agents of both budget types receive the same amount of cash subsidies, i.e., $u_1^* = u_2^*$.*

2. *There exists $\overline{\rho} \leq \frac{\pi}{S(1-\pi)}$ such that the verification probability in an optimal mechanism is zero, i.e., $q(v, b) = 0$ for all v and b, if and only if $\rho \geq \overline{\rho}$. Furthermore, the total welfare is strictly decreasing in ρ over $[0, \overline{\rho}]$ and constant in ρ over $[\overline{\rho}, \infty)$.*

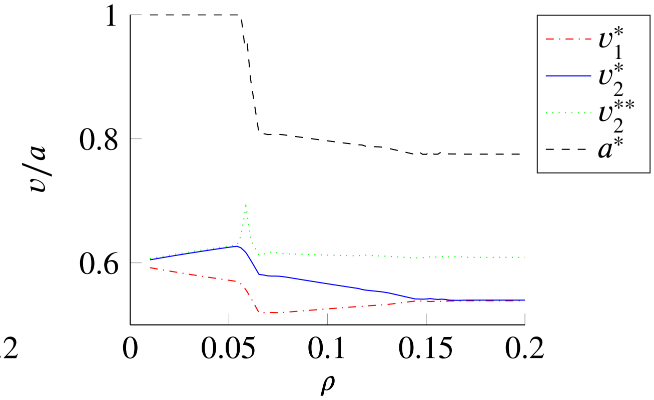
3. *If v is uniformly distributed, then the verification probability is non-increasing in k.*

Figure 4 plots the impact of an increase in the effective verification cost (ρ) on cash subsidies, allocation, verification and welfare in a numerical example. Although an increase

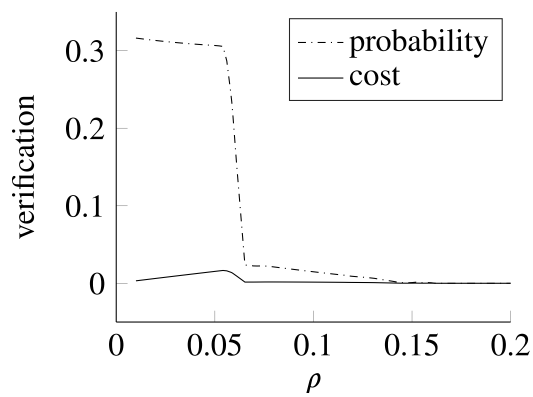
[21] The two lower bounds given in Proposition 3 are not tight, as illustrated in the numerical example in Figure 4.



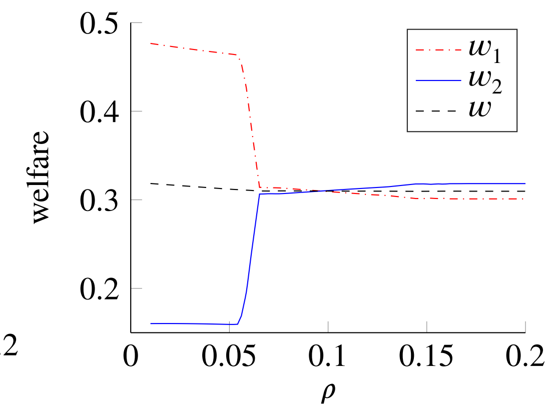
(a) Cash Subsidies



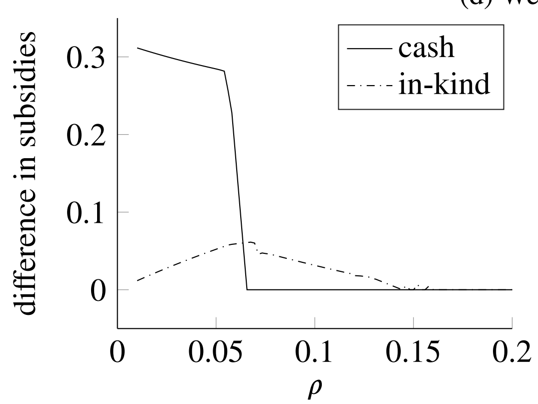
(b) Allocation



(c) Verification



(d) Welfare



(e) Differences in Cash and In-kind Subsidies

Figure 4: The impact of an increase in effective inspection cost (ρ) on cash subsidies, allocation, inspection, welfare and the differences in cash and in-kind subsidies. In this numerical example, v is uniformly distributed on $[0, 1]$, $S = 0.4$, $b_1 = 0.2$, $\pi = 0.5$ and $\rho \in [0, 0.2]$.

in ρ reduces the total welfare, its impacts on different budget types are different. Verification allows the principal to more accurately target low-budget agents and improves their welfare. As a result, as ρ increases, the welfare of low-budget agents declines while that of high-budget agents rises, as seen in Figure 4d.

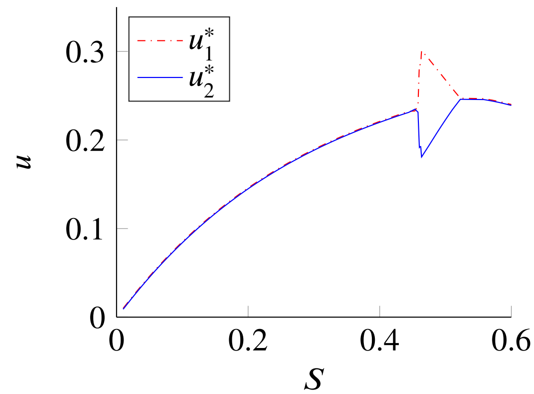
More interestingly, the optimal mechanism makes use of both cash and in-kind subsidies, and the change in verification cost affects that mechanism's reliance on each of them as shown in Figure 4e. If ρ is sufficiently small, then the principal helps low-budget agents mainly by offering them more cash subsidies. As ρ increases, the difference in cash subsidies declines but the difference in in-kind subsidies increases. This is because even though cash subsidy is more efficient in the sense that it does not introduce any distortion in allocation, it is more expensive in terms of verification cost. Cash subsidy is attractive to everyone regardless of their valuations. In contrast, in-kind subsidy is attractive only to agents whose valuations are high enough. Eventually, the difference in in-kind subsidies also declines as verification becomes sufficiently costly.

Supply (S). The impact of an increase in the supply (S) on the optimal mechanism is complicated. On the one hand, the principal becomes less budget constrained, and can direct more subsidies to low-budget agents and inspect them more frequently. On the other hand, low-budget agents also become less budget constrained as S increases,[22] which reduces the needs to subsidize and inspect them. As shown in Propositions 3 and 4, for sufficiently large and small S, agents of both budget types receive the same amount of cash subsidies.

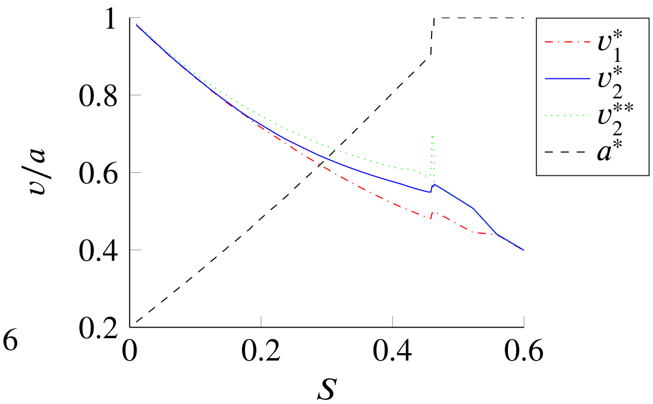
Proposition 4 *Suppose Assumptions 1 and 2 hold. If S is sufficiently small, then agents of both budget types receive the same amount of cash subsidies, i.e.,* $u_1^* = u_2^*$.

These effects can also be seen in Figure 5, which plots cash subsidies, allocation, verification, welfare, the differences in cash and in-kind subsidies and payment in a numerical example. Specifically, Figure 5e plots the differences in cash and in-kind subsidies between high-budget and low-budget agents. If S is sufficiently small, then agents receive the same amount of subsidies regardless of their budgets. As S increases, the principal raises first the difference in in-kind subsidies and then that in cash subsidies. This order occurs because it is less expensive to target only low-budget high-valuation agents than all low-budget agents.

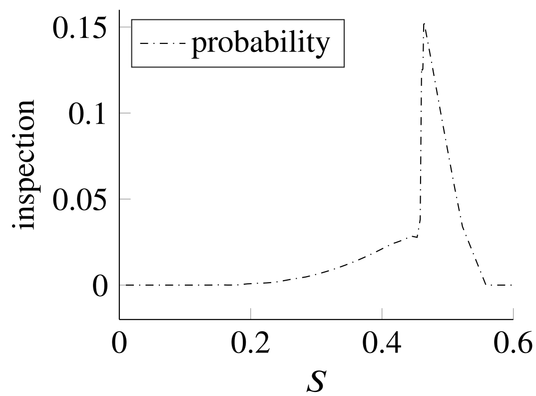
[22] As in Section 4.2.3, I use the additional payment $a^* v_2^* + (1 - a^*) v_2^{**}$ made by a high-budget high-valuation agent a measure of "price". Then this price generally declines as S increases and low-budget agents become less budget constrained in the sense that the gap between this price and their budgets shrinks.



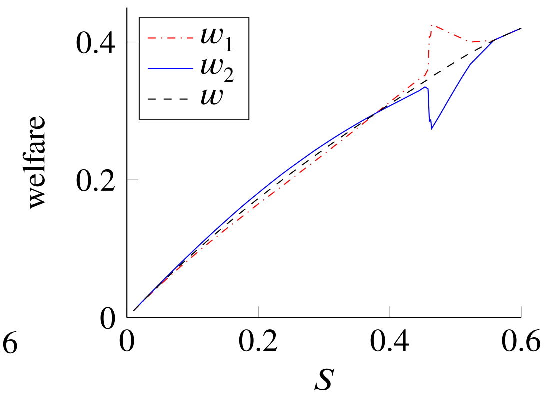
(a) Cash Subsidies



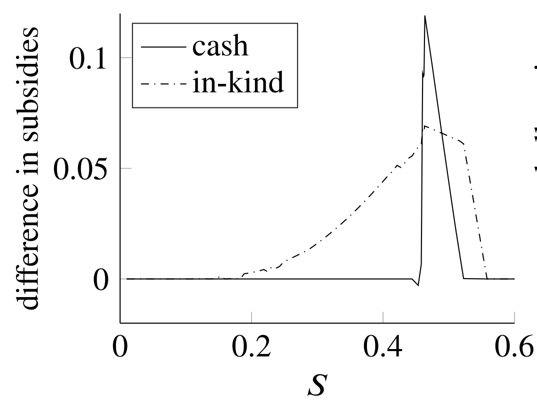
(b) Allocation



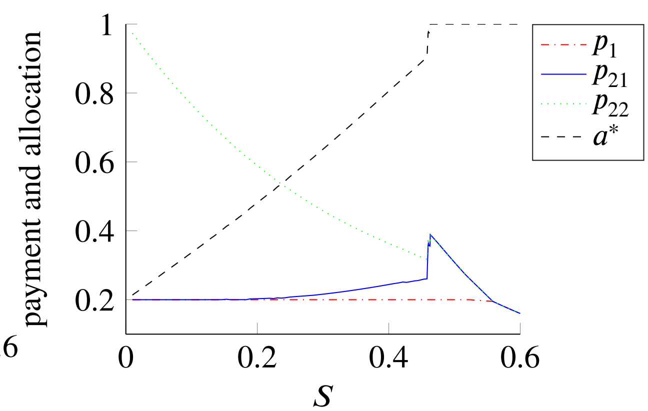
(c) Verification Probability



(d) Welfare



(e) Differences in Cash and In-kind Subsidies



(f) Payment and Allocation

Figure 5: The impact of an increase in the supply (S) on cash subsidies, allocation, verification, welfare, the differences in cash and in-kind subsidies and payment. In the bottom right panel, p_1 denotes the payment by a low-budget agent who receives one unit good with probability a^*, p_{21} denotes the payment by a high-budget agent who receives one unit good with probability a^*, and p_{22} denotes the payment by a high-budget agent who receives one unit good with probability one. In this numerical example, v is uniformly distributed on $[0, 1]$, $\rho = 0.08$, $b_1 = 0.2$, $\pi = 0.5$ and $S \in [0, 0.6]$.

Eventually, the differences in both cash and in-kind subsidies decline as the need to subsidize low-budget agents declines. As a result, the verification probability is hump-shaped as shown in Figure 5c.

Intuitively, the total welfare is strictly increasing in S. More interestingly, the welfare of each type is not monotonic in S. Figure 5d plots the total welfare and the average utility of each budget type as a function of S. Initially, the average utilities of both budget types increase as S increases. When S is large enough that the principal begins to divert more cash subsidies and goods to low-budget agents, the average utility of high-budget agents begins to decrease as S increases. Specifically, high-budget high-valuation agents get worse off because their payments increase as S increases (see Figure 5f). These increases in payments occur because disproportionately more goods will be allocated to low-budget agents and there will be less pooling when S increases. Eventually, the need to subsidize low-budget agents decreases as S increases, and the average utility of low-budget agents begins to decrease while that of high-budget agents begins to increase, until they coincide. Specifically, low-valuation low-budget agents get worse off as they receive less cash subsidies.

7 Extensions and discussions

In this section, I consider several extensions of the model and discuss some issues regarding the model's assumptions. Section 7.1 shows that some of the analysis extends if I replace the budget constraint by a more stringent per-unit price constraint. Section 7.2 shows that the analysis extends to the case where punishment is transferable. Sections 7.3 discusses the robustness of my analysis to weakening the assumptions on verification. Section 7.4 discusses why it is necessary to explicitly model budget constraints.

7.1 Per-unit Price Constraint

In the optimal direct mechanism, agents make payments to the principal regardless of whether they receive goods,[23] which some may consider unrealistic. The question, then, is whether this direct mechanism can be implemented by a mechanism in which agents pay if and only if they receive goods and their payments are within their budgets. Such an implementation is impossible if $a^* < 1$. I can guarantee that such an implementation always

[23] For a finite number of agents, this is similar to an all-pay auction.

exists if I replace (BC) by the following *per-unit price constraint*:

$$p(t) \leq a(t)b, \ \forall t = (\upsilon, b). \tag{PC}$$

(BC) is the same as that found in Che and Gale (2000) and Pai and Vohra (2014), but different from Che, Gale, and Kim (2013), which uses (PC).

Nevertheless, I assume (BC) in the main body of the paper for the following reasons. First, I want to understand how well we can do when we only have the problem of asymmetric information. To answer this question, I do not want to impose any ad hoc constraint on the kind of mechanisms that can be used. Second, as will become clear, the optimal mechanisms in these two settings share qualitatively similar features. For some parameter values (e.g., verification cost is low, resources are relatively abundant or the percentage of budget constrained agents is small), the optimal mechanism under (BC) consists of no rationing (i.e., $a^*(\rho) = 1$) and therefore also satisfies (PC). Third, rationing is realistic if b_1 is close to zero. For example, families with very low income may receive free coverage from Medicaid.

In the rest of this subsection, I consider the principal's problem in which (BC) is replaced by (PC), denoted by $\mathcal{P}_{PC}$. I first make the observation that if (PC) holds for υ' then it holds for all $\upsilon < \upsilon'$. This is trivial if $a(\upsilon, b) = 0$. If $a(\upsilon, b) > 0$, then by the envelope condition we have

$$\begin{aligned}\frac{p(\upsilon', b)}{a(\upsilon', b)} - \frac{p(\upsilon, b)}{a(\upsilon, b)} = &\int_{\upsilon}^{\upsilon'} \left(1 - \frac{a(\nu, b)}{a(\upsilon', b)}\right) \mathrm{d}\nu - \int_{\underline{\upsilon}}^{\upsilon} \left(\frac{a(\nu, b)}{a(\upsilon', b)} - \frac{a(\nu, b)}{a(\upsilon, b)}\right) \mathrm{d}\nu - \frac{u(\underline{\upsilon}, b)}{a(\upsilon', b)} + \frac{u(\underline{\upsilon}, b)}{a(\upsilon, b)} \\ \geq &0,\end{aligned}$$

where the last inequality holds since $a(\upsilon, b)$ is non-decreasing in υ. Hence, (PC) holds if and only if $p(\overline{\upsilon}, b) \leq a(\overline{\upsilon}, b)b$ for all b.

Given this observation, it is straightforward to extend the results of Theorems 1 and 2 to the current setting. Theorem 4 characterizes an optimal mechanism when budgets are common knowledge ($\rho = 0$) (denoted by $\mathcal{P}_{PC,CB}$). Theorem 5 characterizes an optimal mechanism when budget is an agent's private information and the principal cannot verify this information ($\rho = \infty$) (denoted by $\mathcal{P}_{PC,NI}$). The latter theorem extends the results in Section 3 of Che, Gale, and Kim (2013) to the setting where an agent's valuation can take a continuum of possible values under the regularity conditions.

Theorem 4 *Suppose Assumption 2 holds, and budgets are common knowledge. There exists*

$v_1^*(0)$, $v_2^*(0)$, $u_1^*(0)$ and $u_2^*(0)$ such that an optimal mechanism of $\mathcal{P}_{PC,CB}$ is given by

$$a(v,b_1) = \chi_{\{v \geq v_1^*(0)\}} a^*(0), \quad p(v,b_1) = \chi_{\{v \geq v_1^*(0)\}} \left(u_1^*(0) + a^*(0) b_1\right) - u^*(0),$$
$$a(v,b_2) = \chi_{\{v \geq v_2^*(0)\}} 1, \qquad p(v,b_2) = \chi_{\{v \geq v_2^*(0)\}} v_2^*(0),$$

where $a^*(0) = u_1^*(0)/\left[v_1^*(0) - b_1\right]$, $b_1 < v_1^*(0) \leq v_2^*(0) < \overline{v}$ and $0 = u_2^*(0) < u_1^*(0) \leq v_1^*(0) - b_1$.

Theorem 5 *Suppose Assumptions 1 and 2 hold, and the principal does not inspect agents. There exists $v_1^*(\infty)$, $v_2^*(\infty)$, $v_2^{**}(\infty)$, $u_1^*(\infty)$ and $u_2^*(\infty)$ such that an optimal mechanism of $\mathcal{P}_{PC,NI}$ with no verification satisfies*

$$a(v,b_1) = \chi_{\{v \geq v_1^*(\infty)\}} a^*(\infty), \; p(v,b_1) = \chi_{\{v \geq v_1^*(\infty)\}} a^*(\infty) v_1^*(\infty) - u_1^*(\infty),$$
$$a(v,b_2) = \chi_{\{v \geq v_1^*(\infty)\}} a^*(\infty) + \chi_{\{v \geq v_2^{**}(\infty)\}} \left(1 - a^*(\infty)\right),$$
$$p(v,b_2) = \chi_{\{v \geq v_2^*(\infty)\}} a^*(\infty) v_2^*(\infty) + \chi_{\{v \geq v_2^{**}\}} \left(1 - a^*(\infty)\right) v_2^{**}(\infty) - u_2^*(\infty),$$

where $a^*(\infty) = \min\left\{u_1^*(\infty)/\left[v_1^*(\infty) - b_1\right], 1\right\}$, $b_1 < v_1^*(\infty) = v_2^*(\infty) \leq v_2^{**}(\infty) \leq \overline{v}$ and $u_1^*(\infty) = u_2^*(\infty) > 0$.

The analysis is more complex if budget is an agent's private information and the principal can verify this information at a cost. As before, I first consider the principal's relaxed problem $\mathcal{P}'_{PC}$ in which I relax (IC) corresponding to over-reporting of budgets. One can show that Lemmas 3 and 4 and Corollary 1 still hold. Next, I consider the principal's relaxed problem with two modifications: (i) The allocation rule is an M'-step allocation rule for some integer $M' \leq M$ and $M \geq 2$ is a fixed integer; and (ii) the principal's budget balance constraint is relaxed by a constant $d \geq 0$. Denote this problem by $\mathcal{P}'_{PC}(M,d)$ and its value by $V_{PC}(M,d)$. Clearly, $\mathcal{P}'_{PC}(M,d)$ is identical to $\mathcal{P}'(M,d)$ if we replace (BC) by the following (PC) constraint:

$$a^M v_1^{M-1} - \sum_{j=1}^{M-1} a^j (v_1^j - v_1^{j-1}) - u(\underline{v}, b_1) \leq b_1 a^M. \tag{η}$$

One can readily extend the results of Lemma 5 to the current setting, which says that, in an optimal mechanism of $\mathcal{P}'_{PC}(M,d)$, the verification probability is non-decreasing in a low-budget agent's reported value. Using the monotonicity of verification probability and the linearity of $\mathcal{P}'_{PC}(M,d)$ in $u(\underline{v}, b_1)$, $u(\underline{v}, b_2)$ and a^m's, we can prove that $V_{PC}(M,d) = V_{PC}(M-1,d)$ for M sufficiently large. However, it is hard to further improve this result, as in Section 4.2 when we require only the weaker (BC) constraint. In particular, the proof of Lemma 6 does not apply here. It holds if we also assume $a(v,b) = 0$ for all $v < b_1$. This assumption requires that an agent whose valuation is too low (lower than b_1) receives goods with probability zero. Note that optimal mechanisms in Theorems 4 and 5 satisfy this condition. I conjecture this condition also holds in the general case, although I cannot prove it. Under this additional assumption, we can easily extend the result of Theorem 3 to the current setting.

Theorem 6 *Suppose Assumptions 1 and 2 hold and $a(v,b) = 0$ for all $v < b_1$. $a^*(\rho)$, $v_1^*(\rho)$, $v_2^*(\rho)$, $v_2^{**}(\rho)$, $u_1^*(\rho)$ and $u_2^*(\rho)$ exist such that an optimal mechanism of $\mathcal{P}_{PC}$ is given by*

$$\begin{aligned}
&a(v,b_1) = \chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho), \;\; p(v,b_1) = \chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho) v_1^*(\rho) - u_1^*(\rho), \\
&q(v,b_1) = \frac{1}{c}\left[\chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho)\left(v_2^*(\rho) - v_1^*(\rho)\right) + u_1^*(\rho) - u_2^*(\rho)\right], \\
&a(v,b_2) = \chi_{\{v \geq v_2^*(\rho)\}} a^*(\rho) + \chi_{\{v \geq v_2^{**}(\rho)\}}\left(1 - a^*(\rho)\right), \\
&p(v,b_2) = \chi_{\{v \geq v_2^*(\rho)\}} a^*(\rho) v_2^*(\rho) + \chi_{\{v \geq v_2^{**}(\rho)\}}\left(1 - a^*(\rho)\right) v_2^{**}(\rho) - u_2^*(\rho), \\
&q(v,b_2) = 0,
\end{aligned}$$

where $a^*(\rho) = u_1^*(\rho)/\left[v_1^*(\rho) - b_1\right]$, $\underline{v} \leq v_1^*(\rho) \leq v_2^*(\rho) \leq v_2^{**}(\rho) \leq \overline{v}$, $0 < a^*(\rho) \leq 1$ and $u_1^*(\rho) \geq u_2^*(\rho)$.

This optimal mechanism can be implemented by a modified RwRRC scheme introduced in Section 5. In this modified RwRRC scheme, a lottery participant in the first stage pays *only if* he receives one unit good. If verification is sufficiently costly or the principal cannot inspect agents, then this mechanism is identical to the RwRRC scheme in Che, Gale, and Kim (2013).

7.2 Monetary Penalty

In this subsection, I discuss what happens if penalty is transferable. Specifically, the principal can inspect an agent's budget at a cost $k > 0$, and can impose a monetary penalty of up to $c \geq 0$. I also allow the principal to punish an innocent agent and an agent without verification. Nonetheless, as I will show later, it is optimal for the principal to punish an agent if and only if he is found to have lied about his budget. Using this result, the principal's problem can be reduced to the one stated in Section 2, when penalty is not transferable. Hence, all results in previous sections also hold in the case of monetary penalty.

Since I also relax the assumption that an agent is punished if and only if he is found to have lied, a direct mechanism is a quadruple (a, p, q, θ), where a, p and q are defined as before and $\theta : T \times \{b_1, b_2, n\} \to [0, c]$ denotes the penalty imposed on an agent. In particular, $\theta(\hat{t}, n) \in [0, c]$ denotes the penalty imposed on an agent who reports $\hat{t}$ and is not inspected, and $\theta(\hat{t}, b) \in [0, c]$ denotes the penalty imposed on an agent who reports $\hat{t}$ and is inspected, and whose budget is revealed to be b. The utility of an agent who has type $t := (v, b)$ and reports $\hat{t}$ is

$$u(\hat{t}, t) = \begin{cases} a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n) - q(\hat{t})\theta(\hat{t}, b) & \text{if } p(\hat{t}) + \theta(\hat{t}, b) \leq b \text{ and } p(\hat{t}) + \theta(\hat{t}, n) \leq b, \\ -\infty & \text{otherwise.} \end{cases}$$

The principal's problem becomes

$$\max_{a,p,q,\theta} \mathbb{E}_t \left[a(t)v - q(t)k\right], \tag{$\mathcal{P}$}$$

subject to (IR), (S) and

$$u(t) \geq u(\hat{t}, t), \ \forall t \in T, \hat{t} \in \left\{\hat{t} \in T \,\middle|\, p(\hat{t}) + \max\{\theta(\hat{t}, n), \theta(\hat{t}, b)\} \leq b \right\}, \tag{IC}$$

$$p(t) + \max\{\theta(t, n), \theta(t, b)\} \leq b, \ \forall t \in T, \tag{BC}$$

$$\mathbb{E}_t \left[p(t) + (1 - q(t))\theta(t, n) + q(t)\theta(t, b) - q(t)k\right] \geq 0. \tag{BB}$$

Note that (BC) requires that an agent must be able to afford the payment and the punishment. I show that it is without loss of generality to focus on mechanisms in which an agent is penalized if and only if he is found to have lied about his budget, and whenever he is found to have lied he has the maximum possible monetary penalty c imposed upon him.

Lemma 7 *It is without loss of generality to focus on mechanisms in which $\theta(\hat{t}, n) = 0$, $\theta(\hat{t}, b) = 0$ if $\hat{b} = b$ and $\theta(\hat{t}, b) = c$ if $\hat{b} \neq b$.*

Using Lemma 7, the principal's problem can be reduced to the one stated in Section 2 when penalty is not transferable. Hence, all results in previous sections also hold in the case of monetary penalty.

7.3 Costly Disclosure

In this subsection, I study what happens if it is also costly for an agent to have his report verified. For example, agents may bear some costs of providing documentation. Assume that an agent incurs a non-monetary cost only when his report is verified. Let $c^T \geq 0$ denote the cost incurred by an agent from being verified by the principal if he reported his budget truthfully, and let $c^F \geq c^T$ be his cost if he lied.[24] As I will show below, disclosure costs have three effects. Firstly, similar to monetary transfers, disclosure costs can also be used to screen agents with different valuations and help relax agents' budget constraints. Secondly, it is more costly for an agent to lie about his budget because $c^F \geq c^T$. Finally, disclosure costs make verification more costly for the principal whose concern is welfare. Although solving the optimal mechanism is difficult and beyond the scope of this paper, I show that if the difference between c^F and c^T is sufficiently large, then the first two effects dominate and introducing disclosure costs improves welfare.

The utility of an agent who has type $t = (v, b)$ and reports $\hat{t}$ is

$$u(\hat{t}, t) = \begin{cases} a(\hat{t})v - p(\hat{t}) - q(\hat{t})c^T & \text{if } \hat{b} = b \text{ and } p(\hat{t}) \leq b, \\ a(\hat{t})v - p(\hat{t}) - q(\hat{t})\left(c^F + c\right) & \text{if } \hat{b} \neq b \text{ and } p(\hat{t}) \leq b, \\ -\infty & \text{if } p(\hat{t}) > b. \end{cases}$$

The principal's problem becomes

$$\max_{a,p,q} \mathbb{E}_t \left[a(t)v - q(t)k - q(t)c^T\right], \tag{$\mathcal{P}_{DC}$}$$

subject to (IR), (IC), (BC), (BB) and (S). Note that if $c^T = 0$, then ($\mathcal{P}_{DC}$) is equivalent to the original problem $\mathcal{P}$ in which the punishment is $c^F + c$.

Consider the more general case in which $c^T \geq 0$. Let $p^e(t) := p(t) + q(t)c^T$ for all t, $k^e := k + c^T$ and $c^e := c + c^F - c^T$. As in Section 4, I separate (IC) into two categories and ignore those corresponding to over-reporting of budgets. Then the principal's relaxed problem can be written as:

$$\max_{a,p,q} \mathbb{E}_t \left[a(t)v - q(t)k^e\right], \tag{$\mathcal{P}'_{DC}$}$$

subject to

$$a(t)v - p^e(t) \geq 0, \ \forall t \in T, \tag{IR}$$

[24] The analysis goes through as long as $c^F + c \geq c^T$.

$$a(v,b)v - p^e(v,b) \geq a(\hat{v},b)v - p^e(\hat{v},b),\ \forall v,\hat{v},b, \tag{IC-v}$$

$$a(v,b_2)v - p^e(v,b_2) \geq a(\hat{v},b_1)v - q(\hat{v},b_1)c^e - p^e(\hat{v},b_1),\ \forall v,\hat{v}. \tag{IC-b}$$

$$p^e(t) \leq b + q(t)c^T,\ \forall t \in T, \tag{BC}$$

$$\mathbb{E}_t\left[p^e(t) - q(t)k^e\right] \geq 0,, \tag{BB}$$

$$\mathbb{E}_t\left[a(t)\right] \leq S. \tag{S}$$

Compare $\mathcal{P}'_{DC}$ with $\mathcal{P}'$. It is easy to see that the two problems are identical except for the (BC) constraint. In $\mathcal{P}'_{DC}$, a low-budget agent faces a less stringent budget constraint if he expects to be inspected with a non-zero probability. This is because in the presence of disclosure cost the *effective payment* made by an agent who reports his type truthfully is $p^e(t) = p(t) + q(t)c^T$. In addition to the monetary transfer $p(t)$, disclosure cost $q(t)c^T$ can also be used to screen agents with different valuations. Intuitively, an agent with a higher valuation is also willing to bear a higher disclosure cost. Though disclosure cost can be used to relax an agent's budget constraint, it reduces an agent's utility which makes verification more costly from the principal's perspective, i.e., $k^e = k + c^T \geq k$. As a result, the total welfare effect of introducing c^T is ambiguous.

The *effective punishment* perceived by an agent is now $c + c^F - c^T$, which is equal to the original punishment plus the additional disclosure cost one must incur when lying about his budget. Hence, an increase in c^F is always welfare-enhancing, as it discourages agents from misreporting their budgets.

Though solving $\mathcal{P}_{DC}$ is beyond the scope of this paper, Proposition 5 provides a sufficient condition under which introducing disclosure costs c^T and c^F improve the total welfare. Let $V(k,c,b_1)$ denote the value of the principal's original problem, in which verification cost is k, punishment is c and low-budget agent's budget is b_1; and let $V_{DC}(k,c,b_1,c^T,c^F)$ denote the value of the principal's problem in which verification cost is k, punishment is c, low-budget agent's budget is b_1 and disclosure costs are c^T and c^F.

Proposition 5 *Suppose Assumptions 1 and 2 hold. If* $k/c \geq c^T/(c^F - c^T)$, *then* $V_{DC}(k,c,b_1,c^F,c^T) \geq V(k,c,b_1)$. *Furthermore, if* $q(\overline{v},b_1) > 0$ *in the optimal mechanism of* $\mathcal{P}(k + c^T, c + c^F - c^T, b_1)$, *then* $V_{DC}(k,c,b_1,c^F,c^T) > V(k,c,b_1)$.

7.4 Modified Type

In the standard environment, when agents are not budget-constrained, an agent's valuation is defined as the maximum amount of money he is willing to pay for one unit good.

When agents are budget constrained, the natural analogue is the minimum of an agent's valuation v and budget b. I follow Pai and Vohra (2014) and redefine $t := \min\{v, b\}$ as an agent's *modified type*. In this subsection, I explain why it is necessary to explicitly model budget constraint rather than accommodate budgets in the above way.

Let G denote the distribution of the modified type. Then

$$G(t) = \begin{cases} F(t) & \text{if } t < b_1, \\ \pi F(b_1) + 1 - \pi & \text{if } t = b_1, \\ \pi F(t) & \text{if } t > b_1. \end{cases}$$

The principal's ability to inspect an agent's budget implies that she can perfectly learn a low-budget agent's modified type if his valuation exceeds his budget. I first solve the principal's problem by assuming common-knowledge budgets and then verify that no agent has any incentive to misreport his modified type. In other words, inspection is never used in the optimal mechanism. Denote the principal's problem by $\mathcal{P}_{MT}$.

Proposition 6 *Suppose an agent's budget is common knowledge. (i) If* $\pi\left[1 - F(b_1)\right] \leq S < 1 - F(b_1)$*, then the optimal mechanism of* $\mathcal{P}_{MT}$ *is given by*

$$a(t) = \chi_{\{t=b_1\}} \frac{S - \pi\left[1 - F(b_1)\right]}{1 - \pi} + \chi_{\{t>b_1\}} 1,\; p(t) = \chi_{\{t=b_1\}} \frac{S - \pi\left[1 - F(b_1)\right]}{1 - \pi} b_1 + \chi_{\{t \geq b_1\}} b_1.$$

(ii) If $S < \pi\left[1 - F(b_1)\right]$*, then the optimal mechanism is given by* $a(t) = \chi_{\{t>t^*\}} 1$ *and* $p(t) = \chi_{\{t \geq b_1\}} t^*$*, where* t^* *is such that* $\pi\left[1 - F(t^*)\right] = S$*.*

The following corollary is a straightforward corollary of Proposition 6.

Corollary 3 *Suppose an agent's budget is his private information. The mechanism given in Proposition 6 is incentive compatible and therefore optimal in* $\mathcal{P}_{MT}$*.*

Compared with Theorem 3, the mechanism given in Proposition 6 is sub-optimal because (i) it allocates too many resources to high-budget agents; and (ii) it has "too little" rationing for high-budget agents but "too much" rationing for low-budget agents.

What went wrong here? First, consider a low-budget agent with modified type $t = b_1$ and a high-budget agent with modified type $t = b_1 + \epsilon$ for some $\epsilon > 0$. Then the low-budget agent's expected valuation is higher than the high-budget agent's valuation, i.e., $\mathbb{E}[v | t = b_1, b = b_1] > b_1 + \epsilon$, for $\epsilon > 0$ sufficiently small. This implies that the low-budget agent should receive the good with higher probability as in Theorem 3, i.e., $v_1^* \leq v_2^*$.

However, in the current mechanism it is the high-budget agent who receives the good with higher probability. Second, consider two low-budget agents with valuations $v = b_1$ and $v' = b_1 + \epsilon$ for $\epsilon > 0$ sufficiently small, respectively. In the current mechanism, they are pooled. However, their payments are $p(b_1) < b_1$, which suggests that they should be separated as in Theorem 3, i.e., $v_1^* > b_1$. The second observation is also made in Pai and Vohra (2014) in which the principal's objective is maximizing revenue.

8 Conclusion

In this paper, I study the problem of a principal who wishes to distribute a limited supply of indivisible goods to a population of budget-constrained agents. Both valuation and budget are an agent's private information. The principal can inspect an agent's budget through a costly verification process and punish an agent who makes a false statement. I characterize the direct surplus-maximizing mechanism in this environment. This direct mechanism can be implemented by a two-stage mechanism that exhibits some of the features of Singapore's affordable housing program.

Throughout the paper, I impose two regularity assumptions on the distribution of an agent's valuation: monotone hazard rate condition and decreasing density condition. These two assumptions are commonly used in the literature studying mechanism design problem with financially constrained agents. Their primary role is to rule out complicated pooling regions in an optimal mechanism, which greatly simplifies analysis. Several of the paper's results (Lemmas 3, 4 and 5) still hold if I replace these two assumptions by weaker conditions. However, Lemma 6 may not hold anymore as an optimal mechanism is expected to involve more complicated pooling regions.

Another simplifying assumption I make in the paper is that valuation and budget are independent. In some environments, this assumption is reasonable. For example, an individual's valuation of health insurance is largely affected by his or her health risk, which is relatively independent of his or her wealth. In general, an individual's valuation and budget can be either positively or negatively correlated, depending on whether the goods are considered normal goods or inferior goods. The independence assumption is much harder to relax. As Pai and Vohra (2014) show, if valuation and budget are correlated, an optimal mechanism may involve more complicated pooling regions.

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Supplementary Appendix

A The revelation principle

Consider a general mechanism that consists of a message space $\mathscr{M}$ and a quadruplet (a, p, q, θ), where $a : \mathscr{M} \to [0, 1]$ maps a message to the probability an agent obtains the good, $p : \mathscr{M} \to \mathbb{R}$ maps a message to the payment an agent must make, $q : \mathscr{M} \to [0, 1]$ maps a message to the probability an agent is inspected and $\theta : \mathscr{M} \times \{n, b_1, b_2\} \to [0, 1]$ denotes the punishment rule. In particular, $\theta(m, n)$ denotes the probability an agent is penalized if he is not inspected and $\theta(m, b)$ denotes the probability an agent is penalized if he is inspected and his budget is revealed to be b.

Given a mechanism, an agent of type $t = (v, b)$ chooses $m \in \mathscr{M}$ to maximize his expected payoff:

$$a(m)v - p(m) - (1 - q(m))\theta(m, n)c - q(m)\theta(m, b)c$$

subject to the constraint that $p(m) \leq b$. Let $m^*(t)$ denote the solution to the agent's maximization problem. For simplicity, I assume that $m^*(t)$ is deterministic, but it is easy to accommodate mixed strategies. If the agent's problem has multiple solutions, then some deterministic selection rule is used. Consider a new mechanism with message space T. Let $a^*(t) = a(m^*(t))$, $p^*(t) = p(m^*(t))$, $q^*(t) = a(m^*(t))$ and $\theta^*(t, \cdot) = \theta(m^*(t), \cdot)$. Then the new mechanism is incentive compatible. Clearly, an agent has no incentive to report $\hat{t}$ such that $p^*(\hat{t}) > b$. For $\hat{t}$ such that $p^*(\hat{t}) \leq b$, we have

$$\begin{aligned}&a(m^*(t))v - p(m^*(t)) - (1 - q(m^*(t)))\theta(m^*(t), n)c - q(m^*(t))\theta(m^*(t), b)c \\ \geq &a(m^*(\hat{t}))v - p(m^*(\hat{t})) - (1 - q(m^*(\hat{t})))\theta(m^*(\hat{t}), n)c - q(m^*(\hat{t}))\theta(m^*(\hat{t}), b)c.\end{aligned}$$

The inequality simply follows from the fact that $m^*(t)$ is a solution to a type t agent's problem in the original mechanism. Clearly, the principal's payoff in the truthtelling equilibrium is as same as that in the original mechanism.

Hence, it is without loss of generality to focus on direct mechanisms. A direct mechanism is denoted by a quadruplet (a, p, q, θ), where $a : T \to [0, 1]$ denotes the allocation rule, $p : T \to \mathbb{R}$ denotes the payment rule, $q : T \to [0, 1]$ denotes the inspection rule and $\theta : T \times \{n, b_1, b_2\} \to [0, 1]$ denotes the punishment rule. In the main body of the paper, I assume that the principal can only punish an agent who is inspected and found to have lied about his budget. That is $\theta(t, n) = 0$, $\theta(t, \hat{b}) = 1$ if $\hat{b} \neq b$ and $\theta(t, \hat{b}) = 0$ if $\hat{b} = b$.

B Random payment rules

When defining a direct mechanism in Section 2, I implicitly assume that the payment rule is deterministic. I argue that this is without loss of generality. Consider a random payment rule $\tilde{p} : T \to \Delta(\mathbb{R})$. Let $\text{supp}(\tilde{p}(t))$ denote the supremum of payments in the support of $\tilde{p}(t)$. The utility of an agent who has type t and report $\hat{t}$ is

$$u(\hat{t}, t) = \begin{cases} a(\hat{t})v - \mathbb{E}[\tilde{p}(\hat{t})] & \text{if } \hat{b} = b \text{ and } \text{supp}(\tilde{p}(\hat{t})) \leq b, \\ a(\hat{t})v - q(\hat{t})c - \mathbb{E}[\tilde{p}(\hat{t})] & \text{if } \hat{b} \neq b \text{ and } \text{supp}(\tilde{p}(\hat{t})) \leq b, \\ -\infty & \text{if } \text{supp}(\tilde{p}(\hat{t})) > b. \end{cases}$$

In other words, an agent suffers an unbounded dis-utility if his budget constraint is violated with a positive probability. Then IC constraints become

$$u(t) \geq u(\hat{t}, t), \ \forall t \in T, \hat{t} \in \left\{ \hat{t} \in T \, \middle| \text{supp}(\tilde{p}(\hat{t})) \leq b \right\}.$$

The principal's objective function and all the other constraints remain intact.

By a similar argument to that used in Pai and Vohra (2014), for any feasible mechanism $(a, \tilde{p}, q)$, one can construct another feasible mechanism $(a, \hat{p}, q)$ by setting

$$\hat{p}(t) = \begin{cases} \mathbb{E}[\tilde{p}(t)] - \epsilon & \text{with propability } \frac{b - \mathbb{E}[\tilde{p}(t)]}{b - \mathbb{E}[\tilde{p}(t)] + \epsilon}, \\ b & \text{with propability } \frac{\epsilon}{b - \mathbb{E}[\tilde{p}(t)] + \epsilon}, \end{cases}$$

for some $\epsilon > 0$ sufficiently small. Furthermore, both mechanisms have the same welfare. Observe that, under this construction, IC constraints corresponding to over-reporting of budgets are satisfied "for free". Given these observations, it is not hard to see that one can solve the principal's problem (allowing for random payment rules) by restricting attention to deterministic payment rules but relaxing IC constraints corresponding to the over-reporting of budgets. As I will show later, in the optimal mechanism of $\mathcal{P}$ no low-budget agent has any incentive to over report his budget. Hence, it is without loss of generality to focus on deterministic payment rules.

C Common knowledge budgets

Proof of Theorem 1. Since budgets are common knowledge, a direct mechanism can now be denoted by a pair (a, p). Let (a, p) be a feasible mechanism. For each $b \in \boldsymbol{B}$, $a(\cdot, b)$ is non-decreasing and $p(v, b) = va(v, b) - \int_{\underline{v}}^{v} a(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$ for all v. Consider another mechanism (a^*, p^*). Let $a^*(\cdot, b)$ be defined by

$$a^*(v, b) = \begin{cases} a(\overline{v}, b) & \text{if } v \geq v_b^* \\ 0 & \text{otherwise} \end{cases},$$

where v_b^* is such that

$$\int_{\underline{v}}^{\overline{v}} a(v, b) f(v)\mathrm{d}v = a(\overline{v}, b)(1 - F(v_b^*)). \tag{11}$$

Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$ for all v. First, we show that (BC) holds:

$$\begin{aligned} p^*(\overline{v}, b) =& \overline{v}a(\overline{v}, b) - \int_{\underline{v}}^{\overline{v}} a^*(v, b)\mathrm{d}v - u(\underline{v}, b) \\ \leq& \overline{v}a(\overline{v}, b) - \int_{\underline{v}}^{\overline{v}} a(v, b)\mathrm{d}v - u(\underline{v}, b) \leq b. \end{aligned}$$

The first inequality holds if and only if

$$\begin{aligned} & \int_{\underline{v}}^{\overline{v}} [a^*(v, b) - a(v, b)]\mathrm{d}v \geq 0, \\ \iff & \int_{v_b^*}^{\overline{v}} [a^*(v, b) - a(v, b)]\mathrm{d}v \geq \int_{\underline{v}}^{v_b^*} [a(v, b) - a^*(v, b)]\mathrm{d}v. \end{aligned}$$

The last inequality holds since

$$\begin{aligned} \int_{v_b^*}^{\overline{v}} [a^*(v, b) - a(v, b)]\mathrm{d}v &= \int_{v_b^*}^{\overline{v}} [a^*(v, b) - a(v, b)] f(v) \frac{1}{f(v)}\mathrm{d}v \\ &\geq \int_{v_b^*}^{\overline{v}} [a^*(v, b) - a(v, b)] f(v) \frac{1}{f(v_b^*)}\mathrm{d}v \\ &= \int_{\underline{v}}^{v_b^*} [a(v, b) - a^*(v, b)] f(v) \frac{1}{f(v_b^*)}\mathrm{d}v \end{aligned}$$

$$\geq \int_{\underline{v}}^{v_b^*} [a(v,b) - a^*(v,b)] f(v) \frac{1}{f(v)} \mathrm{d}v$$
$$= \int_{\underline{v}}^{v_b^*} [a(v,b) - a^*(v,b)] \mathrm{d}v,$$

where the second and fourth line holds since f is non-increasing by Assumption 2 and the third line holds by (11). Clearly, (a^*, p^*) also satisfies constraints (IR), (IC) and (S) and improves welfare. Finally, the revenue obtained by (a^*, p^*) is

$$\mathbb{E}_t[p^*(t)] = -(1-\pi)u(\underline{v}, b_1) - \pi u(\underline{v}, b_2) + \int_{\underline{v}}^{\overline{v}} \left[v - \frac{1 - F(v)}{f(v)} \right] \left[(1-\pi) a^*(v, b_1) + \pi a^*(v, b_2) \right] \mathrm{d}v.$$

By Assumption 1, $v - [1 - F(v)]/f(v)$ is strictly increasing. Thus, (a^*, p^*) also improves revenue, and therefore satisfies (BB_{CB}). Hence, there exists v_1^* and v_2^* such that the optimal allocation rule satisfies $a(v, b_1) = \chi_{\{v \geq v_1^*\}}(v) \min \left\{ \frac{u(\underline{v}, b_1) + b_1}{v_1^*}, 1 \right\}$ and $a(v, b_2) = \chi_{\{v \geq v_2^*\}}(v)$.
■

D Privately known budgets

This section is organized as follows. Section D.1 consists of proofs in Section 4.1. Section D.2 consists of proofs in Section 4.2 except that of Lemma 6. The proof of Lemma 6 is collected in Section D.3.

D.1 No verification

Proof of Lemma 1. Let (a, p) be an optimal mechanism of $\mathcal{P}'_{NI}$. If $v = \underline{v}$, (3) reduces to $u(\underline{v}, b_2) \geq u(\underline{v}, b_1)$. Suppose, to the contrary, that $u(\underline{v}, b_2) > u(\underline{v}, b_1)$. Let

$$u^*(\underline{v}, b_1) = u^*(\underline{v}, b_2) = (1-\pi)u(\underline{v}, b_1) + \pi u(\underline{v}, b_2).$$

Let v^- and v^+ be such that

$$v^- := \sup \left\{ v \,\middle|\, \int_{\underline{v}}^{v} a(\nu, b_1) \mathrm{d}\nu + u(\underline{v}, b_1) - (1-\pi)u(\underline{v}, b_1) - \pi u(\underline{v}, b_2) \leq 0 \right\}, \quad \text{and}$$

$$\upsilon^+ := \sup \left\{ \upsilon \,\middle|\, (1-\pi)\int_{\underline{\upsilon}}^{\upsilon^-} a(\nu, b_1) f(\nu)\mathrm{d}\nu - \pi \int_{\upsilon^-}^{\upsilon} \left[a(\upsilon, b_2) - a(\nu, b_2)\right] f(\nu)\mathrm{d}\nu \geq 0 \right\}.$$

Assume without loss of generality that

$$(1-\pi)\int_{\underline{\upsilon}}^{\upsilon^-} a(\nu, b_1) f(\nu)\mathrm{d}\nu - \pi \int_{\upsilon^-}^{\upsilon^+} \left[a(\upsilon^+, b_2) - a(\nu, b_2)\right] f(\nu)\mathrm{d}\nu = 0. \tag{12}$$

Let

$$a^*(\upsilon, b_1) := \begin{cases} 0 & \text{if } \upsilon \leq \upsilon^- \\ a(\upsilon, b_1) & \text{if } \upsilon > \upsilon^- \end{cases}, \quad \text{and}$$

$$a^*(\upsilon, b_2) := \begin{cases} a(\upsilon, b_2) & \text{if } \upsilon \leq \upsilon^- \\ a(\upsilon^+, b_2) & \text{if } \upsilon^- < \upsilon \leq \upsilon^+ \\ a(\upsilon, b_2) & \text{if } \upsilon > \upsilon^+ \end{cases}.$$

Clear, $a^*(\upsilon, b)$ is non-decreasing in υ for both b. Let $p^*(\upsilon, b) = \upsilon a^*(\upsilon, b) - \int_{\underline{\upsilon}}^{\upsilon} a^*(\nu, b)\mathrm{d}\nu$ for all υ and b. By the definition of υ^-,

$$\begin{aligned} p^*(\overline{\upsilon}, b_1) &= \overline{\upsilon} a^*(\overline{\upsilon}, b_1) - \int_{\underline{\upsilon}}^{\overline{\upsilon}} a^*(\nu, b_1)\mathrm{d}\nu - u^*(\underline{\upsilon}, b_1) \\ &\leq \overline{\upsilon} a(\overline{\upsilon}, b_1) - \int_{\upsilon^-}^{\overline{\upsilon}} a^*(\nu, b_1)\mathrm{d}\nu - u^*(\underline{\upsilon}, b_1) \\ &\leq \overline{\upsilon} a(\overline{\upsilon}, b_1) - \int_{\upsilon^-}^{\overline{\upsilon}} a(\nu, b_1)\mathrm{d}\nu - \int_{\underline{\upsilon}}^{\upsilon^-} a(\nu, b_1)\mathrm{d}\nu - u(\underline{\upsilon}, b_1) = p(\overline{\upsilon}, b_1) \leq b_1. \end{aligned}$$

Hence, the new mechanism (a^*, p^*) satisfies (BC). Define

$$\Delta(\upsilon) := u^*(\underline{\upsilon}, b_2) + \int_{\underline{\upsilon}}^{\upsilon} a^*(\nu, b_2)\mathrm{d}\nu - u^*(\underline{\upsilon}, b_1) - \int_{\underline{\upsilon}}^{\upsilon} a^*(\nu, b_1)\mathrm{d}\nu, \ \forall \upsilon.$$

For all $\upsilon \leq \upsilon^-$, since $u^*(\underline{\upsilon}, b_2) = u^*(\underline{\upsilon}, b_1)$ and $a^*(\nu, b_2) \geq 0 = a^*(\nu, b_1)$ for all $\nu \leq \upsilon$, $\Delta(\upsilon) \geq 0$. For $\upsilon = \upsilon^+$,

$$\begin{aligned} \Delta(\upsilon^+) =& u^*(\underline{\upsilon}, b_2) + \int_{\underline{\upsilon}}^{\upsilon^+} a^*(\nu, b_2)\mathrm{d}\nu - u^*(\underline{\upsilon}, b_1) - \int_{\underline{\upsilon}}^{\upsilon^+} a^*(\nu, b_1)\mathrm{d}\nu \\ \geq& u(\underline{\upsilon}, b_2) + \int_{\underline{\upsilon}}^{\upsilon^-} a(\nu, b_2)\mathrm{d}\nu + \int_{\upsilon^-}^{\upsilon^+} a(\upsilon^+, b_2)\mathrm{d}\nu - u(\underline{\upsilon}, b_1) - \int_{\upsilon^-}^{\upsilon^+} a(\nu, b_1)\mathrm{d}\nu - \frac{1}{\pi}\int_{\upsilon^-}^{\upsilon^+} a(\nu, b_1)\mathrm{d}\nu \end{aligned}$$

$$
\begin{aligned}
\geq & u(\underline{v}, b_2) + \int_{\underline{v}}^{v^-} a(v, b_2)\mathrm{d}v + \int_{v^-}^{v^+} a(v^+, b_2)\mathrm{d}v - u(\underline{v}, b_1) - \int_{v^-}^{v^+} a(v, b_1)\mathrm{d}v - \int_{v^-}^{v^+} a(v, b_1)\mathrm{d}v \\
& - \int_{v^-}^{v^+} \left[a(v^+, b_2) - a(v, b_2)\right] \mathrm{d}v \\
= & u(\underline{v}, b_2) + \int_{\underline{v}}^{v^+} a(v, b_2)\mathrm{d}v - u(\underline{v}, b_1) - \int_{\underline{v}}^{v^+} a(v, b_1)\mathrm{d}v,
\end{aligned}
$$

where the first inequality holds by the definition of v^-. To see the second inequality holds, observe that

$$
\begin{aligned}
& \pi \int_{v^-}^{v^+} \left[a(v^+, b_2) - a(v, b_2)\right] \mathrm{d}v \\
\geq & \pi \int_{v^-}^{v^+} \left[a(v^+, b_2) - a(v, b_2)\right] f(v) \frac{1}{f(v^-)} \mathrm{d}v \\
= & (1-\pi) \int_{\underline{v}}^{v^-} a(v, b_1) f(v) \frac{1}{f(v^-)} \mathrm{d}v \\
\geq & (1-\pi) \int_{\underline{v}}^{v^-} a(v, b_1)\mathrm{d}v,
\end{aligned}
$$

where the first and the third lines hold by Assumption 2 and the second line holds by (12). Since $a^*(v, b_2) = a(v, b_2)$ for all $v \geq v^+$ and b, $\Delta(v) \geq 0$ for all $v \geq v^+$. Finally, $\Delta'(v) = -a(v, b_1)$ for $v \in (v^-, v^+)$, which is non-increasing. Hence, $\Delta(v)$ is concave over (v^-, v^+). Since $\Delta(v^-) \geq 0$ and $\Delta(v^+) \geq 0$, we have $\Delta(v) \geq 0$ for all $v \in (v^-, v^+)$. Hence, (a^*, p^*) satisfies (3) (i.e., (IC-b) holds.) Clearly, (a^*, p^*) also satisfies constraints (IR), (IC-v) and (S). (a^*, p^*) satisfies (BB) and strictly improves welfare. This is a contradiction to the optimality of (a, p). Hence, it must be that $(u(\underline{v}, b_2), u(\underline{v}, b_1))$. ■

Proof of Lemma 2. Given Lemma 1, (3) becomes

$$
\int_{\underline{v}}^{v} a(v, b_2)\mathrm{d}v \geq \int_{\underline{v}}^{v} a(v, b_1)\mathrm{d}v, \ \forall v. \tag{13}
$$

For each $b \in B$, we have

$$
\begin{aligned}
\int_{\underline{v}}^{v} a(v, b) f(v)\mathrm{d}v &= \int_{\underline{v}}^{v} f(v)\mathrm{d}\int_{\underline{v}}^{v} a(v', b)\mathrm{d}v' \\
&= f(v)\int_{\underline{v}}^{v} a(v', b)\mathrm{d}v' - \int_{\underline{v}}^{v} \left[\int_{\underline{v}}^{v} a(v', b)\mathrm{d}v'\right] f'(v)\mathrm{d}v.
\end{aligned}
$$

Since $f \geq 0$ and $-f' \geq 0$, (4) follows from (13). ■

Proof of Theorem 2. We first solve the optimal mechanism of $\mathcal{P}'$ and then verify that the optimal mechanism also satisfies the (IC) constraint of low-budget agents. Let (a, p) be a feasible mechanism. For each $b \in B$, $a(\cdot, b)$ is non-decreasing and $p(v, b) = va(v, b) - \int_{\underline{v}}^{v} a(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$. Consider another mechanism (a^*, p^*).

Let $\hat{v} := \inf\{v | a(v, b_2) \geq a(\overline{v}, b_1)\}$. Note that $\hat{v} = \overline{v}$ if $a(\overline{v}, b_1) > a(\overline{v}, b_2)$ and $\hat{v} = \underline{v}$ if $a(\overline{v}, b_1) \leq a(\underline{v}, b_2)$. Let a^* be defined by

$$a^*(v, b_1) = \begin{cases} a(\overline{v}, b_1) & \text{if } v \geq v_1^*, \\ 0 & \text{otherwise,} \end{cases}$$

where v_1^* satisfies $a(\overline{v}, b_1)[1 - F(v_1^*)] = \int_{\underline{v}}^{\overline{v}} a(v, b_1) f(v)\mathrm{d}v$, and

$$a^*(v, b_2) = \begin{cases} 1 & \text{if } v \geq v_2^{**}, \\ a(\overline{v}, b_1) & \text{if } v_2^* \leq v < v_2^{**}, \\ 0 & \text{otherwise,} \end{cases}$$

where $v_2^* \leq \hat{v}$ satisfies $a(\overline{v}, b_1)[F(\hat{v}) - F(v_2^*)] = \int_{\underline{v}}^{\hat{v}} a(v, b_2) f(v)\mathrm{d}v$ and $v_2^{**} \geq \hat{v}$ satisfies $1 - F(v_2^{**}) + a(\overline{v}, b_1)[F(v_2^{**}) - F(\hat{v})] = \int_{\hat{v}}^{\overline{v}} a(v, b_2) f(v)\mathrm{d}v$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$.

We show that $v_1^* \geq v_2^*$. If $v_1^* \geq \hat{v}$, then $v_1^* \geq v_2^*$. If $v_1^* < \hat{v}$, then

$$\begin{aligned} a(\overline{v}, b_1)[F(\hat{v}) - F(v_1^*)] &= \int_{\underline{v}}^{\hat{v}} a(v, b_1) f(v)\mathrm{d}v + \int_{\hat{v}}^{\overline{v}} [a(v, b_1) - a(\overline{v}, b_1)] f(v)\mathrm{d}v \\ &\leq \int_{\underline{v}}^{\hat{v}} a(v, b_1) f(v)\mathrm{d}v \\ &\leq \int_{\underline{v}}^{\hat{v}} a(v, b_2) f(v)\mathrm{d}v \\ &= a(\overline{v}, b_1)[F(\hat{v}) - F(v_2^*)], \end{aligned}$$

where the third line holds by Lemma 2. In this case, it must be that $a(\overline{v}, b_1) > 0$ since otherwise $a(\overline{v}, b_1) = 0 \leq a(0, b_2)$, which implies $\hat{v} = \underline{v} \leq v_1^*$. Hence, $v_2^* \leq v_1^*$. Thus, (a^*, p^*) satisfies the (IC-b) constraint.

Clearly, (a^*, p^*) also satisfies constraints (BC), (IR), (IC-v), (S) and (BB) and strictly improves welfare unless $a^* = a$ almost surely. Suppose $v_2^* < v_1^*$, then it is welfare improving to increase v_2^* and reduce v_1^* without affecting any constraint. Hence, it is optimal to set $v_1^* = v_2^* = v^*$. Let $u^* = u(\underline{v}, b_1) = u(\underline{v}, b_2)$. Then the optimal allocation rule satisfies $a(v, b_1) = \chi_{\{v \geq v^*\}} \min\left\{\frac{u^*+b_1}{v^*}, 1\right\}$ and $a(v, b_2) = \chi_{\{v \geq v^*\}} \min\left\{\frac{u^*+b_1}{v^*}, 1\right\} + \chi_{\{v \geq v_2^{**}\}} \left(1 - \min\left\{\frac{u^*+b_1}{v^*}, 1\right\}\right)$.

Clearly, if $u^* + b_1 > v^*$, we can reduce u^* such that $u^* + b_1 = v^*$ without affecting any constraint or the principal's objective function. This completes the characterization of the optimal mechanism of $\mathcal{P}'$. Finally, it is easy to see that the (IC) constraint of low-budget types is satisfied. This completes the proof. ■

D.2 The general case

Proof of Lemma 3. The proof is by construction. Let (a, p, q) be a feasible mechanism. Suppose $a(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$. Consider another feasible mechanism (a^*, p^*, q^*).

Let $\hat{v}_2^m = \inf\{v | a(v, b_2) \geq a^m\}$ for $m = 1, \dots, M$, $\hat{v}_2^0 = 0$ and $\hat{v}_2^{M+1} = \overline{v}$. Given a, the optimal verification rule satisfies $q(v, b_1) = q^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$, where

$$q^m = \frac{1}{c} \max\left\{0, u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v_1^{m-1}) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu\right\}.$$

For each $m = 1, \dots, M+1$, there exists $v_2^{m-1} \in [\hat{v}_2^{m-1}, \hat{v}_2^m]$ such that

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m} a(v, b_2) f(v)\mathrm{d}v = a^{m-1}[F(v_2^{m-1}) - F(\hat{v}_2^{m-1})] + a^m[F(\hat{v}_2^m) - F(v_2^{m-1})]. \tag{14}$$

Consider $a^*(v, b_2)$ such that $a^*(v, b_2) = a^m$ if $v \in (v_2^{m-1}, v_2^m)$ for $m = 1, \dots, M$, $a^*(v, b_2) = 0$ if $v < v_2^0$ and $a^*(v, b_2) = 1$ if $v > v_2^M$. Note that if $a^1 = 0$, then $v_2^0 = \underline{v}$. If $a^M = 1$, then v_2^M is in-determined and we assume $v_2^M = v_2^{M-1}$. Let $a^*(v, b_1) = a(v, b_1)$.

Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$. Let $q^*(v, b_1) = q(v, b_1)$. We show that the (IC-b) constraint is satisfied. That is, for $m = 1, \dots, M$,

$$q^m c \geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v_2^{m-1}} a^*(\nu, b_2)\mathrm{d}\nu + a^m(v_2^{m-1} - v_1^{m-1}) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu.$$

Since $a^*(v, b_2) = a^m$ for $v \in (v_2^{m-1}, \hat{v}_2^m)$, we have

$$\begin{aligned}
&u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v_2^{m-1}} a^*(\nu, b_2)\mathrm{d}\nu + a^m(v_2^{m-1} - v_1^{m-1}) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a^*(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v_1^{m-1}) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v_1^{m-1}) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu,
\end{aligned}$$

where the last inequality holds if and only if

$$\int_{\underline{v}}^{\hat{v}_2^m} [a^*(\nu, b_2) - a(\nu, b_2)]\mathrm{d}\nu \geq 0.$$

To prove this, we prove that for $m = 1, \dots, M$

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m} [a^*(\nu, b_2) - a(\nu, b_2)]\mathrm{d}\nu \geq 0. \tag{15}$$

(15) holds if and only if

$$\int_{v_2^{m-1}}^{\hat{v}_2^m} [a^*(\nu, b_2) - a(\nu, b_2)]\mathrm{d}\nu \geq \int_{\hat{v}_2^{m-1}}^{v_2^{m-1}} [a(\nu, b_2) - a^*(\nu, b_2)]\mathrm{d}\nu. \tag{16}$$

(16) follows from the construction of a^* and Assumption 2:

$$\begin{aligned}
\int_{v_2^{m-1}}^{\hat{v}_2^m} [a^*(\nu, b_2) - a(\nu, b_2)]\mathrm{d}\nu &\geq \int_{v_2^{m-1}}^{\hat{v}_2^m} [a^*(\nu, b_2) - a(\nu, b_2)]f(\nu)\frac{1}{f(v_2^{m-1})}\mathrm{d}\nu \\
&= \int_{\hat{v}_2^{m-1}}^{v_2^{m-1}} [a(\nu, b_2) - a^*(\nu, b_2)]f(\nu)\frac{1}{f(v_2^{m-1})}\mathrm{d}\nu \\
&\geq \int_{\hat{v}_2^{m-1}}^{v_2^{m-1}} [a(\nu, b_2) - a^*(\nu, b_2)]\mathrm{d}\nu.
\end{aligned}$$

By Assumption 1, $\mathbb{E}_t[p^*(t)] \geq \mathbb{E}_t[p(t)]$. Hence, constraint (BB) is satisfied. It is also clear that (a^*, p^*, q^*) satisfies constraints (IR), (IC-v), (BC) and (S), and strictly improves welfare unless $a^* = a$ almost surely. ■

Proof of Lemma 4. The proof is by contradiction. Let (a, p, q) be a feasible mechanism, where a is a M-step allocation rule, p satisfies the envelope condition and q is given by (7). Suppose (a, p, q) satisfies neither (C1) nor (C2). I show that one can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare and satisfies one of the two conditions. Furthermore, a^* is a M'-step function for some $M' \leq M$. I break the proof into three steps.

Step 1. Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) < 0$. Let $m > 1$ be such that $v_2^{m'-1} - v_1^{m'-1} \leq 0$ for all $m' < m$ and $v_2^{m-1} - v_1^{m-1} > 0$. If there is no such m, then (a, p, q) satisfies (C2). Let $\hat{v}$ be defined by $F(\hat{v}) = \pi F(v_2^{m-1}) + (1 - \pi)F(v_1^{m-1})$ if $F(v_1^m) > \pi F(v_2^{m-1}) + (1 - \pi)F(v_1^{m-1})$ and $\hat{v} = v_1^m$ otherwise. Consider two different cases: (i) $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) \geq \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) - a^1 \left(v_2^0 - v_1^0\right)]$ and (ii) $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) < \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) - a^1 \left(v_2^0 - v_1^0\right)]$.

(i) Suppose $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) \geq \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) - a^1 \left(v_2^0 - v_1^0\right)]$. Let $\tilde{v}_1^{m-1} \in [v_1^{m-1}, \hat{v}]$ be such that

$$(a^m - a^{m-1})(\tilde{v}_1^{m-1} - v_1^{m-1}) = \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) - a^1 \left(v_2^0 - v_1^0\right)].$$

Let $\tilde{v}_2^{m-1} \in [\hat{v}, v_2^{m-1}]$ be such that $\pi[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] = (1 - \pi)[F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})]$. Let $\tilde{v}_i^{m'} = v_i^{m'}$ for $i = 1, 2$ and $m' \neq m - 1$. Let $a^*(v, b_1) = a^{m-1}$ if $v \in (v_1^{m-1}, \tilde{v}_1^{m-1})$ and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $a^*(v, b_2) = a^m$ if $v \in (\tilde{v}_2^{m-1}, v_2^{m-1})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $u^*(\underline{v}, b_1) = (1 - \pi)u(\underline{v}, b_1) + \pi u(\underline{v}, b_2) - \pi a^1 \left(v_2^0 - v_1^0\right)$ and $u^*(\underline{v}, b_2) = (1 - \pi)u(\underline{v}, b_1) + \pi u(\underline{v}, b_2) + (1 - \pi)a^1 \left(v_2^0 - v_1^0\right)$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u^*(\underline{v}, b)$. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint holds. For $v \in (\tilde{v}_1^{m'-1}, \tilde{v}_1^{m'})$, $m' = 1, \ldots, m - 1$, (IC-b) holds since

$$u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq 0 \leq q^*(v, b_1)c.$$

For $v \in (\tilde{v}_1^{m'-1}, \tilde{v}_1^{m'})$, $m' = m, \ldots, M$, we have $q^*(v, b_1) = q^m$. Then (IC-b) holds since

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
&= \sum_{j=1}^{m'} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^m - a^{m-1})(\tilde{v}_2^{m-1} - \tilde{v}_1^{m-1} - v_2^{m-1} + v_1^{m-1}) - a^1 \left(v_2^0 - v_1^0\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{m'}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + \frac{(a^m - a^{m-1})(v_1^{m-1} - \tilde{v}_1^{m-1})}{\pi} - a^1\left(v_2^0 - v_1^0\right) \\
&= \sum_{j=1}^{m'}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + u(\underline{v}, b_1) - u(\underline{v}, b_2) \\
&= q^{m'}c,
\end{aligned}
$$

where the third line holds since by Assumption 2

$$
\begin{aligned}
v_2^{m-1} - \tilde{v}_2^{m-1} \geq& \frac{1}{f(\tilde{v}_2^{m-1})}[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] \\
\geq& \frac{1-\pi}{\pi}\frac{1}{f(\tilde{v}_1^{m-1})}[F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})] \\
\geq& \frac{1-\pi}{\pi}\left(\tilde{v}_1^{m-1} - v_1^{m-1}\right).
\end{aligned}
$$

Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare. Note also that the new mechanism satisfies $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = 0$.

If $\tilde{v}_1^{m-1} < v_1^m$, then continue with the argument in step 2. If $\tilde{v}_1^{m-1} = v_1^m < \tilde{v}_2^{m-1}$, by the arguments in Lemma 3, we can first construct a new mechanism which is feasible and strictly increases welfare, and whose allocation rule is a $(M-1)$-step allocation rule; and then continue with the argument in step 2.

(ii) Suppose $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) < \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) + a^1\left(v_2^0 - v_1^0\right)]$. Let $\tilde{v}_1^{m-1} = \hat{v}$. Let $\tilde{v}_2^{m-1} \in [\hat{v}, v_2^{m-1}]$ be such that $\pi[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] = (1-\pi)[F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})]$. Let $\tilde{v}_i^{m'} = v_i^{m'}$ for $i = 1, 2$ and $m' \neq m-1$. Let $a^*(v, b_1) = a^{m-1}$ if $v \in (v_1^{m-1}, \tilde{v}_1^{m-1})$, and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $a^*(v, b_2) = a^m$ if $v \in (\tilde{v}_2^{m-1}, v_2^{m-1})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) + (a^m - a^{m-1})(\hat{v} - v_1^{m-1})$ and $u^*(\underline{v}, b_2) = u(\underline{v}, b_2) - (1-\pi)(a^m - a^{m-1})(\hat{v} - v_1^{m-1})/\pi$. Then $u^*(\underline{v}, b_2) > u^*(\underline{v}, b_1) + a^1\left(v_2^0 - v_1^0\right) \geq 0$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u^*(\underline{v}, b)$. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint is satisfied. By the same argument in Case 1, the (IC-b) constraint is satisfied. Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare.

In this case, by construction, we have $\tilde{v}_1^{m-1} = \min\left\{\tilde{v}_2^{m-1}, v_1^m\right\}$. If $\tilde{v}_1^{m-1} = \tilde{v}_2^{m-1} < v_1^m$, then let $m^* > m$ be such that $\tilde{v}_2^{m'-1} - \tilde{v}_1^{m'-1} \leq 0$ for all $m' < m^*$ and $\tilde{v}_2^{m^*-1} - \tilde{v}_1^{m^*-1} > 0$, and repeat the argument in step 1 for m^*. Note that if there is no such m^*, (a^*, p^*, q^*) satisfies (C2). If $\tilde{v}_1^{m-1} = v_1^m \leq \tilde{v}_2^{m-1}$, then, by the argument in Lemma 3, we can first construct a new mechanism which is feasible and strictly increases welfare, and whose allocation rule is a $(M-1)$-step allocation rule; and then repeat the arguments in step 1 for m.

To summarize, since M is finite, in finite steps we can construct a feasible mechanism (a,p,q) that either satisfies (C2) or $u(0,b_1)-u(0,b_2)+a^1v_2^0 \geq 0$. In the latter case, continue with the argument in step 2.

Step 2. Suppose $u(\underline{v},b_1)-u(\underline{v},b_2)+a^1\left(v_2^0-v_1^0\right) \geq 0$. Consider $m \geq 2$. Suppose (8) holds for all $m' < m$ and

$$u(\underline{v},b_1)-u(\underline{v},b_2)+\sum_{j=1}^{m}(a^j-a^{j-1})(v_2^{j-1}-v_1^{j-1}) < 0.$$

If there is no such m, then (a,p,q) satisfies (C1). It must be the case that $v_2^{m-1} < v_1^{m-1}$. Suppose $v_2^{m-1} < v_2^M$. Let $m^* \geq m$ be the smallest m' such that $v_2^{m'} > v_2^{m-1}$. That is, $v_2^{m^*} > v_2^{m-1}$ and $v_2^{m'} = v_2^{m-1}$ for $m' = m, \ldots, m^*-1$. Let $\hat{v} \in [v_2^{m-1}, v_1^{m-1}]$ be such that

$$u(\underline{v},b_1)-u(\underline{v},b_2)+\sum_{j=1}^{m-1}(a^j-a^{j-1})(v_2^{j-1}-v_1^{j-1})+(a^m-a^{m-1})(\hat{v}-v_1^{m-1}) = 0.$$

We consider two different cases: (i) $(a^{m^*}-a^{m-1})[F(\hat{v})-F(v_2^{m-1})] \leq (a^{m^*+1}-a^{m^*})[F(v_2^{m^*})-F(\hat{v})]$ and (ii) $(a^{m^*}-a^{m-1})[F(\hat{v})-F(v_2^{m-1})] > (a^{m^*+1}-a^{m^*})[F(v_2^{m^*})-F(\hat{v})]$.

(i) Suppose $(a^{m^*}-a^{m-1})[F(\hat{v})-F(v_2^{m-1})] \leq (a^{m^*+1}-a^{m^*})[F(v_2^{m^*})-F(\hat{v})]$**.** Let $\tilde{v}_2^{m^*} \in [\hat{v}, v_2^{m^*})$ be such that

$$(a^{m^*}-a^{m-1})[F(\tilde{v}_2^{m-1})-F(v_2^{m-1})] = (a^{m^*+1}-a^{m^*})[F(v_2^{m^*})-F(\tilde{v}_2^{m^*})]. \tag{17}$$

Let $\tilde{v}_2^{m'} = \hat{v}$ for $m' = m-1, \ldots, m^*-1$ and $\tilde{v}_2^{m'} = v_2^{m'}$ if $m' < m-1$ or $m' > m^*$. Let $\tilde{v}_1^{m'} = v_1^{m'}$ for all m'. Let $a^*(v,b_1) = a(v,b_1)$. Let $a^*(v,b_2) = a^{m-1}$ if $v \in (v_2^{m-1}, \tilde{v}_2^{m-1})$, $a^*(v,b_2) = a^{m^*+1}$ if $v \in (\tilde{v}_2^{m^*}, v_2^{m^*})$ and $a^*(v,b_2) = a(v,b_2)$ otherwise. Let $p^*(v,b) = va^*(v,b) - \int_{\underline{v}}^{v} a^*(\nu,b)d\nu - u(\underline{v},b)$. Clearly, $p^*(\overline{v},b_1) = p(\overline{v},b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v,b_1) = q(v,b_1)$. By Assumption 1, the (BB) constraint holds.

Finally, we show that (a^*,p^*,q^*) satisfies the (IC-b) constraint. That is, for $m' = 1, \ldots, M$

$$u(\underline{v},b_1)-u(\underline{v},b_2)+\sum_{j=1}^{m'}(a^j-a^{j-1})(\tilde{v}_2^{j-1}-\tilde{v}_1^{j-1}) \leq q^{m'}c.$$

This is trivial for $m' \le m$. For $m' = m+1, \ldots, m^*$, we have $\tilde{v}_2^{m'-1} = \tilde{v}_2^{m-1} \le v_1^{m-1} < v_1^{m'-1}$. Hence

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + \sum_{j=m+1}^{m'}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - v_1^{j-1}) < 0 \le q^{m'}c.$$

Finally, consider $m' \ge m^* + 1$. It suffices to show that

$$\begin{aligned} & u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m^*+1}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\ \le & u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m^*+1}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}), \end{aligned}$$

which holds if and only if

$$(a^{m^*} - a^{m-1})(\tilde{v}_2^{m-1} - v_2^{m-1}) \le (a^{m^*+1} - a^{m^*})(v_2^{m^*} - \tilde{v}_2^{m^*}).$$

The last inequality holds by (17) and Assumption 2. Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare. Let $m'' > m$ be such that (8) holds for all $m' < m''$ and is violated for m''. If there is no such m'', then (a^*, p^*, q^*) satisfies (C1). Otherwise repeat the argument in step 2 for m''.

(ii) Suppose $(a^{m^*} - a^{m-1})[F(\hat{v}) - F(v_2^{m-1})] > (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\hat{v})]$**.** Let $\tilde{v}_2^{m-1}$ be such that

$$(a^{m^*} - a^{m-1})[F(\tilde{v}_2^{m-1}) - F(v_2^{m-1})] = (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\tilde{v}_2^{m-1})].$$

Let $\tilde{v}_2^{m'} = \tilde{v}_2^{m-1}$ for $m' = m, \ldots, m^*$ and $\tilde{v}_2^{m'} = v_2^{m'}$ if $m' < m-1$ or $m' > m^*$. Let $\tilde{v}_1^{m'} = v_1^{m'}$ for all m'. Let $a^*(v, b_1) = a(v, b_1)$. Let $a^*(\cdot, b_2)$ such that $a^*(v, b_2) = a^{m-1}$ if $v \in (v_2^{m-1}, \tilde{v}_2^{m-1})$, $a^*(v, b_2) = a^{m^*+1}$ if $v \in (\tilde{v}_2^{m-1}, v_2^{m^*})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$. Clearly, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \le b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint holds. By the same argument in Case 1, the (IC-b) constraint is satisfied. Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare. Note that for (a^*, p^*, q^*) we have $\tilde{v}_2^{m^*} = \tilde{v}_2^{m-1}$. Repeat the argument in step 2 for m with m^* replaced by $m^* + 1$.

To summarize, since M is finite, in finite steps we can construct a feasible mechanism (a, p, q) that either satisfies (C1), or $v_2^M = v_2^{m-1} < v_1^{m-1}$ and

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) < 0.$$

In the latter case, continue with the argument in step 3.

Step 3. Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) \geq 0$,

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m'}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \geq 0$$

for all $m' = 1, \ldots, m-1$, $v_2^M = v_2^{m-1} < v_1^{m-1}$, and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) < 0$.

Let $\tilde{v}_1^{m-1} = v_1^{m-1} - \varepsilon$ for some $\varepsilon > 0$ and $\tilde{v}_2^{m'} = v_2^{m-1} + \delta$ for $m' = m-1, \ldots, M$, where $\delta > 0$ is such that

$$(1-\pi)(a^m - a^{m-1})\left[F(v_1^{m-1}) - F(\tilde{v}_1^{m-1})\right] = \pi(1 - a^{m-1})\left[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})\right]. \tag{18}$$

Let $\tilde{v}_i^{m'} = v_i^{m'}$ if $m' \neq m-1$ for $i = 1, 2$. Let $\varepsilon > 0$ be such that

$$\min\left\{\tilde{v}_1^{m-1} - v_1^{m-2}, u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1})\right\} = 0. \tag{19}$$

Since $\sum_{j=1}^{m-1}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \geq 0$, we have $\tilde{v}_2^{m'} \leq \tilde{v}_1^{m'}$ for all $m' \geq m-1$. Let $a^*(v, b_i) = a^m$ if $v \in (\tilde{v}_i^{m-1}, \tilde{v}_i^m)$ for $i = 1, 2$ and $m = 1, \ldots, M$, $a^*(v, b_2) = 0$ if $v < \tilde{v}_2^0$ and $a^*(v, b_2) = 1$ if $v > \tilde{v}_2^M$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$. Since $a^*(\overline{v}, b_1) = a(\overline{v}, b_1)$ and $a^*(v, b_1) \geq a(v, b_1)$ for all v, we have $p^*(\overline{v}, b_1) \leq p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q^m$ if $v \in (\tilde{v}_1^{m-1}, \tilde{v}_1^m)$ for $m = 1, \ldots, M$. Then the change of the verification cost is

$$k(q^m - q^{m-1})[F(v_1^{m-1}) - F(\tilde{v}_1^{m-1})].$$

Since $q^m = 0 \leq q^{m-1}$, the verification cost is reduced. Furthermore, by Assumption 1, the revenue increases. Hence, the (BB) constraint holds. Finally, we show that the (IC-b) constraint is satisfied. That is, for $m' = 1, \dots, M$

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq q^{m'} c.$$

This is trivial for $m' < m$. For $m' \geq m$, this holds since

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq 0 = q^{m'} c.$$

Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare.

If the first term of (19) reaches zero first, then, by the argument in Lemma 3, we can construct a new mechanism which is feasible and strictly improves welfare, and whose allocation rule is a $(M-1)$-step allocation rule. Then repeat the argument in step 3 for $m-1$. If the second term of (19) reaches zero first and $m < M$, then repeat the argument in step 3 for $m+1$. If the second term of (19) reaches zero first and $m = M$, then (a^*, p^*, q^*) satisfies (C1).

To summarize, since M is finite, in finite steps we can construct a feasible mechanism (a^*, p^*, q^*), which strictly improves welfare and satisfies (C1). Furthermore, a^* is a M'-step allocation rule for some $M' \leq M$. ■

Proof of Lemma 5. We prove Lemma 5 by proving Lemmas 8 and 9, which we introduce and prove below. ■

Lemma 8 *Suppose Assumptions 1 and 2 hold. An optimal mechanism of $\mathcal{P}'(M, d)$ satisfies that $v_2^1 \geq v_1^1$.*

Proof of Lemma 8. Assume without loss of generality that $a^2 > a^1$. Suppose, on the contrary, that $v_2^1 < v_1^1$. Since (8) holds for $m = 2$, it must be that $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0$. Hence, it is either (i) $u(\underline{v}, b_1) > u(\underline{v}, b_2) \geq 0$, or (ii) $a^1 > 0$ and $v_2^0 > v_1^0$. In what follows, we consider these two cases in turn, and show that in both cases we can construct another feasible mechanism which strictly improves welfare.

(i) Suppose $u(\underline{v}, b_1) > u(\underline{v}, b_2) \geq 0$. We construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Let $\varepsilon > 0$ be sufficiently small. Let $\tilde{v}_1^1 = v_1^1 - \pi\varepsilon/(1-\pi)$

and $\tilde{v}_2^1 > v_2^1$ be such that $(1-\pi)\left[F(v_1^1) - F(\tilde{v}_1^1)\right] = \pi\left[F(\tilde{v}_2^1) - F(v_2^1)\right]$. By Assumption 2,

$$\begin{aligned}
\tilde{v}_2^1 - v_2^1 &\leq \left[F(\tilde{v}_2^1) - F(v_2^1)\right]\frac{1}{f(\tilde{v}_2^1)} \\
&\leq \frac{1-\pi}{\pi}\left[F(v_1^1) - F(\tilde{v}_1^1)\right]\frac{1}{f(\tilde{v}_1^1)} \\
&\leq \frac{1-\pi}{\pi}(v_1^1 - \tilde{v}_1^1) = \varepsilon.
\end{aligned}$$

For $\varepsilon > 0$ sufficiently small, $\tilde{v}_2^1 \leq \tilde{v}_1^1$. Let $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$ and $m \neq 1$. Let $u^*(\underline{v}, b_2) = u(\underline{v}, b_2) + (a^2 - a^1)\varepsilon$ and $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) - \pi(a^2 - a^1)\varepsilon/(1-\pi)$. For $\varepsilon > 0$ sufficiently small, $u^*(\underline{v}, b_1) \geq u^*(\underline{v}, b_2) > 0$. Let $a^*(v, b_i) = a^m$ for $v \in (\tilde{v}_i^{m-1}, \tilde{v}_i^m)$ for $i = 1, 2$ and $m = 1, \ldots, M$, $a^*(v, b_2) = 0$ if $v < \tilde{v}_2^0$ and $a^*(v, b_1) = 1$ if $v > \tilde{v}_2^M$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u^*(\underline{v}, b)$. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1)$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b) = q(v, b)$. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare. (a^*, p^*, q^*) satisfies (BB) by Assumption 1.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. If $v < \tilde{v}_1^1$, then

$$u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1(\tilde{v}_2^0 - \tilde{v}_1^0) = u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) - \frac{(a^2 - a^1)\varepsilon}{1-\pi} \leq q^1 c.$$

If $v \in (\tilde{v}_1^1, v_1^2)$, then

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1(\tilde{v}_2^0 - \tilde{v}_1^0) + (a^2 - a^1)(\tilde{v}_2^1 - \tilde{v}_1^1) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) - \frac{(a^2 - a^1)\varepsilon}{1-\pi} + (a^2 - a^1)(\tilde{v}_2^1 - v_2^1 + v_1^1 - \tilde{v}_1^1) \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) - \frac{(a^2 - a^1)\varepsilon}{1-\pi} + (a^2 - a^1)\left(\varepsilon + \frac{\pi\varepsilon}{1-\pi}\right) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \\
\leq&\min\{q^2 c, q^1 c\},
\end{aligned}$$

where the first inequality holds since $\tilde{v}_2^1 - v_2^1 \leq \varepsilon$ and the last inequality holds since $v_2^1 < v_1^1$. If $v \in (v_1^{m-1}, v_1^m)$ for $m \geq 3$, then

$$u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1(\tilde{v}_2^0 - \tilde{v}_1^0) + (a^2 - a^1)(\tilde{v}_2^1 - \tilde{v}_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1})$$

$$\leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1})$$

$$\leq q^m c.$$

Hence, (IC-b) constraint is satisfied. This contradicts to that (a, p, q) is optimal.

(ii) Suppose $a^1 > 0$ and $v_2^0 > v_1^0$. We construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Let $\varepsilon \in (0, a^1]$ be sufficiently small. Let

$$\tilde{v}_1^1 = \frac{(a^2 - a^1)v_1^1 + \varepsilon v_1^0}{a^2 - a^1 + \varepsilon} < v_1^1.$$

By Assumption 2, we have

$$\begin{aligned}
&(a^2 - a^1)\left[F(v_1^1) - F(\tilde{v}_1^1)\right] \\
\leq &(a^2 - a^1)(v_1^1 - \tilde{v}_1^1)f(\tilde{v}_1^1) \\
= &\varepsilon(\tilde{v}_1^1 - v_1^0)f(\tilde{v}_1^1) \\
\leq &\varepsilon\left[F(\tilde{v}_1^1) - F(v_1^0)\right].
\end{aligned}$$

Let $\Delta := (a^2 - a^1)\left[F(v_1^1) - F(\tilde{v}_1^1)\right] - \varepsilon\left[F(\tilde{v}_1^1) - F(v_1^0)\right] \geq 0$. If $v_2^1 > v_2^0$, then let $\tilde{v}_2^0 = v_2^0$ and $\tilde{v}_2^1$ be such that

$$\pi(a^2 - a^1)\left[F(v_2^1) - F(\tilde{v}_2^1)\right] = \pi\varepsilon\left[F(\tilde{v}_2^1) - F(v_2^0)\right] + (1 - \pi)\Delta.$$

For $\varepsilon > 0$ sufficiently small, $\tilde{v}_2^1 \geq \tilde{v}_2^0 \geq v_1^0$. If $v_2^1 = v_2^0$, then let $\tilde{v}_2^1 = \tilde{v}_2^0$ be such that

$$\pi(a^2 - a^1)\left[F(v_2^1) - F(\tilde{v}_2^1)\right] = (1 - \pi)\Delta.$$

For $\varepsilon > 0$ sufficiently small, $\tilde{v}_2^1 = \tilde{v}_2^0 \geq v_1^0$. Let $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$ and $m \geq 2$. For $i = 1, 2$, let $a^*(v, b_i) = a^1 - \varepsilon$ if $v \in (\tilde{v}_i^0, \tilde{v}_i^1)$, $a^*(v, b_i) = a^2$ if $v \in (\tilde{v}_i^1, v_i^1)$, and $a^*(v, b_i) = a(v, b_i)$ otherwise. Let $u^*(\underline{v}, b) = u(\underline{v}, b)$ and $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u^*(\underline{v}, b)$. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1)$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b) = q(v, b)$. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare. (a^*, p^*, q^*) satisfies (BB) by Assumption 1.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. Suppose $v_2^1 > v_2^0$. If

$v < \tilde{v}_1^1$, then

$$u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(v_2^0 - v_1^0) < u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) = q^1 c.$$

If $v \in (\tilde{v}_1^1, \tilde{v}_1^2)$, then

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(v_2^0 - v_1^0) + (a^2 - a^1 + \varepsilon)(\tilde{v}_2^1 - \tilde{v}_1^1) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \\
&+ (a^2 - a^1 + \varepsilon)(\tilde{v}_2^1 - \tilde{v}_1^1) - \varepsilon(v_2^0 - v_1^0) - (a^2 - a^1)(v_2^1 - v_1^1) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \\
&+ (a^2 - a^1 + \varepsilon)\tilde{v}_2^1 - \varepsilon v_2^0 - (a^2 - a^1)v_2^1 \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \leq \min\{q^1 c, q^2 c\},
\end{aligned}$$

where the last inequality holds since $v_2^1 < v_1^1$. To see that the first inequality holds, note that by Assumption 2,

$$\begin{aligned}
(a^2 - a^1)(v_2^1 - \tilde{v}_2^1) &\geq (a^2 - a^1)\left[F(v_2^1) - F(\tilde{v}_2^1)\right]\frac{1}{f(\tilde{v}_2^1)} \\
&\geq \varepsilon\left[F(\tilde{v}_2^1) - F(v_2^0)\right] \\
&\geq \varepsilon(\tilde{v}_2^1 - v_2^0).
\end{aligned}$$

Hence, $(a^2 - a^1 + \varepsilon)\tilde{v}_2^1 \leq (a^2 - a^1)v_2^1 + \varepsilon v_2^0$. Furthermore, $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$ and $m \geq 2$. Hence, the (IC-b) constraint is satisfied. Suppose $v_2^0 = v_2^1$. If $v < \tilde{v}_1^1$, then

$$u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(v_2^0 - v_1^0) < u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) = q^1 c.$$

If $v \in (\tilde{v}_1^1, \tilde{v}_1^2)$, then

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(v_2^0 - v_1^0) + (a^2 - a^1 + \varepsilon)(\tilde{v}_2^1 - \tilde{v}_1^1) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) + a^2(\tilde{v}_2^1 - v_2^1) \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \leq \min\{q^1 c, q^2 c\},
\end{aligned}$$

where the first inequality holds since $\tilde{v}_2^1 \leq v_2^1$ and the second inequality holds since $v_2^1 < v_1^1$. Furthermore, $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$ and $m \geq 2$. Hence, the (IC-b) constraint is satisfied. This contradicts to that (a, p, q) is optimal.

Hence, $v_2^1 \geq v_1^1$. ■

Before introducing Lemma 9, we first write down the first-order conditions of $\mathcal{P}'(M,d)$ for later use. Let $M \geq 3$ be an integer. We note that if a mechanism is a feasible solution to $\mathcal{P}'(M-1,d)$, then it is also a feasible solution to $\mathcal{P}'(M,d)$. Clearly, $V(M-1,d) \leq V(M,d)$. Suppose $V(M-1,d) < V(M,d)$, then in an optimal solution to $\mathcal{P}'(M,d)$ the allocation rule must be a M-step allocation rule, i.e.,

$$0 = a^0 \leq a^1 < a^2 < \cdots < a^M \leq a^{M+1} = 1,$$
$$\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}.$$

Hence $\alpha^2 = \cdots = \alpha^M = 0$ and $\gamma_1^1 = \cdots = \gamma_1^M = 0$. Let $\rho := k/c$. Then the first-order conditions of $\mathcal{P}'(M,d)$ are

$$\begin{aligned}
&\pi\left[\int_{v_2^{m-1}}^{v_2^m}\left[(1+\lambda)v-\lambda\frac{1-F(v)}{f(v)}\right]\mathrm{d}v-\beta[F(v_2^m)-F(v_2^{m-1})]\right]\\
&+(1-\pi)\left[\int_{v_1^{m-1}}^{v_1^m}\left[(1+\lambda)v-\lambda\frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v\right]\\
&-(1-\pi)(1+\lambda)\rho(v_2^{m-1}-v_1^{m-1})[F(v_1^m)-F(v_1^{m-1})]-(1-\pi)\beta[F(v_1^m)-F(v_1^{m-1})]\\
&+(1-\pi)(1+\lambda)\rho(v_2^m-v_1^m-v_2^{m-1}+v_1^{m-1})[1-F(v_1^m)]+\eta(v_1^m-v_1^{m-1})+\mu^m(v_2^{m-1}-v_1^{m-1})\\
&-(v_2^m-v_1^m-v_2^{m-1}+v_1^{m-1})\sum_{j=m+1}^{M}\mu^j+\alpha^m-\alpha^{m+1}=0, \qquad (a^m, 1\leq m\leq M-1)
\end{aligned}$$

$$\begin{aligned}
&\pi\left[\int_{v_2^{M-1}}^{v_2^M}\left[(1+\lambda)v-\lambda\frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v-\beta[F(v_2^M)-F(v_2^{M-1})]\right]\\
&+(1-\pi)\int_{v_1^{M-1}}^{v_1^M}\left[(1+\lambda)v-\lambda\frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v\\
&-(1-\pi)(1+\lambda)\rho(v_2^{M-1}-v_1^{M-1})[F(v_1^M)-F(v_1^{M-1})]-(1-\pi)\beta[F(v_1^M)-F(v_1^{M-1})]\\
&-\eta v_1^{M-1}+\mu^M(v_2^{M-1}-v_1^{M-1})+\alpha^M-\alpha^{M+1}=0, \qquad (a^M)
\end{aligned}$$

$$\begin{aligned}
&(a^{m+1}-a^m)\Big\{(1-\pi)\left[(\beta-(1+\lambda)v_1^m)f(v_1^m)+(\lambda+\rho+\lambda\rho)[1-F(v_1^m)]+(1+\lambda)\rho(v_2^m-v_1^m)f(v_1^m)\right]\\
&-\sum_{j=m+1}^{M}\mu^j-\eta\Big\}=0, \qquad (v_1^m, 1\leq m\leq M-1)
\end{aligned}$$

$$a^1 \left\{ \pi \left[(\beta - (1+\lambda) v_2^0) f(v_2^0) + \lambda [1 - F(v_2^0)] \right] - (1-\pi)(1+\lambda)\rho + \sum_{j=1}^{M} \mu^j \right\} + \gamma_2^0 - \gamma_2^1 = 0, \tag{v_2^0}$$

$$(a^{m+1} - a^m) \left\{ \pi \left[(\beta - (1+\lambda) v_2^m) f(v_2^m) + \lambda [1 - F(v_2^m)] \right] - (1-\pi)(1+\lambda)\rho[1 - F(v_1^m)] + \sum_{j=m+1}^{M} \mu^j \right\}$$
$$+ \gamma_2^m - \gamma_2^{m+1} = 0, \tag{$v_2^m, 1 \leq m \leq M-1$}$$
$$\pi(a^{M+1} - a^M) \left[(\beta - (1+\lambda) v_2^M) f(v_2^M) + \lambda [1 - F(v_2^M)] \right] + \gamma_2^M - \gamma_2^{M+1} = 0, \tag{v_2^M}$$
$$\eta + \sum_{m=1}^{M} \mu^m - (1-\pi)(\lambda + \rho + \lambda\rho) + \xi_1 = 0, \tag{$u(\underline{v}, b_1)$}$$
$$- \sum_{m=1}^{M} \mu^m - \pi\lambda + (1-\pi)(1+\lambda)\rho + \xi_2 = 0. \tag{$u(\underline{v}, b_2)$}$$

The variables in the parentheses denote with respect to which variables the first-order conditions are taken.

Lemma 9 *Suppose Assumptions 1 and 2 hold and $V(M,d) > V(M-1,d)$ for some $M \geq 3$. An optimal mechanism of $\mathcal{P}'(M,d)$ satisfies that $v_2^m - v_1^m$ is strictly increasing in $m = 1, \dots M-1$.*

Proof of Lemma 9. Let (a,p,q) be an optimal solution of $\mathcal{P}'(M,d)$. Since $a^{m+1} > a^m$ for $m = 1, \dots M-1$, the FOCs of v_1^m become

$$(1-\pi) \left[(\beta - (1+\lambda) v_1^m) f(v_1^m) + (\lambda + \rho + \lambda\rho) [1 - F(v_1^m)] + (1+\lambda)\rho(v_2^m - v_1^m) f(v_1^m) \right]$$
$$- \sum_{j=m+1}^{M} \mu^j - \eta = 0,$$

for $m = 1, \dots, M-1$. Then for $m = 1, \dots, M-1$

$$v_2^m - v_1^m = \frac{1}{\rho} v_1^m - \frac{\lambda + \rho + \lambda\rho}{(1+\lambda)\rho} \frac{1 - F(v_1^m)}{f(v_1^m)} - \frac{\beta}{(1+\lambda)\rho} + \frac{\eta + \sum_{j=m+1}^{M} \mu^j}{(1-\pi)(1+\lambda)\rho f(v_1^m)},$$

which is strictly increasing in v_1^m by Assumptions 1 and 2. Next we show that $v_2^{m+1} - v_1^{m+1} > v_2^m - v_1^m$ for all $m = 1, \dots, M-2$.

If $\mu^{m+1} = 0$, then $v_2^{m+1} - v_1^{m+1} > v_2^m - v_1^m$ since $v_1^{m+1} > v_1^m$.

If $\mu^{m+1} > 0$, then $v_2^{m+1} - v_1^{m+1} \geq 0 \geq v_2^m - v_1^m$ since (8) holds for m and $m+2$ and (8) holds with equality for $m+1$.[25] We show that $v_2^{m+1} - v_1^{m+1} > 0 \geq v_2^m - v_1^m$. Suppose, on the contrary, that $v_2^{m+1} - v_1^{m+1} = 0 \geq v_2^m - v_1^m$, By Lemma 8, $v_2^1 \geq v_1^1$. Hence, it must be the case that $v_2^{m+1} - v_1^{m+1} = v_2^m - v_1^m = \cdots = v_2^1 - v_1^1 = 0$. In particular, $v_2^2 - v_1^2 = v_2^1 - v_1^1$. Then we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Let $\hat{v} \in (v_1^1, v_1^2)$ be such that

$$(a^3 - a^2)\left[F(v_1^2) - F(\hat{v})\right] = (a^2 - a^1)\left[F(\hat{v}) - F(v_1^1)\right].$$

Let $a^*(v,b) = a^1$ if $v \in (v_1^1, \hat{v})$, $a^*(v,b) = a^3$ if $v \in (\hat{v}, v_1^2)$ and $a^*(v,b) = a(v,b)$ otherwise. Let $p^*(v,b) = va^*(v,b) - \int_{\underline{v}}^{v} a^*(v,b)\mathrm{d}v - u(\underline{v},b)$. Then the (BC) constraint is satisfied by Assumption 2. Let $q^*(v,b_1) = q(v,b_1)$. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v), (IC-b), (S) and (BB), and strictly improves welfare, which contradicts to the optimality of (a,p,q). ■

Lemma 10 *Suppose Assumptions 1 and 2 hold. Then $V(M,d) = V(5,d)$ for all $M \geq 5$ and $d \geq 0$.*

Proof of Lemma 10. Fix $d \geq 0$ and $M \geq 6$ be an integer. We show that $V(M-1,d) = V(M,d)$. Suppose, on the contrary, that $V(M-1,d) < V(M,d)$, then in an optimal solution to $\mathcal{P}'(M,d)$ the allocation rule must be a M-step allocation rule. In particular, $0 = a^0 \leq a^1 < a^2 < \cdots < a^M \leq a^{M+1} = 1$. By Lemmas 8 and 9, an optimal solution to $\mathcal{P}'(M,d)$ must satisfies

$$v_2^{M-1} - v_1^{M-1} > v_2^{M-2} - v_1^{M-2} > \cdots > v_2^1 - v_1^1 \geq 0. \tag{20}$$

Fix $\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}$ and $0 \geq v_2^0 \leq v_2^1 \leq \cdots \leq v_2^M \leq v_2^{M+1} \leq \overline{v}$ such that (20) holds. Then $\mathcal{P}'(M,d)$ is linear in $u(\underline{v},b_1)$, $u(\underline{v},b_2)$ and a^m for $m = 1,\ldots,M$. Then an optimal solution can be obtained at an extreme point of the feasible region. By (20), inequalities corresponding to μ^m for $m = 2,\ldots M$ holds if the inequality corresponding to μ^1 holds. Hence, the feasible set is characterized by (S), (BC), (BB) and the following inequalities:

$$\begin{aligned} &u(\underline{v},b_1) \geq 0, u(\underline{v},b_2) \geq 0, \\ &u(\underline{v},b_1) - u(\underline{v},b_2) + a^1(v_2^0 - v_1^0) \geq 0, \end{aligned} \tag{21}$$

[25] Recall that μ^m is the Lagrangian multiplier associated with (8).

$$0 \le a^0 \le a^1 \le \cdots \le a^M \le a^{M+1} = 1.$$

Note that if $a^1 = 0$, then $u(\underline{v}, b_1) \ge 0$ is redundant. Hence, in addition to (S), (BC), (BB) and $a^M \le 1$, at most three of the following four inequalities are active at the same time: $u(\underline{v}, b_1) \ge 0$, $u(\underline{v}, b_2) \ge 0$, $a^1 \ge 0$ and (21). Since $M \ge 6$, at least one of the following constraints hold with equality: $a^1 \le a^2 \cdots \le a^{M-1} \le a^M$, a contradiction. ■

Lemma 11 *Suppose Assumptions 1 and 2 hold. For any $d > 0$, there exists $\overline{M}(d)$ such that for all $M > \overline{M}(d)$,*

$$V - V(M, d) \le (1 - \pi)\left(1 + \frac{k}{c}\right)\frac{\mathbb{E}[v]}{M}.$$

Proof of Lemma 11. Let (a, p, q) denote an optimal mechanism of $\mathcal{P}'$. Then $p(v, b) = va(v, b) - \int_{\underline{v}}^{v} a(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$ for all $(v, b) \in T$ and q is defined by (6). Fix $M \ge 2$. Let $a^0 = 0$, $a^{M+1} = 1$ and $a^m = (m-1)a(\overline{v}, b_1)/M$ for $m = 1, \dots, M$. Let $v_1^0 = \underline{v}$, $v_1^M = \overline{v}$ and for $m = 0, \dots, M-1$

$$v_1^m = \inf\left\{ v \,\middle|\, a(v, b_1) \ge a^{m+1} \right\}.$$

Then $\underline{v} = v_1^0 \le v_1^1 \le \cdots \le v_1^M = \overline{v}$ and $0 = a^0 \le a^1 < a^2 < \cdots < a^M \le a^{M+1} = 1$. Let $a^*(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$. Then $a(v, b_1) - 1/M \le a^*(v, b_1) \le a(v, b_1)$. Let $\hat{v}_2^m = \inf\{v | a(v, b_2) \ge a^m\}$ for $m = 1, \dots, M$, $\hat{v}_2^0 = 0$ and $\hat{v}_2^{M+1} = \overline{v}$. For each $m = 1, \dots, M+1$, there exists $v_2^{m-1} \in [\hat{v}_2^{m-1}, \hat{v}_2^m]$ such that

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m} a(v, b_2) f(v)\mathrm{d}v = a^{m-1}[F(v_2^{m-1}) - F(\hat{v}_2^{m-1})] + a^m[F(\hat{v}_2^m) - F(v_2^{m-1})]. \tag{22}$$

Consider $a^*(v, b_2)$ such that $a^*(v, b_2) = a^m$ if $v \in (v_2^{m-1}, v_2^m)$ for $m = 1, \dots, M$, $a^*(v, b_2) = 0$ if $v < v_2^0$ and $a^*(v, b_2) = 1$ if $v > v_2^M$. Note that since $a^1 = 0$, we have $v_2^0 = \underline{v}$. Clearly, a^* satisfies constraint (S). Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$ for $b \in B$. Let q^* be such that

$$cq^*(v, b_1) = cq(v, b_1) + \frac{v}{M}.$$

We show that the (IC-b) constraint is satisfied, i.e., for all $v \in (v_1^{m-1}, v_1^m)$, $m = 1, \dots, M$,

$$cq^*(v, b_1) \ge u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v_2^{m-1}} a(\nu, b_2)\mathrm{d}\nu + a^m(v_2^{m-1} - v) + \int_{\underline{v}}^{v} a^*(\nu, b_1)\mathrm{d}\nu.$$

Recall that for $v \in (v_1^{m-1}, v_1^m)$, we have

$$cq(v, b_1) \geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\mathrm{d}\nu + a(v, b_1)(\hat{v}_2^m - v) + \int_{\underline{v}}^{v} a(\nu, b_1)\mathrm{d}\nu.$$

Then for $v \in (v_1^{m-1}, v_1^m)$

$$\begin{aligned} cq^*(v, b_1) &= cq(v, b_1) + \frac{v}{M} \\ &\geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\mathrm{d}\nu + a(v, b_1)\hat{v}_2^m - \left(a(v, b) - \frac{1}{M}\right) v + \int_{\underline{v}}^{v} a(\nu, b_1)\mathrm{d}\nu \\ &\geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v) + \int_{\underline{v}}^{v} a(\nu, b_1)\mathrm{d}\nu \\ &\geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a^*(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v) + \int_{\underline{v}}^{v} a^*(\nu, b_1)\mathrm{d}\nu \\ &= u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v_2^{m-1}} a^*(\nu, b_2)\mathrm{d}\nu + a^m(v_2^{m-1} - v) + \int_{\underline{v}}^{v} a^*(\nu, b_1)\mathrm{d}\nu, \end{aligned}$$

where the third line holds since $a(v,b) - 1/M \leq a^*(v,b) \leq a(v,b)$ and the fourth line holds by the same argument in the proof of Lemma 3. Then

$$\begin{aligned} &\mathbb{E}_t[p^*(t) - q^*(t)k] - \mathbb{E}_t[p(t) - q(t)k] \\ =&\pi \int_{\underline{v}}^{\overline{v}} \left[v - \frac{1 - F(v)}{f(v)}\right] [a^*(v, b_2) - a(v, b_2)] f(v)\mathrm{d}v \\ &+ (1 - \pi) \int_{\underline{v}}^{\overline{v}} \left[v - \frac{1 - F(v)}{f(v)}\right] [a^*(v, b_1) - a(v, b_1)] f(v)\mathrm{d}v - (1 - \pi) \int_{\underline{v}}^{\overline{v}} k[q^*(v, b_1) - q(v, b_1)] f(v)\mathrm{d}v \\ \geq& - \frac{\mathbb{E}[v]}{M} - (1 - \pi)\frac{\mathbb{E}[v]}{M}\frac{k}{c}. \end{aligned}$$

For any $d > 0$, there exists $\overline{M}(d)$ such that for all $M > \overline{M}(d)$, we have $\frac{\mathbb{E}[v]}{M} + (1-\pi)\frac{\mathbb{E}[v]}{M}\frac{k}{c} < d$. Then (a^*, p^*, q^*) is a feasible solution to $\mathcal{P}'(M, d)$ for $M > \overline{M}(d)$. Hence,

$$\begin{aligned} &V - V(M, d) \\ \leq&(1 - \pi) \left[\int_{\underline{v}}^{\overline{v}} v[a(v, b_1) - a^*(v, b_1)] f(v)\mathrm{d}v - \int_{\underline{v}}^{\overline{v}} \left[q(v, b_1) - q^*(v, b_1)\right] k f(v)\mathrm{d}v\right] \\ \leq&(1 - \pi) \left(1 + \frac{k}{c}\right) \frac{\mathbb{E}[v]}{M}. \end{aligned}$$

■

Proof of Theorem 3. By Lemmas 6 and 11, we have

$$V - V(2,d) = V - V(M,d) \leq (1-\pi)\left(1+\frac{k}{c}\right)\frac{\mathbb{E}[v]}{M}.$$

Let M goes to infinity and we have $V(2,0) \leq V \leq V(2,d)$ for all $d > 0$. By the continuity of $V(2,\cdot)$,[26] $\lim_{d\to 0} V(2,d) = V(2,0)$. Hence, $V = V(2,0)$.

Hence, there exists $u(\underline{v},b_1) \geq 0$, $u(\underline{v},b_2) \geq 0$, $\underline{v} \leq v_1^1 \leq \overline{v}$, $\underline{v} \leq v_2^0 \leq v_2^1 \leq v_2^2 \leq \overline{v}$ and $0 \leq a^1 \leq a^2 \leq \overline{v}$ such that an optimal mechanism of $\mathcal{P}'$ is given by

$$\begin{aligned}
a(v,b_1) &= a^1 + \chi_{\{v\geq v_1^1\}}\left(a^2 - a^1\right),\\
a(v,b_2) &= \chi_{\{v\geq v_2^0\}}a^1 + \chi_{\{v\geq v_2^1\}}\left(a^2-a^1\right) + \chi_{\{v\geq v_2^2\}}(1-a^2),\\
p(v,b_1) &= -u(\underline{v},b_1) + \chi_{\{v\geq v_1^1\}}\left(a^2-a^1\right)v_1^1,\\
p(v,b_2) &= -u(\underline{v},b_2) + \chi_{\{v\geq v_2^0\}}a^1 v_2^0 + \chi_{\{v\geq v_2^1\}}\left(a^2-a^1\right)v_2^1 + \chi_{\{v\geq v_2^2\}}(1-a^2)v_2^2,\\
q(v,b_1) &= \frac{1}{c}\left[u(\underline{v},b_1) - u(\underline{v},b_2) + a^1\left(v_2^0 - v_1^0\right) + \chi_{\{v\geq v_1^1\}}\left(a^2-a^1\right)(v_2^1 - v_1^1)\right],\\
q(v,b_2) &= 0.
\end{aligned}$$

By Lemma 8, $v_2^1 \geq v_1^1$. It is easy to verify that the above mechanism satisfies IC constraints corresponding to agents over reporting their budgets and therefore solves $\mathcal{P}$. To prove Theorem 3, it remains to show that $v_2^0 = \underline{v}$ and $a^1 = 0$.

First, we show that $v_2^0 = \underline{v}$. We consider two different cases: (i) $u(\underline{v},b_1) - u(\underline{v},b_2) + a^1\left(v_2^0 - v_1^0\right) = 0$ and (ii) $u(\underline{v},b_1) - u(\underline{v},b_2) + a^1\left(v_2^0 - v_1^0\right) > 0$.

(i) Suppose $u(\underline{v},b_1) - u(\underline{v},b_2) + a^1\left(v_2^0 - v_1^0\right) = 0$**.** Suppose, on the contrary, that $v_2^0 > \underline{v}$. Then we can construct another feasible mechanism (a^*,p^*,q^*), which strictly improves welfare. Since $v_2^0 > \underline{v} = v_1^0$, we have $u(\underline{v},b_2) > u(\underline{v},b_1)$ and, by construction, $a^1 > 0$ and $v_1^1 > \underline{v}$.

Let $\varepsilon > 0$ be sufficiently small. Let $\tilde{v}_1^0 = \underline{v} + \varepsilon$ and $\tilde{v}_2^0 < v_2^0$ be such that $\pi[F(v_2^0) - F(\tilde{v}_2^0)] = (1-\pi)F(\underline{v}+\varepsilon)$. For $\varepsilon > 0$ sufficiently small, $\tilde{v}_1^0 < \min\{v_1^1, \tilde{v}_2^0\}$. Let $\tilde{v}_i^1 = v_i^1$ for $i = 1,2$. Let $u^*(\underline{v},b_1) = u(\underline{v},b_1) + a^1\varepsilon$ and $u^*(\underline{v},b_2) = u(\underline{v},b_2) - (1-\pi)a^1\varepsilon/\pi$. For $\varepsilon > 0$ sufficiently small, $u^*(\underline{v},b_2) \geq u^*(\underline{v},b_1) > 0$. Let $a^*(v,b_1) = 0$ if $v < \tilde{v}_1^0$ and $a^*(v,b_1) = a(v,b_1)$ otherwise. Let $a^*(v,b_2) = a^1$ if $v \in (\tilde{v}_2^0, v_2^0)$ and $a^*(v,b_2) = a(v,b_2)$ otherwise. Let $p^*(v,b) = va^*(v,b) - \int_{\underline{v}}^{v} a^*(\nu,b)\mathrm{d}\nu - u^*(\underline{v},b)$. Since $u^*(\underline{v},b_1) - a^1\tilde{v}_1^0 = u(\underline{v},b_1) - a^1 v_1^0$,

[26] The proof of continuity is standard and available upon request.

we have $p^*(\bar{v}, b_1) = p(\bar{v}, b_1) \le b_1$. Hence, (BC) is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare. (a^*, p^*, q^*) satisfies (BB) by Assumption 1.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. First, for $v < \underline{v} + \varepsilon$, we have $u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) \le 0 \le q^*(v, b_1)c$. Next, we show that for $m = 1, 2$

$$q^m c \ge u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}).$$

Since

$$\begin{aligned} v_2^0 - \tilde{v}_2^0 &= \int_{\tilde{v}_2^0}^{v_2^0} f(v) \frac{1}{f(v)} \mathrm{d}v \\ &\ge \frac{1}{f(\tilde{v}_2^0)} [F(v_2^0) - F(\tilde{v}_2^0)] \\ &\ge \frac{1-\pi}{\pi} \frac{F(\underline{v}+\varepsilon)}{f(\underline{v}+\varepsilon)} \\ &\ge \frac{1-\pi}{\pi} \varepsilon, \end{aligned}$$

where the inequalities hold by Assumption 2, we have

$$\begin{aligned} & u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{1} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\ =& u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 v_2^0 + \frac{a^1 \varepsilon}{\pi} + a^1(\tilde{v}_2^0 - v_2^0) - a^1(v_1^0 + \varepsilon) \\ \le& u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right). \end{aligned}$$

Furthermore, $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$ and $m \ge 1$. Hence, (IC-b) is satisfied. This contradicts to that (a, p, q) is optimal. Hence, $v_2^0 = \underline{v}$.

(ii) Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) > 0$. Suppose, on the contrary, that $v_2^0 > \underline{v}$. In this case, $\gamma_2^0 = 0$. By construction, we have $a^1 > 0$ and $v_1^1 > \underline{v}$. Hence, $\alpha_1 = 0$. Furthermore, since $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) > 0$ and $v_2^1 \ge v_1^1$, we have $\mu^1 = \mu^2 = 0$. Then v_2^0 satisfies

$$\pi \left[(\beta - (1+\lambda) v_2^0) f(v_2^0) + \lambda [1 - F(v_2^0)] \right] - (1-\pi)(1+\lambda)\rho[1 - F(v_1^0)] = 0, \tag{23}$$

$$
\begin{aligned}
&\pi \int_{v_2^0}^{v_2^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v \\
&+ (1-\pi)\left[\int_{v_1^0}^{v_1^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v - (1+\lambda)\rho(v_2^0 - v_1^0)[1 - F(v_1^0)]\right] \\
&- \eta v_1^0 - \alpha^{M+1} = 0. \qquad (24)
\end{aligned}
$$

Since $v_2^0 \geq v_1^0$, it follows from Claims 4 in the proof of Lemma 6 and (24) that $\int_{v_1^0}^{v_1^M} [v + \lambda\varphi(v)] f(v)\mathrm{d}v \geq \beta[1 - F(v_1^0)]$, i.e., $\hat{v}(\beta) = \underline{v}$, where $\hat{v}(\beta)$ is defined by (35).

It follows from the proof of Lemma 6 that the FOCs of v_1^1 and v_2^1 can be reduced to (27) and (28). Given β, η and λ, (27) and (28) define v_2^1 as functions of v_1^1, denoted by g_1 and g_2, respectively. By a similar argument in Claim 6 in the proof of Lemma 6, $g_1'(v) > 1$, and $g_2'(v) < 1$ if $v > \hat{v}(\beta) = \underline{v}$ and $g_2(v) \geq v$. Let Δ_3 denote the left-hand side of (24), then

$$
\begin{aligned}
\frac{\partial \Delta_3}{\partial v_1^1} &= (1-\pi)\left[(\beta - v_1^1 - \lambda\varphi(v_1^1))f(v_1^1) + (1+\lambda)\rho(v_2^1 - v_1^1)f(v_1^1) + (1+\lambda)\rho[1 - F(v_1^1)]\right] - \eta, \\
\frac{\partial \Delta_3}{\partial v_2^1} &= \pi(\beta - v_2^1 - \lambda\varphi(v_2^1))f(v_2^1) - (1-\pi)(1+\lambda)\rho[1 - F(v_1^1)].
\end{aligned}
$$

Clearly, $\partial\Delta_3(v_1, g_1(v_1))/\partial v_1 = 0$ by (27) and $\partial\Delta_3(v_1, g_2(v_1))/\partial v_2 = 0$ by (28). Since $v_2^1 \geq v_1^1$, then $g_2(v) > g_1(v)$ for all $v < v_1^1$. Then $\partial\Delta_3(v_1, g_2(v_1))/\partial v_1 > \Delta_3(v_1, g_1(v_1))/\partial v_1 = 0$ for all $v_1 < v_1^1$. Hence,

$$
0 = \Delta_3(v_1^1, v_2^1) = \Delta_3(v_1^0, v_2^0) + \int_{v_1^0}^{v_1^1} \frac{\partial \Delta_3(v_1, g_2(v_1))}{\partial v_1}\mathrm{d}v_1 > \Delta_3(v_1^0, v_2^0) = 0,
$$

a contradiction. Hence, $v_2^0 = \underline{v}$.

Next, we show that $a^1 = 0$. Suppose $a^1 > 0$, then $\alpha^1 = 0$. Then v_2^0 satisfies

$$
a^1\left\{\pi\left[\beta - v_2^0 - \lambda\varphi(v_2^0)\right] f(v_2^0) - (1-\pi)(1+\lambda)\rho + \sum_{j=1}^{2}\mu^j\right\} + \gamma_2^0 = 0, \qquad (25)
$$

$$
\pi \int_{v_2^0}^{v_2^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v + (1-\pi)\int_{v_1^0}^{v_1^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v - \eta v_1^0 - \alpha^{M+1} = 0. \qquad (26)
$$

By Claims 4 in the proof of Lemma 6, it follows from (26) that $\int_{v_1^0}^{v_1^M} [v + \lambda\varphi(v)] f(v)\mathrm{d}v - \beta \geq 0$, i.e., $\hat{v}(\beta) = \underline{v}$. Since $g_2'(v) \leq 1$ if $v \geq \hat{v}(\beta)$ and $g_2(v) \geq v$, and $g_2(v_1^1) = v_2^1 \geq v_1^1$, we have $v_2^0 = g_2(v_1^0) > v_1^0 = \underline{v}$, a contradiction. Hence, $a^1 = 0$.

Let $a^*(\rho) = a^2$, $v_1^*(\rho) = v_1^1$, $v_2^*(\rho) = v_2^1$ and $v_2^{**}(\rho) = v_2^2$. Let $u_1^*(\rho) = u(\underline{v}, b_1)$ and $u_2^*(\rho) = u(\underline{v}, b_2)$. This completes the proof. ■

Proof of Corollary 2. This results holds trivially if the first-best can be achieved. For the rest of the proof, we assume that the first-best cannot be achieved. Suppose there are two optimal mechanisms (a, p, q) and $(\hat{a}, \hat{p}, \hat{q})$. By Theorem 3, there exist $(u_1^*, u_2^*, a^*, v_1^*, v_2^*, v_2^{**})$ and $(\hat{u}_1^*, \hat{u}_2^*, \hat{a}^*, \hat{v}_1^*, \hat{v}_2^*, \hat{v}_2^{**})$ that characterize the two different optimal mechanisms, respectively.

Firstly, we show that the convex combination of the two mechanisms $(\kappa a+(1-\kappa)\hat{a}, \kappa p+(1-\kappa)\hat{p}, \kappa+(1-\kappa)\hat{q})$, where $\kappa \in (0, 1)$, is also optimal. Clearly, it satisfies (IR), (BC), (BB) and (S):

$$[\kappa a(t) + (1-\kappa)\hat{a}(t)]v - [\kappa p(t) + (1-\kappa)\hat{p}(t)] = \kappa[a(t)v - p(t)] + (1-\kappa)[\hat{a}(t)v - \hat{p}(t)] \geq 0,$$
$$\kappa p(t) + (1-\kappa)\hat{p}(t) \leq b,$$
$$\mathbb{E}_t\left[\kappa p(t) + (1-\kappa)\hat{p}(t) - [\kappa q(t) + (1-\kappa)\hat{q}(t)]k\right] = \kappa\mathbb{E}_t[p(t) - q(t)k] + (1-\kappa)\mathbb{E}_t[\hat{p}(t) - \hat{q}(t)k] \geq 0,$$
$$\mathbb{E}_t[\kappa a(t) + (1-\kappa)\hat{a}(t)] = \kappa\mathbb{E}_t[a(t)] + (1-\kappa)\mathbb{E}[\hat{a}(t)] \leq S.$$

It satisfies (IC-v) since $\kappa a(v, b) + (1-\kappa)\hat{a}(v, b)$ is non-decreasing in v and the envelope condition holds:

$$\begin{aligned}
&\kappa p(v, b) + (1-\kappa)\hat{p}(v, b)\\
=&\kappa\left[a(v, b)v - \int_{\underline{v}}^{v} a(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)\right] + (1-\kappa)\left[\hat{a}(v, b)v - \int_{\underline{v}}^{v} \hat{a}(\nu, b)\mathrm{d}\nu - \hat{u}(\underline{v}, b)\right]\\
=&[\kappa a(v, b) + (1-\kappa)]\, v - \int_{\underline{v}}^{v} [\kappa a(\nu, b) + (1-\kappa)\hat{a}(\nu, b)]\mathrm{d}\nu - [\kappa u(\underline{v}, b) + (1-\kappa)\hat{u}(\underline{v}, b)].
\end{aligned}$$

Finally, it satisfies (IC-b) since

$$\begin{aligned}
&[\kappa a(v, b_2) + (1-\kappa)\hat{a}(v, b_2)]v - [\kappa p(v, b_2) + (1-\kappa)\hat{p}(v, b_2)]\\
=&\kappa[a(v, b_2)v - p(v, b_2)] + (1-\kappa)[\hat{a}(v, b_2)v - \hat{p}(v, b_2)]\\
\geq&\kappa[a(\hat{v}, b_1)v - p(\hat{v}, b_1) - q(\hat{v}, b_1)c] + (1-\kappa)[\hat{a}(\hat{v}, b_1)v - \hat{p}(\hat{v}, b_1) - \hat{q}(\hat{v}, b_1)c]\\
=&[\kappa a(v, b_1) + (1-\kappa)\hat{a}(v, b_1)]v - [\kappa p(\hat{v}, b_1) + (1-\kappa)\hat{p}(\hat{v}, b_1)] - [\kappa q(\hat{v}, b_1)c + (1-\kappa)q(\hat{v}, b_1)]c.
\end{aligned}$$

Furthermore,

$$\begin{aligned}&\mathbb{E}_t\left[[\kappa a(t)+(1-\kappa)\hat{a}(t)]v-[\kappa q(t)+(1-\kappa)\hat{q}(t)]k\right]\\=&\kappa\mathbb{E}_t[a(t)v-q(t)k]+(1-\kappa)\mathbb{E}_t[\hat{a}(t)v-\hat{q}(t)k]\\=&V.\end{aligned}$$

Hence, $(\kappa a+(1-\kappa)\hat{a}, \kappa p+(1-\kappa)\hat{p}, \kappa+(1-\kappa)\hat{q})$ is an optimal mechanism of $\mathcal{P}$.

Secondly, we show that $v_1^*=\hat{v}_1^*$. Suppose, on the contrary, that $v_1^*<\hat{v}_1^*$. Then

$$\kappa a(v,b_1)+(1-\kappa)\hat{a}(v,b_1)=\chi_{\{v\geq v_1^*\}}\kappa a^*+\chi_{v\geq\hat{v}_1^*}(1-\kappa)\hat{a}^*,$$

which is a 3-step function, a contradiction to Lemma 6.

Thirdly, we show that $v_2^*=\hat{v}_2^*$, $v_2^{**}=\hat{v}_2^{**}$ and $a^*=\hat{a}^*$. Suppose $a^*=\hat{a}^*=1$. By Proposition 7, (S) holds with equality in an optimal mechanism. Hence, $v_2^*=v_2^{**}=\hat{v}_2^*=\hat{v}_2^{**}$.

Suppose $a^*<1$ and $\hat{a}^*=1$. Since (S) holds with equality in both mechanisms, it must be that $v_2^*<\hat{v}_2^*$. In this case, $a(v,b_1)=\chi_{\{v\geq v_1^*\}}[\kappa a^*+(1-\kappa)]$. If $v\in\left(v_2^*,\min\{v_2^{**},\hat{v}_2^{**}\}\right)$, then $a(v,b_2)=\kappa a^*<\kappa a^*+(1-\kappa)$, which is a contradiction to Lemma 3. Hence, it cannot be the case that $a^*<1$ and $\hat{a}^*=1$.

Suppose $a^*<1$ and $\hat{a}^*<1$. In this case, $a(v,b_1)=\chi_{\{v\geq v_1^*\}}[\kappa a^*+(1-\kappa)\hat{a}^*]$. Suppose, on the contrary, that $v_2^*<\hat{v}_2^*$. If $v\in\left(v_2^*,\min\{v_2^{**},\hat{v}_2^*\}\right)$, then $a(v,b_2)=\kappa a^*<\kappa a^*+(1-\kappa)\hat{a}^*$, which is a contradiction to Lemma 3. Hence, $v_2^*=\hat{v}_2^*$. Suppose, on the contrary, that $v_2^{**}<\hat{v}_2^{**}$. If $v\in\left(v_2^{**},\hat{v}_2^{**}\right)$, then $a(v,b_2)=\kappa+(1-\kappa)\hat{a}^*>\kappa a^*+(1-\kappa)\hat{a}^*$, a contradiction to Lemma 3. Hence, $v_2^{**}=\hat{v}_2^{**}$. Finally, since (S) holds with equality in both mechanisms, it must be the case $a^*=\hat{a}^*$.

Lastly, we show that $u_i^*=\hat{u}_i^*$ for $i=1,2$. Proposition 7 shows that if the first-best cannot be achieved, then both (BC) and (BB) hold with equality in an optimal mechanism. Hence, $u_1^*=a^*v_1^*-b_1=\hat{a}^*\hat{v}_1^*-b_1=\hat{u}_1^*$ by (BC). Note that in (BB) the coefficient in front of u_2^* is $(1-\pi)]\rho-\pi$. If $\rho\geq\pi/(1-\pi)$, $u_2^*=u_1^*=\hat{u}_1^*=\hat{u}_2^*$ by Proposition 3. If $\rho<\pi/(1-\pi)$, $u_2^*=\hat{u}_2^*$ by (BB). ■

D.3 Proof of Lemma 6

Let $M\geq 3$ be an integer. We want to show that $V(M-1,d)=V(M,d)$. Suppose, to the contrary, that $V(M-1,d)<V(M,d)$, then an optimal solution to $\mathcal{P}'(M,d)$ satisfies

the first-order conditions given before the proof of Lemma 9 in Appendix 4.2. In what follows, we show that these FOCs imply that $M \leq 2$, which contradicts to the assumption that $M \geq 3$. Hence, it must be the case that $V(M-1,d) = V(M,d)$ for all $M \geq 3$.

I start by providing a proof sketch of Lemma 6. Assume, for ease of exposition, that

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0, \qquad (\mu^1)$$

$$\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}, \qquad (\gamma_1^1, \ldots, \gamma_1^M)$$

$$0 \leq v_2^0 < v_2^1 < \cdots < v_2^M < \overline{v}. \qquad (\gamma_2^0, \ldots, \gamma_2^{M+1})$$

Then $\mu^1 = \cdots = \mu^M = 0$, $\gamma_1^1 = \cdots = \gamma_1^M = 0$ and $\gamma_2^1 = \cdots = \gamma_2^{M+1} = 0$. The first-order conditions for v_1^m and v_2^m $(m = 1, \ldots, M-1)$ are

$$(1-\pi)\left[(\beta - v_1^m - \lambda\varphi(v_1^m))f(v_1^m) + (1+\lambda)\rho[1 - F(v_1^m)] + (1+\lambda)\rho(v_2^m - v_1^m)f(v_1^m)\right] - \eta = 0, \tag{27}$$

$$\pi(\beta - v_2^m - \lambda\varphi(v_2^m))f(v_2^m) - (1-\pi)(1+\lambda)\rho[1 - F(v_1^m)] = 0, \tag{28}$$

where $\varphi(v) := v - [1 - F(v)]/f(v)$ denotes the virtual value function. I show below that if f is "regular", which is to say that it satisfies Assumptions 1 and 2, then the above system of equations has at most one solution. This result is illustrated by Example 1. Hence, I can conclude that $V(M,d) = V(2,d)$.

Example 1 *Let v be uniformly distributed on $[0,1]$ and $\rho < \frac{\pi+\sqrt{\pi}}{1-\pi}$. Then the first-order conditions for v_1^m and v_2^m $(m = 1, \ldots, M-1)$*

$$(1-\pi)\left[\beta + \lambda + (1+\lambda)\rho - (1 + 2\lambda + 2(1+\lambda)\rho)v_1^m + (1+\lambda)\rho v_2^m\right] - \eta = 0, \tag{29}$$

$$\pi(\beta + \lambda) - (1-\pi)(1+\lambda)\rho + (1-\pi)(1+\lambda)\rho v_1^m - \pi(1+2\lambda)v_2^m = 0. \tag{30}$$

Given β, η and λ, (29) and (30) define v_2^m as functions of v_1^m, denoted by g_1 and g_2, respectively. Then

$$g_1'(v_1^m) = 2 + \frac{1+2\lambda}{\rho(1+\lambda)} > \frac{(1-\pi)(1+\lambda)\rho}{\pi(1+2\lambda)} = g_2'(v_1^m).$$

This inequality implies that g_1 can cross g_2 at most once from below. Hence, (29) and (30) have at most one solution.

We break the formal proof into several claims. In all claims, we assume, without explicitly repeating these, that Assumptions 1 and 2 hold, $u(\underline{v}, b_1)$, $u(\underline{v}, b_2)$, $\{a^m\}_{m=1}^M$, $\{v_1^m\}_{m=1}^{M-1}$ and $\{v_2^m\}_{m=0}^M\}$ define an optimal mechanism of $\mathcal{P}'(M, d)$ and β, η, λ, ξ_1, ξ_2, $\{\mu^m\}_{m=1}^M$, $\{\alpha^m\}_{m=1}^{M+1}$, $\{\gamma_1^m\}_{m=1}^M$ and $\{\gamma_2^m\}_{m=0}^{M+1}$ are the associated Lagrangian multipliers.

For later use, we note here that the summation of FOCs of $a^{m'}$, $m+1 \le m' \le M$, $m = 0, \dots, M-1$, gives:

$$\pi\left[\int_{v_2^m}^{v_2^M}\left[(1+\lambda)v - \lambda\frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v - \beta[F(v_2^M) - F(v_2^m)]\right]$$
$$+(1-\pi)\left[\int_{v_1^m}^{v_1^M}\left[(1+\lambda)v - \lambda\frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v - (1+\lambda)\rho(v_2^m - v_1^m)[1-F(v_1^m)] - \beta[1-F(v_1^m)]\right]$$
$$-\eta v_1^m + (v_2^m - v_1^m)\sum_{j=m+1}^{M}\mu^j + \alpha^{m+1} - \alpha^{M+1} = 0. \tag{31}$$

Recall that $\alpha^2 = \cdots = \alpha^M = 0$.

Claim 1 $\gamma_2^m = 0$ *for* $m = 2, \dots, M-1$.

Proof. Since $a^{m+1} > a^m$ for $m = 1, \dots M-1$, the FOCs of v_1^m become

$$(1-\pi)\left[(\beta - (1+\lambda)v_1^m)f(v_1^m) + (\lambda+\rho+\lambda\rho)\left[1-F(v_1^m)\right] + (1+\lambda)\rho(v_2^m - v_1^m)f(v_1^m)\right]$$
$$-\sum_{j=m+1}^{M}\mu^j - \eta = 0,$$

for $m = 1, \dots, M-1$. Then for $m = 1, \dots, M-1$

$$v_2^m = \frac{1+\rho}{\rho}v_1^m - \frac{\lambda+\rho+\lambda\rho}{(1+\lambda)\rho}\frac{1-F(v_1^m)}{f(v_1^m)} - \frac{\beta}{(1+\lambda)\rho} + \frac{\eta + \sum_{j=m+1}^{M}\mu^j}{(1-\pi)(1+\lambda)\rho f(v_1^m)}, \tag{32}$$

which is strictly increasing in v_1^m by Assumptions 1 and 2. Let $m = 1, \dots, M-2$. If $\mu^{m+1} = 0$, then $v_2^{m+1} > v_2^m$ since $v_1^{m+1} > v_1^m$ and (32). If $\mu^{m+1} > 0$, then $v_2^{m+1} \ge v_1^{m+1} > v_1^m \ge v_2^m$ since (8) holds for m and $m+2$ and (8) holds with equality for $m+1$. Hence, $\gamma_2^m = 0$ for $m = 2, \dots, M-1$. ∎

Let

$$\varphi(v) := v - \frac{1-F(v)}{f(v)},$$

denote the "virtual" value, which is strictly increasing in v by Assumption 1.

Claim 2 $v_2^{M-1} > v_1^{M-1}$ *and* $\mu^M = 0$.

Proof. It follows immediately from Lemmas 8 and 9 that $v_2^{M-1} > v_1^{M-1}$. In this case, $\mu^M = 0$. ■

Claim 3 $\overline{v} + \lambda\varphi(\overline{v}) > \beta \geq v_2^{M-1} + \lambda\varphi(v_2^{M-1})$.

Proof. Since $\mu^M = 0$, the FOC of v_2^{M-1} implies that $\beta \geq v_2^{M-1} + \lambda\varphi(v_2^{M-1})$. Since $v_2^{M-1} > v_1^{M-1}$ and $\mu^M = 0$, the FOC of a^M implies that

$$\pi \int_{v_2^{M-1}}^{v_2^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v + (1-\pi) \int_{v_1^{M-1}}^{v_1^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v \geq 0.$$

Hence, it must be the case that $\beta < \overline{v} + \lambda\varphi(\overline{v})$. ■

Claim 4 $\gamma_2^M = \gamma_2^{M+1} = 0$ *and* $v_2^M + \lambda\varphi(v_2^M) \leq \beta$.

Proof. Firstly, we show that $\gamma_2^M = 0$. Suppose $v_2^M + \lambda\varphi(v_2^M) > \beta \geq v_2^{M-1} + \lambda\varphi(v_2^{M-1})$, then $v_2^M > v_2^{M-1}$ and therefore $\gamma_2^M = 0$. Suppose $v_2^M + \lambda\varphi(v_2^M) \leq \beta < \overline{v} + \lambda\varphi(\overline{v})$, then $v_2^M < \overline{v}$ and therefore $\gamma_2^{M+1} = 0$. Since $\gamma_2^{M+1} = 0$ and $v_2^M + \lambda\varphi(v_2^M) \leq \beta$, the FOC of v_2^M implies that $\gamma_2^M = 0$. Hence, $\gamma_2^M = 0$.

Secondly, we show that $v_2^M + \lambda\varphi(v_2^M) \leq \beta$. Suppose $a^{M+1} > a^M$, then the FOC of v_2^M implies that $\beta \geq v_2^M + \lambda\varphi(v_2^M)$. Suppose $a^{M+1} = a^M$, then by construction $v_2^M = v_2^{M-1}$ and therefore $v_2^M + \lambda\varphi(v_2^M) \leq \beta$. Hence, $v_2^M + \lambda\varphi(v_2^M) \leq \beta$.

Finally, since $v_2^M + \lambda\varphi(v_2^M) \leq \beta < \overline{v} + \lambda\varphi(\overline{v})$, $v_2^M < \overline{v}$ and therefore $\gamma_2^{M+1} = 0$. ■

In what follows, we consider two cases: $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) > 0$ and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) = 0$.

Case 1. $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) = 0$**.** In this case, by Lemmas 8 and 9, $\mu^m = 0$ for $m = 3, \ldots, M$.

Claim 5 *Suppose* $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) = 0$, *then* $\gamma_2^1 = 0$.

Proof. Suppose $\gamma_2^0 > 0$, then $v_2^0 = \underline{v}$. Since (8) holds for $m = 2$, we have $v_2^1 \geq v_1^1 > \underline{v} = v_2^0$. Hence, $\gamma_2^1 = 0$.

Suppose $\gamma_2^0 = 0$. Suppose $a^1 = 0$, then the FOC of v_2^0 implies that $\gamma_2^1 = 0$. Suppose $a^1 > 0$. Suppose, on the contrary, that $\gamma_2^1 > 0$, then we can construct another feasible mechanism

(a^*, p^*, q^*), which strictly improves welfare. Since $\gamma_2^1 > 0$, we have $v_2^0 = v_2^1 \geq v_1^1$. We consider two different cases: (i) $v_2^0 = v_2^1 = v_1^1$ and (ii) $v_2^0 = v_2^1 > v_1^1$.

(i) Suppose $v_2^0 = v_2^1 = v_1^1$. Let $\tilde{v}_1^1$ be such that $a^2(v_1^1 - \tilde{v}_1^1) = a^1(v_1^1 - \underline{v})$. Then, by Assumption 2, we have

$$\begin{aligned}
a^2\left[F(v_1^1) - F(\tilde{v}_1^1)\right] &= (a^2 - a^1 + a^1)\left[F(v_1^1) - F(\tilde{v}_1^1)\right] \\
&\leq a^1\left[F(v_1^1) - F(\tilde{v}_1^1)\right] + (a^2 - a^1) f(\tilde{v}_1^1)(v_1^1 - \tilde{v}_1^1) \\
&= a^1\left[F(v_1^1) - F(\tilde{v}_1^1)\right] + a^1 f(\tilde{v}_1^1)\tilde{v}_1^1 \\
&\leq a^1 F(v_1^1).
\end{aligned}$$

Let $\tilde{v}_2^0 = \underline{v}$ and $\tilde{v}_2^1$ be such that $\pi\left[F(v_2^1) - F(\tilde{v}_2^1)\right] = (1-\pi)\left[a^1 F(v_1^1) - a^2\left[F(v_1^1) - F(\tilde{v}_1^1)\right]\right]$. Let $\tilde{v}_1^m = v_1^m$ and $\tilde{v}_2^m = v_2^m$ for all $m \geq 1$. Let $a^*(v, b_i) = a^m$ if $v \in (\tilde{v}_i^{m-1}, \tilde{v}_i^m)$ for $m \geq 2$ and $i = 1, 2$ and $a^*(v, b_i) = 0$ if $v \in (\underline{v}, \tilde{v}_i^1)$ for $i = 1, 2$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$. Then, by construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, (a^*, p^*, q^*) improves revenue and therefore satisfies the (BB) constraint. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. For $v \in (\underline{v}, \tilde{v}_1^1)$, we have

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) < u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = q^1 c.$$

For $v \in (\tilde{v}_1^1, v_1^1)$, we have

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^2(\tilde{v}_2^1 - \tilde{v}_1^1) \leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = q^1 c.$$

The first inequality holds since $a^2(\tilde{v}_2^1 - \tilde{v}_1^1) \leq a^2(v_1^1 - \tilde{v}_1^1) = a^1(v_1^1 - \underline{v}) = a^1\left(v_2^0 - v_1^0\right)$. For $v \in (v_1^{m-1}, v_1^{m-2})$, $m \geq 2$, we have

$$\begin{aligned}
&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^2(\tilde{v}_2^1 - \tilde{v}_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + a^2(\tilde{v}_2^1 - \tilde{v}_1^1) - (a^2 - a^1)(v_2^1 - v_1^1) - a^1\left(v_2^0 - v_1^0\right) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + a^2\tilde{v}_2^1 - a^2\tilde{v}_1^1 - a^2 v_2^1 + (a^2 - a^1)v_1^1 + a^1 v_1^0
\end{aligned}$$

$$\leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) = q^m c,$$

where the last inequality holds by construction. Hence, the (IC-b) constraint is satisfied. Thus, (a^*, p^*, q^*) is feasible. However, this contradicts to that (a, p, q) is optimal.

(ii) Suppose $v_2^0 = v_2^1 > v_1^1$. Let $a^*(v, b_1) = a^1 - \varepsilon$ for some $\varepsilon > 0$ sufficiently small if $v < v_1^1$ and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $\tilde{v}_2^0 < v_2^0$ be such that $\pi(a^1 - \varepsilon)\left[F(v_2^0) - F(\tilde{v}_2^0)\right] = (1 - \pi)\varepsilon F(v_1^1)$. For $\varepsilon > 0$ sufficiently small, $v_1^1 < \tilde{v}_2^0$. Let $\tilde{v}_2^m = v_2^m$ for $m \geq 1$. Let $a^*(v, b_2) = a^1 - \varepsilon$ if $v \in (\tilde{v}_2^0, \tilde{v}_2^1)$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) + \varepsilon(v_1^1 - v_1^0)$ and $u^*(\underline{v}, b_2) = u(\underline{v}, b_2) - (1 - \pi)\varepsilon(v_1^1 - v_1^0)/\pi$. For $\varepsilon > 0$ sufficiently small, $u^*(\underline{v}, b_2) \geq u^*(\underline{v}, b_1) > 0$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$. Then, by construction, we have $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. Then (a^*, p^*, q^*) satisfies (BB) by Assumption 1. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. Note that, by Assumption 2, we have

$$\begin{aligned}
(a^1 - \varepsilon)(v_2^0 - \tilde{v}_2^0) &= (a^1 - \varepsilon)\int_{\tilde{v}_2^0}^{v_2^0} f(v)\frac{1}{f(v)}\mathrm{d}v \\
&\geq (a^1 - \varepsilon)\frac{1}{f(\tilde{v}_2^0)}[F(v_2^0) - F(\tilde{v}_2^0)] \\
&\geq \frac{1-\pi}{\pi}\varepsilon\frac{1}{f(v_1^1)}F(v_1^1) \\
&\geq \frac{1-\pi}{\pi}\varepsilon(v_1^1 - v_1^0).
\end{aligned}$$

Then, for $v < v_1^1$, we have

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(\tilde{v}_2^0 - v_1^0) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 v_2^0 + \frac{\varepsilon(v_1^1 - v_1^0)}{\pi} + (a^1 - \varepsilon)(\tilde{v}_2^0 - v_2^0) - \varepsilon v_2^0 - (a^1 - \varepsilon)v_1^0 \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 v_2^0 + \frac{\varepsilon(v_1^1 - v_1^0)}{\pi} - \frac{(1-\pi)\varepsilon(v_1^1 - v_1^0)}{\pi} - \varepsilon v_2^0 - (a^1 - \varepsilon)v_1^0 \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) + \varepsilon(v_1^1 - v_2^0) \\
<&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = q^1 c.
\end{aligned}$$

For $v \in (v_1^{m-1}, v_1^m)$ for $m = 2, \dots, M$, we have

$$u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(\tilde{v}_2^0 - v_1^0) + (a^2 - a^1 + \varepsilon)(v_2^1 - v_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1})$$

$$\leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + \varepsilon(v_1^1 - v_2^0) + \varepsilon(v_2^1 - v_1^1),$$

$$= u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) = q^m c.$$

Hence, the (IC-b) constraint is satisfied. Thus, (a^*, p^*, q^*) is feasible. However, this contradicts to that (a, p, q) is optimal.

Hence, it must be that $\gamma_2^1 = 0$. ■

By Claims 1, 4 and 5, we have $\gamma_2^m = 0$ for $m = 1, \dots, M+1$. Thus, for $m = 1, \dots, M-1$, v_1^m and v_2^m satisfy

$$(1 - \pi)\left[(\beta - v_1^m - \lambda\varphi(v_1^m))f(v_1^m) + (1 + \lambda)\rho[1 - F(v_1^m)] + (1 + \lambda)\rho(v_2^m - v_1^m)f(v_1^m)\right]$$
$$- \sum_{j=m+1}^{M} \mu^j - \eta = 0, \tag{33}$$

$$\pi(\beta - v_2^m - \lambda\varphi(v_2^m))f(v_2^m) - (1 - \pi)(1 + \lambda)\rho[1 - F(v_1^m)] + \sum_{j=m+1}^{M} \mu^j = 0. \tag{34}$$

where (33) and (34) are the first-order conditions of v_1^m and v_2^m, respectively, for $m = 1, \dots, M - 1$.

Claim 6 *Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = 0$, then $M \leq 2$.*

Proof. Let $\hat{m} = 1$ if $\mu^2 = 0$ and $\hat{m} = 2$ if $\mu^2 > 0$. For $m = \hat{m}, \dots, M - 1$, (33) and (34) become

$$(1 - \pi)\left[(\beta - v_1^m - \lambda\varphi(v_1^m))f(v_1^m) + (1 + \lambda)\rho[1 - F(v_1^m)] + (1 + \lambda)\rho(v_2^m - v_1^m)f(v_1^m)\right] - \eta = 0, \tag{27}$$

$$\pi(\beta - v_2^m - \lambda\varphi(v_2^m))f(v_2^m) - (1 - \pi)(1 + \lambda)\rho[1 - F(v_1^m)] = 0, \tag{28}$$

Let $m = \hat{m}, \dots, M - 1$. Given β, η and λ, (27) and (28) define v_2^m as functions of v_1^m, denoted by g_1 and g_2, respectively. Clearly, by Assumptions 1 and 2, $g_1'(v_1^m) > 1$. By

Lemmas 8 and 9, $v_2^m \geq v_1^m \geq \underline{v} \geq 0$. Furthermore, since $v + \lambda\varphi(v) < \beta$ for all $v < v_2^M$, $\sum_{j=m+1}^{M} \mu^j = 0$, $\eta \geq 0$, $\alpha^{m+1} = 0$ and $\alpha^{M+1} \geq 0$, (31) implies that

$$\int_{v_1^m}^{\bar{v}} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v \geq 0,$$

which holds if and only if $v_1^m \geq \hat{v}(\beta)$, where

$$\hat{v}(\beta) := \inf\left\{ \hat{v} \,\middle|\, \int_{\hat{v}}^{\bar{v}} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v \geq 0 \right\}. \tag{35}$$

Next, we show that $v_1^m \geq \hat{v}(\beta)$ and $g(v_1^m) \leq v_1^m$ implies $0 < g_2'(v_1^m) \leq 1$, where the second inequality holds strictly if $v_1^m > \hat{v}(\beta)$.

By the implicit function theorem, we have

$$g_2'(v_1^m) = \frac{1-\pi}{\pi} \frac{(1+\lambda)\rho f(v_1^m)}{-(\beta - (1+\lambda)v_2^m) f'(v_2^m) + (1+2\lambda) f(v_2^m)} > 0. \tag{36}$$

To see that the last inequality holds, note that $(\beta - v - \lambda\varphi(v)) f(v)$ is strictly decreasing in v for $v < v_2^M$. Taking derivative with respect to v yields $(\beta - (1+\lambda)v) f'(v) - (1+2\lambda) f(v) < 0$ for $v < v_2^M$. Note that Assumption 1 implies that for all $v \geq v_1^m$, we have

$$f(v) \geq f(v_1^m) \frac{1 - F(v)}{1 - F(v_1^m)}. \tag{37}$$

Then for $v_1^m \geq \hat{v}(\beta)$ we have

$$\begin{aligned}
1 - F(v_1^m) &\geq \frac{f(v_1^m)}{1 - F(v_1^m)} \int_{v_1^m}^{\bar{v}} (1 - F(v))\mathrm{d}v \\
&= \frac{f(v_1^m)}{1 - F(v_1^m)} \left[(1+\lambda) \int_{v_1^m}^{\bar{v}} (1 - F(v))\mathrm{d}v - \lambda \int_{v_1^m}^{\bar{v}} (1 - F(v))\mathrm{d}v \right] \\
&= \frac{f(v_1^m)}{1 - F(v_1^m)} \left[-(1+\lambda)v_1^m [1 - F(v_1^m)] + \int_{v_1^m}^{\bar{v}} \left[(1+\lambda)v - \lambda \frac{1 - F(v)}{f(v)} \right] f(v)\mathrm{d}v \right] \\
&\geq (\beta - (1+\lambda)v_1^m) f(v_1^m),
\end{aligned}$$

where the first line holds by (37), the third line holds by integration by parts, and the last line holds since $v_1^m \geq \hat{v}(\beta)$. Combining this and (28) yields

$$\begin{aligned}(\beta - v_2^m - \lambda\varphi(v_2^m))f(v_2^m) &= \frac{1-\pi}{\pi}(1+\lambda)\rho[1-F(v_1^m)] \\ &= \frac{1-\pi}{\pi}\rho\left[[1-F(v_1^m)] + \lambda[1-F(v_1^m)]\right] \\ &\geq \frac{1-\pi}{\pi}\rho\left[(\beta-(1+\lambda)v_1^m)f(v_1^m) + \lambda[1-F(v_1^m)]\right] \\ &= \frac{1-\pi}{\pi}\rho\left[\beta - v_1^m - \lambda\varphi(v_1^m)\right]f(v_1^m).\end{aligned}$$

Hence,

$$\frac{\rho f(v_1^m)}{f(v_2^m)} \leq \frac{\pi}{1-\pi}\frac{\beta - v_2^m - \lambda\varphi(v_2^m)}{\beta - v_1^m - \lambda\varphi(v_1^m)}.$$

Furthermore,

$$\begin{aligned}&-(\beta-(1+\lambda)v_2^m)f'(v_2^m) + (1+2\lambda)f(v_2^m) \\ =&-(\beta - v_2^m - \lambda\varphi(v_2^m))f'(v_2^m) + \lambda\left\{\frac{[1-F(v_2^m)]f'(v_2^m)}{f(v_2^m)} + f(v_2^m)\right\} + (1+\lambda)f(v_2^m) \\ \geq&(1+\lambda)f(v_2^m),\end{aligned}$$

where the last inequality holds since $\beta - v_2^m - \lambda\varphi(v_2^m) > 0$, $f' \leq 0$ by Assumption 2 and $[1-F(v_2^m)]f'(v_2^m) + f^2(v_2^m) \geq 0$ by Assumption 1. Finally, since $v_2^m \geq v_1^m \geq \hat{v}(\beta)$, we have

$$g_2'(v_1^m) = \frac{1-\pi}{\pi}\frac{(1+\lambda)\rho f(v_1^m)}{-(\beta-(1+\lambda)v_2^m)f'(v_2^m) + (1+2\lambda)f(v_2^m)} \leq \frac{\beta - v_2^m - \lambda\varphi(v_2^m)}{\beta - v_1^m - \lambda\varphi(v_1^m)} \leq 1.$$

Note that $g_2'(v_1^m) < 1$ if $v_1^m > \hat{v}(\beta)$ or $v_1^m < v_2^m$.

Thus, there exists at most one $v_1^m \geq \hat{v}(\beta)$ such that $g_1(v_1^m) = g_2(v_1^m) \geq v_1^m$, i.e., (27) and (28) has at most one solution such that $v_2^m \geq v_1^m \geq \hat{v}(\beta)$. Hence, $M \leq \hat{m} + 1$.

If $\hat{m} = 1$, then $M \leq \hat{m} + 1 \leq 2$. Assume for the rest of the proof that $\hat{m} = 2$ and $\mu^2 > 0$. In this case, $M \leq \hat{m} + 1 \leq 3$. Suppose, on the contrary, that $M = 3$. By Claim 4, $v + \lambda\varphi(v) < \beta$ for all $v < v_2^M$. Since $\mu^2 > 0$, $v_2^1 = v_1^1$. Furthermore, $\eta \geq 0$ and $\alpha^{M+1} \geq 0$. Hence, it follows from (31) that

$$\int_{v_1^1}^{\bar{v}} [v + \lambda\varphi(v) - \beta]\, f(v)\mathrm{d}v \geq 0,$$

i.e., $v_1^1 \geq \hat{v}(\beta)$. Then we have $v_1^2 > v_1^1 \geq \hat{v}(\beta)$, and $g_2(v_1^2) = v_2^2 \geq v_1^2$ since $\mu^2 > 0$. Since $g_2'(v) < 1$ if $v > \hat{v}(\beta)$ and $g_2(v) \geq v$, we have $g_2(v) > v$ for all $v_1^1 \leq v < v_1^2$. Hence, $v_2^1 = g_2(v_1^1) > v_1^1$, a contradiction to the fact that $v_2^1 = v_1^1$. Hence, $M = 2$. ■

Case 2. $v_2^{M-1} > v_1^{M-1}$ and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) > 0$. In this case, by Lemmas 8 and 9, $\mu^m = 0$ for $m = 1, \ldots, M$.

Claim 7 $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) > 0$, *then* $\gamma_2^1 = 0$.

Proof. Since $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) > 0$, $\mu^1 = 0$. Suppose, on the contrary, that $\gamma_2^1 > 0$. Then $v_2^1 = v_2^0$.

Suppose $\gamma_2^0 > 0$, then $v_2^1 = v_2^0 = \underline{v} = v_1^0$. Hence, $u(\underline{v}, b_1) > u(\underline{v}, b_2)$. Let $\tilde{v}_2^1 = \underline{v} + \varepsilon$ for some $\varepsilon > 0$ sufficiently small. Let $\tilde{v}_1^1$ be such that $\pi F(\varepsilon) = (1-\pi)\left[F(v_1^1) - F(\tilde{v}_1^1)\right]$. For $\varepsilon > 0$ sufficiently small, $\tilde{v}_2^1 < \tilde{v}_1^1$. Let $\tilde{v}_i^m = v_i^m$ and for $i = 1, 2$ and $m \neq 1$. Let $a^*(v, b_2) = a^1$ for all $v \in (\underline{v}, \tilde{v}_2^1)$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $a^*(v, b_1) = a^2$ for $v \in (\tilde{v}_1^1, v_1^1)$ and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) - (a^2 - a^1)(v_1^1 - \tilde{v}_1^1)$ and $u^*(\underline{v}, b_2) = u(\underline{v}, b_2) + \frac{1-\pi}{\pi}(a^2 - a^1)(v_1^1 - \tilde{v}_1^1)$. For $\varepsilon > 0$ sufficiently small, $u^*(\underline{v}, b_1) \geq u^*(\underline{v}, b_2) > 0$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u^*(\underline{v}, b)$. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint holds. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint holds. For $v \in (\underline{v}, \tilde{v}_1^1)$, (IC-b) holds since

$$u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1(\tilde{v}_2^0 - \tilde{v}_1^0) = u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) - \frac{(a^2 - a^1)(v_1^1 - \tilde{v}_1^1)}{\pi} \leq q^1 c.$$

For $v \in (\tilde{v}_1^1, v_1^1)$, (IC-b) holds since

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1(\tilde{v}_2^0 - \tilde{v}_1^0) + (a^2 - a^1)(\tilde{v}_2^1 - \tilde{v}_1^1) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{2}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \\
&- \frac{(a^2 - a^1)(v_1^1 - \tilde{v}_1^1)}{\pi} + (a^2 - a^1)(v_1^1 - \tilde{v}_1^1 + \tilde{v}_2^1 - v_2^1) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{2}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^2 - a^1)(\tilde{v}_2^1 - v_2^1) - \frac{(1-\pi)(a^2 - a^1)(v_1^1 - \tilde{v}_1^1)}{\pi} \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{2}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1})
\end{aligned}$$

$$\leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) = q^1 c,$$

where the last inequality holds since $v_2^1 = \underline{v} < v_1^1$, and the first inequality holds since by Assumption 2 we have

$$\begin{aligned} \tilde{v}_2^1 - v_2^1 &\leq \frac{F(\varepsilon)}{f(\tilde{v}_2^1)} \\ &\leq \frac{1}{f(\tilde{v}_1^1)} \frac{1-\pi}{\pi} \left[F(v_1^1) - F(\tilde{v}_1^1)\right] \\ &\leq \frac{(1-\pi)(v_1^1 - \tilde{v}_1^1)}{\pi}. \end{aligned}$$

For $v \in (v_1^{m-1}, v_1^m)$, $m = 2, \dots, M$, (IC-b) holds since

$$\begin{aligned} & u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\ =& u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^2 - a^1)(\tilde{v}_2^1 - v_2^1) - \frac{(1-\pi)(a^2 - a^1)(v_1^1 - \tilde{v}_1^1)}{\pi} \\ \leq& u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) = q^m c. \end{aligned}$$

Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare. This contradicts to the optimality of (a, p, q). Hence, $\gamma_2^1 = 0$.

Suppose $\gamma_2^0 = 0$. Suppose $a^1 = 0$, then the FOC of v_2^0 implies that $\gamma_2^1 = 0$. Suppose $a^1 > 0$. Then the FOCs of v_2^0 and v_2^1 imply that

$$\begin{aligned} & \pi(\beta - (1+\lambda)v_2^0) f(v_2^0) + \pi\lambda[1 - F(v_2^0)] \\ \geq& (1-\pi)(1+\lambda)\rho \\ >& (1-\pi)(1+\lambda)\rho[1 - F(v_1^1)] \\ \geq& \pi(\beta - (1+\lambda)v_2^1) f(v_2^1) + \pi\lambda[1 - F(v_2^1)]. \end{aligned}$$

Since $(\beta - (1+\lambda)v)f(v) + \lambda[1 - F(v)]$ is strictly decreasing in v when $v + \lambda\varphi(v) < \beta$, we have $v_2^1 > v_2^0$ and therefore $\gamma_2^1 = 0$. ■

By Claims 1, 4 and 7, we have $\gamma_2^m = 0$ for $m = 1, \dots, M$. Thus, for $m = 1, \dots, M-1$, v_1^m and v_2^m satisfies (27), (28) and (31).

Claim 8 *Suppose* $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) > 0$*, then* $M \leq 2$.

Proof. Suppose, on the contrary, that $M \geq 3$. Then there exists $1 \leq m < M - 1$ such that $v_2^m \geq v_1^m$. It follows from (31) that $\int_{v_1^m}^{v_1^M} [v + \lambda\varphi(v)] f(v)\mathrm{d}v \geq \beta[1 - F(v_1^m)]$, i.e., $v_1^m \geq \hat{v}(\beta)$. Both (v_1^m, v_2^m) and (v_1^{M-1}, v_2^{M-1}) are solutions to (27) and (28), and satisfy $v_2 \geq v_1 \geq \hat{v}(\beta)$. However, by a similar argument in Claim 6, (27) and (28) have at most one solution satisfying $v_2 \geq v_1 \geq \hat{v}(\beta)$, a contradiction. Hence, it must be $M \leq 2$. ■

To summarize, we have shown in both cases that $M \leq 2$. However, this contradicts to the assumption that $M \geq 3$. Hence, it must be that $V(M, d) = V(M-1, d)$ for all $M \geq 3$. This completes the proof of Lemma 6.

E Properties of the optimal mechanism

Let $a^* = a^2$, $v_1^* = v_1^1$, $v_2^* = v_2^1$, $v_2^{**} = v_2^2$, $u_1^* = u(\underline{v}, b_1)$ and $u_2^* = u(\underline{v}, b_2)$ denote an optimal solution of $\mathcal{P}'(2,0)$. Let $\beta, \eta, \lambda, \mu^1, \mu^2, \alpha^3, \xi_1$ and ξ_2 denote the corresponding Lagrangian multipliers.

Proof of Proposition 2. First-best is achieved if the allocation rule satisfies $v^* := v_1^* = v_2^* = F^{-1}(1-S)$ and $a^* = 1$, and verification is zero. Hence, $u_1^* = u_2^* = v^* - b_1$ and (BB) holds if and only if

$$b_1 - v^* F(v^*) \geq 0. \tag{38}$$

Since $v^* = F^{-1}(1-S)$, there exists $\hat{S}(b_1) < 1$ such that (38) holds if and only if $S \geq \hat{S}(b_1)$. Clearly, $\hat{S}(b_1)$ is strictly decreasing in b_1. ■

Before proving Propositions 3 and 4, we first introduce and prove Proposition 7, which is used in later proofs.

Proposition 7 *Suppose Assumptions 1 and 2 hold. Suppose also that* $S < \hat{S}(b_1)$*, i.e., the first-best cannot be achieved. In an optimal mechanism of* $\mathcal{P}$*, (S), (BB) and (BC) hold with equality.*

Proof of Proposition 7. First, we show that (S) holds with equality.[27] Let $S' := (1 - \pi)a^* \left[1 - F(v_1^*)\right] + \pi a^*[F(v_2^{**}) - F(v_2^*)] + \pi[1 - F(v_2^{**})]$. Suppose to the contradiction that $S' < S$. Let $\kappa \in (0,1)$ be such that $\kappa + (1-\kappa)S' = S$. Consider a new mechanism (a^*, p^*, q^*). Let $a^*(v, b) = \kappa + (1-\kappa)a(v, b)$ and $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - (1 -$

[27] I thank Michael Richter for suggesting this proof.

$\kappa)u(\underline{v}, b)$ for all v and b. Finally, let $q(v, b_2) = 0$ for all v, $q(v, b_1) = (1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2)\right]/c$ if $v < v_1^*$ and $q(v, b_1) = (1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^*(v_2^* - v_1^*)\right]/c$ if $v > v_1^*$. Clearly, (a^*, p^*, q^*) strictly improves welfare upon (a, p, q). Now we show that (a^*, p^*, q^*) is also feasible. By construction, (IR) and (IC-v) hold. Note that

$$\begin{aligned}
p^*(v, b) &= va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - (1-\kappa)u(\underline{v}, b) \\
&= (1-\kappa)va(v, b) + \kappa v - \int_{\underline{v}}^{v} \left[\kappa + (1-\kappa)a(\nu, b)\right]\mathrm{d}\nu - (1-\kappa)u(\underline{v}, b) \\
&= (1-\kappa)va(v, b) - (1-\kappa)\int_{\underline{v}}^{v} a(\nu, b)\mathrm{d}\nu - (1-\kappa)u(\underline{v}, b) \\
&= (1-\kappa)p(v, b).
\end{aligned}$$

Hence, $\mathbb{E}\left[p^*(v, b) - kq^*(v, b)\right] = (1-\kappa)\mathbb{E}\left[p(v, b) - kq(v, b)\right] \geq 0$. That is, (BB) holds. Since $p^*(\overline{v}, b_1) = (1-\kappa)p(\overline{v}, b_1) \leq b_1$, (BC) holds. Since $\mathbb{E}[a^*(v, b)] = \kappa + (1-\kappa)\mathbb{E}[a(v, b)] = \kappa + (1-\kappa)S' = S$, (S) holds. Finally, we show that (IC-b) holds. If $v \leq v_1^*$, then

$$(1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2)\right] + \kappa(\underline{v} - \underline{v}) \leq q(v, b_1)c.$$

If $v > v_1^*$, then

$$(1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2)\right] + \kappa(\underline{v} - \underline{v}) + (\kappa + (1-\kappa)a^* - \kappa)(v_2^* - v_1^*) \leq q(v, b_1)c.$$

Thus, we can conclude that (a^*, p^*, q^*) is feasible. However, this contradicts to that (a, p, q) is optimal. Hence, (S) holds with equality.

Second, we show that (BC) holds with equality. Suppose to the contradiction that (BC) holds with strict inequality. We consider four different cases: (i) $v_2^* > v_1^*$, (ii) $v_2^{**} > v_2^* = v_1^*$, (iii) $v_2^{**} = v_2^* = v_1^*$ and $a^* < 1$ and (iv) $v_2^{**} = v_2^* = v_1^*$ and $a^* = 1$.

(i) Suppose $v_2^* > v_1^*$. Let $\varepsilon > 0$ and $\delta > 0$ be such that $(1-\pi)\left[F(v_1^* + \varepsilon) - F(v_1^*)\right] = \pi[F(v_2^*) - F(v_2^* - \delta)]$. For $\varepsilon > 0$ sufficiently small, we have $v_2^* - v_1^* - \varepsilon - \delta \geq 0$. Consider a new mechanism (a^*, p^*, q^*) that satisfies

$$\begin{aligned}
a^*(v, b_1) &= \chi_{\{v \geq v_1^* + \varepsilon\}} a^*, \;\; p(v, b_1) = \chi_{\{v \geq v_1^*\}} a^*(v_1^* + \varepsilon) - u_1^*, \\
q(v, b_1) &= \frac{1}{c}\left[\chi_{\{v \geq v_1^*\}} a^*(v_2^* - v_1^* - \varepsilon - \delta) + u_1^* - u_2^*\right], \\
a(v, b_2) &= \chi_{\{v \geq v_2^* - \delta\}} a^* + \chi_{\{v \geq v_2^{**}\}}(1 - a^*),
\end{aligned}$$

$$p(v, b_2) = \chi_{\{v \geq v_2^* - \delta\}} a^*(v_2^* - \delta) + \chi_{\{v \geq v_2^{**}\}} (1 - a^*) v_2^{**} - u_2^*,$$
$$q(v, b_2) = 0.$$

Clearly, for $\varepsilon > 0$ sufficiently small, (a^*, p^*, q^*) is feasible and strictly improves welfare upon (a, p, q), a contradiction.

(ii) Suppose $v_2^{**} > v_2^* = v_1^*$. Let $\varepsilon > 0$ and $\delta > 0$ be such that $a^* \left[F(v_1^* + \varepsilon) - F(v_1^*)\right] = \pi[F(v_2^{**}) - F(v_2^{**} - \delta)]$. For $\varepsilon > 0$ sufficiently small, we have $v_2^{**} - v_1^* - \varepsilon - \delta \geq 0$. Consider a new mechanism (a^*, p^*, q^*) that satisfies

$$a^*(v, b_1) = \chi_{\{v \geq v_1^* + \varepsilon\}} a^*, \; p(v, b_1) = \chi_{\{v \geq v_1^*\}} a^*(v_1^* + \varepsilon) - u_1^*,$$
$$q(v, b_1) = \frac{1}{c} \left[\chi_{\{v \geq v_1^*\}} a^*(v_2^* - v_1^*) + u_1^* - u_2^* \right],$$
$$a(v, b_2) = \chi_{\{v \geq v_2^* + \varepsilon\}} a^* + \chi_{\{v \geq v_2^{**} - \delta\}} (1 - a^*),$$
$$p(v, b_2) = \chi_{\{v \geq v_2^* + \varepsilon\}} a^*(v_2^* + \varepsilon) + \chi_{\{v \geq v_2^{**} - \delta\}} (1 - a^*)(v_2^{**} - \delta) - u_2^*,$$
$$q(v, b_2) = 0.$$

Clearly, for $\varepsilon > 0$ sufficiently small, (a^*, p^*, q^*) is feasible and strictly improves welfare upon (a, p, q), a contradiction.

(iii) Suppose $v_2^{**} = v_2^* = v_1^*$ and $a^* < 1$. Let $\varepsilon > 0$ and $\delta > 0$ be such that $[(1 - \pi)a^* + \pi] \left[F(v_1^* + \varepsilon) - F(v_1^*)\right] = (1 - \pi)\delta[1 - F(v_1^* + \varepsilon)]$. For $\varepsilon > 0$ sufficiently small, we have $\delta \leq 1 - a^*$. Consider a new mechanism (a^*, p^*, q^*) that satisfies

$$a^*(v, b_1) = \chi_{\{v \geq v_1^* + \varepsilon\}} (a^* + \delta), \; p(v, b_1) = \chi_{\{v \geq v_1^*\}} (a^* + \delta)(v_1^* + \varepsilon) - u_1^*,$$
$$q(v, b_1) = \frac{1}{c} \left(u_1^* - u_2^* \right),$$
$$a(v, b_2) = \chi_{\{v \geq v_2^* + \varepsilon\}}, \; p(v, b_2) = \chi_{\{v \geq v_2^* + \varepsilon\}} - u_2^*,$$
$$q(v, b_2) = 0.$$

Clearly, for $\varepsilon > 0$ sufficiently small, (a^*, p^*, q^*) is feasible and strictly improves welfare upon (a, p, q), a contradiction.

(iv) Suppose $v_2^{**} = v_2^* = v_1^*$ and $a^* = 1$. In this case, the first-best allocation rule is achieved. Hence, it must be the case that the total verification cost is strictly positive, i.e.,

$u_1^* > u_2^* \geq 0$. Let $u_2^* - u_1^* \geq \varepsilon > 0$. Consider a new mechanism (a^*, p^*, q^*) that satisfies

$$\begin{aligned}
&a^*(v, b_1) = \chi_{\{v \geq v_1^*\}}, \; p(v, b_1) = \chi_{\{v \geq v_1^*\}} v_1^* - u_1^* + \varepsilon, \\
&q(v, b_1) = \frac{1}{c}\left(u_1^* - u_2^* - \varepsilon\right), \\
&a(v, b_2) = \chi_{\{v \geq v_2^*\}}, \; p(v, b_2) = \chi_{\{v \geq v_2^*\}} - u_2^*, \\
&q(v, b_2) = 0.
\end{aligned}$$

Clearly, for $\varepsilon > 0$ sufficiently small, (a^*, p^*, q^*) is feasible and strictly improves welfare upon (a, p, q), a contradiction.

Lastly, I show that (BB) holds with equality. Suppose not. Then we can increase u_1^* and u_2^* by the same amount. The resulting new mechanism is feasible and gives the same welfare. In particular, (BC) holds with strict inequality in the new mechanism. Then we can repeat the above argument and construct another feasible mechanism which strictly improves welfare upon (a, p, q), a contradiction. ■

By Theorem 3, $v_1^*, v_2^*, v_2^{**}, a^2, u_1^*, u_2^*, \beta, \eta, \lambda, \mu^1, \mu^2, \alpha^3, \xi_1$ and ξ_2 satisfy the following first-order conditions:

$$(1-\pi)\left[(\beta - v_1^* - \lambda\varphi(v_1^*))f(v_1^*) + (1+\lambda)\rho[1-F(v_1^*)] + (1+\lambda)\rho(v_2^* - v_1^*)f(v_1^*)\right]$$
$$-\eta - \mu^2 = 0, \tag{39}$$
$$\pi(\beta - v_2^* - \lambda\varphi(v_2^*))f(v_2^*) - (1-\pi)(1+\lambda)\rho[1-F(v_1^*)] + \mu^2 = 0, \tag{40}$$
$$(1-a^*)(\beta - v_2^{**} - \lambda\varphi(v_2^{**}))f(v_2^{**}) = 0, \tag{41}$$
$$\pi \int_{v_2^*}^{v_2^{**}} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v$$
$$+ (1-\pi)\left[\int_{v_1^*}^{\bar{v}} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v - (1+\lambda)\rho(v_2^* - v_1^*)[1-F(v_1^*)]\right]$$
$$-\eta v_1^* + \mu^2(v_2^* - v_1^*) - \alpha^3 = 0, \tag{42}$$
$$\eta + \mu^1 + \mu^2 - (1-\pi)(\lambda + \rho + \lambda\rho) + \xi_1 = 0, \tag{43}$$
$$-\mu^1 - \mu^2 - \pi\lambda + (1-\pi)(1+\lambda)\rho + \xi_2 = 0. \tag{44}$$

Furthermore, by Proposition 7, (S) and (BB) become:

$$(1-\pi)a^*[1-F(v_1^*)] + \pi a^*[F(v_2^{**}) - F(v_2^*)] + \pi[1-F(v_2^{**})] = S, \tag{45}$$

$$
\begin{aligned}
&-(1-\pi)u_1^* + (1-\pi)a^* v_1^*[1-F(v_1^*)] - \pi u_2^* + \pi a^* v_2^*[1-F(v_2^*)] + \pi(1-a^*)v_2^{**}[1-F(v_2^{**})] \\
&-(1-\pi)\rho(u_1^* - u_2^*) - (1-\pi)\rho a^*(v_2^* - v_1^*)[1-F(v_1^*)] = 0. \qquad (46)
\end{aligned}
$$

Proof of Proposition 3. We begin by proving the first part of the proposition. Suppose, on the contrary, that $u_1^* > u_2^* \geq 0$. In this case, $\xi_1 = \mu^1 = \mu^2 = 0$. (43) implies that $\eta = (1-\pi)(\lambda + \rho + \lambda\rho)$. (44) implies $\xi_2 = \pi\lambda - (1-\pi)(1+\lambda)\rho$. Since $\xi_2 \geq 0$, we have $\lambda[\pi - \rho(1-\pi)] \geq \rho(1-\pi)$ which implies that $\rho < \pi/(1-\pi)$, a contradiction.

Next, we prove the second part of the proposition. Since $S < 1$, we have $u_1^* = u_2^*$ by the first result of Proposition 3. It suffices to show that $v_1^* = v_2^*$. Suppose, on the contrary, that $v_2^* > v_1^*$. In this case, $\mu^2 = 0$. Combining (43) and (44) yields $\eta - \lambda + \xi_1 + \xi_2 = 0$. Since $\xi_1, \xi_2 \geq 0$, we have $\eta \leq \lambda$. Taking the difference of (39) divided by $(1-\pi)f(v_1^*)$ and (40) divided by $\pi f(v_2^*)$ gives

$$
\begin{aligned}
&[1+(1+\lambda)\rho]\left(v_2^* - v_1^*\right) + \lambda\left[\varphi(v_2^*) - \varphi(v_1^*)\right] + (1+\lambda)\rho\frac{1-F(v_1^*)}{f(v_1^*)} \\
&+ (1+\lambda)\frac{\rho(1-\pi)}{\pi}\frac{1-F(v_1^*)}{f(v_2^*)} - \frac{\eta}{(1-\pi)f(v_1^*)} = 0. \qquad (47)
\end{aligned}
$$

Since $v_2^* > v_1^*$, $f(v_2^*) \leq f(v_1^*)$ and $\eta \leq \lambda$, we have

$$
\begin{aligned}
0 &\geq [1+(1+\lambda)\rho]\left(v_2^* - v_1^*\right) + \lambda\left[\varphi(v_2^*) - \varphi(v_1^*)\right] + (1+\lambda)\frac{\rho}{\pi}\frac{1-F(v_1^*)}{f(v_1^*)} - \frac{\lambda}{(1-\pi)f(v_1^*)} \\
&> \frac{\rho}{\pi}\frac{1-F(v_1^*)}{f(v_1^*)} + \lambda\left[\frac{\rho}{\pi}\frac{1-F(v_1^*)}{f(v_1^*)} - \frac{1}{(1-\pi)f(v_1^*)}\right] \geq 0,
\end{aligned}
$$

where the last inequality holds since $1-F(v_1^*) \geq S$ and $\rho \geq \pi/[S(1-\pi)]$. A contradiction. Hence, $v_1^* = v_2^*$.

The proof of the third part is straightforward and neglected here. ■

Proof of Proposition 4. Suppose, on the contrary, that $u_1^* > u_2^* \geq 0$. In this case, $\xi_1 = \mu^1 = \mu^2 = 0$. (43) implies that $\eta = (1-\pi)(\lambda + \rho + \lambda\rho)$. Taking the difference of (39) divided by $(1-\pi)f(v_1^*)$ and (40) divided by $\pi f(v_2^*)$ gives

$$
\begin{aligned}
&[1+(1+\lambda)\rho]\left(v_2^* - v_1^*\right) + \lambda\left[\varphi(v_2^*) - \varphi(v_1^*)\right] + (1+\lambda)\rho\frac{1-F(v_1^*)}{f(v_1^*)} \\
&+ (1+\lambda)\frac{\rho(1-\pi)}{\pi}\frac{1-F(v_1^*)}{f(v_2^*)} - \frac{\eta}{(1-\pi)f(v_1^*)} = 0. \qquad (47)
\end{aligned}
$$

Suppose $S \leq (1-\pi)\left[1-F(b_1)\right]$. Since (BC) holds with equality and $u_1^* \geq 0$, we have $a^* \geq b_1/v_1^*$. By (S), we have

$$(1-\pi)\frac{b_1}{v_1^*}\left[1-F(v_1^*)\right] \leq S.$$

Since $S \leq (1-\pi)\left[1-F(b_1)\right]$, there exists a unique Let $\hat{v}(S, b_1, \pi) \in [b_1, \overline{v}]$ such that the above inequality holds with equality when $v_1^* = \hat{v}(S, b_1, \pi)$, where $\hat{v}$ is strictly decreasing in S and π and strictly increasing in b_1. Then $v_1^* \geq \hat{v}(S, b_1, \pi)$. Hence, $v_2^* - v_1^* \leq \varphi(v_2^*) - \varphi(v_1^*) \leq \overline{v} - \varphi\left(\hat{v}(S, b_1, \pi)\right)$.

Since $v_2^* \geq v_1^*$, $f(v_2^*) \leq f(v_1^*)$ and $\eta = (1-\pi)(\lambda+\rho+\lambda\rho)$, we have

$$\begin{aligned}
0 \leq & [1+(1+\lambda)\rho]\left(v_2^*-v_1^*\right)+\lambda\left[\varphi(v_2^*)-\varphi(v_1^*)\right]+(1+\lambda)\frac{\rho}{\pi}\frac{1-F(v_1^*)}{f(v_2^*)}-\frac{\lambda+\rho+\lambda\rho}{f(v_1^*)} \\
< & (1+\lambda)(1+\rho)\left[\overline{v}-\varphi\left(\hat{v}(S,b_1,\pi)\right)\right]+(1+\lambda)\frac{\rho}{\pi}\frac{1-F(v_1^*)}{f(v_2^*)}-\frac{(1+\lambda)\rho}{f(v_1^*)} \\
\leq & (1+\lambda)\left\{(1+\rho)\left[\overline{v}-\varphi\left(\hat{v}(S,b_1,\pi)\right)\right]+\frac{\rho}{\pi}\frac{1-F\left(\hat{v}(S,b_1,\pi)\right)}{f(\overline{v})}-\frac{\rho}{f\left(\hat{v}(S,b_1,\pi)\right)}\right\}.
\end{aligned}$$

Note that the term in the braces is strictly increasing in S and converges to $-\rho/f(\overline{v}) < 0$ as S goes to zero. Hence, there exists $\hat{S}$ such that $u_1^* = u_2^*$ if $S < \hat{S}$. ■

F Extensions and discussions

This section consists only the proof of Lemma 7. The proofs of Theorems 4 and 5 are sraightfoward extensions of that Theorems 1 and 2, respectively, and are neglected here. The proof of Theorem 6 is a straightforward extension of that of Theorem 3 and is neglected here. The proof of Proposition 5 is straightforward and is neglected here. The proof of Proposition 6 is a straightforward extension of that of Theorem 1 and is neglected here.

Proof of Lemma 7. Consider types $t := (v, b)$ and $\hat{t}$ such that $p(\hat{t}) + \max\{\theta(\hat{t}, n), \theta(\hat{t}, b)\} \leq b$. Then (IC) requires that

$$\begin{aligned}
& a(t)v - p(t) - (1-q(t))\theta(t,n) - q(t)\theta(t,b) \\
\geq & a(\hat{t})v - p(\hat{t}) - (1-q(\hat{t}))\theta(\hat{t},n) - q(\hat{t})\theta(\hat{t},b)
\end{aligned}$$

Consider an alternative mechanism $(a^*, p^*, q^*, \theta^*)$ with $a^* = a$ and $q^* = q$. Let $\theta^*(t, n) = \theta^*(t, b) = 0$ for all t and $\theta^*(\hat{t}, b) = c$ for all $\hat{t}$ such that $\hat{b} \neq b$. Let $p^*(t) = p(t) + (1 - q(t))\theta(t, n) + q(t)\theta(t, b)$. Since $p(t) + \max\{\theta(t, n), \theta(t, b)\} \leq b$, we have $p^*(t) \leq b$, i.e., (BC) holds. It is easy to see that the new mechanism also satisfies (IR), (BB) and (S) and does not affect the welfare.

Finally, I show that (IC) holds. Consider types $t := (v, b)$ and $\hat{t}$ such that $p^*(\hat{t}) + c \leq b$. If $\hat{b} = b$, then (BC) in the old mechanism implies that $p(\hat{t}) + \max\{\theta(\hat{t}, n), \theta(\hat{t}, b)\} \leq b$. Hence,

$$\begin{aligned}
&a^*(t)v - p^*(t) \\
=&a(t)v - p(t) - (1 - q(t))\theta(t, n) - q(t)\theta(t, b) \\
\geq&a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n) - q(\hat{t})\theta(\hat{t}, b) \\
=&a^*(\hat{t})v - p^*(\hat{t}) + q(\hat{t})\theta(\hat{t}, \hat{b}) - q(\hat{t})\theta(\hat{t}, b) \\
=&a^*(\hat{t})v - p^*(\hat{t}).
\end{aligned}$$

If $\hat{b} \neq b$, then $b \geq p^*(\hat{t}) + c = p(\hat{t}) + (1 - q(\hat{t}))\theta(\hat{t}, n) + q(\hat{t})\theta(\hat{t}, \hat{b}) + c \geq p(\hat{t}) + \max\{\theta(\hat{t}, n), \theta(\hat{t}, b)\}$. Hence,

$$\begin{aligned}
&a^*(t)v - p^*(t) \\
=&a(t)v - p(t) - (1 - q(t))\theta(t, n) - q(t)\theta(t, b) \\
\geq&a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n) - q(\hat{t})\theta(\hat{t}, b) \\
=&a^*(\hat{t})v - p^*(\hat{t}) + q(\hat{t})\theta(\hat{t}, \hat{b}) - q(\hat{t})\theta(\hat{t}, b) \\
\geq&a^*(\hat{t})v - p^*(\hat{t}) - q^*(\hat{t})\theta^*(\hat{t}, b).
\end{aligned}$$

The last inequality holds since $\theta(\hat{t}, \hat{b}) \geq 0$ and $\theta^*(\hat{t}, b) = c \geq \theta(\hat{t}, b)$. ■