How Bayesian Persuasion can Help Reduce Illegal Parking and Other Socially Undesirable Behavior

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Abstract

We consider the question of how best to allocate enforcement resources across different locations with the goal of deterring unwanted behaviour. We rely on "Bayesian persuasion" to improve deterrence. For simplicity, we focus on the problem of how to allocate resources in order to reduce the extent of illegal parking. However, the same model can also be applied to many other types of socially undesirable behaviour. We show that the problem of how to allocate resources and then "persuade," can be represented as a linear programming problem. Notably, optimal persuasion involves the use of only two messages, "high" and "low" that indicate that the amount of expected resources available is high and low, respectively. However, unlike standard results in Bayesian persuasion, it is only possible to achieve a "partially convex" objective function. We also obtain a full solution for a class of "monotone" problems.

1 Introduction

This paper addresses the question of how best to allocate enforcement resources across different locations with the goal of deterring unwanted behaviour. The novelty in our approach is that we employ the techniques of "Bayesian persuasion," namely the use of carefully disseminated truthful communication, in order to maximize deterrence. We show that Bayesian persuasion can significantly improve deterrence and may imply a completely different allocation of resources.

To fix ideas and simplify the presentation, we focus on the problem of how to allocate resources in order to reduce the extent of illegal parking. However, the same model can also be applied to many other types of socially undesirable behaviour.

We show how "Bayesian Persuasion" or in other words carefully disseminated information about parking enforcement in different locations can help, on its own and without strengthening enforcement, to reduce the extent of illegal parking. A city that seeks to deter illegal parking can display such messages on its website, or on electronic street signs. The possibility to disseminate information affects the optimal allocation of enforcement efforts. We also show that for Bayesian persuasion to be effective, it is crucial that the amount of parking enforcement resources be uncertain. Bayesian persuasion provides no advantage if the amount of resources is commonly known. We show that the optimal level of resources allocated to any given location depends both on the state of the world (the total amount of resources available) and the message that is sent. Not conditioning the allocation on the messages sent implies a loss of generality and limits deterrence.

We consider the problem of how to allocate resources and then "persuade," or send signals about the state of the world to motorists in n different neighborhoods, or locations. The amount of resources available in each state of the world introduces a natural resource constrtaint that is not usually imposed in the Bayesian persuasion literature. We show that this problem can be represented as a linear programming problem. Notably, although there are multiple audiences, optimal persuasion involves the use of only two messages, "high" and "low" that indicate that the amount of expected resources available is high and low, respectively. The message "low" may be interpreted as a moratorium on parking enforcement in some clearly defined situations. Our results indicate that such a moratorium can be an important part of an optimal enforcement policy. Intuitively, such a moratorium improves overall deterrence because it achieves stronger deterrence when it is not applied. Indeed, casual empiricism suggests that local governments occasionally experiment with such moratoriums. For example, it is supposedly well known and certainly widely believed among residents of Tel Aviv that the city does not enforce parking violations from Friday to Saturday evenings as well as from the evening before to the evening of state holidays.

For reasons that will become clear below, unlike in standard results in Bayesian persuasion, in our setting it is only possible to achieve a "partially convex" objective function. Finally, we also obtain a full solution for a class of "monotone" problems.

The question of how to allocate resources in order to achieve deterrence is typically analyzed in the context of what is known as a "security game." A security game is a two-player, possibly zero-sum, simultanuous-move game in which an attacker has to decide where to strike while a defender has to decide where to allocate its limited defence resources.[1] Analysis of such games has been applied by political scientists to the question of how to defend against terrorist attacks (Powell, 2007), and by computer scientists to a host of related issues (see Tambe, 2011, and the references therein). Security games are closely related to Colonel Blotto games (Borel, 1921; Roberson, 2006; Hart, 2008). These are zero-sum simultanuous-move two-player games in which players allocate a given number of divisions to n different battlefields. Each battlefield is won by the player who allocated a larger number of divisions there, and the player who wins a larger number of battlefields wins the game. As explained above, we consider a security game in which there is uncertainty about the amount of resources available to the defender, with an added stage in which the defender can send a message about the state of the world.[2]

The question addressed here of how to allocate a given amount of law enforcement resources is different from, and complementary to, the questions famously posed by Becker (1968) about how much resources should be allocated to law enforcement and how to divide these resources between enforcement effort that increases the probability that the offender is caught and the penalty imposed on the offender if caught. Polinsky and Shavell (2000) provide a survey of the theoretical literature on the optimal form of enforcement, and Chalfin and McCrary (forthcoming) provide a survey of the relevant

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[1] The fact that in our formulation, the attacker responds only after observing the defender's signal turns our game into a sequential rather than a simultanuous move game.

[2] Rabinovich et al. (2015) and Xu et al. (2016) have also studied a security game with messages, but in a very different setting.

empirical literature.

Within the law and economics literature, the two papers that are most closely related to our work are by Lando and Shavell (2004) and Eeckhout et al. (2010) who both consider the question of how to allocate enforcement resources. Both papers show that it may be beneficial to concentrate enforcement on a subset of the population. The following example illustrates their idea. Suppose that deterrence of the entire population requires 10 units of resources, but only 5 units are available. In this case, allocation of the 5 units of resources across the entire population fails to achieve deterrence, but concentrating the 5 units on half of the population (say, on those with lightly colored eyes) successfully deters this half. Our paper is more general in that it considers any number of neighborhoods and adds uncertainty, and in that we consider the question of how to further improve deterrence through Bayesian persuasion, or communication.

Finally, a small literature, that started with Aumann and Maschler (1995) and Kamenica and Gentzkow (2011) (see also Sobel (2013), Bergemann and Morris (2017), and the references therein), studies how a sender of information can affect a receiver's beliefs and thereby induce it to act in a way that benefits the sender. We use similar ideas to study how the communication of information about enforcement can improve deterrence. However, unlike the aforementioned literature on Bayesian persuasion, we consider signaling to several receivers simultaneously, which complicates the analysis. And, of course, we consider the joint problem of how best to allocate and communicate. Moreover, as mentioned above, in our problem in it generally impossible to achieve a full convexification of the objective function, which is generally possible in Bayesian persuasion.

The paper proceeds as follows. The model is presented in Section 2. A few examples are presented in Section 3. Section 4 contains the general analysis of the problem, and Section 5 considers the case of "monotone" problems. In Section 6, we briefly address the issue of dynamics, or deterrence over time.

2 Model

Consider a city with $n \geq 1$ different neighborhoods. The set of neighborhoods or locations is denoted $\mathbf{N} = \{1, \ldots, n\}$. Illegal parking is a problem in all of these neighborhoods. The city determines the amount of resources devoted to enforcement in each neighborhood out of the total amount of available resources, denoted r. The amount of available resources is uncertain. We assume that $r = r_k$, $k \in \{1, \ldots, K\}$, with probability π_k, respectively, where $0 \leq r_1 < \cdots < r_K$ and $\sum_{k=1}^{K} \pi_k = 1$. We treat the distribution of resources as exogenously given, but it may obviously depend on the city's decisions, and provides another dimension on which to optimize the allocation of resources. We discuss two ways of endogenizing the distribution of resources in Section 6 below.

We refer to k as the state of the world. The city knows the realization of the state of the world k and hence also the realization r_k, but drivers only know the distribution $\pi = (\pi_1, \ldots, \pi_K)$.

As explained above, we assume that the city may disseminate information about its enforcement effort. We model this possibility by assuming that the city may send a message $m \in \{1, \ldots, M\}$ about the state of the world k.[3] The probability that the city sends message m in state k is denoted by $p_k(m) = \Pr(m|k)$. It follows that

$$p_k(m) \geq 0 \text{ for every } k \text{ and } m, \text{ and } \sum_{m=1}^{M} p_k(m) = 1 \text{ for every } k. \tag{1}$$

The posterior belief that drivers have over the state of the world k upon receiving the message m is denoted

$$\Pr(k|m) = \frac{p_k(m)\,\pi(k)}{\sum_{k'=1}^{K} p_{k'}(m)\,\pi(k')}.$$

Denote the amount of resources allocated to enforcement in neighborhood i in state k when the city sends the message m by $a_k^i(m)$.[4] If message m is sent with probability zero in state k, then $a_k^i(m) \equiv 0$ for every location i.

The city chooses the amounts $a_k^i(m)$ subject to its resource constraint. In every state $k \in \{1, \ldots, K\}$,

$$\sum_{i=1}^{n} a_k^i(m) \leq r_k \tag{2}$$

for every message $m \in \{1, \ldots, M\}$.[5]

The objective of the city is to allocate the amounts of enforcement resources $\left\{a_k^i(m)\right\}$ and send the messages $m \in \{1, ..., M\}$ with probabilities $\{p_k(m)\}$ so as to minimize the extent of illegal parking. The measure of illegal parking in each neighborhood i is given by a function $q^i(a^i(m))$ that is decreasing in the expected amount of enforcement resources $a^i(m) \equiv \sum_{k=1}^{K} a_k^i(m) \Pr(k|m)$ in that neighborhood given message m. For simplicity, we focus on the special case where each q^i is given by a threshold function. Namely, there exists some threshold τ^i such that

$$q^i(a^i(m)) = \begin{cases} 1 & \text{if } a^i(m) < \tau^i \\ 0 & \text{if } \tau^i \leq a^i(m) \end{cases}.$$

Hence, the city's objective is to allocate the amounts of enforcement resources $\left\{a_k^i(m)\right\}$ and send messages with probabilities $\{p_k(m)\}$ so as to minimize the expected social cost of illegal parking as given by

$$\min_{\left\{a_k^i(m)\right\},\{p_k(m)\}} \sum_{k=1}^{K} \sum_{m=1}^{M} \sum_{i=1}^{n} q^i(a^i(m)) s^i p_k(m)\,\pi_k \tag{3}$$

where s^i, $i \in \{1, \ldots, n\}$, denotes the social disutility generated by illegal parking in neighborhood i, subject to the resource constraint (2) and the constaints imposed by the fact that the $p_k(m)$'s are probabilities (1).

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[3] "No signal" is also a signal.

[4] We show below that conditioning the level of enforcement on the signal on top of just the state of the world may contribute to deterrence.

[5] Observe that there is no need to also sum over the messages in the resource constraint because the constraint only requires that resources add up to no more than what is available given a state of the world and the fact that a specific given message has been sent.

For example, if there are just two locations, just two messages m and m', and r_k units are available in state k, then we need to require that $a_k^1(m) + a_k^2(m) \leq r_k$ and $a_k^1(m') + a_k^2(m') \leq r_k$ rather than the weaker requirement that $p_k(m)\left(a_k^1(m) + a_k^2(m)\right) + p_k(m')\left(a_k^1(m') + a_k^2(m')\right) \leq r_k$ because the city may allocate the entire amount of available resources r_k upon sending any message m.

Observe that the constraints (1) and (2) are linear in resources $\left\{a_k^i(m)\right\}$ and probabilities $\{p_k(m)\}$, but the objective function (3) is non-linear both because $q^i(a^i(m))$ is a non-linear function of $a^i(m)$ and because $a^i(m)$ itself is a non-linear function of the probabilities $\{p_k(m)\}$.

Alternatively, it is also useful to consider the city's problem as how to allocate the amounts of enforcement resources $\left\{a_k^i(m)\right\}$ and send messages with probabilities $\{p_k(m)\}$ so as to maximize expected weighted deterrence as given by

$$\max_{\left\{a_k^i(m)\right\},\{p_k(m)\}} \sum_{k=1}^{K} \sum_{m=1}^{M} \sum_{i=1}^{n} d^i(a^i(m))s^i p_k(m)\pi_k \tag{4}$$

where the function $d^i(a^i(m)) = 1 - q^i(a^i(m))$ describes the strength of deterrence and s^i is interpreted as the benefit of deterrence in neighborhood i (which is equal to the decrease in social distutility). Again, the constraints (1) and (2) are linear in $\left\{a_k^i(m)\right\}$ and $\{p_k(m)\}$, but the objective function (4) is not.

It is helpful to represent the allocation of resources in matrix form, as shown in the next example. Suppose that there are three locations and three states of the world. The allocation of resources is given by:

$$\begin{array}{c|c|c|c|c}
\cline{2-4}
\pi_1 & a_1^1(m) & a_1^2(m) & a_1^3(m) & r_1 \\
\cline{2-4}
\pi_2 & a_2^1(m) & a_2^2(m) & a_2^3(m) & r_2 \\
\cline{2-4}
\pi_3 & a_3^1(m) & a_3^2(m) & a_3^3(m) & r_3 \\
\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\tau^1} & \multicolumn{1}{c}{\tau^2} & \multicolumn{1}{c}{\tau^3} & \multicolumn{1}{c}{}
\end{array}$$

If no messages are sent, then we may denote $m = \emptyset$; if the message sent reveals the state of the world, then we may denote $m = m_j$ in row j of the matrix.

The case where two messages m_1 and m_2 are sent is represented as follows:

$$\begin{array}{c|c|c|c|c}
\cline{2-4}
\pi_1 & a_1^1(m_1) & a_1^2(m_1) & a_1^3(m_1) & r_1 \\
\cline{2-4}
 & a_2^1(m_1) & a_2^2(m_1) & a_2^3(m_1) & \\
\hdashline
\pi_2 & a_2^1(m_2) & a_2^2(m_2) & a_2^3(m_2) & r_2 \\
\cline{2-4}
\pi_3 & a_3^1(m_2) & a_3^2(m_2) & a_3^3(m_2) & r_3 \\
\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\tau^1} & \multicolumn{1}{c}{\tau^2} & \multicolumn{1}{c}{\tau^3} & \multicolumn{1}{c}{}
\end{array}$$

Message m_1 is sent in states 1 and 2, and message m_2 is sent in states 2 and 3. This example clarifies the reason that not allowing the allocation to depend on the message sent involves a loss of generality: it does not allow the city to sometimes deter only in neighborhoods 1 and 2 in state 2 (when it sends the message m_1), and sometimes deter in neighborhoods $1, 2, 3$ (when it sends the message m_2). This is something that the city may benefit from if the amount of resources available in state 3 permits deterrence in neighborhoods $1, 2, 3$ ($r_3 > \tau_1 + \tau_2 + \tau_3$) but the amount available in states 1 and 2 only permits deterrence in neighborhoods 1 and 2.

The next example, which is similar to an example in Kamenica and Gentzkow (2011), shows that the city may be able to decrease the extent of illegal parking by disseminating information about the realizations of the amount of enforcement effort $\left\{a_k^i(m)\right\}$. For simplicity, the amounts of enforcement efforts in this example are independent of the messages, so the index m is ommited, and they are denoted by $\left\{a_k^i\right\}$.

Example 1. Consider a city with one neighborhood. Suppose that drivers park illegally if they perceive the expected amount of enforcement to be smaller than $\tau_1 = 2/5$. Suppose that resources are given by $(r_1, r_2) = (0, 1)$ with probabilities $(\pi_1, \pi_2) = (\frac{2}{3}, \frac{1}{3})$, respectively, and that the social cost of illegal parking is $s_1 = 1$. The fact that there is only one neighborhood greatly simplifies the problem of how to allocate the amount of enforcement efforts $\{a_k^i\}$. The city cannot do better than simply allocate its entire enforcement resources in every state of the world to this single neighborhood, so that $a_1^1 = 0$ and $a_2^1 = 1$. All this information is represented in matrix form as follows:

$$\begin{array}{c|c|c}
\cline{2-2}
\frac{2}{3} & 0 & 0 \\
\cline{2-2}
\frac{1}{3} & 1 & 1 \\
\cline{2-2}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\frac{2}{5}} & \multicolumn{1}{c}{}
\end{array}$$

If the city disseminates no information about the state of the world, then drivers park illegally because the expected amount of enforcement is only

$$\frac{2}{3} \cdot a_1^1 + \frac{1}{3} \cdot a_2^1 = \frac{1}{3},$$

which is smaller than the critical threshold $\tau_1 = 2/5$. The expected social cost of illegal parking in this case is 1.

The city can do better by fully revealing the state of the world to the drivers. In this case, when the state of the world is $k = 1$, drivers would realize that there is no enforcement because $a_1^1 = 0$ and would park illegally, but when the state of the world is $k = 2$, drivers would be deterred from parking illegally because $a_2^1 = 1$, which implies that the expected social cost of illegal parking in this case is

$$\frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 0 = \frac{2}{3}.$$

The city can do even better by providing partial information about the state of the world as follows: when $k = 2$ it sends the message H, and when $k = 1$, it sends messages H and L with probability 1/2 each. When drivers receive the message L they know that $k = 1$ and so the amount of enforcement is $a_1^1 = 0$ and so they park illegally. However, when they receive the message H, their posterior belief about the amount of enforcement is

$$\begin{aligned}
E\left[a^1 \,|m = H\right] &= \tfrac{\pi(H|1)\pi_1}{\pi(H|1)\pi_1 + \pi(H|2)\pi_2} \cdot a_1^1 + \tfrac{\pi(H|2)\pi_2}{\pi(H|1)\pi_1 + \pi(H|2)\pi_2} \cdot a_2^1 \\
&= \tfrac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3}} \cdot a_1^1 + \tfrac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3}} \cdot a_2^1 \\
&= \tfrac{1}{2} \cdot a_1^1 + \tfrac{1}{2} \cdot a_2^1 \\
&= \tfrac{1}{2}.
\end{aligned}$$

The fact that this posterior belief is larger than the critical threshold $\tau_1 = 2/5$ implies that drivers don't park illegally. This signaling strategy further decreases the expected social cost of illegal parking from $\frac{2}{3}$ to the probability that the city sends the signal L, or to[6]

$$\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 = \frac{1}{3}.$$

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[6] The city can decrease the expected social cost of illegal parking even further to $\frac{1}{6}$ by sending the signals L and H with probabilities $\frac{1}{4}$ and $\frac{3}{4}$, respectively, when $k = 1$ and just the signal H when $k = 2$. This is the lowest possible value of the expected social cost in this example.

■

It is also possible to illustrate by example that the optimal allocation of enforcement resources depends on whether the city is able to disseminate information or not: a city that can disseminate information about its enforcement allocates its resources differently than a city that does not. The reason that this is so is clarified in the general analysis below, so we do not provide a specific example for this.

3 General Analysis

The Optimal Ratio Rule

For any probabilities and allocations $p_k(m)$ and $\left\{a_k^i(m)\right\}$, each message m achieves deterrence on some set of locations $S(m) \subseteq \{1, \ldots, n\}$. We may thus identify each message m with the set $S(m)$ on which it deters provided we add the following deterrence constraint:

$$a^i(m) \equiv \sum_{k=1}^{K} a_k^i(m) \Pr(k \,|m) \geq \tau^i \tag{5}$$

for every location $i \in S(m)$, and for every message $m \in M \equiv 2^{\{1,\ldots,n\}}$ that is sent with a positive probability. The set of messages includes a message that achieves no deterrence (or that achieves deterrence on the empty set, $\varnothing \in M$). And no loss of generality is implied by the assumption that exactly one message deters on any given set of locations. This is because if two messages m and m' deter on the same set of locations then they can be merged into one message $m \cup m'$.

The identification of messages with the set of locations on which they achieve deterrence clarifies that persuasion, or the sending of messages, can only be useful if there is some underlying uncertainty.

Proposition 1. *Persuasion is ineffective without true underlying uncertainty. If there is only one state of the world, then there exists an optimal solution that does not involve (non-trivial) persuasion.*

Proof. Suppose that there is only one state of the world. Optimality requires that in this state a message m_1 that is such that $S(m_1)$ maximizes the value of deterrence is sent with probability one. Sending another message m_2 that induces the same or less deterrence is either unnecessary or strictly dominated.

$$\begin{array}{c|c|c|c|c}
\cline{2-4}
 & a_2^1(m_1) & a_2^2(m_1) & a_2^3(m_1) & \\
\pi_1 \; - & - - - - & - - - - & - - - - & - \; r_1 \\
 & a_2^1(m_2) & a_2^2(m_2) & a_2^3(m_2) & \\
\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{l}{\tau^1} & \multicolumn{1}{l}{\tau^2} & \multicolumn{1}{l}{\tau^3} & \multicolumn{1}{c}{}
\end{array}$$

■

The next result shows that no loss of generality is implied by restricting attention to a specific class of allocations of resources.

Proposition 2 (the "Optimal Ratio Rule"). *Given probabilities* $\{p_k(m)\}$ *and an allocation* $\left\{a^i_k(m)\right\}$*, the same probabilities together with the allocation* $\left\{a^{i*}_k(m)\right\}$ *such that:*

For every location $i \in S(m)$*, for every state* k*, and for every message* m *that is sent with a positive probability at* k*,*

$$a^{i*}_k(m) = \frac{\tau^i r_k}{\sum_{j\in S(m)} \tau^j};$$

and for every location $i \notin S(m)$*, or messages* m *that are sent with probability zero,*

$$a^{i*}_k(m) = 0;$$

achieves equal or better deterrence than $\left\{a^i_k(m)\right\}$.

Proof. Fix probabilities $\{p_k(m)\}$ and an allocation $\left\{a^i_k(m)\right\}$. For every location $i \in S(m)$ that is deterred by message m,

$$\sum_{k=1}^{K} \Pr(k\,|m)\, a^i_k(m) \geq \tau^i.$$

Summing over $i \in S(m)$ and changing the order of summation yields

$$\begin{array}{rcl} \sum_{i\in S(m)} \tau^i & \leq & \sum_{i\in S(m)} \sum_{k=1}^{K} \Pr(k\,|m)\, a^i_k(m) \\ & \leq & \sum_{k=1}^{K} \Pr(k\,|m) \sum_{i\in S(m)} a^i_k(m) \\ & \leq & \sum_{k=1}^{K} \Pr(k\,|m)\, r_k \end{array}$$

where the last inequality follows from feasibility (1).

It therefore follows that

$$\tau^i \leq \sum_{k=1}^{K} \Pr(k\,|m)\, \frac{\tau^i r_k}{\sum_{j\in S(m)} \tau^j}$$

and so the allocation $a^{i*}_k(m) = \frac{\tau^i r_k}{\sum_{j\in S(m)} \tau^j}$ for every $i \in S(m)$, state k, and message m, and $a^{i*}_k(m) = 0$ for every $i \in \mathbf{N} \setminus S(m)$, state k, and message m, also achieves deterrence of the set $S(m)$. ■

The next example illustrates the intuition for this result.

Example 2. Consider the case in which the city has three neighborhoods with the corresponding thresholds $\tau^1 = 2$, $\tau^2 = 3$ and $\tau^3 = 4$. There are three equally likely states, with the resources $r_1 = 1$, $r_2 = 8$ and $r_3 = 14$, respectively. The city allocates its resources and sends two messages m_1 and m_2 as depicted in the following matrix:

$$\begin{array}{c|c|c|c|c}
\cline{2-4}
 & 1 & & & \\
\frac{1}{3}\;p & - & - & - & 1 \\
 & 1 & & & \\
\cline{2-4}
\frac{1}{3} & 2 & 3 & 5 & 10 \\
\cline{2-4}
\frac{1}{3} & 3 & 6 & 5 & 14 \\
\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{}
\end{array}$$

Message m_1 is sent in state 1 with probability $1-p$, and message m_2 is sent in state 1 with probability p, and in states 2 and 3.

The city achieves deterrence with message m_2 but not with message m_1. Thus, a larger probability p implies a larger probability of deterrence, but if p is too large, then the city loses deterrence in the third location. The maximum probability p that allows the city to deter in all three locations is $p=\frac{1}{2}$. The overall probability of deterrence (in all three locations) with this probability $p=\frac{1}{2}$ is $\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3}+\frac{1}{3}=\frac{5}{6}$.

If however the city allocates its enforcement resources proportionally to the deterrence thresholds in the three locations as implied by the Optimal Ratio Rule, then it can achieve more deterrence. The allocation according to the Optimal Ratio Rule is depicted in the following matrix:

$$\begin{array}{c|c|c|c|c}
\cline{2-4}
 & 1 & & & \\
\cdashline{2-4}
\frac{1}{3}\,p & \frac{2}{9}\times 1 & \frac{3}{9}\times 1 & \frac{4}{9}\times 1 & 1 \\
\cline{2-4}
\frac{1}{3} & \frac{2}{9}\times 10 & \frac{3}{9}\times 10 & \frac{4}{9}\times 10 & 10 \\
\cline{2-4}
\frac{1}{3} & \frac{2}{9}\times 14 & \frac{3}{9}\times 14 & \frac{4}{9}\times 10 & 14 \\
\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{}
\end{array}$$

With this allocation, the city can set $p=\frac{3}{4}$ and achieve deterrence in all three locations with probability $\frac{1}{3}\cdot\frac{3}{4}+\frac{1}{3}+\frac{1}{3}=\frac{11}{12}$.

Linear Programming

The Optimal Ratio Rule implies that the problem can be recast as a problem of choosing the probabilities $\{p_k(m)\}$ so as to maximize deterrence, subject to the probability constraints (1) and the deterrence constraint (5) applied to $\left\{a_k^{i*}(m)\right\}$ as follows:

$$\max_{\{p_k(m)\}}\sum_{k=1}^{K}\sum_{m=1}^{M}\sum_{i\in S(m)}s^i p_k(m)\,\pi_k \tag{6}$$

subject to the probability constraints (1) and the deterrence constraint:

$$a^{i*}(m)\equiv\sum_{k=1}^{K}a_k^{i*}(m)\Pr(k\,|m)\geq\tau^i \tag{7}$$

for every location $i\in S(m)$, and for every message $m\in M$ that is sent with a positive probability.

The objective function (6) is linear, but the deterrence constraint is not because the conditional probabilities $\Pr(k\,|m)$ are not linear in the probabilities $\{p_k(m)\}$, and because the constraint is only imposed on messages that are sent with a positive probability rather than on all messages. Nevertheless, as shown by the next proposition, the problem can be recast as a linear programming problem.

Proposition 3. *The problem* min (3) *subject to the probability and resource constraints* (1) *and* (2)*, respectively, can be recast as the linear programming problem:*

$$\max_{\{p_k(m)\}}\sum_{k=1}^{K}\sum_{m=1}^{M}\sum_{i\in S(m)}s^i p_k(m)\,\pi_k \tag{8}$$

subject to the probability constraints (1) and the deterrence constraints:

$$\sum_{k=1}^{K} p_k\left(m\right)\pi(k)a_k^{i*}\left(m\right) \geq \tau^i \sum_{k=1}^{K} p_k\left(m\right)\pi(k) \tag{9}$$

for every location $i \in S(m)$ and message $m \in M$.

Proof. The problem max (8) subject to the probability and deterrence constraints (1) and (9) is a linear programming problem. The objective function (8) is obtained from (3) upon substitution of the resources according to the Optimal Ratio Rule. The deterrence constraints (9) are obtained from the deterrence constraints (7) upon multiplication of both the right- and left-hand-sides of the constrain by the denominator of the conditional probability $\Pr\left(k \left| m\right.\right) = \frac{p_k(m)\pi(k)}{\sum_{k'=1}^{K} p_{k'}(m)\pi(k')}$. The deterrence constraints can be imposed on all messages because for messages that are not sent with a positive probability in $p_k\left(m\right) = 0$, which trivially satisfies the deterrence constraint. ■

The result that the problem can be recast as a linear programming problem is useful because there are several well known algorithms for solving linear programming problems that work very well in practice. We do not think that the type of problem described here is likely to be very large in practice anyway, but another advantage of linear programming problems is that they can be solved in time that is polynomial in the size of the input of the problem. However, here, the size of the input is the product of the number of states and the number of messages, $k \times 2^n$, which is exponential in the number of locations, n. In the next subsection, we show that it is enough to send just two messages in the optimal solution, but because it is impossible to tell which two messages should be sent, the solution is still exponential in the number of locations, n.

Two Signals are Enough

We next show that it is enough to send just two messages in the optimal solution.

Devoting all the available resources to deterrence on the set of neighbourhoods $S \subseteq \{1, \ldots, n\}$ with no messages produces a non-increasing step-function disutility:

$$D_S(a) = \begin{cases} \sum_{i \in \mathbf{N}} s^i & \text{if } a < \sum_{i \in S} \tau^i \\ \sum_{i \in \mathbf{N} \setminus S} s^i & \text{if } \sum_{i \in S} \tau^i \leq a \end{cases}$$

that maps the amount of available expected resources a into disutility.

It follows that the minimal disutility that can be achieved without persuasion, or without sending any messages, is given by the following non-increasing step-function:

$$D(a) = \min_{S \subseteq \mathbf{N}} D_S(a).$$

It is possible to increase deterrence by sending two messages L and H that "split" the amount of expected resources a into two posterior expected amounts of resources conditional on the messages L and H, denoted $a\left|L\right. < a\left|H\right.$, respectively, such that

$$\pi(L) \cdot a\left|L\right. + \pi(H) \cdot a\left|H\right. = a$$

where $\pi(L) = \sum_{k=1}^{K} p_k(L)\pi_k$ and $\pi(H) = \sum_{k=1}^{K} p_k(H)\pi_k$ denote the overall probability of sending messages L and H, respectively. This can be done by sending the two messages L and H with probabilities $p_k(L)$ and $p_k(H)$, respecetively, in states $k \in \{1, \ldots, K\}$. The posterior expected amounts are given by:

$$a\,|L\, = \frac{\sum_{k=1}^{K} p_k(L)\pi_k r_k}{\sum_{k=1}^{K} p_k(L)\pi_k} \qquad \text{and} \qquad a\,|H\, = \frac{\sum_{k=1}^{K} p_k(H)\pi_k r_k}{\sum_{k=1}^{K} p_k(H)\pi_k}.$$

As illustrated in Figure 1 below, splitting the amount of expected resources a between the posterior expectations $a\,|L$ and $a\,|H$ implies that the optimal level of disutility lies on the straight line that connects the points $(a\,|L\,, D(a\,|L\,))$ and $(a\,|H\,, D(a\,|H\,))$.

– Figure 1 here –

Such spliting convexifies the function $D(a)$ from below. Moreover, it is clear that it is enough to split the expected amount a into just two posterior expected amounts in order to achieve the smallest possible disutility. Note also that the posterior expected amount $a\,|H$ that minimizes expected disutility has to be a discontinuity point of the function $D(a)$. Given that $a\,|H$ is set equal to such a discontinuity point of $D(a)$, it is clear that the posterior expectation $a\,|L$ should be set as low as possible, unless it is optimal to set it equal to another discontinuity of $D(a)$ as shown in the following figure.

– Figure 2 here –

Left panel: Figure 2a ($a\,|L$ is set as low as possible), Left panel: Figure 2b ($a\,|L$ is also set equal to a discontinuity point of $D(a)$)

Proposition 4. *Suppose that the expected amount of enforcement resources is a. Then, maximal deterrence is obtained by splitting a such that*

$$\pi(L)D(a\,|L\,) + \pi(H)D(a\,|H\,)$$

is minimized.

Proof. Follows from the discussion above. ■

"Convexification is Incomplete"

Proposition 4 implies that it is enough to rely on just two messages. We can further show that the "destruction of resources" is inefficient. Namely, the destruction of resources that is necessary to push $a\,|L$ leftwards in order to lower the line that connects $a\,|L$ and $a\,|H$ is inefficient. This result is stated in the next proposition.

Proposition 5. *The "destruction of resources" is inefficient.*

Proof. Splitting the expected amount of enforcement resources a between the posterior expectations $a\,|L$ and $a\,|H$ implies that the optimal level of disutility lies on the straight line that connects the two points $(a\,|L\,, D(a\,|L\,))$ and $(a\,|H\,, D(a\,|H\,))$ where $a\,|H$ is a discontinuity point of the function $D(a)$ as explained above. If $a\,|L$ is also optimally set at a discontinuity point of $D(a)$, then it is clear from Figure 2b that decreasing it by any small $\varepsilon > 0$ is inefficient.

Suppose then that $a\,|L$ is optimally set at a continuity point of $D(a)$. Decreasing it further necessitates the destruction of resources. We show that such destruction of resources is inefficient.

The equation of the line that connects the points $(a\,|L\,, D(a\,|L\,))$ and $(a\,|H\,, D(a\,|H\,))$ is:

$$y = \frac{D(a\,|H\,) - D(a\,|L\,)}{a\,|H\, - a\,|L} \cdot x + D(a\,|L\,) - \frac{D(a\,|H\,) - D(a\,|L\,)}{a\,|H\, - a\,|L} \cdot a\,|L\,.$$

If $a\,|L$ is lowered by a small $\varepsilon > 0$, then the expected amount of resources decreases from a to $a - \varepsilon\pi(L)$ and the line of expected distutility connects the two points: $(a\,|L - \varepsilon, D(a\,|L\,))$ and $(a\,|H\,, D(a\,|H\,))$ is:

$$y = \frac{D(a\,|H\,) - D(a\,|L\,)}{a\,|H\, - a\,|L + \varepsilon} \cdot x + D(a\,|L\,) - \frac{D(a\,|H\,) - D(a\,|L\,)}{a\,|H\, - a\,|L + \varepsilon} \cdot (a\,|L\, - \varepsilon)\,.$$

Algebraic manipulation[7] that the height of the former line at the point where $x = a$ is equal to the height of the second line at the point where $x = a - \varepsilon\pi(L)$. It follows that the destruction of resources does not lower expected disutility. If $a\,|L$ is optimally set at a discontinuity point of $D(a)$ then the destruction of (a small amount of) resources increases expected disutility; if $a\,|L$ is optimally set at a continuity point of $D(a)$ then the destruction of (a small amount of) resources does not affect expected disutility. ■

The fact that splitting the amount of expected resources a into $a\,|L$ and $a\,|H$ where $a\,|H$ is a discontinuity point of the function $D(a)$ and $a\,|L$ is a continuity point of the function $D(a)$ implies that "convexification is incomplete." Namely, the line that connects the points $(a\,|L\,, D(a\,|L\,))$ and $(a\,|H\,, D(a\,|H\,))$ does not fully convexifies the function $D(a)$ from below. This incompleteness stands in contrast to other results in the literature on Bayesian persuasion where persuasion does permit the full convexification of the underlying objective function.

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[7] Observe that

$$\frac{D(a\,|H\,) - D(a\,|L\,)}{a\,|H\, - a\,|L} \cdot (a - a\,|L\,) + D(a\,|L\,) \leq \frac{D(a\,|H\,) - D(a\,|L\,)}{a\,|H\, - a\,|L + \varepsilon} \cdot (a - \varepsilon\pi(L) - a\,|L + \varepsilon) + D(a\,|L\,)$$

if and only if

$$\frac{a - a\,|L}{a\,|H\, - a\,|L} \leq \frac{a - a\,|L + \varepsilon - \varepsilon\pi(L)}{a\,|H\, - a\,|L + \varepsilon}$$

if and only if

$$(a - a\,|L\,)\,(a\,|H\, - a\,|L + \varepsilon) \leq (a - a\,|L + \varepsilon(1 - \pi(L)))\,(a\,|H\, - a\,|L\,)$$

if and only if

$$a - a\,|L \leq (1 - \pi(L))\,(a\,|H\, - a\,|L\,)\,,$$

and given that $a = \pi(L)a\,|L + (1 - \pi(L))a\,|H$, if and only if

$$a\,|L \leq a\,|L\,.$$

4 The Monotone Case

Stronger assumptions permit an explicit solution. We may assume without loss of generality that the locations can be ordered by their importance, or:

$$s^1 \geq s^2 \geq \cdots \geq s^n.$$

In this section, we assume that deterrence thresholds can also be ranked in the same way, or:

$$\tau^1 \leq \tau^2 \leq \cdots \leq \tau^n;$$

This assumption involves loss of generality. In particular, it implies that it is also more effective to deploy resources in more important locations, or:

$$\frac{s^1}{\tau^1} \geq \frac{s^2}{\tau^2} \geq \cdots \geq \frac{s^n}{\tau^n}.$$

It is straightforward to verify that if it is optimal to deter at location i under some message m, then it is also optimal to deters at location $j < i$. It follows that the number of messages that is needed is only $n + 1$. Namely, in the optimal solution, it is enough to restrict attention only to those messages associated with the sets $\varnothing$, $\{1\}$, $\{1, 2\}$, ..., $\{1, \ldots, n\}$. Moreover, the optimal solution satisfies "nesting." Namely, the sets $S(m)$ can be nested in the sense that $n' < n''$ implies $S(\{1, \ldots, n'\}) \subseteq S(\{1, \ldots, n''\})$.

Monotonicty implies that the steps in the function $D(a)$ become longer and less steep with the amount of expected resources a, as depicted in the following figure:

– Figure 3 here ($D(a)$ in the monotone case) –

The monotone case admits a complete solution of the city's problem as follows.

Proposition 6. *If* $r_1 \leq \sum_{i=0}^{m} \tau^i \leq E\ [\ r\ ] < \sum_{i=0}^{m+1} \tau^i \leq r_K$ *for some* $m \leq n - 1$ *then it is possible to achieve full convexification* ($a|H = \sum_{i=0}^{m+1} \tau^i$ *and* $a|L = \sum_{i=0}^{m} \tau^i$) *provided that*

$$\sum_{i=0}^{m} \tau^i \geq \frac{\sum_{k=0}^{k'-1} \pi_k r_k + (1-p)\pi_{k'} r_{k'}}{\sum_{k=0}^{k'-1} \pi_k + (1-p)\pi_{k'}}$$

where $k' \in \{1, \cdots, K\}$ *and* $p \in [0, 1)$ *are the unique solution to:*

$$\sum_{i=0}^{m+1} \tau^i = \frac{\sum_{k=k'+1}^{K} \pi_k r_k + p\pi_{k'} r_{k'}}{\sum_{k=k'+1}^{K} \pi_k + p\pi_{k'}}.$$

Otherwise, convexification is partial, either $a|H = \sum_{i=0}^{m+1} \tau^i$ *but* $a|L > \sum_{i=0}^{m} \tau^i$, *or persuasion is altogether unhelpful.*

Proposition 6 is a corollary of the following generalization of a lemma of Aumann and Maschler (1995).

Lemma. *Given a distribution of resources* $r_1, \ldots, r_K$*, and given any two amounts of resources* $R_L < E\ [\ r\ ] < R_H$ *it is possible to send two messages* L *and* H *such that*

$$a|L = R_L \qquad a|H = R_H$$

provided that $r_1 \leq R_L$, $R_H \leq r_K$, *and*

$$R_L \geq \frac{\sum_{k=0}^{k'-1} \pi_k r_k + (1-p)\pi_{k'} r_{k'}}{\sum_{k=0}^{k'-1} \pi_k + (1-p)\pi_{k'}}$$

where $k' \in \{1, \cdots, K\}$ *and* $p \in [0,1)$ *are the unique solution to:*

$$R_H = \frac{\sum_{k=k'+1}^{K} \pi_k r_k + p\pi_{k'} r_{k'}}{\sum_{k=k'+1}^{K} \pi_k + p\pi_{k'}}.$$

Proof. The maximum difference between R_H and R_L is obtained when message H is sent in states $k \in \{k'+1, \cdots, K\}$, message L in states $k \in \{1, \cdots, k'-1\}$, and in state k' messages H and L are sent with probabilities p, $1-p$, respectively, for some state $k' \in \{1, \ldots, K\}$ and probability p. The condition on R_L reflects the lowest possible value of R_L given a set value for R_H under this message policy. Less extreme messages permit closer values of R_H and R_L. ■

Example 3. Consider a case with three states of the world. Resources are given by $(r_1, r_2, r_3) = (0, \frac{1}{2}, 1)$ and the prior is $(\pi_1, \pi_2, \pi_3) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$. In this case, $E\ [\ r\ ] = \frac{1}{2}$, and

$$R_L \geq \max\left\{\frac{3R_H - 2}{8R_H - 5}, 0\right\}.$$

If $R_H \leq \frac{2}{3}$ then R_L is unrestricted; the lowest possible value of R_L increases monotonically with $\frac{2}{3} < R_H < 1$; and if $R_H = 1$ then $R_L \geq \frac{1}{3}$.

5 Endogenous Distribution of Resources & Deterrence over Time

It is possible to endogenize the prior distribution over the amount of available resources in the following way. Suppose that the city employs n inspectors. Each inspector is allocated to a specific day and time, or to several time slots, depending on how many hours it is required to work per day or week. Each inspector shows up to each asigned time slot with probability $1 - \varepsilon$, independently across the different inspectors.

Any assignment of inspectors to time slots generates a prior distribution of resources available in each time slot. It is then possible to optimize over these prior distributions, given that in each time slot, the city allocates the available resources and disseminates information optimally, as described above. The solution of such a problem provides a theory of enforcement operations.

The ideas presented here suggest that it is possible to achieve better overall deterrence through convexification. That is, the city should allocate its insectors randomly so that in some states of the world their number is small and in other states it is large. This as thhis would allow to improve overall deterrence through convexification. Renault et al. (2016) and Ely (2017) provide solutions of related problems. We are hopeful that the methods they developed can be used to solve the dynamic version of the problem presented here as well.

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