# Efficient Sequential Screening

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**Abstract**

A seller of an item faces a potential buyer whose valuation depends on multiple private signals. When there are informational externalities and the buyer's private signals arrive all at once efficient implementation is impossible. We show that if the buyer's private signals arrive over time in a particular order then the seller can implement efficiency even in the presence of informational externalities. (Keywords: Efficient mechanisms; Sequential screening; Interdependent valuations; Multidimensional information; Informational externalities)

# 1  Introduction

When a government is considering selling an item to a potential buyer, whether it is a permit, a license, or a physical asset, it wants to implement an efficient allocation. That is, it wants to sell the item to the buyer if and only if the social welfare in the case where the buyer receives the item is greater than the social welfare in the case where the government keeps the item. Many times the buyer receives private information that affects social welfare and hence the efficient allocation. It is often the case where the buyer's information is multidimensional and where there are informational externalities, i.e., situations where the buyer receives private information about multiple parameters

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that affect social welfare not only by affecting the buyer's valuation but also by affecting other aspects of social welfare. The externalities may be pecuniary. For example, consider a firm asking for a production right license in an oligopolistic market that has private information about its marginal cost and about its fixed cost. The firm's information about its costs affects the equilibrium price which in turn affects not only the firm's profits but also the profits of other firms and the consumer surplus. The externalities may be real. For example, an energy factory that applies for a drilling permit and holds private information about the amount of waste and air pollution it will produce. This information affects not only the firm's profit but also the quality of the environment. Externalities also arise in the presence of interdependent valuations, i.e., in environments where the buyer's private information also affects the payoff of the government if it decides to keep the item. Unfortunately, Maskin (1992), Dasgupta and Maskin (2000), and Jehiel and Moldovanu (2001) have shown that in environments with multidimensional information and informational externalities it is (generically) impossible to implement an efficient sale.[1]

In this paper we show that this result corresponds to the case where the buyer knows all of her private information before the selling mechanism can be activated. We show that if the buyer's private information arrives over time and if the selling mechanism can be activated before the buyer is exposed to all of her private information, then efficiency can be implemented in natural environments with informational externalities.

Our model considers a seller of an item facing a potential buyer. The buyer receives two payoff-relevant signals in a sequential manner. We first show that in static environments (i.e., environments where the buyer knows all of her signals) a decision rule is implementable if and only if it is monotonic with respect to the buyer's valuation. This means that a necessary condition for implementation is that the buyer's valuation does not change along the boundary of the decision rule (i.e., the boundary between the set of signals that maps "do not sell" and the set of signals that maps "do sell."). We then show that a decision rule is implementable in a sequential environment if and only if it is monotonic with respect to each of the buyer's signals and, in addition, the buyer's valuation moves monotonically along the boundary of the decision rule.

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[1]Jehiel and Moldovanu (2001) show a more general impossibilty result that includes the setting of the allocation of an indivisible good as a special case.

We use these results to compare the possibility of efficient implementation between static and sequential environments. For this purpose we examine the effect of the buyer's information on the social welfare. In situations where the effect of the buyer's information on the social welfare is limited to its effect on the buyer's value, efficiency is implementable in both static and sequential environments. This is because the boundary of the efficient decision rule coincides with one of the buyer's isovalue curves. By contrast, in situations where the buyer's information has other externalities on the social welfare, efficient implementation is typically impossible in static environments. This is because the boundary of the efficient decision rule does not coincide with any of the buyer's isovalue curves. Nonetheless, efficiency can be implemented in sequential environments. This happens in cases where the ratio between the effects of the first and the second signals is greater with respect to the social welfare than with respect to the buyer's valuation. In such cases the buyer's valuation is monotonic along the boundary of the efficient decision rule and efficiency is implementable.

When the buyer's valuation moves monotonically along the boundary of the efficient decision rule, efficient sequential implementation is possible if and only if the buyer's signals arrive in a particular order. When the buyer's signals do not arrive in the right order we show that although the sequential arrival of the buyer's information relaxes the incentive compatibility constraints it does not improve efficiency, i.e., we show that the second-best decision rule of the sequential environment provides the same expected social welfare as the second-best decision rule of the static environment. When the buyer's valuation is not monotonic along the boundary of the efficient decision rule, we present sufficient conditions for the second-best decision rule of the sequential environment to provide a higher expected social welfare than the second-best decision rule of the static environment.

This paper focuses on environments with a single buyer. Nonetheless, it is straightforward to extend the analysis to the case of multiple buyers. We discuss how the analysis in the paper can be carried through to the case of multiple buyers but we do not provide a formal analysis for the sake of not overextending the paper. Most of the proofs are relegated to appendix B.

## Related Literature

This work connects the literature on sequential screening see, e.g., Courty and Li (2000), Esö and Szentes (2007a, b), and Krähmer and Strausz (2011, 2015a, b, 2017),[2] to the literature on the impossibility of efficient implementation in environments with multi-dimensional information and informational externalities, see, Maskin (1992), Dasgupta and Maskin (2000), and Jehiel and Moldovanu (2001). The current sequential screening literature focuses on profit maximization. The standard model is introduced in Courty and Li (2000) who derive a revenue equivalence result and provide regularity conditions that guarantee the implementability of the optimal decision rule. The maximal revenue that the seller can achieve depends on how the information of the buyer in the first period is distributed and on how it affects the distribution that the buyer assigns to her final valuation. The implementation of efficiency, on the other hand, is determined by the relationship between the variation of the buyer's valuation and the variation of the social welfare with respect to both of the buyer's signals. That is, it depends on the payoff effect of both of the buyer's signals and on the order in which they arrive.

Our paper also relates to other works that present positive results on efficient implementation in environments with multidimensional information and informational externalities. Mezzeti (2004) shows that in settings where it is possible to condition transfers on realizations of payoffs, efficiency can be implemented in static environments. Our results show that in sequential environments it is possible to implement efficiency even in settings that require both the allocation and transfers to depend only on agents' signals. Johnson, Miller, and Zeckhauser (2003) show that in static environments where agents' signals are correlated such that different values of an agent's signal imply different distributions of the other agents' signals, efficient Bayesian implementation is (approximately) possible. Our results show that in sequential environments it is possible to implement efficiency even when the buyer's information is independent of

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[2]More works on sequential screening include Baron and Besanko (1984), Dai, Lewis, and Lopomo. (2006), Hoffmann and Inderst (2011), Nocke, Peitz, and Rosar (2011), Bergemann and Wambach (2015), Deb and Said (2015), Li and Shi (2017). More general models of dynamic mechanism design include Battaglini (2005), and Pavan, Segal, and Toikka (2014). For works on efficient dynamic mechanism design in environments with private values see Bergemann and Valimaki (2010) and Athey and Segal (2013).

any other source of information that may be available for the seller. The methods to implement efficiency in static settings that appear in the aforementioned papers have been extended to dynamic settings. Liu (2017) uses correlation among types and He and Li (2016) use transfers that are contingent on payoff realizations, to induce efficiency in dynamic settings.

# 2 The model

Consider a seller (he) of a single item facing a potential buyer (she). There are two periods, 1 and 2. The buyer receives a private signal $\theta^1 \in [0, 1]$ in period 1, and a private signal $\theta^2 \in [0, 1]$ in period 2. These signals are distributed by a probability distribution $F(\theta^1, \theta^2)$, and $F$ is commonly known. We assume that $\{F(\theta^2|\theta^1)\}_{\theta^1 \in [0,1]}$ are ordered by first order stochastic dominance, i.e., $F(\theta^2|\theta^1)$ is strictly decreasing in $\theta^1$. The buyer's valuation $V$ is a function of her signals $V : [0, 1]^2 \to \mathbb{R}_+$. We assume that $V$ is continuously differentiable and strictly increasing in $\theta^1$ and $\theta^2$. We also consider the case where $V$ is independent of $\theta^1$, and the case where $F(\theta^2|\theta^1)$ is independent of[3] $\theta^1$. The buyer's payoff is minus her payment to the seller, plus, in case she gets the item, her value of the item. We denote by $A$ the set of feasible allocations $A = \{0, 1\}$, where 1 is the allocation that assigns the item to the buyer, and 0 is the allocation that assigns the item to the seller. A *decision rule* is a function[4] $q : \Theta \to A$. A *social choice function, s*, assigns an allocation and a payment to the seller for every realization of signals, i.e., $s(\theta) = (q(\theta), t(\theta))$, where $q(\theta) \in A$ and $t(\theta) \in \mathbb{R}$.

# 3 Implementation

In this section we characterize and compare the sets of implementable decision rules in static and sequential environments. We start with an analysis of static mechanisms. Static mechanisms are mechanisms that are activated after the buyer has been exposed

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[3]The first case is the standard setting of sequential screening, see, e.g., Courty and Li (2000) and Krähmer and Strausz (2011, 2017). The second case corresponds with Esö and Szentes (2007a).

[4]We consider deterministic mechanisms because we focus on efficiency, and efficient decision rules are deterministic. In Appendix A we discuss the implications of our focus on deterministic mechanisms.

to both her signals $\theta^1$ and $\theta^2$. By the revelation principle (e.g., Myerson 1981) we restrict our attention to direct mechanisms. We say that a social choice function $(q(\theta), t(\theta))$ is *implementable by a static mechanism* if for every $(\theta^1, \theta^2)$ we have

$$\left(\theta^1, \theta^2\right) \in \underset{(\hat{\theta}^1, \hat{\theta}^2) \in [0,1]^2}{\arg\max} \; V\left(\theta^1, \theta^2\right) \cdot q\left(\hat{\theta}^1, \hat{\theta}^2\right) - t\left(\hat{\theta}^1, \hat{\theta}^2\right)$$

We say that a decision rule $q(\theta)$ is *implementable by a static mechanism* if there exists a transfer function $t(\theta)$ such that $(q(\theta), t(\theta))$ is implementable by a static mechanism.

*Claim* 1. A decision rule $q(\theta)$ is implementable by a static mechanism if and only if it is of the following form:

$$q(\theta) = \begin{cases} 1 \text{ if} & V\left(\theta^1, \theta^2\right) > C \\ 0 \text{ or } 1 \text{ if} & V\left(\theta^1, \theta^2\right) = C \\ 0 & \text{otherwise} \end{cases}$$

for some $C \in \mathbb{R}$.

We proceed to analyze sequential mechanisms. A sequential mechanism maps a pair of the buyer's actions, one in each period, to an allocation and a transfer. By the revelation principle for sequential games (e.g., Myerson 1986) we restrict our attention to direct mechanisms. We say that a social choice function $(q(\theta), t(\theta))$ is *implementable by a sequential mechanism* if the following conditions hold:

(i)
$$E_{\theta^2}\left[V\left(\theta^1, \theta^2\right) \cdot q\left(\theta^1, \theta^2\right) - t\left(\theta^1, \theta^2\right) | \theta^1\right] \geq$$
$$E_{\theta^2}\left[V\left(\theta^1, \theta^2\right) \cdot q\left(\hat{\theta}^1, \hat{\theta}^2\left(\theta^2\right)\right) - t\left(\hat{\theta}^1, \hat{\theta}^2\left(\theta^2\right)\right) | \theta^1\right]$$

for every $\theta^1 \in [0, 1]$ and $\hat{\theta}^1 \in [0, 1]$ and every $\hat{\theta}^2 : [0, 1] \to [0, 1]$.

(ii)
$$V\left(\theta^1, \theta^2\right) \cdot q\left(\theta^1, \theta^2\right) - t\left(\theta^1, \theta^2\right) \geq$$
$$V\left(\theta^1, \theta^2\right) \cdot q\left(\theta^1, \hat{\theta}^2\right) - t\left(\theta^1, \hat{\theta}^2\right)$$

for every $(\theta^1, \theta^2) \in [0, 1]^2$ and $\hat{\theta}^2 \in [0, 1]$.

We say that a decision rule $q(\theta)$ is *implementable by a sequential mechanism* if there exists a transfer function $t(\theta)$ such that $(q(\theta), t(\theta))$ is implementable by a sequential mechanism.

Consider the set $\mathsf{C} := \{C : [0,1] \to [0,1] \text{ s.t. } C \text{ is decreasing}\}$. For each $C \in \mathsf{C}$ we denote $\underline{\theta}^{1,C} := \inf\{\theta^1 \text{ s.t. } C(\theta^1) < 1\}$ and $\overline{\theta}^{1,C} := \sup\{\theta^1 \text{ s.t. } C(\theta^1) > 0\}$.

**Theorem 2.** *A decision rule $q(\theta)$ is implementable by a sequential mechanism if and only if there exists a function $C \in \mathsf{C}$ such that*

$$q(\theta) = \begin{cases} 1 & \text{if} & \theta^2 > C(\theta^1) \\ 0 \text{ or } 1 & \text{if} & \theta^2 = C(\theta^1) \\ 0 & & \text{otherwise} \end{cases}$$

*and in addition $V(\theta^1, C(\theta^1))$ is a decreasing function of $\theta^1$ in the segment*[5] $\left[\underline{\theta}^{1,C}, \overline{\theta}^{1,C}\right]$.

[6][7]The argument of the proof is as follows. Assume the buyer reports her first type $\theta^1$ truthfully. Then in the second period we are facing an implementation problem with respect to a unidimensional signal $\theta^2$, where the buyer's valuation is $V(\theta^1, \theta^2)$. Since $V$ is strictly monotone in $\theta^2$, implementability holds if and only if the decision rule is monotonic with respect to $\theta^2$. The threshold is set at $C(\theta^1)$ and the payment to the seller in case of a sale is

$$\tau(\theta^1) := \begin{cases} V(\theta^1, C(\theta^1)) & \text{if } \underline{\theta}^{1,C} \le \theta^1 \le \overline{\theta}^{1,C} \\ V\left(\overline{\theta}^{1,C}, 0\right) & \text{if } \overline{\theta}^{1,C} < \theta^1 \le 1 \end{cases}$$

This implies that each report of $\theta^1$ in the first period sets a price for the item in the

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[5]For every $\theta^1 < \underline{\theta}^{1,C}$ we have $C(\theta^1) = 1$ and $q(\theta^1, 1) = 0$. For every $\overline{\theta}^{1,C} < \theta^1$ we have $C(\theta^1) = 0$ and $q(\theta^1, 0) = 1$. That is, for these $\theta^1$ the decision rule $q(\theta^1, \cdot)$ is a constant function.

[6]The standard sequential screening model à la Courty and Li (2000) assumes that $V(\theta^1, \theta^2) = \theta^2$ and $F(\theta^2|\theta^1)$ is strictly decreasing in $\theta^1$. The characterization of the set of deterministic decision rules in this setting is established in Krähmer and Strausz (2011). In this case we get that any decreasing threshold function $C(\theta^1) \in \mathsf{C}$ is implementable by a sequential mechanism.

[7]The characterization of Theorem 2 holds in any environment where $V$ is increasing in $\theta^1$ and $\theta^2$, and where the effects of $\theta^1$ on $V$ and on $F(\theta^2|\theta^1)$ constitute a surrounding where higher types in the first period are facing higher probabilities of receiving higher valuations.

second period. In addition, the buyer is charged a fee $p(\theta^1)$ for participating in the mechanism that sets the price $\tau(\theta^1)$. Thus, the transfer function is set as

$$
t(\theta) = \begin{cases} p(\theta^1) + \tau(\theta^1) & \text{if } q(\theta) = 1 \\ p(\theta^1) & \text{if } q(\theta) = 0 \end{cases}
$$

We get that an implementable sequential mechanism provides the buyer in the first period with a menu of options, each sets a strike price, $\tau(\theta^1)$, for the item in the second period. All types of the buyer agree on the ordinal order of these strike prices: the lower the strike price, the better. However, they differ in the intensity of their preferences, such that higher $\theta^1$ types are more willing to pay for lower strike prices. In such a single crossing environment a necessary and sufficient condition for implementation is that higher types are assigned with lower strike prices, i.e., the property that $\tau(\theta^1)$ is decreasing is necessary and sufficient for implementation.

We now present the necessary and sufficient conditions for implementation in both static and sequential environments in terms of the variation of the buyer's valuation along the boundary of the decision rule. Without loss of generality we restrict attention to the set decision rules that are monotonic with respect to each of the buyer's signals, and denote this set by $\mathscr{D}$. Each decision rule in $\mathscr{D}$ is identified with a function $C \in \mathsf{C}$ that maps each type in $\left[\underline{\theta}^{1,C}, \overline{\theta}^{1,C}\right]$ to a threshold type in the second period. We denote the boundary of a decision rule $q(\theta) \in \mathscr{D}$ by $\overline{q}(C)$, i.e., $\overline{q}(C) \coloneqq (\theta^1, C(\theta^1))_{\theta^1 \in \left[\underline{\theta}^{1,C}, \overline{\theta}^{1,C}\right]}$ where $C \in \mathsf{C}$. Using the above characterizations we reach the following conclusion:

**Corollary 3.** *A decision rule $q(\theta) \in \mathscr{D}$ with a boundary $\overline{q}(C)$ is implementable by a static mechanism if and only if for any $\tilde{\theta}^1 < \dot{\theta}^1 \in \left[\underline{\theta}^{1,C}, \overline{\theta}^{1,C}\right]$ we have*

$$
V\left(\tilde{\theta}^1, C\left(\tilde{\theta}^1\right)\right) = V\left(\dot{\theta}^1, C\left(\dot{\theta}^1\right)\right)
$$

*and it is implementable by a sequential mechanism if and only if for any $\tilde{\theta}^1 < \dot{\theta}^1 \in \left[\underline{\theta}^{1,C}, \overline{\theta}^{1,C}\right]$ we have*

$$
V\left(\tilde{\theta}^1, C\left(\tilde{\theta}^1\right)\right) \geq V\left(\dot{\theta}^1, C\left(\dot{\theta}^1\right)\right)
$$

In words, the decision rule is implementable by a static mechanism if and only if its boundary coincides with one of the buyer's isovalue curves and is implementable by a sequential mechanism if and only if the buyer's valuation weakly decreases as we move rightward along its boundary.
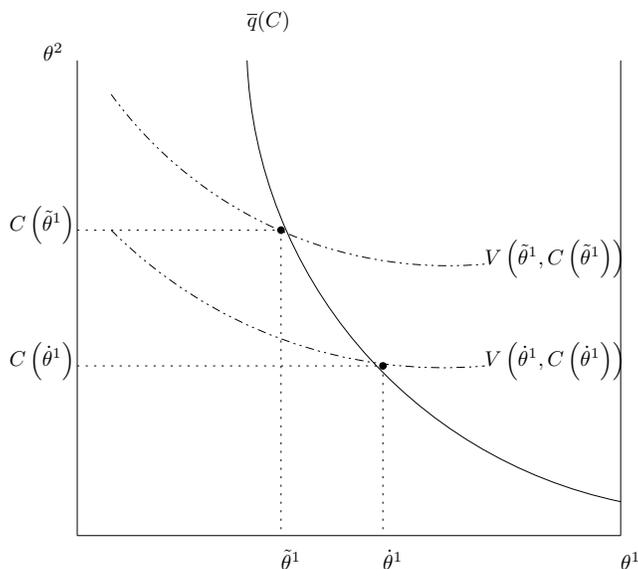


Figure 1: The buyer's valuation decreases along the boundary of the decision rule.

*Remark.* In an implementable sequential mechanism higher types in the first period are facing higher expected utilities. Therefore, individual rationality is satisfied whenever the lowest type in the first period is willing to participate in the mechanism. For example, when the price for the option that offers the highest strike price in the second period is zero.

## 4  Efficiency

This paper considers a seller whose objective is to implement efficiency, namely, who wants to implement the allocation that would produce the greatest social welfare. In situations where the buyer's information affects the social welfare only by its effect on the buyer's valuation, efficiency is implementable in static environments. When there

are informational externalities on the social welfare the possibility of implementing efficiency in static settings depends on the dimensionality of the information. When the buyer's information is unidimensional efficiency is implementable if a single-crossing condition is satisfied.[8] If, however, the buyer's information is multidimensional then it is typically impossible to implement efficiency.[9] In this section we show that in sequential environments efficiency can be implemented even in the latter case.

We consider efficient decision rules that take the following form. There exists a function $U : [0, 1]^2 \to \mathbb{R}$ such that

$$
q^e(\theta) = \begin{cases} 1 \text{ if} & U(\theta^1, \theta^2) > C \\ 0 \text{ or } 1 \text{ if} & U(\theta^1, \theta^2) = C \\ 0 & \text{otherwise} \end{cases}
$$

where $C \in \mathbb{R}$ and $U(\theta^1, \theta^2)$ is continuously differentiable and strictly increasing in $\theta^1$ and $\theta^2$. We define the set $\bar{U}$ to be the boundary of the efficient decision rule, i.e., $\bar{U} := \{(\theta^1, \theta^2) \text{ s.t. } U(\theta^1, \theta^2) = C\}$. We denote by $[\underline{u}, \overline{u}]$ the segment of all $\theta^1$ such that there exists $\theta^2$ where $(\theta^1, \theta^2) \in \bar{U}$. We define $\tilde{\theta}^2(\theta^1)$ to be the function that assigned to any $\theta^1 \in [\underline{u}, \overline{u}]$ the threshold type it inflicts with respect to $\theta^2$, i.e., $\tilde{\theta}^2(\theta^1) := \theta^2$ s.t. $(\theta^1, \theta^2) \in \bar{U}$. Using the results of Section 3 we deduce the main result of the paper.

**Proposition 4.** *The efficient decision rule $q^e(\theta)$ is implementable by a static mechanism if and only if*

$$
\frac{\partial V/\partial \theta^1}{\partial V/\partial \theta^2}(\theta^1, \theta^2) = \frac{\partial U/\partial \theta^1}{\partial U/\partial \theta^2}(\theta^1, \theta^2)
$$

*for every $(\theta^1, \theta^2) \in \bar{U}$, and is implementable by a sequential mechanism if and only if*

$$
\frac{\partial V/\partial \theta^1}{\partial V/\partial \theta^2}(\theta^1, \theta^2) \leq \frac{\partial U/\partial \theta^1}{\partial U/\partial \theta^2}(\theta^1, \theta^2)
$$

*for every $(\theta^1, \theta^2) \in \bar{U}$.*

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[8]See, for example, Cremer and McLean (1985), Maskin (1992), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Perry and Reny (2002).
[9]See, Maskin (1992), Dasgupta and Maskin (2000), and Jehiel and Moldovanu (2001).

Consider the case where $\frac{\partial V/\partial\theta^1}{\partial V/\partial\theta^2}(\theta^1,\theta^2) \geq \frac{\partial U/\partial\theta^1}{\partial U/\partial\theta^2}(\theta^1,\theta^2)$, this inequality is equivalent to $\frac{\partial V/\partial\theta^2}{\partial V/\partial\theta^1}(\theta^1,\theta^2) \leq \frac{\partial U/\partial\theta^2}{\partial U/\partial\theta^1}(\theta^1,\theta^2)$ which implies:

**Corollary 5.** *If the seller controls the order of the buyer's signals, then the monotonicity of the buyer's valuation along the boundary of the efficient decision rule (increasing or decreasing) is a necessary and sufficient condition for implementing efficiency.*

## 4.1  Applications

In this subsection we discuss applications where efficiency is not implementable in a static environment but is implementable in a sequential environment. When we refer to a sequential environment we assume that the order in which the buyer's signals arrive is commonly known and that the seller can activate the selling mechanism before the buyer has learned all of her information. These assumptions are naturally satisfied in environments where the seller can control the timing, and sometimes even the order, in which the buyer is exposed to her private information. For example, environments where the buyer's information arrives from tests conducted on the item. The seller can decide when the buyer would carry out these tests. Or environments where the seller holds some information that only the buyer can infer its meaning. The seller can decide when to release this information to the buyer.

### 4.1.1  Multidimensional Valuation and a Uni-dimensional Externality

An important set of economic environments where efficiency cannot be implemented by static mechanisms but can be implemented by sequential mechanisms is where the buyer's valuation is multidimensional but the social externality is uni-dimensional, i.e., environments where the buyer receives private information that affects her valuation, but only part of this information affects the social welfare. Among the possible examples are: A private firm asking for a license to join an oligopolistic market and has private information about its marginal cost and about its fixed cost. Only the marginal cost affects the equilibrium price which in turn affects the profits of other firms and the consumer surplus. An oil company applying for a drilling license that receives private

information about its drilling cost and about the potential oil spills. The potential oil spills incur potential damages to the quality of the environment.

We denote the part of the buyer's information that affects both the buyer's valuation and the social welfare by $\theta^e$ and the part of the buyer's information that affects only the buyer's valuation by $\theta^b$. The social welfare in the case where the buyer gets the item is equal to the sum of the buyer's valuation and the externality, $\Phi(\theta^e)$, i.e., $U(\theta^e, \theta^b) = V(\theta^e, \theta^b) + \Phi(\theta^e)$. We assume that the social welfare in the case where the seller keeps the item is known and we denote it by $C$. The efficient decision rule is

$$
q^e(\theta) = \begin{cases} 1 & \text{if} & U(\theta^e, \theta^b) > C \\ 0 \text{ or } 1 & \text{if} & U(\theta^e, \theta^b) = C \\ 0 & & \text{otherwise} \end{cases}
$$

now

$$
\frac{\partial U}{\partial \theta^e} = \frac{\partial V}{\partial \theta^e} + \frac{\partial \Phi}{\partial \theta^e} \quad \text{and} \quad \frac{\partial U}{\partial \theta^b} = \frac{\partial V}{\partial \theta^b}
$$

when $\frac{\partial \Phi}{\partial \theta^e} > 0$ we have that

$$
\frac{\partial V/\partial \theta^e}{\partial V/\partial \theta^b}\left(\theta^1, \theta^2\right) < \frac{\partial U/\partial \theta^e}{\partial U/\partial \theta^b}\left(\theta^1, \theta^2\right)
$$

when $\frac{\partial \Phi}{\partial \theta^e} < 0$ we have that

$$
\frac{\partial V/\partial \theta^b}{\partial V/\partial \theta^e}\left(\theta^1, \theta^2\right) < \frac{\partial U/\partial \theta^b}{\partial U/\partial \theta^e}\left(\theta^1, \theta^2\right)
$$

By Proposition 4 efficiency can be implemented by a sequential mechanism as long as signals arrive in the right order, and cannot be implemented by a static mechanism.

### 4.1.2 Interdependent Valuations

Perhaps the most common and important case of informational externalities is the case of interdependent valuations, i.e., where the information of one buyer also affects the

valuation of other potentials buyers.[10] To analyze the case of interdependent valuations we consider a seller who can allocate an item to one of two potential buyers, A and B, and wants to allocate it to the buyer who values it the most. Buyer A has private information, while buyer B does not. The signals of buyer A, $(\theta^1, \theta^2)$, affect the valuations of both buyers. We assume that the buyers' valuations functions are drawn from a set of valuation functions that are ordered in the following two ways. The first is that for every two valuation functions, for every realization of signals, the partial derivatives of one function are bigger than the partial derivatives of the other. The second is that for every two valuation functions, for every realization of signals, one has a weakly smaller MRS than the other. We assume that buyer A is assigned with a valuation that has the higher partial derivatives.[11] An implication of the results in Jehiel and Moldovanu (2001) is that it is generically impossible to implement efficiency in such an environment. Our results imply that in a sequential environment it is possible to implement efficiency on a set of valuations of a positive measure. Moreover, if the seller can control the order in which signals arrive then he can always implement efficiency.

To illustrate this point we consider the set of linear valuations. Assume that the valuation function of buyer A is $V_A(\theta^1, \theta^2) = \alpha^1 \theta^1 + \alpha^2 \theta^2 + a$ and the valuation function of buyer B is $V_B(\theta^1, \theta^2) = \beta^1 \theta^1 + \beta^2 \theta^2 + b$, where $b > a$ and $\alpha^k > \beta^k > 0$ for $k \in \{1, 2\}$. We define $U(\theta^1, \theta^2) := (\alpha^1 - \beta^1)\theta^1 + (\alpha^2 - \beta^2)\theta^2$. The efficient decision rule is

$$
q^e(\theta) = \begin{cases} 1 \text{ if} & U(\theta^1, \theta^2) > b - a \\ 0 \text{ or } 1 \text{ if} & U(\theta^1, \theta^2) = b - a \\ 0 & \text{otherwise} \end{cases}
$$

By Proposition 4 efficiency is implementable by a static mechanism if and only if $\frac{\beta^1}{\beta^2} = \frac{\alpha^1}{\alpha^2}$. This condition is met only for a set of parameters of measure zero in $\mathbb{R}^4_+$, and so efficiency is generically impossible. Efficiency is implementable by a sequential

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[10]There is an extensive literature on mechanism design with interdependent valuations. See, for example, Cremer and McLean (1985), Dasgupta and Maskin (2000), Perry and reny (2000), Jehiel and Moldovanu (2001), Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006), Bikhchandani (2006), Siegel (2014), and Mclean and Postelwaite (2015).

[11]This is a generalization of the single crossing property in the case of a uni-dimensional signal, see Dasgupta and Maskin (2000).

mechanism if and only if $\frac{\beta^1}{\beta^2} \leq \frac{\alpha^1}{\alpha^2}$. This condition is met for a set of parameters of a positive measure in[12] $\mathbb{R}^4_+$. In addition, Corollary 5 implies that if the seller can control the timing in which buyer A's signals arrive, then he can always implement efficiency.

## 4.2 Second-best Analysis

In this subsection we consider the case where the necessary and sufficient condition for implementing efficiency by a sequential mechanism does not hold and analyze whether the use of sequential mechanisms can still enhance the social welfare.[13] We first consider the case where the buyer's valuation is increasing as we move rightward along the boundary of the efficient decision rule. We show that in this case the social welfare cannot increase from applying sequential mechanisms. This means that when the buyer's valuation is monotonic along the boundary of the efficient decision rule and signals do not arrive in the right order, then not only that full efficiency cannot be implemented but the sequential arrival of the buyer's signals does not even improve the second-best outcome.

**Theorem 6.** *Assume that* $\frac{\partial V/\partial \theta^1}{\partial V/\partial \theta^2}(\theta^1, \theta^2) > \frac{\partial U/\partial \theta^1}{\partial U/\partial \theta^2}(\theta^1, \theta^2)$ *for every* $(\theta^1, \theta^2) \in \bar{U}$. *Then there exists a second-best decision rule with the property that* $\tau(\theta^1) = \tau$, *where* $\tau \in \mathbb{R}$; *namely, it sets a single price in the second period.*

*Proof.* We show in Appendix B that there exists a second-best sequential mechanism whose boundary intersects with the boundary of the efficient decision rule, i.e., there exists a type $\tilde{\theta}^1 \in [\underline{u}, \overline{u}]$ for which $\tau\left(\tilde{\theta}^1\right) = V\left(\tilde{\theta}^1, \tilde{\theta}^2\left(\tilde{\theta}^1\right)\right)$. Let's consider this mechanism. For every $\theta^1 < \tilde{\theta}^1$, sequential implementability implies that $\tau(\theta^1) \geq \tau\left(\tilde{\theta}^1\right)$. For any such $\theta^1$, if $\tau(\theta^1) > \tau\left(\tilde{\theta}^1\right)$, then $\{\theta_2 \text{ s.t. } V(\theta^1, \theta^2) \geq \tau(\theta^1)\}$, the set of signals for which a sale is executed, is strictly contained in $\left\{\theta_2 \text{ s.t. } V(\theta^1, \theta^2) \geq \tau\left(\tilde{\theta}^1\right)\right\}$, the set of signals for which a sale would have been executed if the price had been $\tau\left(\tilde{\theta}^1\right)$. Since the MRS of $V$ is steeper than the MRS of $U$, these two sets are contained in

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[12]Same conditions for implementing efficiency also hold in the case of Cobb Douglas valuations, i.e., where $V_A\left(\theta^1, \theta^2\right) = \left(\theta^1\right)^{\alpha^1}\left(\theta^2\right)^{\alpha^2} + a$ and $V_B\left(\theta^1, \theta^2\right) = \left(\theta^1\right)^{\beta^1}\left(\theta^2\right)^{\beta^2} + b$.

[13]We still restrict our analysis to deterministic mechanisms.

$\{\theta_2$ s.t. $U(\theta^1, \theta^2) \geq C\}$, the set of signals for which a sale should be executed according to the efficient decision rule. Therefore, if $\tau(\theta^1) > \tau(\tilde{\theta}^1)$, then the set of signals where a sale does not happen but should happen increases with respect to the case where $\tau(\theta^1) = \tau(\tilde{\theta}^1)$ and the expected social welfare decreases. For every $\theta^1 > \tilde{\theta}^1$ sequential implementability implies that $\tau(\theta^1) \leq \tau(\tilde{\theta}^1)$. For any such $\theta_1$, if $\tau(\theta^1) < \tau(\tilde{\theta}^1)$ then $\{\theta^2$ s.t. $V(\theta^1, \theta^2) \geq \tau(\theta^1)\}$ strictly contains the set $\left\{\theta^2 \text{ s.t. } V(\theta^1, \theta^2) \geq \tau(\tilde{\theta}^1)\right\}$ and since the MRS of $V$ is steeper than the MRS of $U$ these two sets contain the set $\{\theta^2$ s.t. $U(\theta^1, \theta^2) \geq C\}$. Therefore, if $\tau(\theta^1) < \tau(\tilde{\theta}^1)$ then the set of signals where a sale does happen but should not happen increases with respect to the case where $\tau(\theta^1) = \tau(\tilde{\theta}^1)$ and the expected social welfare decreases. We conclude that there exists a second-best mechanism that sets a single price in the second period. Such a mechanism is also implementable in a static environment; hence, the social welfare cannot increase from applying sequential mechanisms. □
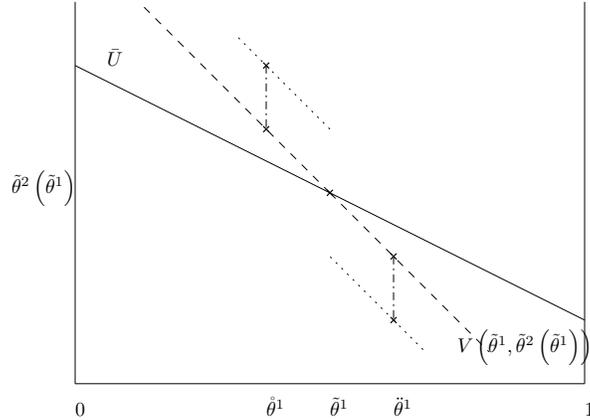


Figure 2: Second-best analysis

The above argument is illustrated in Figure 2. The area above (below) the solid line is where a sale is efficient (inefficient). If $\tau(\theta^1) = \tau(\tilde{\theta}^1)$ for every $\theta^1$, then a sale occurs for every signal in the area above the dashed line. If for types $\mathring{\theta}^1 < \tilde{\theta}^1$ we have $\tau(\mathring{\theta}^1) > \tau(\tilde{\theta}^1)$, then a sale occurs for every signal in the area above the dotted line, and the intersection of the area where the sale is efficient and the area where the

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sale is carried out decreases. If for types $\tilde{\theta}^1 < \ddot{\theta}^1$ we have $\tau\left(\tilde{\theta}^1\right) > \tau\left(\ddot{\theta}^1\right)$, then the intersection of the area where the sale is inefficient and the area where the sale is carried out increases.

We proceed to the case where the buyer's valuation is not monotonic along the boundary of the efficient decision rule. We present sufficient conditions for the second-best solution to provide a higher expected social welfare in a sequential environment than in a static environment. The improvement upon the static second-best mechanism is achieved through the construction of a decision rule whose boundary differs from the boundary of the static second-best decision rule in a way that provides a welfare-improving allocation while maintaining sequential implementability. Consider the second-best decision rule of the static environment. We denote it by $q^{SB}(\theta)$. This decision rule takes the form of

$$q^{SB}(\theta) = \begin{cases} 1 & \text{if} & V(\theta^1, \theta^2) > C^{SB} \\ 0 \text{ or } 1 & \text{if} & V(\theta^1, \theta^2) = C^{SB} \\ 0 & & \text{otherwise} \end{cases}$$

We denote by $\bar{V}^{SB}$ the boundary of the second-best static decision rule $q^{SB}(\theta)$, i.e.,

$$\bar{V}^{SB} := \left\{ (\theta^1, \theta^2) \text{ s.t. } V(\theta^1, \theta^2) = C^{SB} \right\}$$

We note that the boundary of the second-best static decision rule $\bar{V}^{SB}$ and the boundary of the efficient decision rule $\bar{U}$ intersect.[14] We denote by $\dot{\theta}^1$ the rightmost point at which these boundaries intersect, i.e.,

$$\dot{\theta}^1 := \max\left\{ \theta^1 \text{ s.t. } \left(\theta^1, \tilde{\theta}^2(\theta^1)\right) \in \bar{V}^{SB} \cap \bar{U} \right\}$$

We denote by $\ddot{\theta}^1$ the leftmost point at which these boundaries intersect, i.e.,

$$\ddot{\theta}^1 := \min\left\{ \theta^1 \text{ s.t. } \left(\theta^1, \tilde{\theta}^2(\theta^1)\right) \in \bar{V}^{SB} \cap \bar{U} \right\}$$

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[14]In Appendix A we characterize the second-best mechanism in a static environment and show that this property holds.

We now present sufficient conditions for improving the second-best solution by a sequential mechanism.

**Theorem 7.** *Assume one of the following conditions holds: (1) for every $\theta^1 > \dot{\theta}^1$ we have that $V\left(\dot{\theta}^1, \tilde{\theta}^2\left(\dot{\theta}^1\right)\right) > V\left(\theta^1, \tilde{\theta}^2\left(\theta^1\right)\right)$ or (2) for every $\theta^1 < \ddot{\theta}^1$ we have that $V\left(\theta^1, \tilde{\theta}^2\left(\theta^1\right)\right) > V\left(\ddot{\theta}^1, \tilde{\theta}^2\left(\ddot{\theta}^1\right)\right)$. Then there exists a decision rule that is sequentially implementable and provides a higher expected welfare than $q^{SB}\left(\theta\right)$.*

The idea of the proof is as follows. Assume for example that (1) holds. This means that at any point that is to the right of $\dot{\theta}^1$, the boundary of the second-best static decision rule lies above the boundary of the efficient decision rule. Therefore, we can construct a decision rule $\tilde{q}\left(\theta\right)$ with two properties. The first is that to the left of $\dot{\theta}^1$ the boundary of the decision rule $\tilde{q}\left(\theta\right)$ coincides with the boundary of $q^{SB}\left(\theta\right)$, while to the right of $\dot{\theta}^1$ the boundary of the decision rule $\tilde{q}\left(\theta\right)$ is below the boundary of $q^{SB}\left(\theta\right)$ and above the boundary of the efficient decision rule. This property implies that $\tilde{q}\left(\theta\right)$ provides a higher expected welfare than $q^{SB}\left(\theta\right)$. The second property is that the buyer's valuation is decreasing as we move rightward along the boundary of $\tilde{q}\left(\theta\right)$. This property implies that $\tilde{q}\left(\theta\right)$ is sequentially implementable. Such a construction is illustrated in Figure 3.
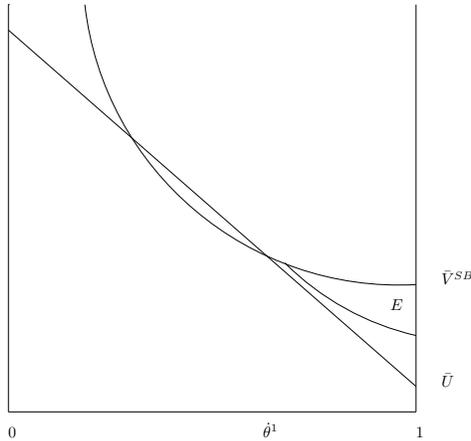


Figure 3: The set where $q^{SB}\left(\theta\right)$ and $\tilde{q}\left(\theta\right)$ do not coincide is denoted by E.

# 5   Discussion

We have considered the problem of efficient allocation of a single item in environments with two periods where the buyer receives a uni-dimensional signal in each period. We characterized necessary and sufficient conditions for implementation in terms of simple monotonicity, and used this characterization to deduce tractable necessary and sufficient conditions for implementing efficiency. A straightforward way to extend the analysis in this paper to the case of multiple buyers is to define a notion of *ex-post implementation by a sequential mechanism* that requires that the conditions of sequential implementation in the single buyer case would hold for every buyer, for every realization of signals of the other buyers. The set of ex-post implementable decision rules in the multiple buyers case is characterized as the set of decision rules for which the necessary and sufficient conditions for implementation in the single-buyer case apply for every buyer for every realization of signals of the other buyers.

A natural direction for future research is to explore how the sequential arrival of information affects the possibility of implementing efficiency in other environments with multidimensional information and informational externalities. In environments with two periods where the buyer receives multidimensional information in each period it seems that the logic of the impossibility result of Jehiel and Moldovanu (2001) holds and that typically full efficiency cannot be implemented. In environments with more than two periods and\or with more than one item, the set of types in the first period does not entail a single crossing structure (unless strong assumptions are made) and the characterization of the set of implementable decision rules becomes less tractable.

Another direction is to consider detail-free mechanisms, i.e., mechanisms that are, in the spirit of Wilson's critique (see Wilson 1987), independent of the agents' valuation functions and of the joint distribution of their private information.[15] Our work provides the theoretical insight that in the environments we have considered there exist efficient sequential mechanisms, and opens the way for the investigation of efficient detail-free mechanisms in these environments.

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[15]Dasgupta and Maskin (2000) and Perry and Reny (2002) present efficient detail-free mechanisms in environments of interdependent valuations where each agent's information is uni-dimensional.

# Appendix

## A    Generality of the Results

In the present paper we have restricted our attention to direct deterministic mechanisms. In this appendix we analyze for which results of this paper this restriction is without loss of generality. We consider two generalizations. The first is to the set of indirect deterministic mechanisms. In an indirect deterministic mechanism if agents play mixed strategies then the direct mechanism, that mimic the equilibrium strategies of the indirect mechanism, is not deterministic (see Strausz 2003). We find that all the results of the paper still hold even if we consider indirect deterministic mechanisms. This outcome is based on the observation that for every implementable indirect deterministic mechanism there exists an implementable direct deterministic mechanism that yields equal or greater social welfare.

An implementable deterministic static mechanism yields two alternatives to the buyer (buy the item or don't buy the item) and assigns to each alternative a single price. Therefore, the only set of signals where the buyer can be indifferent between the two alternatives (and play mixed strategies) is the isovalue curve where the buyer's valuation equals the difference in transfers. This set is of measure zero and does not affect the expected social welfare. Hence, a direct mechanism that arbitrarily assigns to the signals in this set a single alternative yields the same expected welfare. An implementable sequential deterministic mechanism yields two alternatives in the second period (buy the item or don't buy the item) and assigns to each alternative a single price. Therefore, there is a single signal in the second period where the buyer can be indifferent between the two alternatives. Of course, allowing for mixed strategies at this point would not change the expected social welfare. In the first period of an implementable sequential deterministic mechanism every type of the buyer may randomize between several options that this type is indifferent among. Each option is composed of a single price in the second period and a payment today. Denote by $I\left(\theta^1\right)$ the support of options that type $\theta^1$ is mixing. Implementability implies that every $a \in I\left(\theta^1\right)$ is preferred by type $\theta^1$ to every $b \in I\left(\tilde{\theta}^1\right)$ and every $b \in I\left(\tilde{\theta}^1\right)$ is preferred

by type $\tilde{\theta}^1$ to every $a \in I(\theta^1)$. Therefore, every mechanism that offers some arbitrary $a$ in $I(\theta^1)$ to type $\theta^1$ is implementable by a direct mechanism. Now, each second-period price sets an expected social welfare given $\theta^1$. Consider the option $a^*(\theta^1) \in I(\theta^1)$ that sets the second-period price that maximizes this expected social welfare given $\theta^1$ out of all the options in[16] $I(\theta^1)$. The deterministic mechanism for which $a^*(\theta^1) = I(\theta^1)$ yields equal or greater expected welfare than the original mechanism and is implementable by a direct mechanism.

The second generalization is to stochastic mechanisms.[17] A deterministic decision rule is implementable by a stochastic mechanism if and only if it is implementable by a deterministic mechanism. Since efficient decision rules are (almost everywhere) deterministic, all the result about the possibility of implementing full efficiency are without loss of generality. We now show that the second-best decision rule in a static environment is also deterministic. This implies that Theorem 7 is without loss of generality. Denote the valuation of the seller if he keeps the item by $V_s$ and the valuation of the buyer if she gets the item by $V_b$. We assume that the following condition holds: $V_b(\theta') - V_b(\theta) > V_s(\theta') - V_s(\theta)$ for every $\theta' > \theta$ where[18] $\theta', \theta \in [0,1]^2$. Consider the buyer's isovalue curves in $[0,1]^2$ and let

$$V_I(V) = \left\{ \theta \in [0,1]^2 \text{ s.t } V_b(\theta) = V \right\}$$

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[16]Such an option exists because the support of the second-period prices in $I(\theta^1)$ is a closed set. Denote by $I_h(\theta^1)$ the set of options in $I(\theta^1)$ that set a price that is greater than or equal to the price that is set by the efficient decision rule. Denote by $I_l(\theta^1)$ the set of options in $I(\theta^1)$ that set a price that is less than or equal to the price that is set by the efficient decision rule. We have that $a^*(\theta^1)$ is either the option that sets the minimum price in $I_h(\theta^1)$ or the option that sets the maximal price in $I_l(\theta^1)$.

[17]The restriction to deterministic mechanisms can be justified by practical considerations that derive from the commitment assumption. Laffont and Martimort (2002) note that: "Ensuring this verifiability is a more difficult problem than ensuring that a deterministic mechanism is enforced, because any deviation away from a given randomization can only be statistically detected once sufficiently many realizations of the contracts have been observed. [...] The enforcement of such stochastic mechanisms is thus particularly problematic."

[18]Let $x' > x$ denote that $x'$ is at least as large as $x$ in every coordinate and $x' \neq x$.

We define the following function:

$$W\left(V\right) := E_{\,\theta \in V_I\left(V\right)}\left[V_b\left(\theta\right) - V_s\left(\theta\right)\right]$$

This function is strictly increasing in $V$. We denote by $V^*$ the value for which[19] $W\left(V^*\right) = 0$. In that case we get that the second-best decision rule out of the set of all stochastic decision rules is

$$q\left(\theta\right) = \begin{cases} 1 & \text{if } \theta \in V_I\left(V\right) \text{ s.t. } V > V^* \\ r \in [0,1] & \text{if } \theta \in V_I\left(V^*\right) \\ 0 & \text{if } \theta \in V_I\left(V\right) \text{ s.t. } V < V^* \end{cases}$$

where $r$ is the probability that the item is assigned to the buyer. That is, the second best decision rule is (almost everywhere) deterministic.

# B    Proofs

### Proof of Claim 1

Implementability implies that the buyer pays one price if she wins the item, $t(1)$, and another price if she does not win the item, $t(0)$. Let $\theta$ and $\theta'$ be two pairs of signals on the same buyer's isovalue curve such that $q\left(\theta\right) = 1$ and $q\left(\theta'\right) = 0$; then implementability implies that $V\left(\theta\right) - t\left(1\right) \geq -t\left(0\right)$ and $V\left(\theta'\right) - t\left(1\right) \leq -t\left(0\right)$ and so $V\left(\theta\right) = V\left(\theta'\right) = t\left(1\right) - t\left(0\right)$. That is, there can be at most one isovalue curve for which two pairs of signals that lie on this isovalue curve are assigned with different alternatives. This means that the decision rule maps according to values of $V$. Assume that there exists a valuation $V\left(\theta\right)$ such that $q\left(\theta\right) = 1$ and a valuation $V\left(\theta'\right)$ such that $q\left(\theta'\right) = 0$ and $V\left(\theta'\right) > V\left(\theta\right)$. Implementability implies that $V\left(\theta\right) - t\left(1\right) \geq -t\left(0\right)$ and $V\left(\theta'\right) - t\left(1\right) \leq -t\left(0\right)$ and so we get $V\left(\theta\right) \geq V\left(\theta'\right)$, a contradiction. This proves necessity. We now prove sufficiency. We set $t\left(0\right) = 0$ and $t\left(1\right) = C$; then we get that for every $\theta$ such that $V\left(\theta\right) < C$ we have $V\left(\theta\right) - t\left(1\right) < 0$ and for every $\theta$ such that $V\left(\theta\right) \geq C$ we have $V\left(\theta\right) - t\left(1\right) \geq 0$. ∎

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[19]If no such value exists then the efficient decision rule is trivial and implementable.

**Proof of Theorem 2**

**Lemma.** *Condition (ii) in the definition of "implementation by a sequential mechanism" is satisfied iff for every $\theta^1$ there exists $C(\theta^1)$ such that*

$$q(\theta) = \begin{cases} 1 \ if & \theta^2 > C(\theta^1) \\ 0 \ or \ 1 \ if & \theta^2 = C(\theta^1) \\ 0 & otherwise \end{cases}$$

*and the transfers $t(q(\theta),\theta^1)+p(\theta^1)$ are set as follows: $t(1,\theta^1) = V(\theta^1,C(\theta^1))+p(\theta^1)$ and $t(0,\theta^1) = p(\theta^1)$*

*Proof.* Consider some $C(\theta^1) \in [0,1]$. If the buyer reports $\theta^2 > C(\theta^1)$ then she receives a utility of $V(\theta^1,\theta^2) - V(\theta^1,C(\theta^1)) - p(\theta^1)$ and if the buyer reports $\theta^2 < C(\theta^1)$ she receives a utility of $-p(\theta^1)$. By the monotonicity of $V$ we have that if $\theta^2 > C(\theta^1)$ then $V(\theta^1,\theta^2) - V(\theta^1,C(\theta^1)) - p(\theta^1) > -p(\theta^1)$ and if $\theta^2 < C(\theta^1)$ then $V(\theta^1,\theta^2) - V(\theta^1,C(\theta^1)) - p(\theta^1) < -p(\theta^1)$. Assume that the mechanism is incentive compatible in the second period. Then for every $\theta^2$ such that $q(\theta^1,\theta^2) = 1$ we have $V(\theta^1,\theta^2) \geq t(1,\theta^1)-t(0,\theta^1)$, and for every $\theta^2$ such that $q(\theta^1,\theta^2) = 0$ we have $V(\theta^1,\theta^2) \leq t(1,\theta^1)-t(0,\theta^1)$. Since $V$ is continuous and monotonic there exists a single number $C(\theta^1)$ that satisfies $V(\theta^1,C(\theta^1)) = t(1,\theta^1) - t(0,\theta^1)$. IC and the monotonicity of $V$ imply that if $\theta^2 > C(\theta^1)$ then $q(\theta^1,\theta^2) = 1$ and if $\theta^2 < C(\theta^1)$ then $q(\theta^1,\theta^2) = 0$. $\square$

We now proceed to prove that given that the condition in the above Lemma is satisfied, it is necessary and sufficient for implementation that $V(\theta^1,C(\theta^1)) := \tau(\theta^1)$ is a decreasing function of $\theta^1$ in the segment $\left[\underline{\theta}^{1,C},\overline{\theta}^{1,C}\right]$. We show that given the assumptions of the model the set of types of the first period entails a single crossing structure, i.e., all types prefer lower second period prices and higher types are more willing to pay for lower prices. Consider some type $\theta^1$ facing a price $\tau$ in the second period, this type's expected valuation is

$$\int_{V^{-1}(\theta^1,\tau)}^{1} \left(V(\theta^1,s) - \tau\right) f(s|\theta^1)ds$$

this function is decreasing in $\tau$, i.e., all types prefer lower $\tau$'s. We now move on to show

that higher $\theta^1$ have higher willingness to pay for lower $\tau$'s. Consider some type $\theta^1$ and two prices $\tau' < \tau$ we define the following function

$$h_{\tau,\tau'}\left(\theta^1, \theta^2\right) := \begin{cases} 0 & \text{if } \theta^2 < V^{-1}(\theta^1, \tau') \\ (V(\theta^1, s) - \tau') & \text{if } V^{-1}(\theta^1, \tau') \leq \theta^2 \leq V^{-1}(\theta^1, \tau) \\ \tau - \tau' & \text{if } V^{-1}(\theta^1, \tau) < \theta^2 \leq 1 \end{cases}$$

We now define the function $WTP\left(\theta^1, \tau, \tau'\right)$ which is type $\theta^1$ willingness to pay from moving from price $\tau$ to price $\tau'$

$$WTP\left(\theta^1, \tau, \tau'\right) = \int_0^1 h_{\tau,\tau'}\left(\theta^1, s\right) f(s|\theta^1) ds$$

Consider two types $\mathring{\theta}^1 < \tilde{\theta}^1$ we have that $F\left(\theta^2|\tilde{\theta}^1\right)$ strictly first order stochastically dominates $F\left(\theta^2|\mathring{\theta}^1\right)$ and since $h_{\tau,\tau'}\left(\theta^1, \theta^2\right)$ is a non-constant increasing function in $\theta^2$ we get that

$$\int_0^1 h_{\tau,\tau'}\left(\mathring{\theta}^1, s\right) f(s|\tilde{\theta}^1) ds > \int_0^1 h_{\tau,\tau'}\left(\mathring{\theta}^1, s\right) f(s|\mathring{\theta}^1) ds$$

in addition $h_{\tau,\tau'}\left(\theta^1, \theta^2\right)$ is increasing in $\theta^1$ and therefore

$$\int_0^1 h_{\tau,\tau'}\left(\tilde{\theta}^1, s\right) f(s|\tilde{\theta}^1) ds \geq \int_0^1 h_{\tau,\tau'}\left(\mathring{\theta}^1, s\right) f(s|\tilde{\theta}^1) ds$$

we conclude that

$$WTP\left(\tilde{\theta}^1, \tau, \tau'\right) > WTP\left(\mathring{\theta}^1, \tau, \tau'\right)$$

In the case where $V\left(\theta^1, \theta^2\right)$ is strictly increasing in $\theta^1$ and where $F\left(\theta^2|\theta^1\right) = F\left(\theta^2\right)$ we get that $h_{\tau,\tau'}\left(\theta^1, \theta^2\right)$ is a non-constant increasing function in $\theta^1$ and so for two types $\mathring{\theta}^1 < \tilde{\theta}^1$ we get that

$$\int_0^1 h_{\tau,\tau'}\left(\tilde{\theta}^1, s\right) f(s) ds > \int_0^1 h_{\tau,\tau'}\left(\mathring{\theta}^1, s\right) f(s) ds$$

so in this case we also get

$$WTP\left(\tilde{\theta}^1, \tau, \tau'\right) > WTP\left(\mathring{\theta}^1, \tau, \tau'\right)$$

Given that the set of types of the first period entails a single crossing structure the monotonicity of $\tau\left(\theta^1\right)$ is necessary and sufficient for implementation (see, Theorem 2.1.3 in Vohra 2007)

**Proof of Theorem 6**

We show that there exists a sequential second-best mechanism in which there exists $\tilde{\theta}^1 \in [\underline{u}, \overline{u}]$ for which $\tau\left(\tilde{\theta}^1\right) = V\left(\tilde{\theta}^1, \tilde{\theta}^2\left(\tilde{\theta}^1\right)\right)$. That is, the boundary of the second-best decision rule intersects with the boundary of the efficient decision rule. Consider a mechanism in which there is no $\theta^1$ such that $\tau\left(\theta^1\right) = V\left(\theta_1, \tilde{\theta}_2\left(\theta^1\right)\right)$. I.e., we are in one of the four following cases:

(a) $\tau\left(\theta^1\right) > V\left(\theta^1, \tilde{\theta}_2\left(\theta^1\right)\right)$ for every $\theta^1 \in [\underline{u}, \overline{u}]$. Consider the price function $\tau'\left(\theta_1\right)$:

$$\tau'\left(\theta_1\right) := \begin{cases} \tau\left(\theta^1\right) & \text{if } \theta^1 < \overline{u} \\ V\left(\overline{u}, \tilde{\theta}_2\left(\overline{u}\right)\right) & \text{if } \theta^1 \geq \overline{u} \end{cases}$$

(b) $\tau\left(\theta^1\right) < V\left(\theta^1, \tilde{\theta}_2\left(\theta^1\right)\right)$ for every $\theta^1 \in [\underline{u}, \overline{u}]$. Consider the price function $\tau'\left(\theta_1\right)$:

$$\tau'\left(\theta_1\right) := \begin{cases} V\left(\underline{u}, \tilde{\theta}_2\left(\underline{u}\right)\right) & \text{if } \theta^1 \leq \underline{u} \\ \tau\left(\theta^1\right) & \text{if } \theta^1 > \underline{u} \end{cases}$$

(c) There exists $\hat{\theta}^1 \in (\underline{u}, \overline{u})$ such that $\tau\left(\hat{\theta}^1\right) > V\left(\hat{\theta}^1, \tilde{\theta}^2\left(\hat{\theta}^1\right)\right)$ and for every $\hat{\theta}^1 < \theta^1$ we have that $\tau\left(\theta^1\right) < V\left(\theta^1, \tilde{\theta}^2\left(\theta^1\right)\right) < V\left(\hat{\theta}^1, \tilde{\theta}^2\left(\hat{\theta}^1\right)\right)$. Consider the price function $\tau'\left(\theta_1\right)$:

$$\tau'\left(\theta_1\right) := \begin{cases} V\left(\hat{\theta}^1, \tilde{\theta}_2\left(\hat{\theta}^1\right)\right) & \text{if } \theta^1 = \hat{\theta}^1 \\ \tau\left(\theta^1\right) & \text{otherwise} \end{cases}$$

(d) There exists $\hat{\theta}^1 \in (\underline{u}, \bar{u})$ such that $\tau\left(\hat{\theta}^1\right) < V\left(\hat{\theta}^1, \tilde{\theta}^2\left(\hat{\theta}^1\right)\right)$ and for every $\theta^1 < \hat{\theta}^1$ we have that $V\left(\hat{\theta}^1, \tilde{\theta}^2\left(\hat{\theta}^1\right)\right) < V\left(\theta^1, \tilde{\theta}^2\left(\theta^1\right)\right) < \tau\left(\theta^1\right)$. Consider the price function $\tau'\left(\theta_1\right)$:

$$\tau'\left(\theta_1\right) := \begin{cases} V\left(\hat{\theta}^1, \tilde{\theta}^2\left(\hat{\theta}^1\right)\right) & \text{if } \theta^1 = \hat{\theta}^1 \\ \tau\left(\theta^1\right) & \text{otherwise} \end{cases}$$

In all of the four cases the mechanism that set $\tau'\left(\theta_1\right)$ is sequentially implementable. In addition, it yields an expected social welfare that is at least as high as the expected social welfare in the original mechanism. This implies that there exists a second best mechanism with the property that there is $\theta^1$ such that $\tau\left(\theta^1\right) = V\left(\theta_1, \tilde{\theta}_2\left(\theta^1\right)\right)$. The rest of the proof appears in the body of the text.

**Proof of Theorem 7**

We now show the formal proof for the case where (1) holds; the case where (2) holds is proven by a similar argument. First, we denote by $[\underline{v}, \bar{v}]$ the segment of all $\theta^1$ such that there exists $\theta^2$ where $(\theta^1, \theta^2) \in \bar{V}^{SB}$. We define $\mathring{\theta}^2\left(\theta^1\right)$ to be the function that assigns to any $\theta^1 \in [\underline{v}, \bar{v}]$ the threshold type it inflicts with respect to $\theta^2$, i.e., $\mathring{\theta}^2\left(\theta^1\right) := \theta^2$ s.t. $(\theta^1, \theta^2) \in \bar{V}^{SB}$. Assume that (1) holds and consider some $\varepsilon$ such that $\dot{\theta}^1 + \varepsilon < \bar{u}$. Let

$$V' := \max_{\theta^1 \in \left[\dot{\theta}^1 + \varepsilon, \bar{u}\right]} V\left(\theta^1, \tilde{\theta}^2\left(\theta^1\right)\right)$$

and we have that $V' < V\left(\dot{\theta}^1, \tilde{\theta}^2\left(\dot{\theta}^1\right)\right)$. We define $\hat{\theta}^2\left(\theta^1\right)$ as follows:

$$\hat{\theta}^2\left(\theta^1\right) = \begin{cases} \theta^2 \text{ s.t. } V_i\left(\theta^1, \theta^2\right) = V' & \text{if such } \theta^2 \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

Define the function $\tilde{C}\left(\theta^1\right)$ as follows:

$$
\tilde{C}\left(\theta^1\right) = \begin{cases}
1 & \text{if } 0 \leq \theta^1 < \underline{v} \\
\mathring{\theta}^2\left(\theta^1\right) & \text{if } \underline{v} \leq \theta^1 \leq \dot{\theta}^1 + \varepsilon \\
\hat{\theta}^2\left(\theta^1\right) & \text{if } \dot{\theta}^1 + \varepsilon < \theta^1 \leq \overline{u} \\
0 & \text{if } \overline{u} < \theta^1 \leq 1
\end{cases}
$$

Consider a decision rule $\tilde{q}\left(\theta\right)$ that takes the following form:

$$
\tilde{q}\left(\theta\right) = \begin{cases}
1 \text{ if} & \theta^2 \geq \tilde{C}\left(\theta^1\right) \\
0 & \text{otherwise}
\end{cases}
$$

The function $V\left(\theta^1, \tilde{C}\left(\theta^1\right)\right)$ is decreasing in the segment $\left[\underline{\theta}^{1,\tilde{C}}, \overline{\theta}^{1,\tilde{C}}\right]$ and therefore $\tilde{q}\left(\theta\right)$ is implementable by a sequential mechanism. To see that the social welfare under $\tilde{q}\left(\theta\right)$ is greater than under $q^{SB}\left(\theta\right)$, note that $q^{SB}\left(\theta\right)$ and $\tilde{q}\left(\theta\right)$ coincide except for a set of positive measure that lies above the boundary of the efficient decision rule in which $\tilde{q}\left(\theta\right)$ allocates the item to the buyer and $q^{SB}\left(\theta\right)$ allocates the item to the seller.

# References

[1] S. Athey and I. Segal. An efficient dynamic mechanism. *Econometrica*, 81(6):2463–2485, 2013.

[2] D. P. Baron and D. Besanko. Regulation and information in a continuing relationship. *Information Economics and policy*, 1(3):267–302, 1984.

[3] M. Battaglini. Long-term contracting with markovian consumers. *American Economic Review*, 95(3):637–658, 2005.

[4] D. Bergemann and J. Välimäki. The dynamic pivot mechanism. *Econometrica*, 78(2):771–789, 2010.

[5] D. Bergemann and A. Wambach. Sequential information disclosure in auctions. *Journal of Economic Theory*, 159:1074–1095, 2015.

[6] S. Bikhchandani. Ex post implementation in environments with private goods. *Theoretical Economics*, 1(3):369–393, 2006.

[7] P. Courty and L. Hao. Sequential screening. *The Review of Economic Studies*, 67(4):697–717, 2000.

[8] J. Cremer and R. P. McLean. Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdepen0 denty. *Econometrica*, 53:345–361, 1985.

[9] C. Dai, T. R. Lewis, and G. Lopomo. Delegating management to experts. *The Rand Journal of Economics*, 37(3):503–520, 2006.

[10] P. Dasgupta and E. Maskin. Efficient auctions. *The Quarterly Journal of Economics*, 115(2):341–388, 2000.

[11] R. Deb and M. Said. Dynamic screening with limited commitment. *Journal of Economic Theory*, 159:891–928, 2015.

[12] P. Eső and B. Szentes. Optimal information disclosure in auctions and the handicap auction. *The Review of Economic Studies*, 74(3):705–731, 2007.

[13] P. Eső and B. Szentes. The price of advice. *The Rand Journal of Economics*, 38(4):863–880, 2007.

[14] W. He and J. Li. Efficient dynamic mechanisms with interdependent valuations. *Games and Economic Behavior*, 97:166–173, 2016.

[15] F. Hoffmann and R. Inderst. Pre-sale information. *Journal of Economic Theory*, 146(6):2333–2355, 2011.

[16] P. Jehiel, M. Meyer-ter Vehn, B. Moldovanu, and W. R. Zame. The limits of ex post implementation. *Econometrica*, 74(3):585–610, 2006.

[17] P. Jehiel and B. Moldovanu. Efficient design with interdependent valuations. *Econometrica*, 69(5):1237–1259, 2001.

[18] S. Johnson, N. H. Miller, and R. Zeckhauser. *Efficient design with multidimensional, continuous types, and interdependent valuations*. Research Programs, John F. Kennedy School of Government, Harvard University, 2003.

[19] D. Krähmer and R. Strausz. Optimal procurement contracts with pre-project planning. *The Review of Economic Studies*, 78(3):1015–1041, 2011.

[20] D. Krähmer and R. Strausz. Ex post information rents in sequential screening. *Games and Economic Behavior*, 90:257–273, 2015.

[21] D. Krähmer and R. Strausz. Optimal sales contracts with withdrawal rights. *The Review of Economic Studies*, 82(2):762–790, 2015.

[22] D. Krähmer and R. Strausz. Sequential versus static screening: An equivalence result. *Games and Economic Behavior*, 106:317–328, 2017.

[23] J. Laffont. J. and d. martimort (2002) the theory of incentives: The principal! agent model.

[24] H. Li and X. Shi. Discriminatory information disclosure. *American Economic Review*, 107(11):3363–85, 2017.

[25] H. Liu. Efficient dynamic mechanisms in environments with interdependent valuations: the role of contingent transfers. *Theoretical Economics*, 2017.

[26] E. Maskin and H. Siebert. Auctions and privatization. *Privatization*, 1992.

[27] R. P. McLean and A. Postlewaite. Implementation with interdependent valuations. *Theoretical Economics*, 10(3):923–952, 2015.

[28] C. Mezzetti. Mechanism design with interdependent valuations: Efficiency. *Econometrica*, 72(5):1617–1626, 2004.

[29] R. B. Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981.

[30] R. B. Myerson. Multistage games with communication. *Econometrica: Journal of the Econometric Society*, pages 323–358, 1986.

[31] V. Nocke, M. Peitz, and F. Rosar. Advance-purchase discounts as a price discrimination device. *Journal of Economic Theory*, 146(1):141–162, 2011.

[32] A. Pavan, I. Segal, and J. Toikka. Dynamic mechanism design: A myersonian approach. *Econometrica*, 82(2):601–653, 2014.

[33] M. Perry and P. J. Reny. An efficient auction. *Econometrica*, 70(3):1199–1212, 2002.

[34] R. Siegel. Asymmetric all-pay auctions with interdependent valuations. *Journal of Economic Theory*, 153:684–702, 2014.

[35] R. Strausz. Deterministic mechanisms and the revelation principle. *Economics Letters*, 79(3):333–337, 2003.

[36] R. Vohra. Paths, cycles and mechanism design. *Preprint*, 2007.

[37] B. Wilson. Game-theoretic analysis of trading processes. *In: Advances in Economic Theory: Fifth World Congress. Cambridge University Press, Cambridge*, pages pp. 33–70, 1987.