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Extreme Points and Majorization: Economic Applications

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Extreme Points and Majorization: Economic Applications*

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Abstract

We characterize the set of extreme points of monotonic functions that are either majorized by a given function f or themselves majorize f and show that these extreme points play a crucial role in many economic design problems. Our main results show that each extreme point is uniquely characterized by a countable collection of intervals. Outside these intervals the extreme point equals the original function f and inside the function is constant. Further consistency conditions need to be satisfied pinning down the value of an extreme point in each interval where it is constant. We apply these insights to a varied set of economic problems: equivalence and optimality of mechanisms for auctions and (matching) contests, Bayesian persuasion, optimal delegation, and decision making under uncertainty.

1 Introduction

In this paper we show that many well-known optimal design and decision problems have a basic common structure: all these problems can be reduced to the choice of an optimal element - that maximizes a given functional - from the set of monotonic functions that are

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either majorized by, or majorize a given monotonic function f. We apply our results to the determination of feasible and optimal auctions and matching contests, of feasible and optimal delegation mechanisms, and of optimal mechanisms for Bayesian persuasion.[1] Our main goal is to reveal the common underlying role of majorization, and to offer a unified treatment to well-known but complex problems that have been previously attacked by separate, "ad-hoc" methods. We also show how both novel and classical results in the relevant literatures are straightforward corollaries of our findings.

The *majorization* relation, due to Hardy, Littlewood and Polya (1929), embodies an elegant notion of "variability" and defines a partial order among vectors in Euclidean space, or among integrable functions.[2] Our main results characterize the extreme points of the sets of monotonic functions that are majorized by, or majorize a given monotonic function f. The monotonicity constraint, a novel feature of our work, is not standard in the mathematical literature: the set of extreme points that respect monotonicity is quite different from the set of extreme points obtained without imposing it (see Ryff (1967) for the latter).[3] In addition, every extreme point is *exposed*, i.e., it can be obtained as the unique maximizer of some linear functional. Hence, no extreme point can be a-priori dismissed as potentially irrelevant for maximization.

Any linear or convex functional will attain a maximum on an extreme point, and this explains the major role played by extreme points for maximization. But, information about the extreme points is very useful besides their role for optimization: any property that is satisfied by the extreme points and that is preserved under averaging will also be satisfied by all elements of a majorization set. This follows from Choquet's theorem:[4] any feasible element in a relevant majorization set can be expressed as an integral with respect to a measure that is supported on the extreme points of that set. Since the sets of extreme points of majorization sets are much smaller than the original sets, and since they can be easily parametrized (see Theorems 1 and 2), the integral representation drastically simplifies the task of establishing a given property for the original set. In Section 2.1 we illustrate this methodology in the classical context of auctions: our insights almost immediately imply both a generalized version of Border's Theorem about reduced auctions, and the equivalence of Bayesian and Dominant Strategy incentive compatible mechanisms in the symmetric case. For the latter, we note that every extreme point can be implemented by a DIC mechanism.

[1] In an Online Appendix we also briefly discuss applications to decision making under uncertainty.

[2] In Economics, a related order has been popularized and applied, most famously to the theory of choice under risk, under the name *second-order stochastic dominance.*

[3] For the discrete case and the differences to the celebrated Birkhoff-von Neumann theorem, see Dahl (2001).

[4] See Phelps (2001) for an excellent introduction.

By Choquet's theorem, the interim allocation associated with every BIC mechanism can be represented as a mixture over extreme points; the result follows since DIC incentive compatibility is equivalent to a monotonicity condition that is preserved under averaging.[5]

Consider the set of non-decreasing functions that majorize, or are majorized by, a non-decreasing function f. Roughly speaking, each extreme point of this set is characterized by its specific, countable collection of intervals. Outside these intervals an extreme point must equal f, and inside each interval the extreme point is a step function that takes at most three different values determined by specific, local "equal-areas" consistency conditions (such that the majorization constraints become tight). We relate these flat areas to the classical *ironing* procedure, and show how our majorization/extreme points focus illuminates it and its uses in applications.

We also identify specialized conditions on the objective functional such as super-modularity that allow us to compare feasible outcomes and to infer features of particular extreme points where the objective functional will attain its maximum. A functional that respects the majorization order (or its converse) will have an optimum on an element that is the least variable (most variable) in a given set. Thus, under conditions that are often present in applications and that can be easily checked, the optimum is either achieved at the a-priori fixed function f or at a step function g with at most two steps. This is a consequence of an elegant theorem due to Fan and Lorentz (1954) that identifies necessary and sufficient conditions for a large class of convex functionals to respect the majorization order.

The paper contains a varied array of illustrations. The majorization constraint is not always explicit in the description of the applied economic problems, and it arises for different reasons. For example, in the theory of auctions it stems from a feasibility condition related to the availability of a limited supply (i.e., reduced-form auctions), in the theory of optimal delegation it is a consequence of incentive compatibility, and in Bayesian persuasion it is induced by information garbling together with Bayesian consistency. The monotonicity constraint also arises for various reasons, for example because of incentive compatibility constraints, or because a cumulative distribution function is non-decreasing.

We cover optimality of mechanisms for auctions and (matching) contests in Sections 4.1 and 4.2. The characterization of extreme points and the Fan-Lorenz inequality immediately yield the revenue - and welfare maximizing mechanisms in multi-prize contests where agents spend resources in order to obtain prizes. In Section 4.3 we formulate optimal delegation as a linear maximization problem under a majorization constraint. Using our characterization of extreme points, this yields a novel characterization of those (potentially stochastic)

[5]An argument similar to the one used in Theorem 3 also shows that, for any convex objective function, there exists an optimal mechanism that is non-randomized.

delegation mechanisms that can be optimal. Moreover, we use our results to characterize when particularly simple delegation mechanisms are optimal, significantly extending earlier results in the delegation literature. We obtain analogous results for the Bayesian persuasion problem in Section 4.4. In recent, independent work, Arieli et al. (2020) also study a Bayesian persuasion problem via an extreme points approach and consider maximization on a majorizing set of functions.[6]

Our majorization approach thereby clearly reveals the close connection between delegation and Bayesian persuasion and their respective optimal mechanisms, and shows that the equivalence between delegation and persuasion mechanisms obtained for a subset of mechanisms by Kolotilin and Zapechelnyuk (2019) extends to all randomized mechanisms. We also illustrate how results obtained in one strand can be immediately applied to the other.

1.1 Majorization Preliminaries

Throughout, we consider right-continuous functions that map the unit interval $[0,1]$ into the real numbers. For two non-decreasing functions $f, g \in L^1$ we say that f *majorizes* g, denoted by $g \prec f$, if the following two conditions hold:

$$\begin{aligned} \int_x^1 g(s)\,\mathrm{d}s &\leq \int_x^1 f(s)\,\mathrm{d}s \quad \text{for all } x \in [0,1] \\ \int_0^1 g(s)\,\mathrm{d}s &= \int_0^1 f(s)\,\mathrm{d}s. \end{aligned} \tag{1}$$

We say that f *weakly majorizes* g, denoted by $g \prec_w f$, if the first condition above holds (but not necessarily the second). For non-monotonic functions f, g majorization is defined analogously by comparing their non-decreasing *rearrangements* f^*, g^*, i.e. f majorizes g if $g^* \prec f^*$.[7]

Majorization is closely related to other concepts from Economics and Statistics. Let X_F and X_G be now random variables with distributions F and G, respectively, defined on the interval $[0,1]$. Define also

$$G^{-1}(x) = \sup\{s : G(s) \leq x\},\ x \in [0,1]$$

to be the generalized inverse (or quantile function) of G, and analogously for F. It follows

[6] See also Section 2. We thank Itai Arieli for bringing this paper to our attention.

[7] Given a function f, let $m(x)$ denote the Lebesgue measure of the set $\{s \in [0,1] : f(s) \leq x\}$. The non-decreasing rearrangement of f, f^*, is defined by $f^*(t) = \inf\{x \in \mathbb{R} : m(x) \geq t\}$ for all $t \in [0,1]$.

from Shaked and Shanthikumar (2005, Section 3.A) that

$$G \prec F \;\Leftrightarrow F^{-1} \prec G^{-1} \Leftrightarrow X_F \leq_{cx} X_G \Leftrightarrow X_G \leq_{ssd} X_F \text{ and } \mathbb{E}[X_G] = \mathbb{E}[X_F],$$

where cx denotes the *convex stochastic order* among random variables, and where ssd denotes the standard *second-order stochastic dominance.*[8] Thus, F majorizes G if and only if G is a *mean preserving spread* of F, i.e., one can construct random variables X, Y, jointly distributed on some probability space, such that $X \sim F$, $Y \sim G$ and such that $Y = \mathbb{E}[X|Y]$.[9]

2 Extreme Points and Majorization

An *extreme* point of a convex set A is a point $x \in A$ that cannot be represented as a convex combination of two other points in A.[10] The *Krein–Milman Theorem* states that any convex and compact set A in a locally convex space is the closed, convex hull of its extreme points. In particular, such a set has extreme points. The usefulness of extreme points for optimization stems from *Bauer's Maximum Principle*: a convex, upper-semicontinuous functional on a non-empty, compact and convex set A of a locally convex space attains its maximum at an extreme point of A.

Let L^1 denote the real-valued and integrable functions defined on $[0,1]$. Given $f \in L^1$, let the *orbit* of f, be the set of all functions that are majorized by f:

$$\{g \in L^1 \mid g \prec f\}.$$

Ryff (1967) has shown that g in the orbit is an extreme point of this set if and only if $g = f \circ \Psi$ where Ψ is a *measure preserving* transformation of $[0,1]$ into itself. This generalizes the discrete case analyzed by Hardy, Littlewood and Polya where the extreme points correspond, by the *Birkhoff-von Neumann Theorem*, to permutation matrices.

In economic applications we are often interested in functional maximizers that are non-decreasing, e.g., a cumulative distribution function in Bayesian persuasion, or an incentive compatible allocation in mechanism design. Thus, we study the subset of non-decreasing

[8]A non-decreasing density $f = F'$ majorizes another non-decreasing density $g = G'$ if and only if the associated distribution F dominates G in *first-order stochastic dominance.*

[9]See Strassen (1965).

[10]Formally $x \in A$ is an extreme point of A if $x = \alpha y + (1-\alpha)z$, for $z, y \in A$ and $\alpha \in [0,1]$ imply together that $y = x$ or $z = x$.

functions in the orbit

$$\mathrm{MPS}(f) = \{g \in L^1 \mid g \text{ non-decreasing such that } g \prec f\}.^{11}$$

Similarly, we denote by $\mathrm{MPS}_w(f)$ the set of non-negative, non-decreasing functions that are weakly majorized by f. Finally, let

$$\mathrm{MPC}(f) = \{g \in L^1 \mid g \text{ non-decreasing such that } g \succ f \text{ and } f(0) \leq g \leq f(1)\}.^{12}$$

Proposition 1 (Representation)**.**

1. *Let $f \in L^1$ be non-decreasing. Then, the sets $\mathrm{MPS}(f)$, $\mathrm{MPS}_w(f)$, and $\mathrm{MPC}(f)$ are convex and compact in the norm topology, and hence the respective sets of extreme points are non-empty.*[13]

2. *For any $g \in \mathrm{MPS}(f)$ there exists a probability measure λ_g supported on the set of extreme points of $\mathrm{MPS}(f)$, $\operatorname{ext}\mathrm{MPS}(f)$, such that $g = \int_{\operatorname{ext}\mathrm{MPS}(f)} h \,\mathrm{d}\lambda_g(h)$ (and analogously for any $g \in \mathrm{MPS}_w(f)$ and $g \in \mathrm{MPC}(f)$).*[14]

The second part of the Proposition is a consequence of Choquet's celebrated theorem, a powerful strengthening of the Krein-Milman insight. Immediate implications are a generalized Jensen inequality, and the Bauer's Maximum Principle for the respective majorization sets. While applications of Choquet's result in infinite-dimensional function spaces are often hampered by the difficulty to identify all relevant extreme points, we offer below relatively simple characterizations:

Theorem 1. *Let f be non-decreasing. Then g is an extreme point of $\mathrm{MPS}(f)$ if and only if there exists a collection of disjoint intervals $[\underline{x}_i, \overline{x}_i)$ indexed by $i \in I$ such that for a.e. $x \in [0,1]$*

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_{i \in I}[\underline{x}_i, \overline{x}_i) \\ \frac{\int_{\underline{x}_i}^{\overline{x}_i} f(s)\,\mathrm{d}s}{\overline{x}_i - \underline{x}_i} & \text{if } x \in [\underline{x}_i, \overline{x}_i). \end{cases} \tag{2}$$

[11] We use the suggestive MPS in order to remind the reader of the relation to more familiar mean-preserving spreads. But note that our functions are not necessarily distributions.

[12] Analogously, the suggestive MPC stands for Mean-Preserving Contractions. The additional constraint $f(0) \leq g \leq f(1)$ ensures compactness, and is suitable for our applications below.

[13] For linear maximization it is enough to establish compactness in the weak topology. We need the stronger result in order to apply Choquet's Theorem.

[14] The integral in the statement is a *Bochner integral* (see, for example, Phelps (2001)). The equality means that $V(g) = \int V(h)\,\mathrm{d}\mu(h)$ for any continuous, linear functional V.

Intuitively, if a function g is an extreme point of $\mathrm{MPS}(f)$ then, at any point in its domain, either the majorization constraint binds, or the monotonicity constraint binds. This implies either that $g(x) = f(x)$ or that g is constant at x. An analogous result for the discrete case is in Dahl (2001).

An element x of a convex set A is *exposed* if there exists a linear functional that attains its maximum on A uniquely at x.[15] Every exposed point is extreme, but the converse is not true in general. Our next result establishes that all extreme points of $\mathrm{MPS}(f)$ are exposed. Thus, we cannot a-priori exclude any extreme point from consideration when maximizing a linear functional.

Corollary 1. *Every extreme point of* $\mathrm{MPS}(f)$ *is exposed.*

Following the approach in Horsley and Wrobel (1987) (who, like Ryff, did not impose monotonicity), we can extend our characterization of extreme points to the set of weakly majorized functions. For $A \subseteq [0,1]$, denote by $\mathbf{1}_A(x)$ the indicator function of A: it equals 1 if $x \in A$ and it equals 0 otherwise.

Corollary 2. *Suppose that f is non-decreasing and non-negative. A function g is an extreme point of* $\mathrm{MPS}_w(f)$ *if and only if there is $\theta \in [0,1]$ such that g is an extreme point of* $\mathrm{MPS}(f \cdot \mathbf{1}_{[\theta,1]})$ *and $g(x) = 0$ for a.e. $x \in [0,\theta)$.*

Finally, we characterize the extreme points of the set of non-decreasing functions that majorize f and that have the same range as f, denoted by $\mathrm{MPC}(f)$.

Theorem 2. *Let f be non-decreasing and continuous. Then $g \in \mathrm{MPC}(f)$ is an extreme point of $\mathrm{MPC}(f)$ if and only if there exists a collection of intervals $[\underline{x}_i, \overline{x}_i)$, (potentially empty) sub-intervals $[\underline{y}_i, \overline{y}_i) \subset [\underline{x}_i, \overline{x}_i)$, and numbers v_i indexed by $i \in I$ such that for a.e. $x \in [0,1]$*

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_{i \in I} [\underline{x}_i, \overline{x}_i) \\ f(\underline{x}_i) & \text{if } x \in [\underline{x}_i, \underline{y}_i) \\ v_i & \text{if } x \in [\underline{y}_i, \overline{y}_i) \\ f(\overline{x}_i) & \text{if } x \in [\overline{y}_i, \overline{x}_i) \end{cases} \tag{3}$$

Moreover, a function g as defined in (3) *is in* $\mathrm{MPC}(f)$ *if the following three conditions are*

[15] Formally, x is exposed if there exists a supporting hyperplane H such that $H \cap A = \{x\}$.

satisfied:

$$(\overline{y}_i - \underline{y}_i)v_i = \int_{\underline{x}_i}^{\overline{x}_i} f(s)\,\mathrm{d}s - f(\underline{x}_i)(\underline{y}_i - \underline{x}_i) - f(\overline{x}_i)(\overline{x}_i - \overline{y}_i) \tag{4}$$

$$f(\underline{x}_i)(\overline{y}_i - \underline{x}_i) + f(\overline{x}_i)(\overline{x}_i - \overline{y}_i) \leq \int_{\underline{x}_i}^{\overline{x}_i} f(s)\,\mathrm{d}s \leq f(\underline{x}_i)(\underline{y}_i - \underline{x}_i) + f(\overline{x}_i)(\overline{x}_i - \underline{y}_i)\,. \tag{5}$$

If $v_i \in (f(\underline{y}_i), f(\overline{y}_i))$ then for an arbitrary point m_i satisfying $f(m_i) = v_i$ it must hold that

$$\int_{m_i}^{\overline{x}_i} f(s)\,\mathrm{d}s \leq v_i(\overline{y}_i - m_i) + f(\overline{x}_i)(\overline{x}_i - \overline{y}_i). \tag{6}$$

Condition (4) in the Theorem ensures that g and f have the same integrals for each subinterval $[\underline{x}_i, \overline{x}_i)$, analogously to the condition imposed in Theorem 1. Condition (5) ensures that $v_i \in (f(\underline{x}_i), f(\overline{x}_i))$, ensuring that g is non-decreasing. If f crosses g in the interval $[\underline{y}_i, \overline{y}_i]$ then there is $m_i \in [\underline{y}_i, \overline{y}_i]$ such that $f(m_i) = v_i$. In this case, Condition (6) ensures that $\int_s^{\overline{x}_i} f(t)\,\mathrm{d}t \leq \int_s^{\overline{x}_i} g(t)\,\mathrm{d}t$ for all $s \in [\underline{x}_i, \overline{x}_i)$ and thus that $f \prec g$. If $v_i \notin (f(\underline{y}_i), f(\overline{y}_i))$ Condition (5) is enough to ensure that $f \prec g$ and thus Condition (6) is not necessary.

We note here that the instance of Bayesian persuasion studied by Arieli et al. (2020) corresponds to a maximization exercise over a set of majorizing functions of the form MPC (see also Section 4.4 for details). Analogously to the first part of our Theorem 2, these authors identify the extreme points in their problem and further show that all extreme points are exposed.

Extreme Points: An Intuitive Description Let f be a cumulative distribution function (CDF) and recall that a CDF admits a jump at a given value if the distribution assigns a mass point to that value. As h majorizes g if and only if g is a mean-preserving spread of h, it follows that MPS(f) is the set of *mean preserving spreads* of f and MPC(f) is the set of *mean preserving contractions* of f. These properties are also reflected in the extreme points: Each extreme point $g \in$ MPS(f) is obtained by taking the mass in each interval $[\underline{x}_i, \overline{x}_i]$ and spreading it out into two mass points at the boundaries of the interval, $\underline{x}_i$ and $\overline{x}_i$ (see Figure 1). There is a unique way to do so while preserving the mean determined by (2). In contrast, each extreme point $g \in$ MPC(f) is obtained by contracting the mass in each interval $[\underline{x}_i, \overline{x}_i]$ into two mass points placed at $\underline{y}_i$ and $\overline{y}_i$. If $v_i \in (f(\underline{y}_i), f(\overline{y}_i))$ the CDFs g and f intersect at $m_i \in (\underline{y}_i, \overline{y}_i)$. Mass to the left of $m_i = f^{-1}(v_i)$ is moved to $\underline{y}_i$ and mass to the right of m_i is moved to $\overline{y}_i$ (see Figure 1).[16] Condition (4) determines the mass at these

[16]If f is not strictly increasing, then f is constant on the interval $\{s\colon f(s) = v\}$, which implies that the distribution assigns no mass to that interval. Thus, any choice of m in that interval will lead to mass being

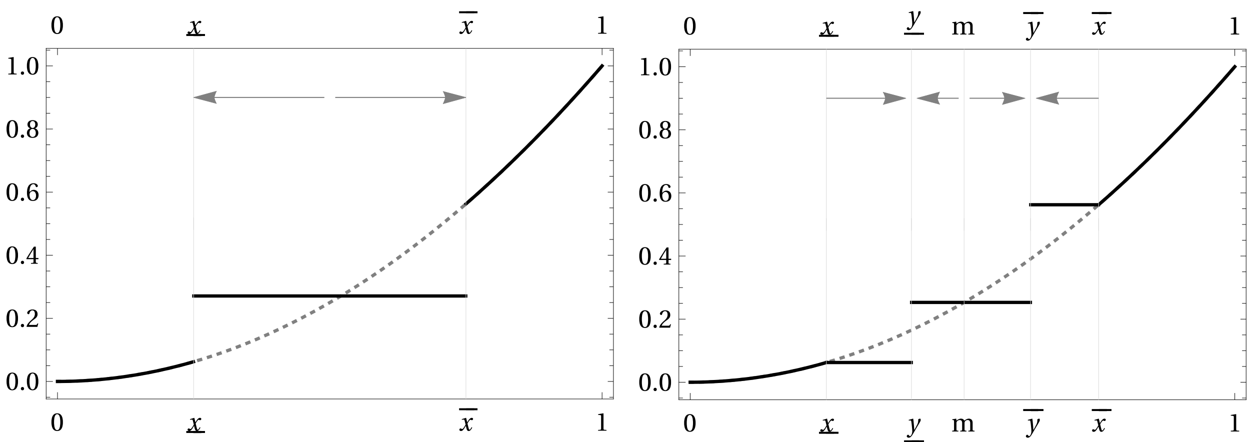

Figure 1: This figure illustrates the differences between the extreme points of MPS(f) and MPC(f). Here $f(s) = s^2$, and there is a single interval $[\underline{x}, \overline{x}] = [1/4, 3/4]$ with $[\underline{y}, \overline{y}] = [13/32, 10/16]$. On the left is the corresponding extreme point in MPS(f) and on the right is the corresponding extreme point in MPC(f). The arrows indicate how mass is moved to transform f into the extreme point.

mass points, and ensures that the mean is preserved on the interval $[\underline{x}_i, \overline{x}_i]$. Condition (5) ensures that g can be obtained from f by moving mass, and Condition (6) ensures that g is a contraction of f.[17]

The main insight of Theorems 1 and 2 is that the mean-preserving spreads (or contractions) of f described there *cannot* be represented as convex combinations of other functions in MPS and in MPC, respectively, and that these are the *only* functions with this property.[18]

2.1 An Application to Ranked-Item Auctions

We illustrate the usefulness of the extreme-point characterization obtained in the previous section by showing that it immediately implies a generalization of the symmetric version of Border's (1991) theorem and the BIC-DIC equivalence for symmetric mechanisms (Manelli and Vincent 2010, Gerhskov et al 2013).[19]

moved in the same way.

[17]A simpler characterization where each interval $[\underline{x}_i, \overline{x}_i]$ is split into the two intervals $[\underline{x}_i, m]$ and $[m, \overline{x}_i]$ each containing only a single mass point is not valid since the mean on these sub-intervals need not be preserved by an extreme point.

[18]Winkler (1988) shows that every extreme point of a set of probability measures characterized by n constraints is the sum of at most $n+1$ mass points. Thus, if there is a unique constraint on the mean, any extreme point is a sum of at most two mass points. Winkler's characterization does not hold here since we impose uncountably many majorization constraints.

[19]Manelli and Vincent (2010) analyze the one-object auction case, and Gerhskov et al. (2013) general social choice problems. Both papers also treat the asymmetric case. Manelli and Vincent use the weaker Krein-Milman Theorem and an approximation argument. Gershkov et al. use a result about measures with

There are n agents with types $\theta_1, \ldots, \theta_n$ that are independently and identically distributed on $[0,1]$ according to a common distribution F, with density $f > 0$. Each agent wants at most one object. There are n objects with qualities $0 \leq q_1 \leq q_2 \leq \ldots \leq q_n = 1$, and we define $\mathcal{A} \subset \{0, q_1, q_2 \ldots, q_n\}^n$ to be the set of feasible allocations, i.e. $\alpha_i = q_k \neq 0 \Rightarrow \alpha_j \neq q_k$ for all $j \neq i$.[20] If agent i with type θ_i receives an object with quality q and pays t for it, then his utility is given by $\theta_i q - t$.

Fix a (potentially random) allocation rule $\alpha : [0,1]^n \times \Omega \to \mathcal{A}$ that depends on the agents' types $\theta_1, \ldots, \theta_n$ and on randomness generated by the mechanism ω. For each i, let[21]

$$\varphi_i(\theta_i) = \mathbb{E}[\alpha_i(\theta_i, \theta_{-i}, \omega) \mid \theta_i]$$

denote the expected quality obtained by agent i, conditional on his type - this is also called the *interim allocation rule.* It is useful to also consider the quantile $s_i = F(\theta_i)$, and to define the *interim quantile allocation* functions

$$\psi_i(s_i) = \varphi_i(F^{-1}(s_i))$$

It is straightforward to show that an allocation α is part of a *Bayesian incentive compatible* (IC) mechanism if and only if each induced *interim quantile allocation* ψ_i is non-decreasing.[22]

Denote by α^* the *assortative allocation* of agents to objects where the highest type gets highest quality, etc. and ties are broken by fair randomization. In our symmetric model, assortative matching α^* is incentive compatible, and induces the symmetric interim quantile allocation

$$\psi_i^*(s_i) = \psi^*(s_i) = \sum_{k=1}^{n} q_k \left[\frac{(n-1)!}{(k-1)!(n-k)!} (s_i)^{k-1} (1-s_i)^{n-k} \right].$$

A vector of interim allocations $\varphi = (\varphi_1, \ldots, \varphi_n)$ where $\varphi_i : [0,1] \to \mathbb{R}$, is *feasible* if there exists an allocation rule α that induces φ as its set of interim allocations, conditional on type. We restrict attention to symmetric interim allocation rules where $\varphi_i = \varphi$, $i = 1, 2, \ldots, n$ and thus $\psi_i = \psi, i = 1, 2, \ldots, n$.

In our terminology, Border's Theorem for the single object case, i.e., $q_n = 1$ and $q_k = 0$

monotonic marginals. See also Goeree and Kushnir (2020).

[20] It is without loss of generality to assume that $m = n$ as we can always add objects with zero quality if $m < n$, or only sell the n highest-quality objects if $m > n$.

[21] A random allocation rule is a random variable defined on a probability space with outcomes (θ, ω) where ω describes the randomization in the mechanism and the probability measure prescribes that the agents types $\theta_1, \ldots, \theta_n$ are i.i.d. distributed according to F and types θ are independent of the randomization of the mechanism ω.

[22] See Gershkov and Moldovanu (2010) who use discrete majorization in a dynamic mechanism design framework with several qualities.

for $k < n$, says that a symmetric and monotonic interim allocation φ is feasible if and only if the associated quantile interim allocation satisfies $\psi \prec_w s^{n-1}$.[23] In this case, the assortative matching interim allocation $\varphi^*(\theta_i) = [F(\theta_i)]^{n-1}$ is the *efficient* allocation and hence $\psi^*(s_i) = (s_i)^{n-1}$.

Theorem 3 (Border's Theorem and BIC-DIC Equivalence)**.** *In the ranked-items auction model the following holds:*

1. *A symmetric and monotonic interim allocation rule φ is feasible if and only if its associated quantile interim allocation $\psi(s) = \varphi(F^{-1}(s))$ is weakly majorized by the interim quantile allocation rule associated with the assortative allocation ψ^*.*

2. *For any symmetric, BIC mechanism there exists an equivalent, symmetric DIC mechanism that yields all agents the same interim utility, and that creates the same social surplus.*

Proof: 1. We first show that $\psi \prec_w \psi^*$ is necessary for feasibility. Consider a monotonic and symmetric interim quantile allocation rule ψ generated by $\alpha \neq \alpha^*$. As switching to the assortative rule takes high-quality objects from lower types and gives them to higher types, we have that for each agent i and for every $\tau \in [0, 1]$

$$\mathbb{E}[\alpha_i(\theta) \mid \theta_i \geq \tau] \leq \mathbb{E}[\alpha_i^*(\theta) \mid \theta_i \geq \tau]\,,$$

If we define $s = F(\tau)$ to be the quantile associated with τ we have that

$$\mathbb{E}[\alpha_i(\theta) \mid \theta_i \geq \tau] = \frac{1}{1 - F(\tau)} \int_\tau^1 \varphi(\theta_i) f(\theta_i)\, \mathrm{d}\theta_i = \frac{1}{1-s} \int_s^1 \psi(t_i)\, \mathrm{d}t_i$$

Since this holds for any $\tau \in [0, 1]$, we obtain that $\psi \prec_w \psi^*$.

To show that weak majorization is also sufficient for feasibility and prove part (2) of the theorem we will construct, for any allocation whose interim quantile allocation rule is weakly majorized by the efficient allocation rule, a DIC mechanism that implements it. We first construct such a DIC mechanism for every extreme point of $\mathrm{MPS}_w(\psi^*)$. Recall that, by Corollary 2, every extreme point ψ of $\mathrm{MPS}_w(\psi^*)$ is described by $\widetilde{s} \in [0, 1]$ and by a collection

[23] See also Maskin and Riley (1984) and Matthews (1984). This is not the original formulation. For connections to majorization see Hart and Reny (2015) (one-object) and Gershkov et al. (2019) (identical objects). Hart and Reny's proof is direct, while Gerskov et al. use a result by Che et al. (2013) based on a network-flow approach.

of intervals $[\underline{s}_l, \overline{s}_l) \subseteq [\widetilde{s}, 1]$ such that

$$\psi(s) = \begin{cases} 0 & \text{if } s < \widetilde{s} \\ \frac{1}{\overline{s}_l - \underline{s}_l} \int_{\underline{s}_l}^{\overline{s}_l} \psi^*(r)\, \mathrm{d}r & \text{if } s \in [\underline{s}_l, \overline{s}_l) \\ \psi^*(s) & \text{if } s \geq \widetilde{s} \text{ and } s \notin \bigcup_{l \in I} [\underline{s}_l, \overline{s}_l) \end{cases} .$$

Any such extreme point is implemented by a random allocation $\alpha : [0,1]^n \times [0,1]^n \to \mathcal{A}$ that: 1) does not allocate to types below $F^{-1}(\widetilde{s})$; 2) uses uniformly i.i.d. drawn priorities $(\pi_1, \ldots, \pi_m)$ to determine the allocation between agents with values in the same interval $[\underline{\theta}_l, \overline{\theta}_l) = [F^{-1}(\underline{s}_l), F^{-1}(\overline{s}_l))$, and 3) is otherwise assortative, i.e.

$$\alpha_i(\theta, \pi) = \begin{cases} 0 & \text{if } \theta_i \leq F^{-1}(\widetilde{s}) \\ q_{r_i(\theta,\pi)} & \text{if } \theta_i \leq F^{-1}(\widetilde{s}) \end{cases}$$

where the rank r_i of agent i is given by

$$r_i(\theta, \pi) = \left| \{ j \colon \psi(F^{-1}(\theta_j)) < \psi(F^{-1}(\theta_i)) \text{ or } \psi(F^{-1}) = \psi(F^{-1})(\theta_i) \text{ and } \pi_j < \pi_i \} \right| .$$

As $\alpha_i(\theta, \pi)$ – the object given to agent i – increases in θ_i for every type profile of the other agents θ_{-i} and for all priorities π, this allocation is implementable in dominant strategies. We thus established that every extreme point of $\mathrm{MPS}_w(\psi^*)$ is implementable in dominant strategies.

It follows from Proposition 1 that, for any $\psi \in \mathrm{MPS}_w(\psi^*)$, there exists a probability measure λ_ψ supported on the extreme points of $\mathrm{MPS}_w(\psi^*)$ such that

$$\psi = \mathbb{E}\left[\widetilde{\psi} \mid \widetilde{\psi} \sim \lambda_\psi \right] .$$

The designer can thus implement the interim allocation ψ by randomizing over mechanisms that each implement extreme points of $\mathrm{MPS}_w(\psi^*)$. As we have shown that each extreme point is implementable in dominant strategies and as dominant strategy incentive compatibility is preserved under randomization, this completes the proof. □

The above argument generalizes to many other problems. For example, consider the case where n is even, and where half of the agents are men and half are women. Suppose that the planner is constrained to allocate at least $\beta\%$ of objects to men and $\beta\%$ of objects to women.[24] Which interim allocations are feasible if all men and all women must be treated

[24] $\beta = 0$ corresponds to the unconstrained problem analyzed before.

equally as individuals, and if men and women needed to be treated equally as groups?[25] Exactly the same proof as the one for Theorem 3 yields that an interim allocation is feasible if and only if it is weakly majorized by the interim allocation induced by the respective constrained efficient allocation.[26] The constrained efficient allocation maximizes the sum of the agents' utilities subject to the constraint that each group receives at least β of the objects. Furthermore, each interim allocation can be implemented in dominant strategies. This example illustrates that our approach can be readily used in other settings, yielding the new economic insight that interim feasibility relates to majorization with respect to the efficient allocation.

Finally, we also note that standard approaches to Border's theorem in the discrete case - see for example Vohra (2011) - use neither majorization, nor Dahl's (2001) characterization of extreme points.

3 Maximization of Special Objective Functionals

Our previous characterizations of extreme points determines all functions that can arise as a unique maximizer of some convex functional over a set described by monotonicity and majorization constraints. In many applications, further monotonicity or super-modularity conditions are either naturally satisfied or can be imposed on the objective function. We show below how such conditions can be used to further shrink the set of relevant extreme points.

3.1 Convex, Super-modular Functionals

A functional $V : L^1 \to \mathbb{R}$ that is monotonic with respect to the majorization order is called *Schur-concave*. An integral inequality, due to Fan and Lorentz (1954) identifies a large set of convex and Schur-concave functionals.

Theorem 4 (Fan and Lorentz 1954)**.** *Let* $K : [0,1] \times [0,1] \to \mathbb{R}$ *. Then*

$$\int_0^1 K(f(t), t)\,\mathrm{d}t \le \int_0^1 K(g(t), t)\,\mathrm{d}t$$

[25]This model is similar to the one considered in Che et al. (2013), but not covered by their setup since they assume identical objects. Notably, no relation between implementability and the efficient allocation is established in Che et al. (2013).

[26]The expected quality conditional on having a type above a threshold τ, but unconditional on gender, is always maximized by efficiently allocating objects subject to the constraint.

holds for any two non-decreasing functions $f, g : [0,1] \to [0,1]$ *such that* $f \prec g$ *if and only if the function* $K(u,t)$ *is convex in* u *and super-modular in* (u,t)*.*

Theorem 4 is extremely useful for the applications below since it provides conditions on the objective function such that a maximum over majorization sets determined by a function f is attained either at f itself (highest variability), or at a particular function g with at most two steps (lowest variability). In the Online Appendix we offer more explanations about Schur-concave functionals and the Fan-Lorenz inequality and briefly illustrate applications to decision under uncertainty (with and without expected utility), and to portfolio choice.

3.2 Linear Optimization under Majorization Constraints

We now consider optimization problems where the objective is a linear functional, and where the constraint set is defined by majorization and by monotonicity. The classical *Riesz Representation Theorem* says that, for every continuous, linear functional V on L^1, there exists a unique, essentially bounded function $c \in L^\infty(0,1)$ such that for every $f \in L^1$

$$V(f) = \int_0^1 c(x) f(x) \, \mathrm{d}x \tag{7}$$

A linear kernel of the form $K(f,x) = c(x)f(x)$ is super-modular (sub-modular) in (f,x) (and hence the linear functional given in (7) is Schur-concave (convex)) if and only if c is non-decreasing (non-increasing). In these cases the Fan-Lorenz inequality provides simple solutions to the optimization problem. We repeatedly apply this observation below.

3.2.1 Maximizing a Linear Functional on $\mathrm{MPS}(f)$

Given a non-decreasing function f and a bounded function c consider then the problem

$$\max_{h \in \mathrm{MPS}(f)} \int_0^1 c(x) h(x) \, \mathrm{d}x. \tag{8}$$

There are three cases:

1. If c is non-decreasing, f itself is a solution for the optimization problem.

2. If c is non-increasing, then a solution for the optimization problem is the overall constant function g that is equal to $\mu_f = \int_0^1 f(x) \, \mathrm{d}x$. This follows since $h \succ g$ for any $h \in \mathrm{MPS}(f)$.

3. If c is not monotonic, other extreme points of $\mathrm{MPS}(f)$ may be optimal.

The next result essentially characterizes the conditions under which an arbitrary extreme point is optimal. The ironing technique, originally used in Myerson (1981) (see also Toikka, 2011) for an optimization problem formulated without majorization constraints, can be used if the constraint set includes all non-decreasing functions in a given orbit.[27]

Define

$$C(x) = \int_0^x c(s)\,\mathrm{d}s$$

and let $\operatorname{conv} C$ denote the convex hull of C, i.e., the largest convex function that lies below C.

Proposition 2. *Let g be an extreme point of $\mathrm{MPS}(f)$, and let $\{[\underline{x}_i, \overline{x}_i)|i \in I\}$ be the collection of intervals described in Theorem 1. If $\operatorname{conv} C$ is affine on $[\underline{x}_i, \overline{x}_i)$ for each $i \in I$ and if $\operatorname{conv} C = C$ otherwise, then g is optimal. Moreover, if f is strictly increasing then the converse holds.*

3.2.2 Maximizing a Linear Functional on $\mathrm{MPC}(f)$

We now analyze the problem[28]

$$\max_{h \in \mathrm{MPC}(f)} \int_0^1 c(x)h(x)\,\mathrm{d}x\,. \tag{9}$$

Again, there are three cases:

1. If c is non-increasing then f solves this problem.

2. If c is non-decreasing, then an optimum is obtained at the step function g defined by

$$g(x) = \begin{cases} f(0) & \text{for } x < \overline{x} \\ f(1) & \text{for } x \geq \overline{x}, \end{cases}$$

   where $\overline{x}$ solves

$$\int_0^{\overline{x}} f(0)\,\mathrm{d}s + \int_{\overline{x}}^1 f(1)\,\mathrm{d}s = \int_0^1 f(s)\,\mathrm{d}s$$

   Indeed, it holds that $g \in \mathrm{MPC}(f)$ and that $g \succ h$ for all $h \in \mathrm{MPC}(f)$. Therefore, the Fan-Lorentz Theorem 4 implies that g is optimal in this case.

[27] In the discrete case, the set of all vectors that are majorized by a given vector is a base polyhedron (Dahl (2010)), which implies useful combinatorial properties. But the set of monotone vectors that are majorized by a given vector is not a base polyhedron, necessitating an ironing procedure.

[28] In the discrete case, the set of all vectors that majorize a given vector is not a base polyhedron. This suggests that problem (9) differs fundamentally from problem (8) and requires different tools for its solution.

3. If c is non-monotonic we cannot directly use the Fan-Lorentz result, but the following observations suggests an approach to solve the problem:

Lemma 1. *Let*

$$C(x) = \int_0^x c(s)\,\mathrm{d}s.$$

A function $g \in \mathrm{MPC}(f)$ is optimal if and only if there exists a concave function $\overline{C}(x) \leq C(x)$ such that:

1. $\int \overline{C}(x)\,\mathrm{d}g(x) = \int C(x)\,\mathrm{d}g(x)$, $\overline{C}(0) = C(0)$ *and* $\overline{C}(1) = C(1)$ *and*
2. $\int_0^1 \overline{C}'(x) g(x)\,\mathrm{d}x = \int_0^1 \overline{C}'(x) f(x)\,\mathrm{d}x$.

In general, there is no pointwise largest concave function below a given function. In order to verify that g is optimal, one therefore has to construct a concave function $\overline{C}$ that is specific to g.[29] This contrasts the situation in the previous subsection, where the convex hull provided a largest convex function below a given function.

Using our previous characterization, we can now determine when particular extreme points are optimal.[30]

Proposition 3. *Suppose that f is strictly increasing, and that f and c are continuous. Let g be an extreme point of $\mathrm{MPC}(f)$, and let $\{[\underline{x}_i, \overline{x}_i)\}_{i=1}^n$ and $\{[\underline{y}_i, \overline{y}_i)\}_{i=1}^n$ be finite collections of intervals as described in Theorem 2 that satisfy $\overline{x}_i < \underline{x}_{i+1}$.*

Then g is optimal if and only if

1. *the complement of $\bigcup_{i \in I} [\underline{x}_i, \overline{x}_i)$ is a subset of the set where c is non-increasing,*
2. $c(\underline{y}_i)(x - \underline{y}_i) \leq \int_{\underline{y}_i}^x c(t)\mathrm{d}t$ *for all $i \in I$ and $x \in [\underline{x}_i, \overline{x}_i]$, and*
3. *equality holds in the previous inequality whenever $x = \underline{x}_i, \overline{y}_i, \overline{x}_i$ such that $x \neq 0, 1$.*

Note that the first condition can be understood via the Fan-Lorentz inequality: the solution g is strictly increasing on an interval only if c is non-increasing on this interval. The equalities in condition 3 are the first-order conditions with respect to local changes of the interval boundaries. Recall that on each interval $[\underline{x}_i, \overline{x}_i)$, we have

$$g(x) = \begin{cases} f(\underline{x}_i) & \text{for } x \in [\underline{x}_i, \underline{y}_i) \\ v_i & \text{for } x \in [\underline{y}_i, \overline{y}_i) \\ f(\overline{x}_i) & \text{for } x \in [\overline{y}_i, \overline{x}_i), \end{cases}$$

[29] The existence of such a function was shown in Dworczak and Martini (2019) and Dizdar and Kovac (2020).

[30] Partial characterizations of solutions to related problems appeared in Kolotilin et al. (2017) and Saeedi and Shourideh (2020).

where v_i is defined in (4). Consider a marginal increase in $\underline{y}_i$. This decreases $g(\underline{y}_i)$ from v_i to $f(\underline{x}_i)$ and, by implicitly differentiating (4), we obtain that this increases v_i by $\frac{v_i - f(\underline{x}_i)}{\overline{y}_i - \underline{y}_i}$. The first-order condition requires then that these two changes do not affect the objective function:

$$c(\underline{y}_i)[v_i - f(\underline{x}_i)] = \frac{v_i - f(\underline{x}_i)}{\overline{y}_i - \underline{y}_i} \int_{\underline{y}_i}^{\overline{y}_i} c(t)\mathrm{d}t.$$

This argument shows that the equality in condition 3 for $x = \overline{y}_i$ is necessary for g to be optimal. The other equalities can be obtained as first-order conditions with respect to changes in $\underline{x}_i$ and $\overline{x}_i$. Combined with the inequality in condition 2, these necessary conditions are also sufficient for g to be optimal.

4 Maximization under Majorization Constraints: Economic Applications

We now show how seemingly different and well-known economic problems share a common structure: they all involve maximization of functionals over majorization sets.

4.1 The Revenue Maximizing Ranked-Item Auction

We first return to the ranked-items auction model of Section 2.1. Consider incentive compatible mechanisms where the utility of the lowest type is zero (as required by individual rationality and revenue optimality). Denote by $J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$ the "virtual value" function. Then the expected revenue generated by a symmetric mechanism with interim allocation rule φ equals

$$n \int_0^1 J(\theta)\varphi(\theta)\, f(\theta)\,\mathrm{d}\theta = n \int_0^1 J(F^{-1}(s))\psi(s)\,\mathrm{d}s\,.$$

Thus, by Theorem 3, the revenue maximization problem becomes

$$\max_{\psi \in \mathrm{MPS}_w(\psi^*)} \int_0^1 J(F^{-1}(s))\psi(s)\,\mathrm{d}s$$

where ψ^* is the interim quantile allocation induced by assortative matching. The maximum is attained at an extreme point of $\mathrm{MPS}_w(\psi^*)$, and by Corollary 2 there is $\widehat{s} \in [0,1]$ such that this extreme point is an extreme point of $\mathrm{MPS}(\psi^* \cdot \mathbf{1}_{[\hat{s},1]})$ and equals zero on $[0, \hat{s}]$. Assuming an increasing virtual value function J, the type $\hat{\theta} = F^{-1}(\hat{s})$ must solve the equation $J(\theta) = 0$.

The Fan-Lorenz Theorem 4 immediately yields then that an optimal allocation $\hat{\psi}$ satisfies[31]

$$\widehat{\psi}(s) = \begin{cases} \psi^*(s) & \text{for } s \geq \widehat{s} \\ 0 & \text{otherwise} \end{cases}.$$

This can be implemented by an auction with a reserve price (say pay-your-bid, or all-pay) where the highest bidder gets the highest quality, and so on[32]. If the virtual value is not increasing, other extreme points may be optimal, corresponding to the outcome of an "ironing procedure", as described in Proposition 2.

4.2 Matching Contests

We now analyze the same basic model as in Section 2.1, but assume that there is a continuum of agents and prizes. Let F denote the distribution of types on $[0,1]$, and let G denote the distribution of prizes awarded, also on $[0,1]$. For simplicity, we assume that both F and G are strictly increasing, and consider allocation schemes where all prizes are distributed. If an agent with type θ obtains prize q and pays t for it, then her utility is given by $\theta q - t$.[33]

We analyze contests where each agent makes an effort (or submits a bid), and where agents are matched to prizes according to their bids. The assortative allocation is given by $\varphi^*(\theta) = G^{-1}(F(\theta))$, and is strictly increasing. It is implemented by the strictly increasing bidding equilibrium

$$t(\theta) = \theta\varphi^*(\theta) - \int_0^\theta \varphi^*(s)\,\mathrm{d}s$$

The induced interim quantile allocation is given here by

$$\psi^*(s) = \varphi^*(F^{-1}(s)) = G^{-1}(F(F^{-1}(s)) = G^{-1}(s)$$

The agents' expected utility from the physical allocation of prizes is maximized by the assortative scheme,[34] but agents need to waste resources (e.g., signaling costs, payments to a designer) in order to achieve it. Another feasible scheme is *random* matching where, inde-

[31]See also Gershkov et al. (2019) who look at a revenue maximization problem with several identical goods where the objective is convex rather than linear. The convexity stems there from investments undertaken prior to the auction.

[32]Iyengar and Kumar (2006) study several variants of the ranked-item model and applications to revenue maximization in keyword auctions. Ulku (2013) allows for interdependent values and for agents that have values over sets of objects (while keeping one-dimensional types).

[33]This formulation is easily generalized to other multiplicative, super-modular production functions and also (at least for some questions) to non-linear costs.

[34]This follows from the rearrangement inequality of Hardy, Littlewood and Polya (1929). Under complete information, the set of feasible allocations is the set of measure-preserving mappings such that each subset of prizes is matched to a subset of agents of equal measure.

pendently of bids, everyone gets a prize equal to the expected value of the prize distribution μ_G. Expected utility from the physical allocation is smaller than under assortative matching, but random matching can be implemented without costs. The induced quantile distribution of prizes is given by

$$G_r(x) = \begin{cases} 0 & \text{if } x \leq \mu_G \\ 1 & \text{otherwise} \end{cases}$$

and thus $G_r \succ G \Leftrightarrow G_r^{-1} \prec G^{-1}$. Intermediate schemes can be obtained by *coarse* matching: for example, an agent with a bid in given quantile is randomly matched to a prize in the same quantile, i.e., he expects to obtain the average prize in that quantile. Coarse matching schemes balance output and bidding costs in less extreme ways than random or assortative matching, and may be superior for some objectives.

The Proposition below generalizes and complements several well-known, existing results in the contest and matching literature (see Damiano and Li (2007), Hoppe, Moldovanu, Sela [HMS] (2009), Condorelli (2012) and Olszewski and Siegel (2018)).[35] These are obtained as immediate consequences of our theoretical insights together with the Fan-Lorenz Theorem.

Proposition 4.

1. *A matching scheme is feasible and incentive compatible if and only if the induced distribution of prizes G_{ic} satisfies $G_{ic}^{-1} \prec G^{-1}$.*

2. *Assume that the distribution of types F is convex. Then each type of the agent prefers random matching to any other scheme.*[36]

3. *Random matching (assortative matching) maximizes the agents' expected utility if the distribution of types F has an Increasing (Decreasing) Failure Rate.*[37]

4. *If F has an Increasing Failure Rate, the revenue (i.e., average bid) to a designer is maximized by assortative matching.*[38]

[35] In recent work, Akbarpour, Dworczak and Kominers (2020) discuss the main role played by the extreme points identified above for problems where a designer maximizes a weighted sum of revenue and social surplus given an arbitrary set of Pareto weights.

[36] F being convex implies, in particular, that F first-order stochastically dominates the uniform distribution on on $[0, 1]$. The present result generalizes the one in HMS, who did not consider intermediate schemes. See also Olszewski and Siegel (2018) for a derivation that includes coarse matching. If F is concave, there is a uniquely defined interval $[\theta^*, 1]$ such that all types in this interval prefer assortative matching while all types in $[0, \theta^*)$ prefer random matching (see HMS).

[37] This generalizes one of the main results of HMS (2009) who only compared the two extreme cases (random and assortative matching). See also Condorelli (2012). Conversely, random matching (assortative matching) minimizes average welfare if the distribution of types F has a Decreasing (Increasing) Failure rate.

[38] See also Damiano and Li (2007).

4.3 Optimal Delegation

We now study a model of optimal delegation.[39] The state of the world θ is distributed according to a distribution F with support $[0,1]$ and with density f. Its realization is privately observed by an agent. The action space is the real line.

The agent's utility from a deterministic action a in state θ is given by $U_A(\theta,a) = -(\theta-a)^2$, and the principal's utility is given by $U_P(\theta,a) = -(\gamma(\theta)-a)^2$, where $\gamma : [0,1] \to \mathbb{R}$ is bounded.[40] We denote by $\Lambda = \sup_{\theta\in[0,1]} |\theta - \gamma(\theta)|$ the maximal disagreement between the agent and the principal. Both agent and principal have expected utility preferences.

A *direct mechanism* $M : [0,1] \to \Delta(\mathbb{R})$ assigns to each agent's report a lottery over actions with finite mean and variance. The principal can implement any incentive compatible (IC) direct mechanism by offering a menu of lotteries, out of which the agent chooses a preferred one; conversely, any menu of lotteries induces an IC direct mechanism.[41]

For a direct mechanism M denote by $\mu_M : [0,1] \to \mathbb{R}$ its type-dependent mean action function and by $\sigma_M^2 : [0,1] \to \mathbb{R}_+$ its type-dependent variance. Since indirect utilities can be expressed as a function of μ_M and σ_M^2,

$$\begin{aligned}U_A(\theta) &= -(\theta - \mu_M(\theta))^2 - \sigma_M^2(\theta)\,,\\ U_P(\theta) &= -(\gamma(\theta) - \mu_M(\theta))^2 - \sigma_M^2(\theta),\end{aligned}$$

we identify each mechanism with its induced mean and variance functions $M = (\mu_M, \sigma_M^2)$.

In general, the set of IC mechanisms cannot be satisfactorily characterized by majorization.[42] But, we show below that, for maximizing the principal's utility, it is without loss of generality to only consider a subset of IC mechanisms that can be characterized in this way. We call a mechanism *undominated* if there does not exist a mechanism where the menu of lotteries is a singleton (i.e., $\mu(\theta)$ and $\sigma^2(\theta)$ are constant), and that yields a higher utility for the principal.

Proposition 5. *Define an interval of actions* $[\underline{a}, \overline{a}]$ *by*

$$[\underline{a}, \overline{a}] = [-\sqrt{2Var(\gamma(\theta) + 2\Lambda^2)}, 1 + \sqrt{2Var(\gamma(\theta) + 2\Lambda^2)}]\,.$$

[39] Variants have been analyzed, for example, by Holmström (1984), Melumad and Shibano (1991), Alonso and Matouschek (2008), and Amador and Bagwell (2013).

[40] Our approach can easily be extended to more general utilities. In particular, we obtain analogous results if $U_A(\theta,a) = \theta a + b(a)$ and $U_P(\theta,a) = \gamma(\theta)a + b(a)$ for a strongly concave function b. Closely related utility functions have been used e.g. by Amador and Bagwell (2013) and Kolotilin and Zapechelnyuk (2019).

[41] This is the familiar *taxation principle*, but note that there are no monetary transfers here.

[42] For example, the mechanism (μ_0, σ_0) that always implements the deterministic action 0 and (μ_1, σ_1) that always implements the deterministic action 1 satisfy $\int_0^1 \mu_0(\theta)\,\mathrm{d}F(\theta) = \int_0^1 0\,\mathrm{d}F(s) = 0 \neq 1 = \int_0^1 1\,\mathrm{d}F(\theta) = \int_0^1 \mu_1(\theta)\,\mathrm{d}F(\theta)$. Thus, μ_0 and μ_1 are *not* comparable to any other function by majorization simultaneously.

A (potentially randomized) undominated mechanism $M = (\mu_M, \sigma_M^2)$ is incentive compatible if and only if there exists an extension[43] $(\mu_{\tilde{M}}, \sigma_{\tilde{M}}^2)$ *of the functions* μ_M, σ_M^2 *to the interval* $[\underline{a}, \overline{a}]$ *such that* $\mu_{\tilde{M}}(\underline{a}) = \underline{a}$, $\mu_{\tilde{M}}(\overline{a}) = \overline{a}$, $\sigma_{\tilde{M}}^2(\underline{a}) = \sigma_{\tilde{M}}^2(\overline{a}) = 0$, *and such that:*

1. $\mu_{\tilde{M}} \in \mathrm{MPC}(a^*)$ *where* $a^* : [\underline{a}, \overline{a}] \to [\underline{a}, \overline{a}]$ *is the identity, and*
2. $\sigma_{\tilde{M}}^2(\theta) = -(\mu_{\tilde{M}}(\theta) - \theta)^2 - 2\int_{\underline{a}}^{\theta} (\mu_{\tilde{M}}(s) - s)\,\mathrm{d}s$ *for all* $\theta \in [\underline{a}, \overline{a}]$.

Proof: Necessity. Let $M = (\mu_M, \sigma_M^2)$ be an undominated IC mechanism. Define a new mechanism on the extended type space $[\underline{a}, \overline{a}]$ by the menu that consists of all options $(\mu_M(\theta), \sigma_M^2(\theta))_{\theta \in [0,1]}$ available in the original mechanism M and, in addition, the two deterministic actions $\underline{a}, \overline{a}$. Any such menu induces an IC direct mechanism $\tilde{M} = (\mu_{\tilde{M}}, \sigma_{\tilde{M}}^2)$ that assigns to every agent in the extended type space $[\underline{a}, \overline{a}]$ his most preferred option.

By Lemma A.2 in the Appendix, the agent's utility in M is bounded from below by $-2Var(\gamma(\theta)) - 2\Lambda^2$. This implies that any original type θ prefers the allocation assigned to her in M to the deterministic actions $\underline{a}$ and $\overline{a}$, and thus that $\mu_{\tilde{M}}(\theta) = \mu_M(\theta)$ for any $\theta \in [0, 1]$. Clearly, it is also optimal for an agent of type $\underline{a}$ $(\overline{a})$ to pick the deterministic action $\underline{a}$ $(\overline{a})$ in $\tilde{M}$, and hence, $\mu_{\tilde{M}}(\underline{a}) = \underline{a}$, $\mu_{\tilde{M}}(\overline{a}) = \overline{a}$ and $\tilde{\sigma}_M(\underline{a}) = \tilde{\sigma}_M(\overline{a}) = 0$. As a consequence, an agent with hypothetical type $\underline{a}$ $(\overline{a})$ obtains utility 0 in $\tilde{M}$.

Since type $\underline{a}$ obtains utility 0, it follows from the envelope theorem and from the supermodularity of the agent's utility in (θ, μ) that the mechanism $\tilde{M} = (\mu_{\tilde{M}}, \sigma_{\tilde{M}}^2)$ is IC if and only if $\mu_{\tilde{M}}$ is non-decreasing and satisfies the envelope condition for all $\theta \in [\underline{a}, \overline{a}]$:

$$-(\theta - \mu_{\tilde{M}}(\theta))^2 - \sigma_{\tilde{M}}^2(\theta) = 2\int_{\underline{a}}^{\theta} [\mu_{\tilde{M}}(s) - s]\,\mathrm{d}s \tag{10}$$

Since

$$-(\theta - \mu_{\tilde{M}}(\theta))^2 - \sigma_{\tilde{M}}^2(\theta) \leq 0,$$

the envelope condition (10) implies that

$$\int_{\underline{a}}^{\theta} \mu_{\tilde{M}}(s)\,\mathrm{d}s \leq \int_{\underline{a}}^{\theta} a^*(s)\,\mathrm{d}s,$$

where $a^*(s) = s$. Since $\mu_{\tilde{M}}(\overline{a}) = \overline{a}$ and $\sigma_{\tilde{M}}^2(\overline{a}) = 0$, we obtain by (10) that

$$\int_{\underline{a}}^{\overline{a}} [\mu_{\tilde{M}}(s) - a^*(s)]\,\mathrm{d}s = 0.$$

[43] A function $\tilde{g} : [\underline{a}, \overline{a}] \to \mathbb{R}$ is an extension of a function $g : [0, 1] \to \mathbb{R}$ to the interval $[\underline{a}, \overline{a}]$ if $\tilde{g}(\theta) = g(\theta)$ for all $\theta \in [0, 1]$.

We conclude that $\mu_{\tilde{M}} \in \mathrm{MPC}(a^*)$. Thus, $(\mu_{\tilde{M}}, \sigma_{\tilde{M}})$ is an extension of (μ_M, σ^2_M) to $[\underline{a}, \overline{a}]$ with the desired properties.

Sufficiency. Conversely, suppose that $(\mu_{\tilde{M}}, \sigma^2_{\tilde{M}})$ are such that $\mu_{\tilde{M}} \in \mathrm{MPC}(a^*)$ and such that $\sigma^2_{\tilde{M}}$ satisfies the condition of the Proposition. Then, we can define a stochastic mechanism $M = (\mu_M, \sigma^2_M)$ by the restriction of $(\mu_{\tilde{M}}, \sigma^2_{\tilde{M}})$ to the set of types $[0, 1]$. This mechanism is well-defined since its variance is non-negative:

$$
\begin{aligned}
\sigma^2_M(\theta) &= -(\mu_{\tilde{M}}(\theta) - \theta)^2 - 2\int_{\underline{a}}^{\theta} (\mu_{\tilde{M}}(s) - s)\,\mathrm{d}s \\
&= -2\int_{\theta}^{\tilde{\mu}_M(\theta)} (\mu_{\tilde{M}}(\theta) - s)\mathrm{d}s - 2\int_{\underline{a}}^{\theta} (\mu_{\tilde{M}}(s) - s)\,\mathrm{d}s \\
&\geq -2\int_{\underline{a}}^{\tilde{\mu}_M(\theta)} (\mu_{\tilde{M}}(s) - s)\,\mathrm{d}s \geq 0,
\end{aligned}
$$

where the first inequality follows since $\mu_{\tilde{M}}$ is non-decreasing, and the second follows since $\mu_{\tilde{M}} \succ a^*$. Since $\mu_{\tilde{M}}$ is non-decreasing and $(\mu_{\tilde{M}}, \sigma^2_{\tilde{M}})$ satisfies the envelope condition by assumption, it follows that the mechanism M is IC. □

Kovac and Mylovanov (2009) characterized IC mechanisms by: 1) monotonicity of the mean action function; 2) the envelope condition determining the variance functions, and 3) a non-negativity constraint on the variance. This imposes a joint constraint on the mean action function and on the variance of the lowest type. In contrast, our condition $\mu_{\tilde{M}} \in \mathrm{MPC}(a^*)$ encompasses the monotonicity constraint on the mean action function, and ensures that the variance derived by the envelope condition is non-negative for all types if $\sigma^2_{\tilde{M}}(\underline{a}) = 0$. This new formulation allows us to reduce the problem to a linear maximization problem where we optimize only over mean action functions subject to the majorization constraint.

Similar to the revenue equivalence result for auctions, we now use Proposition 5 to show that the value of the principal in different, undominated, IC delegation mechanisms only depends on the implemented mean action function:[44]

Proposition 6 (Value Equivalence). *Fix an undominated, IC delegation mechanism $M = (\mu_M, \sigma^2_M)$ and let $\mu_{\tilde{M}}, \sigma^2_{\tilde{M}}$ be an extension satisfying the conditions of Proposition 5. The principal's expected utility in M is only a function of $\mu_{\tilde{M}}$ and is given by*

$$
V_P(\mu_{\tilde{M}}) = 2\int_{\underline{a}}^{\overline{a}} J(\theta)\,\mu_{\tilde{M}}(\theta)\,\mathrm{d}\theta + C\,, \tag{11}
$$

[44]Note though that the present majorization constraint is the opposite of that for auctions, and that the envelope condition characterizing the variance is non-linear due to the agent's quadratic utility.

where the "virtual value" $J : [\underline{a}, \overline{a}] \to \mathbb{R}$ is defined as

$$J(\theta) = \begin{cases} 1 & \text{for } \theta \in [\underline{a}, 0) \\ 1 - F(\theta) + (\gamma(\theta) - \theta) f(\theta) & \text{for } \theta \in [0, 1] \\ 0 & \text{for } \theta \in (1, \overline{a}] \end{cases}$$

and where

$$C = \int_0^1 (\theta^2 - \gamma(\theta)^2) f(\theta) - 2\theta(1 - F(\theta)) \, \mathrm{d}\theta + \underline{a}^2 \, .$$

Proof: The principal's expected utility from using an IC mechanism M can be written as

$$V_P(\mu_{\tilde{M}}) = \int_{\underline{a}}^{\overline{a}} \left[-\gamma(\theta)^2 + 2\gamma(\theta)\mu_{\tilde{M}}(\theta) - \mu_{\tilde{M}}(\theta)^2 - \sigma^2_{\tilde{M}}(\theta) \right] \, \mathrm{d}F(\theta).$$

Substituting for $\sigma^2_{\tilde{M}}(\theta)$ by the characterization of IC, we obtain that:

$$V_P(\mu_{\tilde{M}}) = \int_{\underline{a}}^{\overline{a}} \left[-\gamma(\theta)^2 + 2\gamma(\theta)\mu_{\tilde{M}}(\theta) - \mu_{\tilde{M}}(\theta)^2 - \left(-(\mu_{\tilde{M}}(\theta) - \theta)^2 - 2\int_{\underline{a}}^{\theta} (\mu_{\tilde{M}}(s) - s) \, \mathrm{d}s \right) \right] \mathrm{d}F(\theta).$$

Integration by parts yields

$$\begin{aligned} \int_{\underline{a}}^{\overline{a}} \int_{\underline{a}}^{\theta} (\mu_{\tilde{M}}(s) - s) \, \mathrm{d}s f(\theta) \, \mathrm{d}\theta &= \left[\int_{\underline{a}}^{\theta} (\mu_{\tilde{M}}(s) - s) \, \mathrm{d}s F(\theta) \right]_{\theta = \underline{a}}^{\theta = \overline{a}} - \int_{\underline{a}}^{\overline{a}} (\mu_{\tilde{M}}(\theta) - \theta) F(\theta) \, \mathrm{d}\theta \\ &= \int_{\underline{a}}^{\overline{a}} (\mu_{\tilde{M}}(\theta) - \theta)(1 - F(\theta)) \, \mathrm{d}\theta \, . \end{aligned}$$

Plugging this back into the above equation and simplifying yields

$$\begin{aligned} V_P(\mu_{\tilde{M}}) &= \int_{\underline{a}}^{\overline{a}} \left[-\gamma(\theta)^2 f(\theta) + 2(\gamma(\theta) - \theta) f(\theta) \mu_{\tilde{M}}(\theta) + \theta^2 f(\theta) + 2(\mu_{\tilde{M}}(\theta) - \theta)(1 - F(\theta)) \right] \, \mathrm{d}\theta \\ &= \int_{\underline{a}}^{\overline{a}} \left[2 \left((\gamma(\theta) - \theta) f(\theta) + (1 - F(\theta)) \right) \mu_{\tilde{M}}(\theta) + f(\theta)(\theta^2 - \gamma(\theta)^2) - 2\theta(1 - F(\theta)) \right] \, \mathrm{d}\theta \, . \end{aligned}$$

□

What is remarkable about the above "virtual value" characterization is that the objective of the principal (i) does not depend on the choice of the extension $\mu_{\tilde{M}}$ (as long as it satisfies the conditions in Proposition 5) and (ii) becomes linear in the extension of the mean allocation rule $\mu_{\tilde{M}}$ despite the fact that the original objective of the principal was strictly concave in μ_M.

Corollary 3. *The principal's problem is given by*

$$\max_{\mu_{\tilde{M}} \in \mathrm{MPC}(a^*)} V_P(\mu_{\tilde{M}})$$

and therefore an extreme point of $\mathrm{MPC}(a^*)$ *must be optimal.*

We start with some insights into the nature of optimal delegation mechanisms:

Remark 1: Recall that an extreme point $\mu_{\tilde{M}}$ of $\mathrm{MPC}(a^*)$ is characterized by a collection of intervals $[\underline{\theta}_i, \overline{\theta}_i)$ with sub-intervals $[\underline{y}_i, \overline{y}_i)$ indexed by $i \in I$ such that:

1. If, for some $i \in I$, $\theta \in [\underline{\theta}_i, \overline{\theta}_i)$ and $\underline{y}_i = \overline{y}_i$ then

$$\mu_{\tilde{M}}(\theta) = \begin{cases} \underline{\theta}_i & \text{for } \theta < \frac{\overline{\theta}_i + \underline{\theta}_i}{2} \\ \overline{\theta}_i & \text{for } \theta > \frac{\overline{\theta}_i + \underline{\theta}_i}{2} \end{cases}.$$

2. If, for some $i \in I$, $\theta \in [\underline{\theta}_i, \overline{\theta}_i)$ and $\underline{y}_i < \overline{y}_i$ then

$$\mu_{\tilde{M}}(\theta) = \begin{cases} \underline{\theta}_i & \text{for } \theta < \underline{y}_i \\ v_i & \text{for } \theta \in [\underline{y}_i, \overline{y}_i) \\ \overline{\theta}_i & \text{for } \theta > \overline{y}_i \end{cases},$$

   where v_i is defined in equation (4).

3. If $\theta \notin \bigcup_{i \in I} [\underline{\theta}_i, \overline{\theta}_i)$ then $\mu_{\tilde{M}}(\theta) = \theta$.

Such a mechanism is implemented by letting the agent choose any action $a \in [\underline{a}, \overline{a}] \setminus \bigcup_{i \in I} (\underline{\theta}_i, \overline{\theta}_i)$ and, for each $i \in I$ such that $\underline{y}_i < \overline{y}_i$, adding to the agent's choice set an additional option with mean v_i and variance $(\underline{\theta}_i - \underline{y}_i)^2 - (\underline{y}_i - v_i)^2$. In particular, a delegation mechanism corresponding to an extreme point is deterministic if $\underline{y}_i = \overline{y}_i$ for each $i \in I$.

Optimal delegation mechanisms sometimes involve deliberate randomization by the principal (see Kovac and Mylovanov (2009) and Alonso and Matouschek (2008) for examples). But, our result above significantly reduces the class of uniquely optimal stochastic mechanisms: any extreme (and thus exposed) point will use at most one non-degenerate lottery on each of the intervals $(\underline{\theta}_i, \overline{\theta}_i)$, and any stochastic extreme point will have a discontinuous mean-action function.

Remark 2: Certain Bayesian persuasion problems give rise to the same class of optimization problems (see Section 4.4 below), and this allows us to extend the equivalence observed in Kolotilin and Zapechelnyuk (2019) to stochastic delegation and to general persuasion mechanisms. As an illustration of this equivalence, we now provide a sufficient condition for a deterministic delegation mechanism to be optimal by applying a result in Dworczak and Martini (2019) about the optimality of *monotone partitional signals* in Bayesian persuasion.[45]

Corollary 4. *Suppose that there are $a_1, a_2 \in [\underline{a}, \overline{a}]$ such that J is non-increasing on the intervals $[\underline{a}, a_1]$ and $[a_2, \overline{a}]$, and non-decreasing on the interval $[a_1, a_2]$. Then a deterministic mechanism is optimal.*

Proof: Using integration by parts for the Riemann-Stieltjes integral,[46] the principal's objective becomes

$$\max_{\mu_{\tilde{M}} \in \mathrm{MPC}(a^*)} \int_{\underline{a}}^{\overline{a}} \left(-\int_{\underline{a}}^{\theta} J(s)\,\mathrm{d}s \right) \mathrm{d}\mu_{\tilde{M}}(\theta).$$

The assumption implies that the integrand, as a function of θ, is convex on $[\underline{a}, a_1]$ and on $[a_2, \overline{a}]$, and concave on $[a_1, a_2]$. It is therefore an *affine-closed function* (see Definition 2 in Dworczak and Martini (2019)). Their Theorem 3 implies then that the principal's problem is solved by an extreme point such that, in the notation of our Theorem 2, $\underline{y}_i = \overline{y}_i$ for all $i \in I$ (see also Section 4.4). Any such mechanism corresponds to a deterministic delegation mechanism. $\square$

Remark 3: Our results can be used to characterize when particular extreme points are optimal. The Fan-Lorentz Theorem (Theorem 4) immediately yields a result obtained by Kovac and Mylovanov (2009) who used a rather different approach:

Corollary 5. *Full delegation, i.e., allowing the agent to chose any action in $[0, 1]$ is optimal if $J(\theta) = 1 - F(\theta) + (\gamma(\theta) - \theta)f(\theta)$ is non-increasing on $[0, 1]$, and if $\gamma(0) \leq 0$ and $\gamma(1) \geq 1$.*

Proof: The assumptions imply that J is non-increasing on $[\underline{a}, \overline{a}]$, and thus the objective is linear, sub-modular and thus Schur-convex. The function a^* itself is then a maximizer over $\mathrm{MPC}(a^*)$ for any such functional. As a consequence, each type gets a mean allocation equal to his type $\mu_M(\theta) = \theta$. In turn, Proposition 5 implies that the variance for each type, $\sigma_M(\theta)$, equals zero. $\square$

More generally, Proposition 3 characterizes, for any delegation mechanism in a large class, when this particular mechanism is optimal. This result applies to all deterministic mechanisms in which the agent can choose out of a finite union of non-degenerate intervals, significantly extending previous results. In addition, it applies to a large class of stochastic extreme points, yielding novel characterizations when particular stochastic delegation mechanisms are optimal.

[45] Our result also extends a result by Kovac and Mylovanov (2009). Recently, Kartik et al. (2020) provided sufficient conditions for the optimality of deterministic mechanisms in a related veto bargaining model.

[46] Note that $\mu_{\tilde{M}}(\theta)$ is non-decreasing and hence has bounded variation.

Finally, note that, due to a failure of revenue equivalence on discrete type spaces, there is no obvious characterization of the set of feasible mechanisms using majorization.[47] Thus, without analyzing the continuous-type case, we cannot prove the equivalence between delegation and Bayesian persuasion (see below) via our simple majorization techniques.

4.4 Persuasion with Preferences over the Posterior Mean

We consider here the persuasion problem studied by Kolotilin (2018) and Dworczak and Martini (2019).

The state of the world ω is distributed according to a continuous distribution F on the interval $[0,1]$, and a sender can reveal information about the state to an uninformed receiver. The sender chooses a signal (or Blackwell experiment) π that consists of a signal realization space S and a family of distributions $(\pi_\omega)_\omega$ over S, conditional on the state. By Bayes' rule each signal induces a distribution of posteriors, and hence a distribution of posterior means. The receiver observes the choice of signal and the signal realization, and then chooses an optimal action that depends on the mean of the posterior, denoted here by x. The sender's indirect utility v is state independent and only depends on the posterior mean x.[48] Note that the posterior mean could take a continuum of values even if the underlying state space is discrete. Requiring that the mean takes only one of finitely many values is an unnecessary, exogenous restriction on an endogenous object.

Any signal is a "garbling" of the prior, and thus, for any signal π, the prior F is a mean-preserving spread of the generated distribution of posterior means G_π, i.e. $G_\pi \succ F$. Conversely, it is well known that, for any G such that $G \succ F$, there exists a signal π such $G_\pi = G$. Hence, formally, the sender's problem is to choose a distribution over posterior mean beliefs of the receiver G that solves:

$$\max_{G \in \mathrm{MPC}(F)} \int_0^1 v(x)\, \mathrm{d}G(x)\,.$$

As the objective is linear, a maximum is attained at one of the extreme points characterized in Theorem 2.[49] This immediately implies that an optimal signal structure partitions the states in intervals such that, in each interval:

1. Either all states are perfectly revealed.

[47] Consider the delegation problem with 3 types, $\{0,1,2\}$, together with a mechanism that chooses action 0 for type 0, action a for type 1, and action 2 for type 2. Such a mechanism is IC if and only if a is in $[0,2]$.

[48] This allows for the sender's payoff to depend on the action taken by the receiver.

[49] In the discussion paper version we also treat a Bayesian persuasion problem with an ex-ante informed agent, and we show how the insights of our Theorem 1 become relevant.

2. Or states are pooled, so that only one (deterministic) signal is sent for all states in this interval.

3. Or two different (potentially random) signals are sent for states in that interval, inducing two possible posterior means on this interval.

A signal structure is called *monotone partitional* if it partitions the state space into intervals such that each interval is either of type 1 or type 2; such an information structure either reveals the state perfectly, or sends the same signal for all states in an interval. While other information structures may be optimal, our result implies that an optimal signal structure can still be implemented in a simple way by sending at most two signals on each interval. Arieli et al. (2020) independently obtained the same result - they call signal structures of type 3 *bi-pooled*.[50]

Equivalence to Optimal Delegation

Our majorization/extreme points approach highlights the close connections between Bayesian persuasion and delegation. Although the delegation problem is a-priori non-linear, we have shown that both exercises can be reduced to a maximization of a linear functional over a set of majorizing functions. Hence, the basic structure of their respective optimal mechanisms is identical.

Kolotilin and Zapechelnyuk (2019) have recently established a formal equivalence between optimal delegation and Bayesian persuasion for the case where the set of policies for the principal was exogenously restricted to deterministic delegation mechanisms and to monotone partitional signals, respectively. Our majorization characterization immediately implies that this equivalence holds without **any** restrictions on the policy space: optimal signal structures for Bayesian persuasion that are not monotone partitional correspond to randomized optimal delegation mechanisms.

5 Conclusion

We provided characterizations of the extreme points of the sets of all monotonic functions that are either majorized by, or themselves majorize a given function. We have also shown that many well-known optimization exercises in Economics can be rephrased as maximizing a convex functional over such sets. Hence, a maximum must be attained at one of the extreme points.

[50] The optimality of such a structure in a particular example has already been established Gentzkow and Kamenica (2016) and for general piecewise linear objective functions in Candogan (2019).

Together with an integral representation result due to Choquet, the characterization of extreme points directly imply many results, both novel and well-known. For example, in the context of auctions it implies both, a new generalization of Border's Theorem and the known equivalence between Bayes and dominant strategy incentive compatible mechanisms. For optimal delegation and Bayesian persuasion, our results imply that it is without loss of generality to restrict attention to a small class of mechanisms, and reveal a novel, general equivalence result between these two problems and their (possibly randomized) solutions.

An interesting question for future research is if an analogous extreme point characterization could be obtained for notions of multivariate majorization. Such a result would be potentially useful in various other applications, e.g., information revelation in auctions where the state is naturally multi-dimensional.

A Appendix

Throughout, we assume for any bounded non-decreasing function that it is right-continuous and, moreover, left-continuous at $x = 1$.[51]

Proof of Proposition 1: We first establish that $\text{MPS}(f)$ is a compact subset of L^1 in the norm topology. For any $g \in \text{MPS}(f)$, $f(0) \leq g(x) \leq f(1)$, and the total variation of g is uniformly bounded by $f(1) - f(0)$. *Helly's Selection Theorem* therefore implies that any sequence $\{g_n\}$ in $\text{MPS}(f)$ has a subsequence that converges pointwise, and in L^1, to some function g with bounded variation. Since $\int_x^1 g_n(s)\text{d}s \leq \int_x^1 f(s)\text{d}s$, we obtain that $\int_x^1 g(s)\text{d}s \leq \int_x^1 f(s)\text{d}s$, with equality for $x = 0$. Also, since each g_n is non-decreasing, g is non-decreasing and we conclude that $\text{MPS}(f)$ is compact in the topology induced by the L^1-norm. Analogous arguments establish compactness of $\text{MPS}_w(f)$ and $\text{MPC}(f)$.

It is clear from the definitions that the sets $\text{MPS}(f)$, $\text{MPS}_w(f)$ and $\text{MPC}(f)$ are convex. It then follows from Choquet's theorem that, for any $g \in \text{MPS}(f)$, there is a probability measure λ_g that puts measure 1 on the extreme points of $\text{MPS}(f)$ such that $g = \int h \text{d}\lambda_g(h)$. The same argument applies to $\text{MPS}_w(f)$ and $\text{MPC}(f)$. $\square$

Preparations for the Proof of Theorem 1.

Fix $g \in \text{MPS}(f)$ and for any function h let $h(x_-) = \lim_{x' \uparrow x} h(x')$ and $h(x_+) = \lim_{x' \downarrow x} h(x')$ whenever the limits exist. Given $s_1, s_2 \in [0, 1]$ such that $s_1 < s_2$ and given $y \in [g(s_1), g(s_2)]$,

[51] Recall that in L^1 we identify functions that are equal almost everywhere. A bounded, non-decreasing function $f : [0, 1] \to \mathbb{R}$ has at most countably many discontinuities, limits from the right are defined for each $x \in [0, 1)$, and the limit from the left is defined for $x = 1$.

define

$$u(s) := median\{g(s) - g(s_1),\ g(s) - g(s_2),\ y - g(s)\} \text{ for } s \in [s_1, s_2] \text{ and } u(s) = 0 \text{ else.} \quad (12)$$

Lemma A.1.

1. $g \pm u$ is non-decreasing, and $g(s_1) \le (g \pm u)(s) \le g(s_2)$ for all $s \in [s_1, s_2]$.
2. If $g(s_1) < g(s)$ for all $s > s_1$, then $u \not\equiv 0$.
3. If $g(s) < g(s_2)$ for all $s < s_2$ and if g is continuous at s_2, then $u \not\equiv 0$.
4. There exists $y \in [g(s_1), g(s_2)]$ such that $\int_{s_1}^{s_2} u(s)\mathrm{d}s = 0$.

Proof of Lemma A.1:
(1) Let

$$s_a := \inf\left\{x \ \mid \ g(x) \ge \frac{g(s_1) + y}{2}\right\} = \inf\{\ x \ \mid \ g(x) - g(s_1) \ge y - g(x)\}$$

and

$$s_b := \inf\left\{x \ \mid \ g(x) \ge \frac{g(s_2) + y}{2}\right\} = \inf\{\ x \ \mid \ g(x) - g(s_2) \ge y - g(x)\}$$

It follows that

$$u(s) = \begin{cases} g(s) - g(s_1) & \text{for } s \in (s_1, s_a) \\ y - g(s) & \text{for } s \in (s_a, s_b) \\ g(s) - g(s_2) & \text{for } s \in (s_b, s_2). \end{cases}$$

and hence that

$$(g + u)(s) = \begin{cases} 2g(s) - g(s_1) & \text{for } s \in [s_1, s_a) \\ y & \text{for } s \in [s_a, s_b) \\ 2g(s) - g(s_2) & \text{for } s \in [s_b, s_2). \end{cases}$$

By the definition of s_a, and because $g + u$ is right-continuous, we obtain

$$(g + u)(s_a^-) = 2g(s_a^-) - g(s_1) \le y = (g + u)(s_a)$$

Similarly,

$$(g + u)(s_b^-) = y \le 2g(s_b^+) = (g + u)(s_b)$$

by definition of s_b. Since, in addition, $u(s_1) = u(s_2) = 0$ we conclude that $g + u$ is non-decreasing. Similar arguments show that $g - u$ is non-decreasing as well. Since $u(s) = 0$ for $s \notin (s_1, s_2)$ the inequalities follow.

(2) Note that the first argument of the median function in (12) is strictly positive for $s > s_1$ since, by assumption, $g(s_1) < g(s)$ for all $s > s_1$.

If $y = g(s_1)$ then the third argument in the definition of u is strictly negative for $s > s_1$, and the second argument is also strictly negative for a sufficiently small interval $s \in (s_1, s_1 + \delta)$. Hence, $u \neq 0$ on a set of positive measure and therefore $u \not\equiv 0$.

If $y > g(s_1)$ then the right-continuity of g implies that there exists $\delta > 0$ such that the third argument is strictly positive on $[s_1, s_1 + \delta]$; similarly, there exists $\delta' > 0$ such that the second term is strictly negative on $[s_1, s_1 + \delta']$. Hence, $u \neq 0$ on a set of positive measure and therefore $u \not\equiv 0$.

(3) If $y = g(s_2)$ then the third argument in the definition of u is strictly positive for $s < s_2$ since $g(s) < g(s_2)$ for all $s < s_2$; if $y < g(s_2)$ then continuity of g at s_2 implies that there is $\delta > 0$ such that the third argument is strictly positive on $[s_2 - \delta, s_2]$; the second argument is strictly negative for $s < s_2$; and continuity of g at s_2 implies that there is $\delta' > 0$ such that the first argument is strictly positive on $[s_2 - \delta', s_2]$. Hence, $u \neq 0$ on a set with positive measure and therefore $u \not\equiv 0$.

(4) In order to emphasize the fact that the definition of u in (12) depends on the parameter y we write $u(s, y)$ in this part. Note that, for all s, the function $u(s, y)$ is continuous in y, and that, for all $y \in [g(s_1), g(s_2)]$, $u(\cdot, y)$ is integrable in s and uniformly bounded. Hence, $\int_0^1 u(s, y)\mathrm{d}s$ is continuous in y. If $y = g(s_1)$ then $u(s, y) \leq 0$ for all s; if $y = g(s_2)$ then $u(s, y) \geq 0$ for all s. The intermediate value theorem implies therefore that there exists $y \in [g(s_1), g(s_2)]$ such that $\int_0^1 u(s, y)\mathrm{d}s = 0$. $\square$

Proof of Theorem 1: "$\Rightarrow$": Suppose that g is an extreme point. The proof proceeds in two steps: Step 1 shows that, if g is non-constant in an interval around x, then $f(x) = g(x)$. Step 2 argues that, if g constant on an interval around x, then it has the same average as f on this interval.

Step 1: Fix an arbitrary $s_1 \in [0, 1)$ and suppose that $g(s_1) < g(s)$ for all $s > s_1$. Since g is right-continuous, if $g(s_1) < f(s_1)$, then there exists $s_2 > s_1$ such that $g(s_2) < f(s_1)$. Define u according to (12) such that $\int_{s_1}^{s_2} u(s)\mathrm{d}s = 0$. Then $(g \pm u)(s) < f(s)$ holds on $[s_1, s_2]$ as

$$g(s) \pm u(s) \leq g(s_2) < f(s_1) \leq f(s).$$

Also, $\int_{s_2}^1 f(s) - g(s)\mathrm{d}s \geq 0$ holds since $f \succ g$. This implies that $\int_x^1 f(s) - (g \pm u)(s)ds \geq 0$ for all x, and hence that $g \pm u \in \mathrm{MPS}(f)$. Lemma A.1 (ii) implies that $u \not\equiv 0$, contradicting the assumption that g is an extreme point of $\mathrm{MPS}(f)$.

Similarly, if $g(s_1) > f(s_1)$ then there exists $s_2 > s_1$ such that $f(s_2) < g(s_1)$. Define u according to (12) such that $\int_{s_1}^{s_2} u(s)\mathrm{d}s = 0$. Then $(g \pm u)(s) > f(s)$ holds on $[s_1, s_2]$. Since

$$\int_{s_1}^{1} (f(s) - g(s))\mathrm{d}s = \int_{s_1}^{1} [f(s) - (g \pm u)(s)]\mathrm{d}s \geq 0$$

we conclude that $\int_x^1 [f(s) - (g \pm u)(s)]\mathrm{d}s \geq 0$ for all x. Hence, $g \pm u \in \mathrm{MPS}(f)$. Lemma A.1 (ii) implies that $u \not\equiv 0$, contradicting the assumption that g is an extreme point of $\mathrm{MPS}(f)$. We conclude that, if for an arbitrary $x \in [0, 1)$ the inequality $g(x) < g(s)$ holds for all $s > x$, then $g(x) = f(x)$.

Step 2: It follows from Step 1 that, for any $x \in [0, 1)$ such that $f(x) \neq g(x)$, there exists a non-degenerate interval containing x where g is constant. Hence, there exists a countable collection of intervals $\{[\underline{x}_i, \overline{x}_i) | i \in \mathcal{I}\}$ such that, for each i, $g(s) = g(\underline{x}_i)$ for $s \in [\underline{x}_i, \overline{x}_i)$, $g(s) < g(\underline{x}_i)$ for $s < \underline{x}_i$, $g(s) > g(\underline{x}_i)$ for $s > \overline{x}_i$, and $f(x) = g(x)$ for $x \neq 1$ with $x \notin \bigcup_i [\underline{x}_i, \overline{x}_i)$.

Suppose now that $\int_{\underline{x}_i}^{\overline{x}_i} (f(s) - g(s))\mathrm{d}s < 0$ for some $i \in \mathcal{I}$. This implies that $\int_{\overline{x}_i}^{1} (f(s) - g(s))\mathrm{d}s > 0$ and, since g is constant on $[\underline{x}_i, \overline{x}_i)$, that $f(\underline{x}_i) < g(\underline{x}_i)$. If $g(\underline{x}_i^-) = g(\underline{x}_i)$ we can choose $s_2 = \underline{x}_i$ and $s_1 < s_2$ large enough such that u defined according to (12) satisfies $g \pm u \in \mathrm{MPS}(f)$ and $u \not\equiv 0$, contradicting that g is an extreme point. Hence, $g(\underline{x}_i^-) < g(\underline{x}_i)$. Also, if $g(s) > g(\overline{x}_i)$ for all $s > \overline{x}_i$ we can choose $s_1 = \overline{x}_i$ and $s_2 > s_1$ small enough such that u defined according to (12) satisfies $g \pm u \in \mathrm{MPS}(f)$ and $u \not\equiv 0$, contradicting that g is an extreme point. Hence, g is constant to the right of $\overline{x}_i$. Let $b = \sup\{\ x \ \mid\ g(x) = g(\overline{x}_i)\}$. There are two cases to consider:

Case 1: $\int_b^1 (f(s) - g(s))\mathrm{d}s > 0$. If g is continuous at b, then we can choose $s_1 = b$ and $s_2 > s_1$ small enough such that u satisfies $g \pm u \in \mathrm{MPS}(f)$ and $u \not\equiv 0$. Hence, $g(b^-) < g(b)$. We can therefore choose $\varepsilon > 0$ and $\delta > 0$ such that

$$g \pm (\varepsilon \mathbf{1}_{[\underline{x}_i, \overline{x}_i)} - \delta \mathbf{1}_{[\overline{x}_i, b)}) \in \mathrm{MPS}(f),$$

contradicting the fact that g is an extreme point.

Case 2: $\int_b^1 (f(s) - g(s))\mathrm{d}s = 0$. Since, by assumption, $\int_{\underline{x}_i}^{\overline{x}_i} (f(s) - g(s))\mathrm{d}s < 0$ and $\int_{\underline{x}_i}^{1} (f(s) - g(s))\mathrm{d}s \geq 0$ are true, we obtain $\int_{\overline{x}_i}^{1} (f(s) - g(s))\mathrm{d}s > 0$. This implies that $\int_{\overline{x}_i}^{b} (f(s) - g(s))\mathrm{d}s > 0$, and hence that $g(b^-) < f(b)$. Since $\int_b^1 (f(s) - g(s))\mathrm{d}s = 0$, $f(b) > g(b)$ would imply $\int_{b+\varepsilon}^{1} (f(s) - g(s))\mathrm{d}s < 0$ for $\varepsilon > 0$ small enough, which contradicts $f \succ g$. Therefore,

$g(b^-) < f(b) \leq g(b)$. We can therefore choose $\varepsilon > 0$ and $\delta > 0$ such that

$$g \pm (\varepsilon \mathbf{1}_{[\underline{x}_i, \overline{x}_i)} - \delta \mathbf{1}_{[\overline{x}_i, b)}) \in \mathrm{MPS}(f)$$

contradicting the fact that g is an extreme point.

We can conclude that $\int_{\underline{x}_i}^{\overline{x}_i} (f(s) - g(s))\mathrm{d}s \geq 0$ for all $i \in \mathcal{I}$. Since $\int_0^1 (f(s) - g(s))\mathrm{d}s = 0$ and $f(s) = g(s)$ for $s \notin \bigcup_i [\underline{x}_i, \overline{x}_i)$, we obtain $\int_{\underline{x}_i}^{\overline{x}_i} (f(s) - g(s))\mathrm{d}s = 0$ for all $i \in \mathcal{I}$.

"$\Leftarrow$": Suppose that g has the form in the statement, and suppose that there exists $u \in L^1$ such that $g \pm u \in \mathrm{MPS}(f)$. Let $\{[\underline{x}_i, \overline{x}_i) \mid i \in \mathcal{I}\}$ be a countable collection of intervals such that g is constant on each of the intervals, and such that $f(x) = g(x)$ for $x \notin \bigcup_{i \in \mathcal{I}} [\underline{x}_i, \overline{x}_i)$. Since $g \pm u$ is nondecreasing, for every $i \in \mathcal{I}$ there is a constant function that equals u for a.e. $x \in [\underline{x}_i, \overline{x}_i)$. Also, the properties of g imply that $\int_x^1 (f(s) - g(s))\mathrm{d}s = 0$ for $x \notin \bigcup_i (\underline{x}_i, \overline{x}_i)$, and therefore $\int_x^1 u(s)\mathrm{d}s = 0$ must hold for $x \notin \bigcup_i (\underline{x}_i, \overline{x}_i)$. Together this implies $u(s) = 0$ for a.e. $s \in \bigcup_i (\underline{x}_i, \overline{x}_i)$ and therefore $\int_x^1 u(s)\mathrm{d}s = 0$ holds for all x and hence $u(x) = 0$ for a.e. x. We conclude that g is an extreme point of $\mathrm{MPS}(f)$. $\square$

Proof of Corollary 1: Fix an arbitrary $h \in \mathrm{MPS}(f)$ and define

$$c(x) = \begin{cases} x & \text{if } x \notin \bigcup_{i \in I} [\underline{x}_i, \overline{x}_i) \\ \overline{x}_i - x + \underline{x}_i & \text{if } x \in [\underline{x}_i, \overline{x}_i). \end{cases}$$

Moreover, let

$$\overline{c}(x) = \begin{cases} c(x) & \text{if } x \notin \bigcup_{i \in I} [\underline{x}_i, \overline{x}_i) \\ \frac{\int_{\underline{x}_i}^{\overline{x}_i} c(s)\mathrm{d}s}{\overline{x}_i - \underline{x}_i} & \text{if } x \in [\underline{x}_i, \overline{x}_i) \end{cases}$$

and

$$\overline{h}(x) = \begin{cases} h(x) & \text{if } x \notin \bigcup_{i \in I} [\underline{x}_i, \overline{x}_i) \\ \frac{\int_{\underline{x}_i}^{\overline{x}_i} h(s)\mathrm{d}s}{\overline{x}_i - \underline{x}_i} & \text{if } x \in [\underline{x}_i, \overline{x}_i). \end{cases}$$

It follows that

$$\int_0^1 (g(x) - h(x))c(x)\mathrm{d}x \geq \int_0^1 (g(x) - \overline{h}(x))c(x)\mathrm{d}x = \int_0^1 (g(x) - \overline{h}(x))\overline{c}(x)\mathrm{d}x \tag{13}$$

$$= \int_0^1 \int_x^1 g(s) - \overline{h}(s)\mathrm{d}s \; \mathrm{d}\overline{c}(x) = \int_0^1 \int_x^1 f(s) - \overline{h}(s)\mathrm{d}s \; \mathrm{d}\overline{c}(x) \geq 0, \tag{14}$$

where the first inequality and the first equality follow from Chebyshev's inequality, the second equality follows from integration by parts, the third follows from Theorem 1, and the final inequality holds since $\overline{h} \prec f$. Consequently, c determines a supporting hyperplane for

MPS(f) through g. Moreover, this hyperplane contains no other point in MPS(f): equality holds in (13) only if h is constant on each of the intervals $[\underline{x}_i, \overline{x}_i)$ (see Fink and Jodeit (1984)), which yields $h = \overline{h}$. Equality holds in (14) only if $\int_x^1 \overline{h}(s)\mathrm{d}s = \int_x^1 f(s)\mathrm{d}s$ for all $x \notin \bigcup_i [\underline{x}_i, \overline{x}_i)$. This implies that

$$\begin{aligned}\overline{h}(x) &= f(x) = g(x) \text{ for all } x \notin \bigcup_{i\in I} [\underline{x}_i, \overline{x}_i) \text{ and} \\ \overline{h}(x) &= \frac{\int_{\underline{x}_i}^{\overline{x}_i} f(s)\mathrm{d}s}{\overline{x}_i - \underline{x}_i} = g(x) \text{ for } x \in [\underline{x}_i, \overline{x}_i)\end{aligned}$$

Therefore, g is the only point of MPS(f) contained in the hyperplane. □

Proof of Corollary 2: Observe first that $\mathrm{MPS}_w(f) = \bigcup_{\theta\in[0,1]} \mathrm{MPS}(f \cdot \mathbf{1}_{[\theta,1]})$: Since f is non-negative, $g \prec f \cdot \mathbf{1}_{[\theta,1]}$ implies that $g \prec_w f$. Conversely, if $g \prec_w f$ then there exists $\theta \in [0, 1]$ such that

$$\int_0^1 g(x)\mathrm{d}x = \int_0^1 f(x) \cdot \mathbf{1}_{[\theta,1]}(x)\mathrm{d}x$$

and therefore $g \prec f \cdot \mathbf{1}_{[\theta,1]}$.

Suppose g is an extreme point of $\mathrm{MPS}_w(f)$. It follows that there exists $\theta \in [0, 1]$ such that $g \in \mathrm{MPS}(f \cdot \mathbf{1}_{[\theta,1]})$. Since $\mathrm{MPS}(f \cdot \mathbf{1}_{[\theta,1]}) \subseteq \mathrm{MPS}_w(f)$, g must be an extreme point of $\mathrm{MPS}(f\cdot\mathbf{1}_{[\theta,1]})$. If $\int_0^1 g(s)\mathrm{d}s = \int_0^1 f(s)\mathrm{d}s$ we can assume $\theta = 0$ which trivially implies $g(x) = 0$ for all $x \in [0, \theta)$. Otherwise, $\int_0^1 g(s)\mathrm{d}s < \int_0^1 f(s)\mathrm{d}s$ and Theorem 1 implies that if $g(x) > 0$ for some $x \in [0, \theta)$ then g is constant on some interval $[a, b)$ with $x, \theta \in [a, b)$ and g has jump discontinuities at a and b. It follows that there exists $\varepsilon > 0$ such that $g \pm \varepsilon\mathbf{1}_{[a,b)} \in \mathrm{MPS}_w(f)$, contradicting that g is an extreme point of $\mathrm{MPS}_w(f)$.

Conversely, assume there is $\theta \in [0, 1]$ such that g is an extreme point of $\mathrm{MPS}(f \cdot \mathbf{1}_{[\theta,1]})$ and $g(x) = 0$ for a.e. $x \in [0, \theta)$. Suppose there exists $u \in L^1$ such that $g \pm u \in \mathrm{MPS}_w(f)$. Since $g(x) \pm u(x) \geq 0$ for a.e. x, we obtain $u(x) = 0$ for a.e. $x \in [0, \theta)$. Therefore, $g \pm u \in \mathrm{MPS}_w(f \cdot \mathbf{1}_{[\theta,1]})$. Also, since

$$\int_\theta^1 g(s)\mathrm{d}s = \int_\theta^1 f(s) \cdot \mathbf{1}_{[\theta,1]}(s)\mathrm{d}s,$$

we obtain $\int_\theta^1 u(s)\mathrm{d}s = 0$. We conclude that

$$\int_\theta^1 (g \pm u)(s)\mathrm{d}s = \int_\theta^1 f(s) \cdot \mathbf{1}_{[\theta,1]}(s)\mathrm{d}s$$

and therefore that $g \pm u \in \mathrm{MPS}(f \cdot \mathbf{1}_{[\theta,1]})$. Since g is an extreme point of $\mathrm{MPS}(f \cdot \mathbf{1}_{[\theta,1]})$,

$u \equiv 0$, and hence g is an extreme point of $\mathrm{MPS}_w(f)$. $\square$

Proof of Theorem 2: "$\Rightarrow$": Recall that f is continuous by assumption and let g be an extreme point of $\mathrm{MPC}(f)$.

Step 1: Fix any x such that $g(x) < g(s)$ for all $s > x$. If $\int_x^1 (g(s) - f(s))\mathrm{d}s > 0$, then we can choose $s_1 = x$ and $s_2 > s_1$ small enough such that u defined in (12) satisfies $g \pm u \in \mathrm{MPC}(f)$ and $u \not\equiv 0$, a contradiction; hence, $\int_x^1 (g(s) - f(s))\mathrm{d}s \leq 0$ for any such x.

Now if $g(x) > f(x)$, then right-continuity of g and of f implies that there exists $\varepsilon > 0$ such that $\int_{x+\varepsilon}^1 (g(s) - f(s))\mathrm{d}s < 0$, which contradicts $g \succ f$. Therefore, $g(x) \leq f(x)$.

If $g(x) < f(x)$,[52] then this inequality holds on $[x, x+\varepsilon)$ for some $\varepsilon > 0$, and hence we can choose $s_1 = x$ and $s_2 > s_1$ small enough such that $g \pm u \in \mathrm{MPC}(f)$, and such that $u \not\equiv 0$, contradicting that g is an extreme point.

We conclude that, if $g(x) < g(s)$ for all $s > x$, then $g(x) = f(x)$.

Step 2: Hence, for all x, either $g(x) = f(x)$ or there exists $y > x$ such that g is constant on $[x, y]$. Since $[x, y]$ contains a rational number, there is a countable collection of intervals I_j such that g is constant on I_j for each j, and such that $f = g$ outside of $\bigcup_j I_j$. Let

$$Y = \left\{ y \in \bigcup_j cl(I_j) \mid \int_y^1 (f(s) - g(s))\mathrm{d}s = 0 \right\}$$

and observe that, since f is strictly increasing, the set Y is countable. Then Y defines a partition of $\bigcup_j I_j$ into non-degenerate intervals. Consider an arbitrary such interval, say $[\underline{x}_i, \overline{x}_i)$. We have

$$\begin{aligned} \int_{\underline{x}_i}^1 (f(s) - g(s))\mathrm{d}s &= 0 \ , \int_{\overline{x}_i}^1 (f(s) - g(s))\mathrm{d}s = 0, \text{ and} \\ \int_x^1 (f(s) - g(s))\mathrm{d}s &< 0 \quad \text{for all } x \in (\underline{x}_i, \overline{x}_i) \text{ (since } g \succ f), \end{aligned}$$

and g is piece-wise constant on $[\underline{x}_i, \overline{x}_i)$.

We now prove that g consists of either two or three pieces on this interval. Indeed, if $[\underline{x}_i, \overline{x}_i)$ is partitioned into more than three intervals, then there are non-empty intervals $[a, b)$ and $[c, d)$ with $a > \underline{x}_i$ and $d < \overline{x}_i$ such that g is constant on these intervals and increases strictly at a, b, c, d (i.e., $g(a) > g(s)$ for all $s < a$, $g(s) > g(a)$ for all $s > b$, $g(c) > g(s)$ for all $s < c$, and $g(s) > g(c)$ for all $s > d$). Moreover, since $\int_x^1 (f(s) - g(s))\mathrm{d}s$ is continuous in the

[52] If $x = 1$ then left-continuity of f and g at 1 imply that there is $\varepsilon > 0$ such that $\int_{1-\varepsilon}^1 f(s) - g(s)\mathrm{d}s > 0$, contradicting $g \succ f$. Hence, $x < 1$.

variable x, it achieves its maximum on $[a,d]$, which is strictly negative by assumption. Now if g were continuous at $x \in \{a,b,c,d\}$ we could choose s_1 and s_2 such that u defined by (12) satisfies $g \pm u \in \mathrm{MPC}(f)$ and $u \not\equiv 0$ (Lemma A.1), a contradiction. Hence, g must have a discrete jump at $x \in \{a,b,c,d\}$. But, this implies that we can choose $\delta, \varepsilon > 0$ small enough such that u defined by

$$u(s) = \delta \mathbf{1}_{[a,b)}(s) - \varepsilon \mathbf{1}_{[c,d)}(s)$$

satisfies $g \pm u \in \mathrm{MPC}(f)$, contradicting the assumption that g is an extreme point.

Finally, we show that $\lim_{s \uparrow \overline{x}_i} g(s) = f(\overline{x}_i)$. Observe that $g(\overline{x}_i) \leq f(\overline{x}_i)$ since the right-continuity of g and f would otherwise imply that $\int_y^1 (g(s) - f(s)) \mathrm{d}s < 0$ for some y whenever $\overline{x}_i < 1$, and that $g(1) \leq f(1)$ by assumption. By an analogous argument, it must hold that $f(\overline{x}_i) \leq \lim_{s \uparrow \overline{x}_i} g(s)$. Since $\lim_{s \uparrow \overline{x}_i} g(s) \leq g(\overline{x}_i)$, we obtain

$$\lim_{s \uparrow \overline{x}_i} g(s) \leq g(\overline{x}_i) \leq f(\overline{x}_i) \leq \lim_{s \uparrow \overline{x}_i} g(s)$$

and thus all terms are equal. Similar arguments establish that $g(\underline{x}_i) = f(\underline{x}_i)$.

"$\Leftarrow$": Suppose that $g \in \mathrm{MPC}(f)$ satisfies the conditions in the first part of the theorem and $u \in L^1$ satisfies $g \pm u \in \mathrm{MPC}(f)$. Then, for all $i \in \mathcal{I}$ it must hold that

$$\int_{\underline{x}_i}^1 u(s) \mathrm{d}s = 0 \text{ and } \int_{\overline{x}_i}^1 u(s) \mathrm{d}s = 0.$$

If $i \in \mathcal{I}$ is such that

$$g(x) = \begin{cases} f(\underline{x}_i) & \text{if } x \in [\underline{x}_i, \underline{y}_i) \\ f(\overline{x}_i) & \text{if } x \in [\underline{y}_i, \overline{x}_i) \end{cases}$$

then u is constant on $(\underline{x}_i, \underline{y}_i)$ and on $[\underline{y}_i, \overline{x}_i)$. If $\underline{x}_i = 0$ then $u = 0$ on $[\underline{x}_i, \underline{y}_i)$ since $g(0) = f(0)$ and $(g \pm u)(0) \geq f(0)$. So suppose $\underline{x}_i > 0$ and $u < 0$ on $[\underline{x}_i, \underline{y}_i)$ (and otherwise consider $-u$). Then

$$(g + u)(\underline{x}_i) < f(\underline{x}_i)$$

Since f is continuous and since $g + u$ is non-decreasing, we obtain for some $\varepsilon > 0$ that

$$\int_{\underline{x}_i - \varepsilon}^1 [(g + u)(s) - f(s)] \mathrm{d}s < 0$$

which yields a contradiction. Therefore, $u = 0$ on $[\underline{x}_i, \underline{y}_i)$ and since $\int_{\underline{x}_i}^{\overline{x}_i} u(s) \mathrm{d}s = 0$ we obtain that $u = 0$ on $[\underline{x}_i, \overline{x}_i)$.

On the other hand, if $i \in \mathcal{I}$ is such that g satisfies

$$g(x) = \begin{cases} f(\underline{x}_i) & \text{if } x \in [\underline{x}_i, \underline{y}_i) \\ v_i & \text{if } x \in [\underline{y}_i, \overline{y}_i) \\ f(\overline{x}_i) & \text{if } x \in [\overline{y}_i, \overline{x}_i) \end{cases}$$

for some v_i then u is constant on $[\underline{x}_i, \underline{y}_i)$, on $[\underline{y}_i, \overline{y}_i)$ and on $[\overline{y}_i, \overline{x}_i)$. The same arguments as in the preceding paragraph imply that $u = 0$ on $[\underline{x}_i, \underline{y}_i)$. Now assume $u < 0$ on $[\overline{y}_i, \overline{x}_i)$ (otherwise consider $-u$). Then $(g + u)(\overline{y}_i) < f(\overline{x}_i)$ and there exists $\varepsilon > 0$ such that

$$\int_{\overline{x}_i - \varepsilon}^{1} (g + u)(s) - f(s)\mathrm{d}s < 0,$$

a contradiction. We conclude that $u = 0$ on $[\overline{y}_i, \overline{x}_i)$ and since $\int_{\underline{x}_i}^{\overline{x}_i} u(s)\mathrm{d}s = 0$ we obtain that $u = 0$ on $[\underline{x}_i, \overline{x}_i)$.

Observe that $\int_x^1 (f(s) - g(s))\mathrm{d}s = 0$ for $x \notin \bigcup[\underline{x}_i, \overline{x}_i)$ and hence $\int_x^1 u(s)\mathrm{d}s = 0$ for $x \notin \bigcup[\underline{x}_i, \overline{x}_i)$. Since $u(x) = 0$ for all $x \in \bigcup_i[\underline{x}_i, \overline{x}_i)$, we conclude that $\int_x^1 u(s)\mathrm{d}s = 0$ for all $x \in [0, 1]$, and therefore that $u \equiv 0$.

To prove the second part of the theorem, we show that Conditions (4), (5), (6) are equivalent to $\int_{\underline{x}_i}^{\overline{x}_i} f(s) - g(s)\mathrm{d}s = 0$, $v_i \in [f(\underline{y}_i), f(\overline{y}_i)]$, and $g \succ f$, respectively. We begin by showing that (4) is equivalent to $\int_{\underline{x}_i}^{\overline{x}_i} f(s) - g(s)\mathrm{d}s = 0$. Plugging in the definition of g yields that this condition is equivalent to

$$0 = \int_{\underline{x}_i}^{\overline{x}_i} f(s)\mathrm{d}s - f(\underline{x}_i)(\underline{y}_i - \underline{x}_i) - f(\overline{x}_i)(\overline{x}_i - \overline{y}_i) - v_i(\overline{y}_i - \underline{y}_i)$$

and thus equivalent to (4).

We next show that (5) is equivalent to $v_i \in [f(\underline{x}_i), f(\overline{x}_i)]$ and thus to the monotonicity of g. It follows from (4) that $v_i \leq f(\overline{x}_i)$ is equivalent to

$$\int_{\underline{x}_i}^{\overline{x}_i} f(s)\mathrm{d}s - f(\underline{x}_i)(\underline{y}_i - \underline{x}_i) - f(\overline{x}_i)(\overline{x}_i - \overline{y}_i) \leq f(\overline{x}_i)(\overline{y}_i - \underline{y}_i).$$

Adding $f(\underline{x}_i)(\underline{y}_i - \underline{x}_i) - f(\overline{x}_i)(\overline{x}_i - \overline{y}_i)$ yields

$$\int_{\underline{x}_i}^{\overline{x}_i} f(s)\mathrm{d}s \leq f(\underline{x}_i)(\underline{y}_i - \underline{x}_i) + f(\overline{x}_i)(\overline{x}_i - \underline{y}_i)\,.$$

The other side of the inequality follows from an analogous argument for $f(\underline{x}_i) \leq v_i$ and we thus have that (5) is equivalent to $v_i \in [f(\underline{x}_i), f(\overline{x}_i)]$.

Finally, we show that (6) ensures that $g \succ f$ if $v_i \in (f(\underline{y}_i), f(\overline{y}_i))$ and that $g \succ f$ is automatically satisfied if $v_i \notin (f(\underline{y}_i), f(\overline{y}_i))$. As $\int_x^1 f(s) - g(s)\mathrm{d}s = 0$ for all $x \notin \bigcup[\underline{x}_i, \overline{x}_i)$ it suffices to show that $\int_x^1 f(s) - g(s)\mathrm{d}s \le 0$ for all $x \in [\underline{x}_i, \overline{x}_i)$.

Consider the case where $v_i \in (f(\underline{y}_i), f(\overline{y}_i))$. Since f is continuous and since $v_i \in [f(\underline{y}_i), f(\overline{y}_i)]$, there exists a point $m_i \in (\underline{y}_i, \overline{y}_i)$ such that $f(m_i) = v_i$. As $g(x) \le f(x)$ for $x \in [\underline{x}_i, \underline{y}_i]$, we obtain for all $x \in [\underline{x}_i, \underline{y}_i]$ that

$$0 = \int_{\underline{x}_i}^1 f(s) - g(s)\mathrm{d}s \ge \int_x^1 f(s) - g(s)\mathrm{d}s\,.$$

Furthermore, as $g(x) \ge f(x)$ for $x \in [\underline{y}_i, m]$ we get that for all $x \in [\underline{y}_i, m]$

$$\int_x^1 f(s) - g(s)\mathrm{d}s \le \int_m^1 f(s) - g(s)\mathrm{d}s\,.$$

A symmetric argument shows that the same conclusion holds for all $x \in [m, \overline{y}_i]$ and that $\int_x^1 f(s) - g(s)\mathrm{d}s \le 0$ for all $x \in [\overline{y}_i, \overline{x}_i)$. We thus have that $\int_x^1 f(s) - g(s)\mathrm{d}s \le 0$ for all $x \in [\underline{x}_i, \overline{x}_i)$ if and only if $\int_m^1 f(s) - g(s)\mathrm{d}s \le 0$, which is equivalent to (6).

If $v_i \notin (f(\underline{y}_i, \overline{y}_i))$ then $x \mapsto \int_x^1 f(s) - g(s)\mathrm{d}s$ is quasi-concave on the interval $[\underline{x}_i, \overline{x}_i]$ and thus maximized at either $\underline{x}_i$ or $\overline{x}_i$. Condition (4) ensures that this integral equals zero at both points and thus $f \prec g$. □

Proof of Proposition 2: To simplify notation, let $\overline{C}$ denote the convex hull of C. Note that, since C is continuous, $\overline{C}(0) = C(0)$ and $\overline{C}(1) = C(1)$. Also, if $\overline{C}(x) < C(x)$ for all $x \in (a, b) \subset [0, 1]$, then $\overline{C}$ is affine on (a, b).

For every non-decreasing function h that satisfies $h \prec f$ we obtain[53]

$$\int c(x)h(x)\mathrm{d}x = C(1)h(1) - \int_0^1 C(x)\mathrm{d}h(x) \le \overline{C}(1)h(1) - \int_0^1 \overline{C}(x)\mathrm{d}h(x) \tag{15}$$

$$= \int_0^1 \overline{C}'(x)h(x)\mathrm{d}x \le \int_0^1 \overline{C}'(x)f(x)\mathrm{d}x, \tag{16}$$

where the equalities follow from integration by parts for the Riemann-Stieltjes integral, where the first inequality follows since $\overline{C}(x) \le C(x)$, and where the final inequality follows from the Fan-Lorentz Theorem since $\overline{C}'$ is non-decreasing.

Since, by assumption, $\overline{C}(x) = C(x)$ for $x \notin \bigcup_{i \in I}[\underline{x}_i, \overline{x}_i)$ and since g is constant on $[\underline{x}_i, \overline{x}_i)$,

[53] Since $\overline{C}(x)$ is convex, $\overline{C}'(x)$ exists a.e. and we extend its definition by right-continuity to all x.

we obtain that

$$\int_0^1 C(x)\mathrm{d}g(x) = \int_0^1 \overline{C}(x)\mathrm{d}g(x);$$

and hence, (15) holds as an equality for $h = g$. Also, since $f(x) = g(x)$ for $x \notin \bigcup_{i\in I}[\underline{x}_i, \overline{x}_i)$, since $\overline{C}$ is affine on $[\underline{x}_i, \overline{x}_i)$, and since g is constant on $[\underline{x}_i, \overline{x}_i)$ with $g(x) = \int_{\underline{x}_i}^{\overline{x}_i} f(s)\mathrm{d}s$, we obtain

$$\int_0^1 \overline{C}'(x)g(x)\mathrm{d}x = \int_0^1 \overline{C}'(x)f(x)\mathrm{d}x.$$

Hence, setting $h = g$ also satisfies (16) as an equality, and we conclude that g is optimal.

For the converse, assume that f is strictly increasing. Observe first that there is $h \in \mathrm{MPS}(f)$ that satisfies (15) as an equality: Let $\{\ [\underline{y}_j, \overline{y}_j] \mid j \in J\}$ be a minimal collection of intervals such that $\overline{C}$ is affine on $[\underline{y}_j, \overline{y}_j]$ for each $j \in J$ and such that $\overline{C}(x) = C(x)$ for all $x \notin \bigcup_{j\in J}[\underline{y}_j, \overline{y}_j]$. Define h to be constant on $[\underline{y}_j, \overline{y}_j]$ for each j with

$$\int_{\underline{y}_j}^{\overline{y}_j} h(s)\mathrm{d}s = \int_{\underline{y}_j}^{\overline{y}_j} f(s)\mathrm{d}s,$$

and set $h(x) = f(x)$ for $x \notin \bigcup_{j\in J}[\underline{y}_j, \overline{y}_j]$. It follows from the previous step that h satisfies (15) and (16) with equality.

If $\overline{C}$ is not affine on $[\underline{x}_i, \overline{x}_i)$ for some $i \in I$, then $\overline{C}'$ is non-decreasing and it is not constant on $[\underline{x}_i, \overline{x}_i)$. Since f is strictly increasing and g is constant on $[\underline{x}_i, \overline{x}_i)$, an application of Chebyshev's inequality (see Theorem 1 in Fink and Jodeit (1984)) yields

$$\begin{aligned}\int_{\underline{x}_i}^{\overline{x}_i} 1\mathrm{d}x \int_{\underline{x}_i}^{\overline{x}_i} \overline{C}'(x)f(x)\mathrm{d}x &> \int_{\underline{x}_i}^{\overline{x}_i} f(x)\mathrm{d}x \int_{\underline{x}_i}^{\overline{x}_i} \overline{C}'(x)\mathrm{d}x \\ = \int_{\underline{x}_i}^{\overline{x}_i} g(x)\mathrm{d}x \int_{\underline{x}_i}^{\overline{x}_i} \overline{C}'(x)\mathrm{d}x &= \int_{\underline{x}_i}^{\overline{x}_i} 1\mathrm{d}x \int_{\underline{x}_i}^{\overline{x}_i} g(x)\overline{C}'(x)\mathrm{d}x.\end{aligned}$$

Hence, g satisfies (15) with strict inequality, and therefore g cannot be optimal.

If $\overline{C}(x) < C(x)$ for some $x \notin \bigcup_{i\in I}[\underline{x}_i, \overline{x}_i)$ then there is $\varepsilon > 0$ such that $\overline{C}(z) < C(z)$ for all $z \in [x, x+\varepsilon]$ and $g(x) < g(x+\varepsilon)$ (since f is strictly increasing). Hence,

$$\int_x^{x+\varepsilon} \overline{C}(s)\mathrm{d}g(x) < \int_x^{x+\varepsilon} C(s)\mathrm{d}g(x)$$

and g satisfies (15) as a strict inequality and therefore g cannot be optimal. $\square$

Proof of Lemma 1: For any $h \in \mathrm{MPC}(f)$,

$$\begin{aligned}\int_0^1 c(x)[g(x)-h(x)]\mathrm{d}x &\geq -\int_0^1 C(x)\mathrm{d}[g(x)-h(x)] \\ &\geq \int_0^1 \overline{C}(x)\mathrm{d}h(x) - \int_0^1 \overline{C}(x)\mathrm{d}g(x) \\ &= \int_0^1 \overline{C}(x)\mathrm{d}h(x) - \int_0^1 \overline{C}(x)\mathrm{d}f(x) \geq 0.\end{aligned}$$

where the first inequality follows from integration by parts since $g(1) \geq h(1)$ and $C(0) = 0$; the second inequality follows from $\overline{C}(x) \leq C(x)$ and condition 1; the equality follows from condition 2; and the final inequality follows since $\overline{C}$ is concave and $h \in \mathrm{MPC}(f)$.

The existence of such $\overline{C}$ for an optimal g is proven in Theorem 1 of Dizdar and Kovac (2020). To see this, set $u = -C, \mu = f$ in their notation, and observe that u is Lipschitz continuous as c is bounded. □

Proof of Proposition 3:
"$\Leftarrow$": For each i and $x \in [\underline{x}_i, \overline{x}_i)$, let $\underline{y}_i$ denote the first jump point of g on $[\underline{x}_i, \overline{x}_i]$ and define $\overline{c}(x) = c(\underline{y}_i)$; for $x \notin \bigcup_i [\underline{x}_i, \overline{x}_i)$, define $\overline{c}(x) = c(x)$ and let $\overline{C}(x) = \int_0^x \overline{c}(s)\mathrm{d}s$. We claim that $\overline{C}$ is concave: By definition, $\overline{c}$ is non-increasing outside the set $\bigcup_i [\underline{x}_i, \overline{x}_i)$, and $\overline{c}$ is constant on $[\underline{x}_i, \overline{x}_i)$ for each i. Moreover, it follows from conditions 2 and 3 that, for each i,

$$\begin{aligned}&\lim_{x\uparrow \underline{x}_i} \overline{c}(x) = c(\underline{x}_i) \geq c(\underline{y}_i) = \lim_{x\downarrow \underline{x}_i} \overline{c}(x) \text{ and} \\ &\lim_{x\uparrow \overline{x}_i} \overline{c}(x) = c(\underline{y}_i) \geq c(\overline{x}_i) = \lim_{x\downarrow \overline{x}_i} \overline{c}(x).\end{aligned}$$

We conclude that $\overline{c}$ is non-increasing. Since $\overline{C}$ is absolutely continuous, it follows that $\overline{C}$ is concave.

Letting $C(x) = \int_0^x c(s)\mathrm{d}s$, it also follows from condition 2 that $\overline{C}(x) \leq C(x)$. Moreover,

$$\int_0^1 \overline{C}(x)\mathrm{d}g(x) = \int_0^1 C(x)\mathrm{d}g(x)$$

since, by construction, $\overline{C}(x) = C(x)$ whenever $x \notin \bigcup_i (\underline{x}_i, \overline{x}_i)$, and g is constant on $(\underline{x}_i, \overline{x}_i)$ for each i.

Finally, note that $\int_{\underline{x}_i}^{\overline{x}_i} \overline{C}'(x)\mathrm{d}x = \int_{\underline{x}_i}^{\overline{x}_i} C'(x)\mathrm{d}x$ by condition 3. Since g is an extreme point, $\int_{\underline{x}_i}^{\overline{x}_i} (g(x) - f(x))\mathrm{d}x = 0$ and $g(x) = f(x)$ for $x \notin \bigcup [\underline{x}_i, \overline{x}_i)$. We therefore obtain $\int \overline{C}'(x)g(x)\mathrm{d}x = \int \overline{C}'(x)f(x)\mathrm{d}x$. Hence, all conditions in Lemma 1 are satisfied, and it follows that g is optimal.

"⇒": Let g be an optimal extreme point. It follows from Lemma 1 that there exists a concave function $\overline{C}$ with $\overline{C}(0) = C(0)$ and $\overline{C}(1) = C(1)$ that satisfies

$$\overline{C}(x) \leq C(x) \text{ for all } x \tag{17}$$

$$\int_0^1 \overline{C}(x)\mathrm{d}g(x) = \int_0^1 C(x)\mathrm{d}g(x) \tag{18}$$

$$\int_0^1 \overline{C}'(x)g(x)\mathrm{d}x = \int_0^1 C'(x)f(x)dx. \tag{19}$$

Since $\overline{C}$ is concave, $\overline{c}(x) = \overline{C}'(x)$ is well-defined almost everywhere and non-decreasing, and we extend its definition to all of $[0,1]$ by right-continuity.

Since f is strictly increasing and since $f(x) = g(x)$ for $x \notin \bigcup_i(\underline{x}_i, \overline{x}_i)$, it follows from (18) that $\overline{C}(x) = C(x)$ for $x \notin \bigcup_i(\underline{x}_i, \overline{x}_i)$. Since $\overline{C}$ is concave, we conclude that $c(x) = C'(x) = \overline{C}'(x)$ is non-increasing for $x \notin \bigcup_i(\underline{x}_i, \overline{x}_i)$, establishing the first condition in Proposition 3.

Next, we establish that, for each i, $\overline{c}(x)$ is constant on $(\underline{x}_i, \overline{x}_i)$. Since g is an extreme point, it follows from our characterization that, for each i, $g(\underline{x}_i) = f(\underline{x}_i)$, $g(\overline{x}_i) = f(\overline{x}_i)$, and $\int_{\underline{x}_i}^{\overline{x}_i} g(s) - f(s)\mathrm{d}s = 0$. Because $\overline{C}$ is concave, it follows that $\int_{\underline{x}_i}^{\overline{x}_i} \overline{C}(x)\mathrm{d}g(x) \geq \int_{\underline{x}_i}^{\overline{x}_i} \overline{C}(x)\mathrm{d}f(x)$.[54] Since $g(x) = f(x)$ outside $\bigcup_i(\underline{x}_i, \overline{x}_i)$, (19) implies then that, for each i, $\int_{\underline{x}_i}^{\overline{x}_i} \overline{C}(x)\mathrm{d}g(x) = \int_{\underline{x}_i}^{\overline{x}_i} \overline{C}(x)\mathrm{d}f(x)$. Using integration by parts twice, this yields

$$\begin{aligned}
0 &= \int_{\underline{x}_i}^{\overline{x}_i} \overline{C}(x)\mathrm{d}f(x) - \int_{\underline{x}_i}^{\overline{x}_i} \overline{C}(x)\mathrm{d}g(x) \\
&= \left[\overline{C}(x)\left(f(x) - g(x)\right)\right]_{x=\underline{x}_i}^{x=\overline{x}_i} + \int_{\underline{x}_i}^{\overline{x}_i} [g(x) - f(x)]\overline{C}'(x)dx \\
&= \left[\overline{C}'(x)\int_x^{\overline{x}_i}[f(s) - g(s)]ds\right]_{x=\underline{x}_i}^{x=\overline{x}_i} - \int_{\underline{x}_i}^{\overline{x}_i}\left[\int_x^{\overline{x}_i} g(s) - f(s)ds\right]\overline{c}(x)\mathrm{d}x \\
&= \int_{\underline{x}_i}^{\overline{x}_i}\left[\int_x^{\overline{x}_i} f(s) - g(s)ds\right]\overline{c}(x)\mathrm{d}x
\end{aligned}$$

Observe that $\int_x^{\overline{x}_i} g(s) - f(s)\mathrm{d}s > 0$ for $x \in (\underline{x}_i, \overline{x}_i)$ because g is an extreme point. Since $\overline{c}$ is non-increasing, $\int_{\underline{x}_i}^{\overline{x}_i}\left[\int_x^{\overline{x}_i} f(s) - g(s)ds\right]\overline{c}(x)\mathrm{d}x = 0$ holds only if $\overline{c}$ equals some constant for all $x \in (\underline{x}_i, \overline{x}_i)$.[55]

[54] Note that $\frac{1}{f(\overline{x}_i)-f(\underline{x}_i)}[f(x) - f(\underline{x}_i)]$ and $\frac{1}{f(\overline{x}_i)-f(\underline{x}_i)}[g(x) - f(\underline{x}_i)]$ are CDF's. Since $\int_x^{\overline{x}_i} g(s) - f(s)\mathrm{d}s \geq 0$ with equality for $x = \underline{x}_i$, $\frac{1}{f(\overline{x}_i)-f(\underline{x}_i)}[f(x) - f(\underline{x}_i)]$ is a mean-preserving spread of $\frac{1}{f(\overline{x}_i)-f(\underline{x}_i)}[g(x) - f(\underline{x}_i)]$. For any concave function $\overline{C}$, this implies $\int_{\underline{x}_i}^{\overline{x}_i} \overline{C}(x)\mathrm{d}g(x) \geq \int_{\underline{x}_i}^{\overline{x}_i} \overline{C}(x)\mathrm{d}f(x)$.

[55] Suppose to the contrary that there are $y, z \in (\underline{x}_i, \overline{x}_i)$ such that $\overline{c}(y) \neq \overline{c}(z)$. Suppose w.l.o.g. that $y < z$.

Next, since $\overline{C}(x) \leq C(x)$ and, by (18), $\overline{C}(\underline{y}_i) = C(\underline{y}_i)$, we obtain

$$\lim_{x \uparrow \underline{y}_i} \overline{c}(x) \geq c(\underline{y}_i) \geq \lim_{x \downarrow \underline{y}_i} \overline{c}(x).$$

Because $\overline{c}$ is constant on $(\underline{x}_i, \overline{x}_i)$, this implies $\overline{c}(x) = c(\underline{y}_i)$ for all $x \in (\underline{x}_i, \overline{x}_i)$ and therefore

$$\int_{\underline{y}_i}^{x} \overline{c}(t) \mathrm{d}t = c(\underline{y}_i)(x - \underline{y}_i).$$

Since $\overline{C}(\underline{y}_i) = C(\underline{y}_i)$ and $\overline{C}(x) \leq C(x)$, we obtain $\int_{\underline{y}_i}^{x} c(t) \mathrm{d}t \geq c(\underline{y}_i)(x - \underline{y}_i)$, which establishes condition 2.

Finally, since f is strictly increasing by assumption, $f(x) = g(x)$ outside $\bigcup_i [\underline{x}_i, \overline{x}_i)$, and $\overline{x}_i < \underline{x}_{i+1}$ for all i, we obtain that $x \in \operatorname{supp} g$ whenever $x = \underline{x}_i, \overline{y}_i, \overline{x}_i$, but $x \neq 0, 1$ (where g is interpreted as a distribution function). Condition 3 then follows from (18). □

Proof of Proposition 4: 1) This follows from the first statement in Theorem 3 and generalizes **all** matching schemes considered in the literature.[56]

2) Assuming that the distribution of prizes is G_{ic}, the expected utility of the agent with type θ in the contest is given by

$$U(\theta) = \int_0^{\theta} G_{ic}^{-1}(F(\tau)) \, \mathrm{d}\tau$$

This is the standard payoff-equivalence result a la Myerson. Let us first maximize $U(1)$, the utility of the highest type. Substituting $F(\theta) = s$, yields the problem

$$\max_{G_{ic}^{-1} \in \mathrm{MPS}(G^{-1})} \int_0^1 G_{ic}^{-1}(s) f(s) \, \mathrm{d}s \, .$$

We immediately obtain from the Fan-Lorentz Theorem 4 that a maximizer is G_r^{-1} (G^{-1}) if the density f is non-increasing (non-decreasing), i.e. if F is convex (concave).[57] Thus, the

Then $\overline{c}(y) < \overline{c}(z)$ and, letting $b = \min_{t \in [y,z]} \int_t^{\overline{x}_i} g(s) - f(s) \mathrm{d}s$, we obtain

$$\int_{\underline{x}_i}^{\overline{x}_i} \left[\int_x^{\overline{x}_i} g(s) - f(s) \mathrm{d}s \right] \overline{c}(x) \mathrm{d}x \geq b[\overline{c}(z) - \overline{c}(y)] > 0$$

since $b > 0$.

[56]See for example the schemes considered by Olszewski and Siegel (2018)) - these are extreme points of the majorization set, and our result shows that the restriction they make is without loss for determining Pareto optimal allocations.

[57]If the distribution F is uniform, then the highest type is indifferent among all feasible schemes since his utility is $\int_0^1 G_{ic}^{-1}(s) \, \mathrm{d}s = \mu_G$.

highest type prefers the random allocation if the distribution of types is convex. But, then it is easy to see that all types prefer the random allocation.

3) Consider now the average contestant utility (welfare) given by

$$\begin{aligned}\int_0^1 U(\theta)f(\theta)\,\mathrm{d}\theta &= \int_0^1\left(\int_0^\theta G_{ic}^{-1}(F(\tau))\,\mathrm{d}\tau\right)f(\theta)\,\mathrm{d}\theta = \\ \int_0^1 G_{ic}^{-1}(F(\theta))(1-F(\theta))\,\mathrm{d}\theta &= \int_0^1 G_{ic}^{-1}(s)(1-s)\,\mathrm{d}F^{-1}(s)\end{aligned}$$

where the second equality follows by integration by parts, and the last equality by substituting $s = F(\theta)$.

Observe that $F^{-1}(s) = -\ln(1-s)$ and that $(1-s)\,\mathrm{d}F^{-1}(s) = 1$ for the *exponential distribution*. We obtain by Theorem 4 that random matching (assortative matching) maximizes average welfare if the distribution of types F is more convex (concave) on its domain than the exponential distribution, which yields the result.

4) If F has an increasing failure rate, the revenue (i.e., average bid) to a designer is maximized by assortative matching because assortative matching maximizes aggregate welfare while, by the above result, it also minimizes the agents' welfare. □

Lemma A.2. *A mechanism is* undominated *if there does not exists a mechanism where the set of actions is a singleton that yields a higher utility for the principal. The utility of the agent in any undominated, IC mechanism satisfies* $U_A(\theta) \geq -2Var(\gamma(\theta)) - 2\Lambda^2$

Proof: A first observation is that, in any undominated mechanism M, the utility of the principal is bounded from below by the utility she obtains in the mechanism where she takes the ex-ante optimal action $\mathbb{E}[\gamma(\theta)]$, and where she does not ask the agent to report. The principal's utility in that mechanism is given by $-Var(\gamma(\theta))$. Hence, in mechanism M there must exist at least one type $\hat{\theta}$ such that $U_p(\hat{\theta}) \geq -Var(\gamma(\theta))$. As the agent can always pretend to be of type $\hat{\theta}$, a lower bound on the utility of the agent is given by[58]

$$\begin{aligned}U_A(\theta) &\geq -(\theta-\mu_M(\hat{\theta}))^2 - \sigma_M^2(\hat{\theta}) = -([\gamma(\theta)-\mu_M(\hat{\theta})] + [\theta-\gamma(\theta)])^2 - \sigma_M^2(\hat{\theta}) \\ &\geq -2(\gamma(\theta)-\mu_M(\hat{\theta}))^2 - \sigma_M^2(\hat{\theta}) - 2(\theta-\gamma(\theta))^2 \\ &\geq -2U_P(\hat{\theta}) - 2\Lambda^2 \geq -2Var(\gamma(\theta)) - 2\Lambda^2\,.\end{aligned}$$

□

[58]The second inequality follows as for all $a, b \in \mathbb{R}$ we have $(a-b)^2 \geq 0 \Leftrightarrow \frac{a^2}{2}+\frac{b^2}{2} \geq ab \Leftrightarrow 2a^2+2b^2 \geq (a+b)^2$.

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B Online Appendix

B.1 Schur-Convex Functions and Functionals

Consider X_F and X_G to be uniform, discrete random variables, each taking n values $x_F = (x_F^1, ..., x_F^n)$ and $x_G = (x_G^1, .., x_G^n)$, respectively. Then

$$x_F \prec_{dm} x_G \Leftrightarrow F^{-1} \prec G^{-1} \Leftrightarrow G \prec F$$

where $\prec_{dm}$ denotes the classical discrete majorization relation due to Hardy, Littlewood and Polya. Thus, discrete majorization is equivalent to the present majorization relation applied to quantile functions. A function $V : \mathbb{R}^n \to \mathbb{R}$ is *Schur-convex* (*concave*) if $V(\mathbf{x}) \geq V(\mathbf{y})$ $(V(\mathbf{x}) \leq V(\mathbf{y}))$ whenever $\mathbf{x} \succ_{dm} \mathbf{y}$. If V is a symmetric function, and if all its partial derivatives exist, then the *Schur-Ostrovski criterion* says that V is *Schur-convex (concave)* if and only if

$$(x_i - x_j)\left(\frac{\partial V}{\partial x_i} - \frac{\partial V}{\partial x_j}\right) \geq (\leq)\ 0 \text{ for all } x.$$

It is useful to have a similar characterization for continuous majorization. Chan et al. (1987) showed that a law-invariant[1], Gâteaux-differentiable functional $V : L^1(0,1) \to \mathbb{R}$ respects the majorization relation on $L^1(0,1)$, if and only if its *Gâteaux-derivatives* in specially defined directions are non-positive. The considered directions are of the form

$$h = \lambda_1 \mathbf{1}_{(a,b)} + \lambda_2 \mathbf{1}_{(c,d)}$$

with $0 \leq a < b < c < d \leq 1$ and $\lambda_1 \geq 0 \geq \lambda_2$ such that $\lambda_1(b-a) + \lambda_2(d-c) = 0$. Note that the function h takes at most two values that are different from zero, and is decreasing on $[a,b] \cup [c,d]$. Moreover, $\int_0^1 h(t)\,\mathrm{d}t = 0$.

This result also yields a simple intuition for the Fan Lorentz Theorem in the case where K is differentiable. Consider a monotonic f and note that, for any direction h, the Gâteaux-derivative of the functional $V(f) = \int_0^1 K(f(t),t)\,\mathrm{d}t$ is given by

$$\delta V(f,h) = \frac{d}{d\varepsilon}\int_0^1 K(f(t) + \varepsilon h(t), t)\,\mathrm{d}t\ |_{\varepsilon=0} = \int_0^1 K_f(f(t),t)h(t)\,\mathrm{d}t,$$

where the last equality follows by interchanging the order of differentiation and integration.[2]

[1]This means that the functional is constant over the equivalence class of functions with the same non-decreasing re-arrangement. This replaces the symmetry in the discrete formulation.

[2]This is allowed since K is convex in f.

The Fan-Lorentz conditions imply together that

$$\frac{\mathrm{d}K_f}{\mathrm{d}t} = f_t \cdot K_{ff} + K_{ft} \geq 0.$$

For a direction h such that $\int_0^1 h(t)\,\mathrm{d}t = 0$, and such that h is a decreasing two-step function as defined above, we obtain that

$$\delta V(f,h) = \int_0^1 K_f(f(t),t)h(t)\,\mathrm{d}t \leq 0.$$

Hence the Fan-Lorentz functional $V(f) = \int_0^1 K(f(t),t)\,\mathrm{d}t$ is Schur-concave by the result of Chan et al. (1987)

B.2 Decision-Making Under Uncertainty

We briefly illustrate here how our insights can be applied in order to understand how agents with non-expected utility preferences choose among risky prospects.

B.2.1 Rank-Dependent Utility and Choquet Capacities

Quiggin (1982) and Yaari (1987) axiomatically derived utility functionals with rank-dependent assessments of probabilities of the form[3]

$$U(F) = \int_0^1 v(t)\,\mathrm{d}(g \circ F)(t)$$

where F is the distribution of a random variable on the interval $[0,1]$, $v : [0,1] \to R$ is continuous, strictly increasing and bounded, and where $g : [0,1] \to [0,1]$ is strictly increasing, continuous and onto. The function v represents a transformation of monetary payoffs, while the function g represents a transformation of probabilities[4].

The case $g(x) = x$ yields the classical von-Neumann and Morgenstern expected utility model where risk-aversion is equivalent to v being concave. The case $v(x) = x$ yields Yaari's (1987) dual utility theory, where risk aversion is equivalent to g being concave. Because of the possible interactions between v and g, it is not clear what properties yield risk aversion

[3]Their theory is a bit more general (for example it allows a more general domain for the functions v and F) . We keep here a framework that is compatible with the rest of the paper.

[4]For the sake of brevity we assume below that both g an v are twice differentiable. Since the Fan-Lorentz result does not require differentiability, the observations below generalize.

in the general rank-dependent model. Using integration by parts, we can also write:

$$\begin{aligned} U(F) &= \int_0^1 v(t)\,\mathrm{d}(g\circ F)(t) = v(1) - \int_0^1 v'(t)(g\circ F)(t)\,\mathrm{d}t \\ &= v(1) + \int_0^1 K(F(t),t)\,\mathrm{d}t \end{aligned}$$

where

$$K(F,t) = -v'(t)(g\circ F)$$

and where we used $g(0)=0$ and $g(1)=1$. Then

$$\frac{\partial^2 K(F,t)}{\partial F \partial t} = -g'(F(t))v''(t) \geq 0$$

for all t if and only if v is concave. Similarly

$$\frac{\partial^2 K(F,t)}{\partial^2 F} = -g''(F(t))v'(t) \geq 0$$

for all t if and only if g is concave.

Hence, the Fan-Lorentz conditions are satisfied if and only if $v'' \leq 0$ and $g" \leq 0$. As a consequence, the utility functional $U = \int_0^1 v(t)\,\mathrm{d}(g\circ F)(t)$ is Schur-concave, and the agent whose preferences are represented by U is *risk averse* , exactly as under standard expected utility[5].

Another important strand of the literature on non-expected utility considers ambiguity aversion. The main tool is the *Choquet integral* with respect to a (convex) *capacity* (this is unrelated to the Choquet representation used above!) Analogously to the derivations above, it can be shown that the Choquet integral yields a Schur-concave functional if and only if it is computed with respect to a convex capacity.

B.2.2 A Portfolio Choice Problem

Dybvig (1988) studies a simplified version of the following problem:

$$\begin{gathered} \min_X \mathbb{E}[\ XY\ ] \\ \text{s.t. } X \geq_{cv} Z \end{gathered}$$

[5] The equivalence between the concavity of the functions v and g, and risk-aversion has been pointed out by Hong et al (1987), who build on Machina (1982).

where Y and Z are given random variables. Y represents here the distribution of a pricing function over the states of the world, and the goal is to choose, given Y, the cheapest contingent claim X that is less risky than a given claim Z. To make the problem well-defined, Y needs to be essentially bounded and X, Z must be integrable. Recalling that

$$X \geq_{cv} Z \Leftrightarrow F_X \succ F_Z \Leftrightarrow F_X^{-1} \prec F_Z^{-1}.$$

we obtain that:

$$\mathbb{E}[XY] \geq \int_0^1 F_Y^{-1}(1-t) F_X^{-1}(t)\,\mathrm{d}t \geq \int_0^1 F_Y^{-1}(1-t) F_Z^{-1}(t)\,\mathrm{d}t$$

where the first inequality follows by the rearrangement inequality of Hardy, Littlewood and Polya (1929) (the anti-assortative part!), and where the second inequality follows by the Fan-Lorentz Theorem.

By choosing a random variable X that has the same distribution as Z and that is anti-comonotonic with Y,[6] the lower bound $\int_0^1 F_Y^{-1}(1-t) F_Z^{-1}(t)\,\mathrm{d}t$ is attained, and hence such a choice solves the portfolio choice problem.[7]

If $Y' \leq_{cv} Y$, we obtain by the Fan-Lorentz inequality (now applied to the functional with argument F_Y^{-1}) that

$$\sup_{X \succ_{cv} Z} \mathbb{E}[XY] = \int_0^1 F_Y^{-1}(1-t) F_Z^{-1}(t)\,\mathrm{d}t \geq \int_0^1 F_{Y'}^{-1}(1-t) F_Z^{-1}(t)\,\mathrm{d}t = \sup_{X \succ_{cv} Z} \mathbb{E}[XY']$$

In other words, a decision maker that becomes more informed (in the Blackwell sense) will bear a lower cost.

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[6]This can always be done if the underlying probability space is non-atomic. A random vector (X, Y) is anticomonotonic if there exists a random variable W and non-decreasing functions h_1, h_2 such that $(X, Y) =^{dist} (h_1(W), -h_2(W)$.

[7]For more details on this problem see Dana (2005) and the literature cited there. It does not use the Fan-Lorentz inequality.

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