Discussion Paper Series – CRC TR 224

Discussion Paper No. 275
Project C 03

Market Depth, Leverage, and Speculative Bubbles

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March 2021

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Funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 is gratefully acknowledged.

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Abstract

We develop a model of rational bubbles based on leverage and the assumption of an imprecisely known maximum market size. In a bubble, traders push the asset price above its fundamental value in a dynamic way, driven by rational expectations about future price developments. At a previously unknown date, the bubble will endogenously burst. Households optimally decide whether to lend to traders with limited liability. Bubbles increase welfare of the initial asset holders, but reduce welfare of future households. We provide general conditions for the possibility of bubbles depending on uncertainty about market size, traders' degree of leverage and the risk-free rate. This allows us to discuss several policy measures. Capital requirements and a correctly implemented Tobin tax can prevent bubbles. Implemented incorrectly, however, these measures may create the possibility of bubbles and can reduce welfare.

Keywords: Bubbles, Rational Expectations, Market Size, Liquidity, Financial Crises, Leveraged Investment, Capital Structure.

JEL-Codes: E44, G01, G12.

*We thank Felix Bierbrauer, Olivier Blanchard, John Boyd, Martin Hellwig, Andreas Irmen, Dirk Krüger, Benny Moldovanu, George Pennacchi, Eva Schliephake, Mark Brisske, our anonymous referees, workshop and conference participants at the Austrian National Bank, the Deutsche Bundesbank, the German Economic Association, the German Finance Association, the IMF, the Max Planck Institute Bonn, and the Universities of Bonn, Freiburg, Göttingen, Hannover, Heidelberg, Innsbruck, Karlsruhe, Luxembourg, Mainz, Mannheim, and Tübingen for helpful discussions. All errors are our own. Financial support from the German Research Foundation (DFG, through EN892/1 and CRC TR 224 TP C03 and EXC2126/1-390838866) is gratefully acknowledged. A former version of this paper was called "On the Existence and Prevention of Speculative Bubbles".

1. Introduction

Which conditions can lead to bubbles? How can they be prevented? To answer these questions, we propose a fundamental mechanism that allows for the emergence of bubbles, based on an unknown maximum market size and limited liability of traders. Analyzing optimal lending of households to traders and the investment decision of traders, we derive conditions under which bubbles can occur. These conditions allow us to evaluate policy measures for the prevention of bubbles, such as a Tobin tax or capital requirements. We also analyze who benefits from these policies and who loses.

The model features traders and households, both with endowments they want to invest. Households have a relative investment disadvantage, which incentivizes them to lend to traders. We make two crucial assumptions in the model setup. First, traders face limited liability towards households. Second, the number of households and traders is fixed but unknown. We call the aggregate volume of traders' and households' endowments *market depth*, as it represents the maximum amount of investment that a specific asset market can attract. Rational traders in this market are willing to invest in an overpriced asset, as long as there is a sufficiently high probability that they will be able to sell the asset, at an even higher price, to yet another future market participant.[1] In this case there can be price paths above the steady-state price, which we call bubbles. At some point, a bubble will exceed market depth, leading to an immediate burst. The risk of holding the asset at this moment deters traders from buying the asset on their own account. In equilibrium, however, traders are leveraged with loans from households. This creates incentives to invest in a risky, overpriced asset. Whether this rationale is sufficient to create a bubble depends on the thickness of the tail of the ex-ante distribution of market depth and on the degree of leverage.

By assuming an unknown market depth and limited liability, we combine core elements from the models of bubbles of Blanchard and Watson (1982) and Weil (1987) on the one hand, and Allen and Gale (2000) on the other. Blanchard and Watson show, in a nutshell, that asset-price bubbles can always emerge if market depth is implicitly assumed to be large enough.[2] Weil (1987)

1. In the model, traders are aware that they are investing in a bubble. Conlon (2004) argues that in many bubble periods the overvaluation of assets was widely discussed. Referring to the dot-com bubble, Brunnermeier and Nagel (2004) provide evidence that hedge funds were riding the bubble, a result similar to a previous finding by Wermers (1999). The authors relate this to a short-term horizon of the managers, among other elements. Our model is consistent with this notion.

2. Tirole (1982) takes the opposite position and shows that in a simple finite economy bubbles are impossible. Santos and Woodford (1997) also demonstrate that the conditions for the existence of bubbles are very restrictive if one assumes a fixed number of households that participate in the asset market and own finite aggregate endowments. The model of Zeira (1999) is similar to our model as he also assumes an unknown market size after, e.g.,

and related papers demonstrate that bubbles can also develop if the economy, and hence market depth, is growing in a dynamically inefficient way. Due to the assumption of a *fixed* market depth, in contrast, our model applies to short-run dynamics. That is, bubbles can emerge in a stationary economy and their expected lifespan may be brief. Moreover, and different from Blanchard and Watson (1982), we explicitly spell out the conditions that beliefs about market depth have to fulfill for bubbles to be possible. Crucially, no matter how large the bubble has become, there must always be at least some probability that it will survive until the next period. Thus, the distribution function of ex-ante expectations regarding market depth needs to have unbounded support and a sufficiently thick tail. Yet, market depth is finite and will be revealed once it is exhausted. Traders learn from each price increase that this critical value has not yet been reached and update their expectations accordingly. For a low degree of uncertainty about market depth, the bubble size that will soak up all market liquidity can be computed with sufficient accuracy. When the asset price is close to this maximum, traders will be unwilling to continue investing. By backward induction, no bubble can then exist in the first place. For a higher degree of uncertainty, bubbles can emerge but will endogenously burst at an unknown date. Our assumption of finite but imprecisely known market depth therefore endogenizes a time-varying probability of bursting; in Blanchard and Watson (1982) and Weil (1987), this probability is constant and exogenous. Unknown market depth in our model is meant to capture, in a stylized way, uncertainty regarding the size of future investments into a specific market. The financial crisis of 2007 has forcefully shown that domestic and, even more so, international capital flows can swell and dry up very quickly. The size and end point of these flows are not precisely predictable, just like the fraction that will be channeled into a specific asset market.[3] Hence, in light of increasingly complex and opaque financial markets, the maximum amount of resources that a particular market might attract can only be estimated, with significant uncertainty always remaining.

Yet, even for relatively high uncertainty regarding market depth, traders in the model are only willing to invest in a bubble if they do not bear the full financial risk in case of a burst. Like Allen and Gale (2000), we find that traders are willing to invest in overpriced assets if some of the risk is

a financial liberalization. This uncertainty, however, creates asset-price booms and crashes by moving the fundamental value, above which the price cannot rise. Similarly, Allen and Gale (2000) show in a two-period model that expected expansions in credit can generate uncertainty about the steady-state price, which influences prices in previous periods. Prices can then also fall, depending on the realized expansion of credit.

3. Before the crisis, many foreign investors were buying collateralized debt obligations issued in the US. How much they had invested was not even clear for market participants after the crisis had unfolded (see, e.g., Carrington et al. 2008). More recently, many observers were wondering how much Chinese investors in particular will continue to invest in the Australian housing market, see UBS (2017) and Punwasi (2017), among others.

shifted towards households. Different from Allen and Gale, these overvaluations may increase dynamically above the steady-state price, which is constant and known to all agents. In our model, limited liability arises because traders borrow from rational, risk-neutral households, in addition to investing their own funds.[4] Households know whether the market is in a bubble but cannot observe the investment decisions of traders. Instead, we assume that households can monitor true project returns only at a cost. This endogenously makes debt the preferred form of a financial contract, although households know that leveraged traders might invest in assets they themselves would deem too risky. Traders profit from rising asset prices, but potential losses are limited to their own invested funds. The model directly applies to any type of intermediated finance with limited liability, such as investment through banks, investment banks, insurance companies, and private equity firms as well as to non-intermediated, debt-financed investments. We show that, depending on traders' investment opportunities compared with those of households, the latter are willing to lend to traders under similar conditions as those that let traders invest in bubbles. We also demonstrate that traders who hold a bubbly asset at the time a bubble emerges benefit in terms of realized consumption, while households that invest during a bubble have lower expected consumption levels. This is due to the limited liability of traders, which leads to a socially undesirably large amount of resources flowing into the risky asset; this amount increases as the bubble grows. An additional welfare-reducing effect arises if households have heterogeneous alternatives for investing their funds instead of lending to traders. Similar to a lemons problem, a bubble may lead to a partial breakdown of the lending market: some households prefer to invest elsewhere, despite the superior investment opportunity of traders. Because of these welfare results, policy interventions intended to prevent bubbles, if successful, increase expected welfare of future investors. At the same time, they hurt initial investors who cannot start a bubble by selling an overpriced asset.

Summing up, the first main contribution of the paper is the development of the bubble-generating mechanism by using elements from Blanchard and Watson (1982), Weil (1987), and Allen and Gale (2000).[5] In this setup,

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4. According to the OECD database on institutional investors' assets, in 2007 institutional investors in the US managed assets worth 211.2% of GDP, showing investors' prominent role in investment decisions. Furthermore, this amount has grown steadily over the last decade with a yearly average growth rate of 6.6% from 1995-2005 within the OECD(17), see Gonnard et al. (2008).

5. Brunnermeier (2001) provides an extensive survey of alternative bubble models based on asymmetric information. From the vast literature on bubbles, Allen and Gorton (1993) is closest to our model in spirit. In their finite-horizon model in continuous time, traders are exiting the bubbly market one after another. They are willing to take on the risk of being the last market participant because of limited liability. Heterogeneous types of traders and stocks induce investors, who cannot buy the same assets, to lend despite limited liability. For each trader, the number of remaining other traders is uncertain; the bubble bursts when the market ceases to exist. Different to our model, higher risk does not imply a higher potential return for speculating traders.

the possible existence of bubbles depends on the economic environment, because first, bubbles might become too risky for traders to invest, and second, households' participation constraint (governing their decision to lend to traders) can be violated, today or at a future date. We provide necessary and sufficient conditions for both constraints. Hence, depending on the interaction of leverage, uncertainty about market depth, expected lifetime of the asset, and the risk-free interest rate, the prerequisites for bubbles may be fulfilled or not.[6] The model can hence answer questions like 'Does high leverage foster the emergence of bubbles?' Some policy implications ensue immediately; they form the second main contribution of the paper.

The remainder of this paper is organized as follows. Section 2 introduces the model and the equilibrium concept. Section 3 constructs the unique steady-state (rational-expectations) equilibrium price process. Section 4 provides a necessary and sufficient condition for the existence of bubbles. The section begins with the construction of a special type of bubble, which then serves as an example for the general case. We then provide a necessary and sufficient condition for households' participation in lending. The conditions lend themselves to basic policy analysis, which is performed in Section 6, preceded by a welfare analysis of homogeneous- and heterogeneous-household setups in Section 5. Section 7 concludes. All proofs are in the appendix.

2. The Model

2.1. Setup

Consider an infinite-horizon economy with a series of cohorts of risk-neutral households and traders.[7] In each period, a continuum $[0, N]$ of traders and a continuum $[0, N] \times [0, 1]$ of households are born. The assumption that for each trader there is a continuum of households implies that no single household can influence a trader's balance sheet, or investment choice. Each household has an initial endowment of $l \in (0, 1)$, and each trader owns $1-l$ non-storable consumption goods, such that the total endowment of one unit measure of

6. Using the latest US housing bubble as an example, we find that conditions that are favorable for the emergence of bubbles in our model were fulfilled. Increasingly international financial flows and more complex financial instruments obscured potential market depth. Furthermore, the Securities and Exchange Commission's 2004 decision to allow large investment banks to assume more debt raised their leverage and further increased uncertainty about market depth. Kaminsky and Reinhart (1999), among others, also indicate an empirical connection between financial liberalization, credit expansion, and bubble emergence. Moreover, Adelino et al. (2016) find evidence that house buyers were indeed attracted by the prospect of higher future prices.

7. Traders enter and exit the market in an OLG fashion to generate trade each period. We do not see this OLG structure as representing actual generations, but as a shortcut for non-modeled market imperfections, such as heterogeneous liquidity preferences of traders.

households and one trader is normalized to unity. Thus, N is the amount of wealth in the economy and the maximum market capitalization of any asset. We therefore call N market depth. It is fixed over time but unknown. Traders are thus uncertain about how many resources other traders can invest in a certain market.[8] Assume that the distribution $F(N)$ is a Pareto,

$$F(N) = 1 - (N/N_0)^{-\gamma} \tag{1}$$

for $N \in [N_0, \infty)$, with $N_0 > 0$. For our analysis, the most relevant statistic of the distribution function is the parameter γ, which denotes the probability that market depth will be exhausted for a marginal percentage increase in the price. Hence, γ measures the thinness of the tail or equivalently the precision of the information on market depth N.[9] If $\gamma \to \infty$, the tail becomes arbitrarily thin and $N = N_0$ is the only possible realization. We will see that in that case, bubbles cannot emerge. N_0 represents the lowest possible N, it drops out of the analysis at an early stage.

There are two types of assets, safe assets (short: storage) of unlimited supply and a single risky asset (short: the asset) with a supply of one. Storage bears a risk-free return of $Y > 1$ to a trader. The return of the storage asset to a household is only λY, with $0 \leq \lambda < 1$. The inverse of λ thus measures the investment advantage of traders, which may arise because of higher abilities or better information. Because of these different returns, households can gain from lending to traders. However, households can observe the traders' return only at a monitoring cost $c > 0$, but they can commit to monitoring. Only traders have access to the risky asset.[10] The asset can be interpreted as shares of a firm. It cannot be sold short. The firm pays a dividend of d in each period until it goes bankrupt. There is a probability $1 - q \geq 0$ in each period that this happens;

8. In a model with stochastic growth, we would similarly obtain a non-zero conditional probability for future increases of total resources. In this case, N would vary stochastically over time. The present setup with a fixed N, in contrast, corresponds to an analysis of short-run dynamics. Note that infinite horizons are not central to this model setup in which traders hope to sell a risky asset before the bubble bursts, if combined with an asymmetric information framework similar to Allen et al. (1993), Conlon (2004, 2015), and Doblas-Madrid (2016).

9. If the distribution is not Pareto, our results still hold, γ needs to be replaced by the so-called stable index $\lim_{N\to\infty} N f(N)/(1 - F(N))$.

10. Even if households could invest in the risky asset, they would not do so because in equilibrium the asset will be overpriced (as we will show in Section 3.1). Note that the reason for trade between agents is the higher return to traders, as in Allen and Gale (2000), Allen and Gorton (1993), and Barlevy (2014), instead of some form of risk sharing, as in Allen et al. (1993) and Conlon (2004, 2015).

in this event the firm's shares are then worthless.[11] Hence q determines the expected lifetime of the asset.

Households and traders have a life span of two dates. They invest at date t. At date $t+1$, they liquidate their investment and consume. The duration of a period between these two dates stands for the investment horizon of a trader. At each date, old traders sell the asset to young traders on a competitive market. Expected lifetime utility of each trader and household depends linearly on consumption,

$$U_i^H = EC_i^H \qquad \text{and} \qquad U_j^T = EC_j^T. \tag{2}$$

C_i^H is consumption of household $i \in [0,N] \times [0,1]$ and C_j^T that of trader $j \in [0,N]$ in the next period. E is the expectational operator based on information available in the current period, common to households and traders.

The optimization problems of traders and households can be divided in two parts. In Section 2.2, we develop the optimal contract in case households lend to traders. We then define the corresponding equilibrium in Section 2.3 and derive the equilibrium investment decisions of households and traders in sections 3 and 4.

2.2. The contract between households and traders

Households can lend to traders in order to profit from their better investment opportunities. A household can only observe the return from the traders' investment if it exerts a cost c. As a consequence, each household lends to one trader (or not). The index i then denotes a household and the trader she lends to. Let x_i^H represent the amount that household i lends, and x_i^T the amount that trader i borrows, with $0 \leq x_i^H, x_i^T \leq l$. Both parties have to agree upon a repayment structure. In general, if the gross return from investing in the safe and the risky asset to trader i, who borrows from households i, is R_i, then the financial contract between the two can stipulate any repayment $z(R_i) \leq R_i$. Without loss of generality, we assume that if the trader is unable to fulfil her contractual obligations, the households are repaid a fraction x_i^H/x_i^T of the trader's total assets, i.e., pro rata. Because each household and each trader is small, they do not internalize the externality of their contract choice on equilibrium prices. Commitment to monitoring, contingent on the trader's repayment, is possible.[12] In analogy to Townsend (1979) and Gale and Hellwig

11. The asset can also be interpreted as real estate, where $1-q$ is the probability that the land becomes uninhabitable in the next period (e.g., because of a flooding). If the asset represents art, q is the probability that the piece of art is not destroyed. For some assets, like gold, $q=1$, or $q<1$ if it can get stolen.

12. Like Townsend (1979) and Gale and Hellwig (1985), we consider commitment only in pure strategies. Implicitly, we thus assume that a commitment device exists, but only for pure strategies. If stochastic monitoring were allowed, households could save costs by using

(1985), we can then show that the optimal contract under pure strategies has the shape of a pure debt contract. The rest of the model is independent from this specific choice for endogenizing the debt contract. For the emergence of bubbles, any other micro-foundation would be equivalent.

LEMMA 1 (Optimal Contract). *There is a $\bar{c} > 0$ such that for $c \geq \bar{c}$ each household maximizes its utility U_i^T in (2) by choosing a standard debt contract with a repayment z of the form*

$$z(R_i) = \min\{\beta_i; R_i x_i^H / x_i^T\}. \tag{3}$$

The trader keeps the residual from this individual contract

$$\max\{R_i x_i^H / x_i^T - \beta_i; 0\} \quad \textit{with} \tag{4}$$

$$\beta_i = Y x_i^H. \tag{5}$$

The variable β_i summarizes the contract: it represents the contracted repayment of a loan, including interest payments. The gross loan rate is $\beta_i / x_i^H = Y$. We assume that traders are in perfect competition, i.e., they compete for the funds of households. This forces traders to their participation constraint. That is, β_i is set such that investing their own funds in the safe asset (with a return of $Y(1-l)$) or investing their own plus borrowed resources in the safe asset (with a return of $Y(1-l+x_i^T)-\beta_i$) yields the same payoff, where the risky asset has the same expected return for traders in equilibrium. This implies equation (5). To analyze the consequences of the contract design, we will discuss how bubbles depend on β_i, but bear in mind that β_i is an endogenous object in this setup.[13] Comparative statics with respect to β_i, holding Y constant, are then equivalent to comparative statics regarding leverage of traders x_i^H, which will equal l if households lend (shown below).

2.3. Equilibrium and Definitions

The price path $\{p_t\}_{t \in \mathbb{N}}$ of the asset is the realization of a stochastic process $\{\tilde{p}_t\}_{t \in \mathbb{N}}$.[14] Given an optimal contract and expectations induced by a stochastic

an expected monitoring frequency just high enough to implement truth telling on the side of the trader. The optimal contract may then involve rebates, like in Border and Sobel (1987). Importantly, the trader's expected return is convex in her investment success. Hence, in neither of these modeling choices would leverage, our key ingredient for bubbles, disappear, but the algebra would be much more involved.

13. The parameter β_i can alternatively be interpreted as a hurdle: if the return exceeds β_i, the trader collects a bonus, otherwise not. With such an interpretation, other policy measures can be analyzed, like the relation between the traders' compensation package and the emergence of bubbles.

14. To be precise, $\{p_t\}_{t \in \mathbb{N}}$ follows a stochastic process $(\Omega, \mathcal{F}, P)$. Here, Ω is the sample space, consisting of all possible price paths $\{p_t\}_{t \in \mathbb{N}}$. $\mathcal{F}$ is a filtration, that is, a family of

process $\{\tilde{p}_t\}_{t\in\mathbb{N}}$, the young household i's optimization problem is

$$\max_{x_i^H} U_i^H \quad \text{s.t.} \quad C_i^H = (l - x_i^H)\lambda Y + x_i^H \min\{Y; R_i/x_i^T\}. \tag{6}$$

The optimization problem of the young trader i is

$$\max_{x_i^T, y_i^T} U_i^T \qquad \text{s.t.} \quad C_i^T = R_i - \min\{x_i^T Y; R_i\} \tag{7}$$
$$\text{and} \quad R_i = (1 - l + x_i^T)[(1 - y_i^T)Y + y_i^T Y^r],$$

where $0 \leq y_i^T \leq 1$ denotes the share of total assets under control of trader i invested in the risky asset and Y^r is the return to the risky asset, as induced by the stochastic price process $\{\tilde{p}_t\}_t$. Since lifetime utilities U_i^H and U_i^T consist of expected consumption, see equation (2), traders and households need to form expectations about Y^r. We can derive a basic insight from this optimization problem that is valid in all following sections: traders will not mix storage and investment in the asset, due to their limited liability.

LEMMA 2 (Trader Behavior). *For an optimal contract, traders either invest all funds under their control in the safe asset, $y_i^T = 0$, or all in the risky asset, $y_i^T = 1$.*

We are interested only in stochastic processes that form rational-expectations equilibria for both traders and households.

DEFINITION 1 (Rational-Expectations Equilibrium). *A stochastic process $\{\tilde{p}_t\}_{t\in\mathbb{N}}$ is an equilibrium if at each date t,*

1. *given the expectations induced by the stochastic process $\{\tilde{p}_t\}_t$, young households maximize their optimization problem (6) by choosing the amount x_i^H they lend to traders,*
2. *given the expectations induced by the stochastic process $\{\tilde{p}_t\}_t$, young traders maximize their optimization problem (7) by choosing the amount x_i^T they borrow from households and the share y_i^T of funds under their control they invest in the risky asset,*
3. *the market for intermediated funds clears,*

$$\iint_{i\in[0,N]\times[0,1]} x_i^T \, di = \int_{j\in[0,N]} x_j^H \, dj, \tag{8}$$

σ-algebras that contains past information. P is the probability measure. For each current price p_t, the measure P gives the probability of future prices. In that sense, if agents know the stochastic process, the probability distribution of future prices is induced by P. To avoid clutter, we write $\{\tilde{p}_t\}_{t\in\mathbb{N}}$ for the stochastic process.

4. *and the market for the risky asset clears,*

$$\int_{j\in[0,N]} y_j^T(1-l+x_j^T)dj = p_t. \tag{9}$$

Old households and old traders are not included in the equilibrium definition because they do not have a choice to make. Both consume the returns from their investment.

Importantly, depending on the model parameters there may be only one stochastic process compatible with a rational-expectations equilibrium as defined in Definition 1, or there may be several. One possible stochastic process is the steady-state stochastic process.

Definition 2 (Steady State). *A steady-state price path is a price path where the asset price is constant for each date, until it switches to zero at some point. The constant is called the steady-state price $\bar{p}$.*

A steady-state stochastic process is thus a Markov process with just two states, and zero as an absorbing state. Any price movement is caused by a change in the fundamentals. From now on, we assume that market depth is sufficient to reach at least the steady-state price, i.e., we rule out cash-in-the-market pricing by assuming $N_0 \geq \bar{p}$.[15] We demonstrate in Section 3 that a rational-expectations equilibrium steady-state stochastic process always exists and is uniquely determined. Hence, if there is just one stochastic process compatible with a rational-expectations equilibrium, then only one initial price is compatible with a rational-expectations equilibrium: the steady-state price $\bar{p}$. In Section 4, we show that if there are several stochastic processes compatible with a rational-expectations equilibrium, bubbles in the sense of the following definition are possible.

Definition 3 (Bubble). *A bubble is a price path with prices above the steady-state price $\bar{p}$.*

Note that, because a steady-state price path always exists, the existence of a bubble always implies multiple equilibria. Which price path realizes depends on the expectations of traders. As will be shown, bubbles are only possible if traders expect rising prices. Our definition thus isolates speculative and dynamic bubbles. The steady-state price will typically be above the

15. If $N_0 < \bar{p}$, and thus possibly $N < \bar{p}$, we could obtain $p = N$. Market depth N would then be immediately revealed and the price could never rise above the initial price, an uninteresting case. The assumption $N_0 \geq \bar{p}$ does not change the bubble conditions.

fundamental value, i.e., the present discounted value of future dividends, but it is not a bubble according to our definition. In particular, our aim is to discuss bubbles with the following properties: a) they will burst for sure, even if the underlying asset does not break down, and b) they involve rising price paths. A constantly overpriced asset does not fulfill these requirements. Note that this definition deviates from that of, e.g., Allen and Gale (2000) who call any price path above the fundamental price a bubble.

Before analyzing the model, let us summarize the key imperfections and their implications. First, households have an investment disadvantage relative to traders, i.e., they receive a lower return when investing in the safe asset. They hence try to benefit from the better investment opportunity of traders. Second, ex interim, households cannot monitor traders' investment choices. Third, households cannot monitor investment returns without costs (costly state verification). These three assumptions result in an endogenous debt contract. The latter two constitute a moral hazard problem, while the first assumption assures that the lending market does not break down. Instead of these three assumptions, one could alternatively have assumed that there is only debt finance. Fourth, and most importantly, market depth N is unknown. Each of these four imperfections is necessary for bubbles to emerge. With equity finance instead of debt, the upside risk of a potential price movement can never be enough to compensate a trader for the risk of a bursting bubble. If the true value of N was known, the maximum market size could be calculated and backward induction would prevent a bubble from taking off. This highlights the crucial interaction between limited liability, present in a debt contract, and uncertainty about market depth.

3. The Steady State

Our goal in this section is to construct, and thus prove the existence, of a steady-state price process.

Proposition 1 (Steady State). *A steady-state equilibrium price process always exists and is uniquely determined.*

We start by investigating asset-market clearing in Section 3.1 under the assumption that all households lend their endowment to traders, $x_i^H = l$ for all i. We then analyze when this is the case in Section 3.2. Given this assumption and symmetry of the optimization problem (7) for all traders, we can drop the index i in Section 3.1.

3.1. Asset-Market Clearing

Consider a steady-state stochastic process $\{\tilde{p}_t\}_{t\geq 0}$. With probability q, the price of the asset equals some constant, $\tilde{p}_t = \bar{p}$. If the underlying firm goes bankrupt (with probability $1-q$), cash ceases to flow, and the price drops to zero. That is, the price follows a Markov process with $\Pr_t\{\tilde{p}_{t+1}=\bar{p}|p_t=\bar{p}\}=q$. Let us now derive the price $\bar{p}$ for which the asset market clears. Traders expect the steady-state price $\bar{p}$ to be realized with probability q. Each trader invests an amount of one ($1-l$ own and l borrowed endowment goods). At the price $\bar{p}$, she can buy a volume $1/\bar{p}$ of the asset. She then benefits from the dividend d and hence earns $d/\bar{p}$ with probability q. In the absence of a bankruptcy the price remains at $p_{t+1}=\bar{p}$ and the trader additionally receives $p_{t+1}/p_t=\bar{p}/\bar{p}=1$ from selling the asset. A trader's expected payoff is therefore

$$\mathrm{E}\ \max\left\{0;\, Y^r-\beta\right\} = \mathrm{E}\ \max\left\{0;\, \frac{\tilde{p}_{t+1}}{p_t}+\frac{d}{p_t}-\beta\right\} = q\ \max\left\{0;\, \frac{\bar{p}+d}{\bar{p}}-Yl\right\}$$

because of the assumption that $x^H=l$ and hence $\beta=Yl$. Furthermore, due to Lemma 2, traders do not mix both forms of investment. This implies that, in equilibrium, a number of $\bar{p}$ traders buy $1/\bar{p}$ shares, while the others just store their endowment. As an additional consequence, traders cannot infer the number N from market information. If a trader opts for storage, her payoff is $Y-\beta=(1-l)Y$. For the market to clear, both options have to yield the same return,

$$(1-l)Y = q\ \max\left\{0;\, \frac{\bar{p}+d}{\bar{p}}-Yl\right\} \tag{10}$$

$$\Rightarrow \quad \bar{p} = \frac{d\,q}{Y-q-Yl\,(1-q)}. \tag{11}$$

The steady-state price $\bar{p}$ depends on traders' leverage l and exceeds the fundamental value,

$$\underline{p} := \sum_{t=1}^{\infty}\frac{q^t\,d}{Y^t} = \frac{d\,q}{Y-q} \leq \bar{p}. \tag{12}$$

Only if $l=0$ (no leverage) or if $q=1$ (unlimited lifetime, no fundamental risk), the fundamental value and the steady-state price are equal, $\underline{p}=\bar{p}$. Any $l>0$ makes the traders' target function convex, which raises the asset price in the presence of risk. The deviation between the two is static and driven by fundamentals, but not by traders' expectations about increasing prices. In our definition, this deviation does not constitute a bubble. The effect of leveraged traders pushing prices of risky assets above their fundamental levels has been analyzed previously by Allen and Gale (2000).[16]

16. Malamud and Petrov (2014) refer to this effect as mispricing, not as a bubble.

3.2. Household Participation in Lending

We now check whether households want to lend to traders in the steady-state. An individual household i investing x_i^H could alternatively obtain $\lambda Y x_i^H$ from storage. When the household delegates investment to a trader, where traders are ex-ante symmetric, it does not know whether the trader uses the borrowed funds to buy the risky asset. Let $\overline{x}$ denote the total funds that a trader has under her control, i.e., $\overline{x} = 1 - l + x^T$, where x^T equals the average x_i^H under market clearing for intermediated funds (8). At an asset price $\bar{p}$, a number $\bar{p}/\overline{x}$ traders (equivalent to the share $y^s = \bar{p}/(N\overline{x})$ of traders) buy the asset, while the remaining $N - \bar{p}/\overline{x}$ invest safely (storage). The probability that a trader invests riskily is thus $\bar{p}/(N\overline{x})$. In this case, the household is repaid $\beta_i = Y x_i^H$ only with probability q. With probability $1-q$, the trader defaults and the household pays the verification cost c. If the trader stores the capital and earns a gross return Y, the household receives β_i with certainty. Households, however, do not know N. They know that N must be at least the current price $p_t = \bar{p}$ divided by $\overline{x}$. The conditional distribution of N is then $F(N|p = \bar{p}) = 1 - (N\overline{x}/\bar{p})^{-\gamma}$, and thus $f(N|p = \bar{p}) = \gamma\, (N)^{-\gamma-1}\, (\bar{p}/\overline{x})^{\gamma}$. The expected return to household i in the steady state is then

$$\int_{\bar{p}/\overline{x}}^{\infty} \left[\frac{\bar{p}}{N\overline{x}}\, (q\,\beta - (1-q)\,c) + \left(1 - \frac{\bar{p}}{N\overline{x}}\right) \beta_i \right] f(N)\,\mathrm{d}N = \frac{(1+q\,\gamma)\,\beta_i - (1-q)\,\gamma\,c}{1+\gamma}. \tag{13}$$

In a symmetric equilibrium, all households take the same decision and we can again drop the index i. The expected return exceeds the household's private return $\lambda Y x^H$ whenever

$$\lambda \leq \hat{\lambda} := \frac{1}{\beta}\, \frac{(1+q\,\gamma)\,\beta - (1-q)\,\gamma\,c}{1+\gamma}. \tag{14}$$

Note that $\hat{\lambda}$ depends positively on $\beta = Y x^H$ and reaches its maximum at $x^H = l$. Hence, if $\lambda > \hat{\lambda}$ for $x^H = l$, gains from trade are small, relative to the risk of a speculative investment, such that households refrain from lending to traders, $x^H = 0$ and no intermediation takes place. As a consequence, traders are not leveraged and the asset trades at its fundamental value. Yet, the intermediation market is in equilibrium: traders are indifferent between borrowing or not, since they would be pushed to their participation constraint in case they could borrow. Any x^T in condition (8) is therefore compatible with market clearing of the intermediation market. The asset market is in equilibrium as traders buy the asset with their own funds. The price process is a steady-state process, with $\bar{p} = \underline{p}$ and $\beta = 0$. The price process is thus always uniquely determined.

In short, households hence solve their optimization problem (6) by either setting $x^H = l$, if (14) holds, or $x^H = 0$, if not. In all cases, there is a fixed price until the underlying asset breaks down, which proves Proposition 1.

4. Bubbles

We now come to the main question of the paper. For given fundamentals, is a price path above the steady-state price $\bar{p}$ possible? We answer this question in two theorems. The first, stated below, shows under which conditions bubbles can emerge on the asset market, given the assumption of $x_i^H = l$ for all i. The second, in Section 4.3, states when this assumption is fulfilled. Again, given $x_i^H = l$ for all i and symmetry of the optimization problem (7) for all traders, we can drop the index i until Section 4.3.

THEOREM 1. *The set of stochastic processes in a rational-expectations equilibrium contains bubbles if and only if*

$$(\gamma + 1)(1 - \beta) < 1 \qquad \text{and} \tag{15}$$

$$Y - \beta \leq q \, \frac{\gamma^\gamma}{\beta^\gamma (\gamma + 1)^{\gamma+1}}. \tag{16}$$

This set is non-empty. For endogenous contracts, $\beta = Yl$, such that the above inequalities become

$$(\gamma + 1)(1 - Yl) < 1 \quad \text{and} \quad Y(1 - l) \leq q \, \frac{\gamma^\gamma}{(Yl)^\gamma (\gamma + 1)^{\gamma+1}}. \tag{17}$$

The proof of Theorem 1 proceeds in two steps. In Section 4.1, we first assume that (15) and (16) hold and construct a bubble. We then show that, if the conditions do not hold, the steady state is the unique price path, such that bubbles are not possible. The intuition for the proof is in the main text; some of the formalism is relegated to the appendix. The following proposition summarizes the comparative statics for Theorem 1.

PROPOSITION 2. *For an endogenous contract, i.e., $\beta = Yl$, if the bubble conditions* (17) *hold for a given set of parameters (Y, q, γ, l), then they also hold for any lower values of Y and γ, and for any larger values of q and l.*[17]

In this sense, bubbles tend to be possible for a low risk-free yield Y, long expected lifetime (large q), high leverage (high l, see Figure 1), and large uncertainty about market depth (low γ, see Figure 1). For $\gamma \to 0$, bubble equilibria exist if $\beta = Yl \geq Y - q$, that is, $l \geq 1 - q/Y$. This shows that in our framework at least some leverage is necessary for the existence of bubble

17. If β represents an exogenous contract parameter and the bubble conditions (15) and (16) hold for given parameters Y and β, then they also hold for any lower values of Y and for any larger values of β.

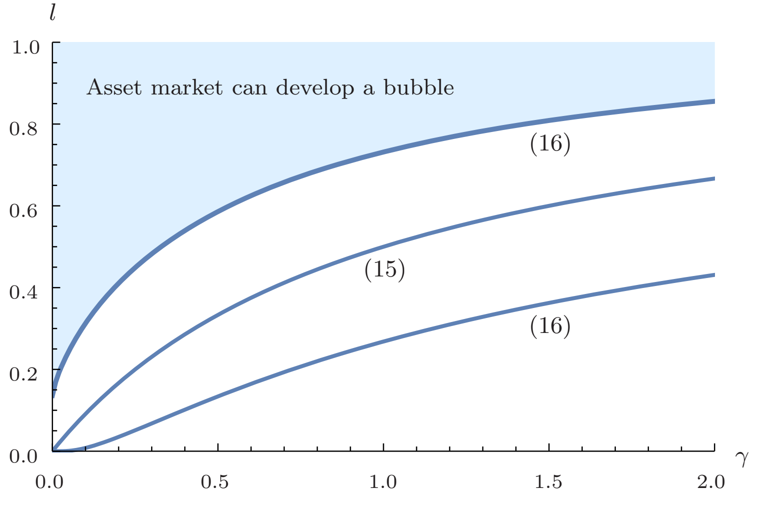

FIGURE 1. Parameter Range where Bubbles are Possible on the Asset Market
The figure is based on a numerical example with $Y = 1.1$ and $q = 0.95$. Condition (15) holds above the middle curve. Condition (16) holds above the upper curve and below the lower curve. In between, the overpriced asset is dominated by storage at some point in time, such that no bubble can emerge. Combining conditions (15) and (16) shows that—considering only traders' actions—bubbles are possible in the shaded parameter region. Below, the information on market depth is too precise (high γ), i.e., the probability of a burst of a bubble is too large for given leverage and traders are not willing to buy the overpriced asset.

equilibria, even with a high degree of uncertainty about N. For $\gamma \to \infty$ (known market depth), we obtain Tirole (1982)'s impossibility result, see footnote 2. Observe the difference between a *dynamic* price deviation from the steady-state price in a bubble and the *static* deviation of the steady-state price from the fundamental value. The static deviation is larger for assets with a shorter expected lifetime, i.e., higher fundamental risk, but bubbles tend to emerge for assets with a longer expected lifetime.

4.1. Asset-Market Clearing

We now construct an example bubble, that way proving that the asset market clears and a bubble can emerge if (15) and (16) hold. Consider a special ("trinomial") class of price processes with

$$\tilde{p}_{t+1} = \begin{cases} 0, & \text{with probability } 1 - q \\ \bar{p}, & \text{with probability } q - Q_t \\ p_{t+1}, & \text{with probability } Q_t \end{cases} \tag{18}$$

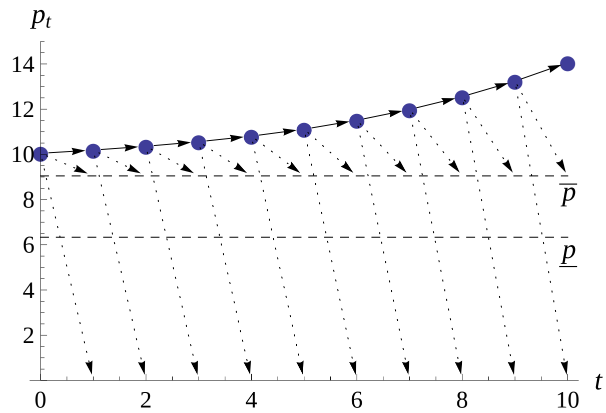

FIGURE 2. A Trinomial Price Process with a Bubble

Here and in the following figures, the parameters are $q=0.95$, $d=1$, $Y=1.1$, $l=0.81$, such that $\beta=0.9$, and $\gamma=1$.

where $Q_t \leq q$ is the probability of a continuation of the bubble.[18] The sequence of variables $\{p_t, Q_t\}_{t\geq 0}$ will be determined endogenously. Trinomial processes are the simplest ones that allow for a fundamental default of the firm (first case in equation (18)), a bursting of the bubble (second case), and the continuation of the bubble (third case). A possible price process is depicted in Figure 2. In the figure, the process begins at some price $p_0 > \bar{p}$. In fact, prices below the steady-state price are not possible. Given the expected dividend payment and the expected lifetime of the asset, traders are always willing to pay up to $\bar{p}$, except after a bankruptcy (see the proof of Theorem 1).

If a bubble follows a trinomial price path, as long as the bubble does not burst, the price grows further and further, $p_0 < p_1 < p_2 < \ldots$ At any price p_t, because each trader disposes of one unit of the endowment good, p_t traders invest in the bubble. This implies that more and more resources will be absorbed by the bubble. At some date, p_t will reach N. As N is unknown, this date is also not known. The investments of all traders are insufficient to absorb the entire asset for prices above N, i.e., the asset market cannot clear according to condition (9). Traders realize by backward induction that no price above $\bar{p}$ can be sustained. They hence stop demanding the asset at any price $p_t > \bar{p}$, and so do all future generations.[19] In short, from the moment that an upper ceiling

18. Note the notational difference between $\tilde{p}_{t+1}$ and p_{t+1}. $\tilde{p}_{t+1}$ is the stochastic price at date $t+1$ that can assume three different values. p_{t+1} is one of these realizations.

19. In reality, the information on N may become outdated over time, as new noise, e.g., about changes in N, interferes with this information, generating new uncertainty about market depth. In a potential extension of the model, one could model this process and get an empirical prediction about the minimum gap in time that must occur between bubble

for N is revealed, the only possible price is the steady-state price. The date at which the bubble bursts is (and must be) unknown, but the ceiling will be reached with certainty at some date.

The price that a trader is willing to pay depends on the expected future price increase and on the probability of a burst, which can be calculated from the distribution of N. Additional to the distribution function $F(N)$, traders have two additional pieces of information. First, for a current price p_t at date t, traders know that $N \geq p_t$. Second, if the number of traders is large, the probability that a specific trader can buy the asset is small. Hence, a trader gets a piece of information on the size of N at the time she can buy a share (or not). That is, we have a version of the winner's curse problem in the market. When a trader can buy a share, she learns that the market is smaller than expected and the bubble is thus more likely to burst than previously thought.[20] Formally, let X mark the event that the asset is allocated to the trader. Then

$$f(N|X) = \frac{\Pr(X|N)\, f(N)}{\int_{p_t}^{\infty} \Pr(X|N)\, f(N)\, dN} = \frac{\frac{p_t}{N}\, \gamma\, \frac{N^{-\gamma-1}}{p_t^{-\gamma}}}{\int_{p_t}^{\infty} \frac{p_t}{N} \gamma\, \frac{N^{-\gamma-1}}{p_t^{-\gamma}} dN} = (\gamma+1)\frac{N^{-\gamma-2}}{p_t^{-\gamma-1}}. \tag{19}$$

The conditional probability that N is between p_t and $p_{t+1} > p_t$ (which means that the bubble will burst in the next period) is then

$$\Pr\{N \leq p_{t+1} | N \geq p_t, X\} = \frac{F(p_{t+1}|X) - F(p_t|X)}{1 - F(p_t|X)} = 1 - \left(\frac{p_t}{p_{t+1}}\right)^{\gamma+1}. \tag{20}$$

If p_{t+1} was below p_t, the probability of reaching the ceiling and thus a burst of the bubble would be zero because $N \geq p_t$ for sure. Remember that the bubble can also burst because the firm goes bankrupt (probability $1-q$). In this case, the profit of the trader is 0. The probability that the bubble does *not* burst (i.e., there's no bankruptcy, and the ceiling is also not reached), given that the trader gets the asset, is

$$Q_t = q\left(1 - \frac{F(p_{t+1}|X) - F(p_t|X)}{1 - F(p_t|X)}\right) = q\,\frac{p_t^{\gamma+1}}{p_{t+1}^{\gamma+1}}. \tag{21}$$

for $p_{t+1} > p_t$; for $p_{t+1} \leq p_t$ it is q. The asset market can only be in equilibrium if a modified version of the arbitrage condition (10) holds: a trader's profit from storage $Y - \beta$ must equal the probability Q_t (bubble does not burst) times the

episodes. Along the same line, one can get a prediction about bubbles in different asset markets, if they draw liquidity from the same pool of market participants.

20. This is a version of the winner's curse, as traders do not get the asset because no one else wants it, but because there are not many other buyers. This is a negative sign for the continuation of the bubble.

expected profit, including appreciation and dividend yields,

$$Y-\beta=q\min\left(1;\frac{p_t^{\gamma+1}}{p_{t+1}^{\gamma+1}}\right)\max\Big(0;\frac{p_{t+1}+d}{p_t}-\beta\Big). \tag{22}$$

If the asset price falls because market depth is exhausted, it drops to $\bar{p}$ and the return is $(\bar{p}+d)/p_t$. Let us temporarily assume that $p_0>(\bar{p}+d)/\beta$, which means that the trader can keep nothing if the bubble bursts. The general case is discussed in the Appendix. Note that for $p_{t+1}<p_t$, (22) cannot be fulfilled. In this case, its right-hand side turns into

$$q\max\Big(0;\frac{p_{t+1}}{p_t}+\frac{d}{p_t}-\beta\Big). \tag{23}$$

At the price $\bar{p}$, dividend payments $d/\bar{p}$ just compensate traders for the default risk of the asset, see equation (10). Since in a bubble $p_t>\bar{p}$, d/p_t in (23) is smaller than $d/\bar{p}$. Thus, traders must be compensated otherwise, and the only possible additional compensation are increasing prices.[21] We can hence concentrate on rising prices and drop the min and max operators, so (22) becomes

$$Y-\beta=q\,\frac{p_t^{\gamma+1}}{p_{t+1}^{\gamma+1}}\Big(\frac{p_{t+1}+d}{p_t}-\beta\Big). \tag{24}$$

This equation gives an implicit recursive rule for the evolution of the price process. Starting with some $p_0>\bar{p}$, the equation implicitly defines p_1, which is just high enough that the appreciation compensates a trader for buying an overpriced asset, i.e., for the risk of a bursting bubble. In the next step we can use (24) to calculate p_2 from p_1, and so on. One process constructed in this way is shown in Figure 2. Starting from p_0, the complete process $\{\tilde{p}_t\}_{t\geq 0}$ is implicitly defined by forward induction—if equation (24) has a solution.[22] Below, we will derive the conditions under which this is the case for bubble paths.

4.2. Existence of a Bubble Process.

Equation (24) does not necessarily yield a solution p_{t+1} for all starting points p_t. The higher the potential future price increase, the more likely it is that the ceiling N is reached and that the bubble bursts. The more likely the bubble is to burst, however, the larger the expected price increase must be to compensate

21. Specifically, since $Y-\beta$ equals the right-hand side of (10) and in a bubble $d/p_t<d/\bar{p}$, (23) is smaller than $Y-\beta$ for falling prices. Inequality (22) is not fulfilled, traders strictly prefer storage over the asset.

22. Note that, starting with the steady-state value $p_0=\bar{p}$, the path $p_t=\bar{p}$ for all t is always a solution of the implicit equation—we remain in the steady state.

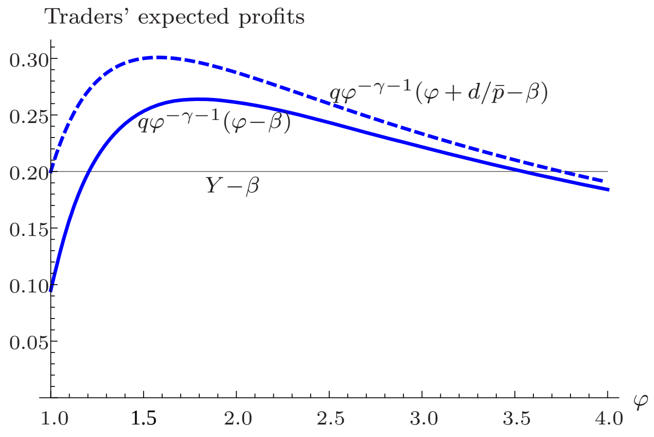

FIGURE 3. Possibility of a Bubble—Expected Returns

As in the previous figures, parameters are $\gamma=1$, $d=1$, $q=95\%$, $Y=1.1$, and $l=0.81$, such that $\beta=0.9$.

traders for the risk they face. Hence, there might be no equilibrium price p_{t+1} for all starting points p_t. If so, there exists a critical price above which the risk of a bursting bubble outweighs the potential gains from a price increase. Because all market participants can calculate the date t at which this critical price is reached (if it exists), a bubble would burst with certainty at t. By backward induction, the bubble is not sustainable right from the beginning—there is no bubble, the price path is unique. We are thus interested in conditions under which a bubble can or cannot be sustained. To be sustainable, the implicit equation (24) must have a solution at any date t or, equivalently, for any price p_t. Rewrite (24) by defining the auxiliary variable $\varphi_t=p_{t+1}/p_t$ as the relative price increase,

$$Y-\beta=q\,\varphi_t^{-\gamma-1}\left(\varphi_t+\frac{d}{p_t}-\beta\right). \tag{25}$$

The left-hand side of the equation is independent of p_t, but the right-hand side is not. Figure 3 shows the traders' expected profit from storage (left-hand side of (25), thin solid line) and the expected profit from buying the asset (right-hand side of (25), thick solid and dashed curves), which depends on φ_t. The expected profit from the asset depends also on the price at which the asset is bought. The dashed thick curve corresponds to the lowest possible price, the steady-state price $\bar{p}$. The solid thick curve corresponds to the highest possible price, $p_t\to\infty$. Any intermediate price leads to a curve in between these two curves. For $p_t \to \infty$, (25) simplifies to[23]

$$Y - \beta = q\,\varphi^{-\gamma-1}\,(\varphi - \beta). \tag{26}$$

This equation does not depend on time, we have hence dropped the index t. Let us define the right-hand side of the equation, i.e., the solid thick curve, as

$$h(\varphi) \equiv q\,\varphi^{-\gamma-1}\,(\varphi - \beta), \tag{27}$$

and the value of φ that maximizes this expression as $\varphi^* = \beta(1 + 1/\gamma)$.

Figure 3 contains a lot of intuition. *First*, at an intersection, the trader is indifferent between the asset and storage. The steady state is defined by the fact that the price does not move, $\varphi = 1$, and that the asset market clears. Hence, the dashed curve for the steady-state price must intersect with the straight horizontal line at $\varphi = 1$. At this point, the expected return from the asset in steady state equals that of storage, $Y - \beta$. *Second*, when the asset price increases, the curve shifts downward. An intersection between the line and the curve can cease to exist, that is $Y - \beta > h(\varphi^*)$, depending on the parameters. Hence, for some values of p_t, there might be no equilibrium price p_{t+1}. Condition (16) in Theorem 1 states whether $Y - \beta \leq h(\varphi^*)$. *Third*, $h(\varphi)$ is hump-shaped. This means that there can be two price factors φ that lead to market clearing. If φ is low, the probability of a burst is also low, and if the risk of a burst is high, the expected capital gain is high, too. We are, however, interested in the mere existence of a bubble process, i.e., the existence of at least one φ that satisfies (26); multiple solutions for φ thus do not matter.

As discussed above, an intersection at a $\varphi < 1$ is not possible. Geometrically, this would require a negative slope at least somewhere in the region $\varphi \leq 1$. There, however, the expected asset return no longer corresponds to $h(\varphi)$, which was derived for $\varphi > 1$ and $p_t \to \infty$. Instead, equation (23) applies. The curve therefore has a positive slope for $\varphi < 1$. By requiring a positive slope at $\varphi = 1$, condition (15) thus rules out solutions of (25) at a $\varphi < 1$, which would mathematically be based on probabilities larger than one.

Given the above properties, let us discuss the intuition for the comparative statics of Proposition 2, which follow from equations (15) and (16) in Theorem 1. If the risk-free rate Y is higher, storage becomes more attractive to traders. Thus, they only hold the risky asset if it displays a larger potential price increase. A larger increase, however, corresponds to a higher likelihood of a burst, which might impede the existence of a fixed point. Hence, for a larger risk-free yield Y, bubbles might cease to be possible. This finding is consistent with the idea that central banks can puncture bubbles by raising interest rates and that bubbles are particularly likely if interest rates are low. Furthermore,

23. The factor $\varphi_t = p_{t+1}/p_t$ does not converge to infinity but to a limit φ implicitly defined by (26). As a consequence, the continuation probability of a bubble also does not converge to zero but to $q\,\varphi^{-\gamma-1}$.

bubbles can exist particularly if q is high, that is, if the underlying asset's expected lifetime is relatively long, which reduces the likelihood of a burst due to a disappearance of the asset. The parameter γ captures the uncertainty in the market. The smaller the value of γ, the larger are the mean and the variance of the distribution, and the more uncertain is the potential market size. For $\gamma \to 0$, the expected market depth becomes infinite. On the other hand, if $\gamma \to \infty$, the market depth is almost surely N_0 and a bubble can never be sustained, independently of the values of other parameters. Finally, $\beta = Yl$ captures the degree of leverage and thus the importance of limited liability. The larger the value of β (i.e., the higher the leverage l of traders), the more traders rely on external financing and the more prominent the effect of limited liability becomes. Hence, the emergence of bubbles may become possible in the context of a high degree of leverage.

General Price Paths. We have demonstrated that the asset market can exhibit a bubble if both (15) and (16) hold by constructing an example bubble, the trinomial bubble. We now argue that these conditions are not only sufficient, but also necessary for bubbles to exist, such that Theorem 1 is complete. Traders are not willing to buy the risky asset if its expected return is lower than that of storage. That is, if there is no value of φ that lets the expected return of the asset rise above $Y-\beta$, bubbles will not be possible. This, in turn, is the case if condition (16) is violated: its right-hand side represents the highest expected possible return $h(\varphi^*)$ of the asset (that is, the highest point of the curve in Figure 3), which needs to be equal or above the payoff $Y-\beta$ of storage. Crucially, this argument does not hinge on the trinomial price path. Instead, it is valid for any distribution of probability mass across different values of φ. Intuitively, the trinomial bubble with all probability mass concentrated on φ^* represents the highest possible expected payoff for traders. Geometrically, the rest of the curve $h(\varphi)$ is below $h(\varphi^*)$ because $h(\varphi)$ is concave. Hence, if no trinomial bubble can exist because $h(\varphi^*) < Y-\beta$, then $\mathrm{E}[h(\varphi)] \leq h(\varphi^*) < Y-\beta$ and no other bubble can exist. Condition (16) is therefore necessary for traders to buy the risky asset and thus for bubbles to emerge for general price paths. Regarding condition (15), we have already shown that it is necessary: the curve $h(\varphi)$ will be below the line $Y-\beta$ in Figure 3 if the condition does not hold. To sum up, for a given set of parameters, if no trinomial bubble path exists, no bubble can exist at all. However, if a bubble exists, its path depends on the evolution of price expectations; it cannot be unique.

4.3. Household Participation in Lending

We have analyzed the conditions under which traders are willing to invest in a bubble. Traders, however, borrow from households. If households know that traders might invest their funds into overpriced assets, will they lend in the first place? We have assumed that the risk-free return is Y for traders but only

$\lambda Y < Y$ for households. If there was no risky asset, households would always invest through traders. In the presence of the risky asset, however, there are two reasons why households might not do so.

First, even in the absence of a bubble, households anticipate and dislike the fact that traders invest in the risky asset, since they lose their investment if the underlying firm goes bankrupt. The probability that the commissioned trader buys the asset depends on the asset's price and thus on its market capitalization. If the asset is expensive, it soaks up a lot of funds and this probability is large. The willingness to lend also depends on households' expectations regarding market depth N. More uncertainty about N also implies a higher expected N. For a given price, market capitalization is then smaller than total wealth and the probability that a trader invests in the asset is lower. These considerations lead to inequality (14), households' participation condition in steady state.

Second, within a bubble, households are even more reluctant to lend because they suffer from a bursting bubble. This implies that households might be willing to lend in the absence of a bubble but not in the presence of a bubble. As the bubble evolves, the probability of a burst increases and households become more reluctant to lend. If households reduce their lending or stop lending completely at some point because their participation constraint becomes binding, the bubble bursts since unleveraged traders do not invest in a bubble. By backward induction, the bubble can then not emerge in the first place. We can show that, given all other parameters, a critical $\bar{\lambda} > 0$ exists such that for any $\lambda \leq \bar{\lambda}$, the participation constraint of households does *not* rule out bubble equilibria.

We hence have a 'lemons problem,' resulting in the existence (all households lend) or non-existence (no household lends) of the lending market. If households do not even lend in steady state, there are foregone gains from trade between traders with a profitable investment opportunity and less productive households. If they lend in steady state but not in bubbles, bubbles cannot emerge in the first place. In Section 5.1, we extend the model to a setup with heterogeneous households. There, some households might stop participating in the lending market because they consider lending too risky, given their alternative investment opportunity. In that case, bubbles also destroy gains from trade.

Lending in the Example Bubble. Consider the case of the example bubble with a trinomial price path. As we are now interested in the conditions for clearing of the intermediation market, we no longer assume $x^H = l$. Instead, we analyze the decisions of a specific household i. The probability that a (ex-ante symmetric) trader invests in the asset is

$$\int_{p_t/\bar{x}}^{\infty} \frac{p_t}{\bar{x}N} f(N)\,\mathrm{d}N = \int_{p_t/\bar{x}}^{\infty} \frac{p_t}{N\bar{x}} \frac{\gamma\bar{x}}{p_t} \left(\frac{N\bar{x}}{p_t}\right)^{-\gamma-1} \mathrm{d}N = \frac{\gamma}{\gamma+1}, \tag{28}$$

where $\bar{x}$ are, as before, total funds under control of a trader. Otherwise, with probability $1/(\gamma+1)$, the trader purchases the safe asset and the household's return is β_i. We know from Lemma 1 that $\beta = Y x_i^H$. If the trader has bought the asset, there are the three cases of the trinomial bubble tree (18). First, the firm may default with probability $1-q$, in which case the household gets nothing and pays the verification cost c. Second, given that the trader has bought the asset, the bubble continues with probability $Q_t = q\, p_t^{\gamma+1}/p_{t+1}^{\gamma+1} = q\,\varphi^{-\gamma-1}$ and the household receives β_i. Third, the bubble bursts and the price drops to the steady state $\bar{p}$. The corresponding probability, conditional on the fact that the trader has bought the asset, equals $Q_t = q\,(1-\varphi^{-\gamma-1})$. Household i, whose trader has invested in the risky asset, then get a fraction $x_i^H/x^T \times \bar{p}/p_t$ of the trader's funds and pays the verification cost c. Summing up, the expected return to a lending household is

$$\frac{1}{\gamma+1}\, x_i^H Y + \frac{\gamma}{\gamma+1}\left[(1-q)\,(-c) + q\varphi^{-\gamma-1} x_i^H Y + q\,\left(1-\varphi^{-\gamma-1}\right)\left(\frac{x_i^H}{x^T}\frac{\bar{p}}{p_t}\bar{x} - c\right)\right].$$

The household optimizes (6) by choosing the x_i^H that maximizes U_i^H in (2), thus the maximum of the above expected return to lending and the alternative investment return $\lambda Y(l - x_i^H)$. Since the derivatives of both expressions with respect to x_i^H have opposing signs, the household either lends its entire endowment to the trader or nothing at all: $x_i^H \in \{0, l\}$. We can therefore focus on the case of $x_i^H = l$, such that $\beta_i = Yl$, and check whether the households participates in the loan market in this case. The relevant expression for the expected return to lending is

$$\frac{1}{\gamma+1}\,\beta_i + \frac{\gamma}{\gamma+1}\left[(1-q)\,(-c) + q\,\varphi^{-\gamma-1}\,\beta_i + q\,\left(1-\varphi^{-\gamma-1}\right)\left(\frac{l}{x^T}\frac{\bar{p}}{p_t}\bar{x} - c\right)\right]. \tag{29}$$

This term has to exceed $\lambda Y l$ for the household to participate. We can hence calculate a critical $\bar{\lambda}_i$. If λ is above this critical point, the gains from trade are too small; households are better off buying the safe asset themselves instead of lending to traders. The critical $\bar{\lambda}_i$ depends on p_t and $p_{t+1} = \varphi_t\, p_t$. As the bubble evolves, both p_t and φ_t increase. Both effects reduce households' expected return (29). Consequently, $\bar{\lambda}_i$ decreases and households might become unwilling to lend at a certain stage: the expected negative return from a potential investment in the bubbly asset can no longer be compensated with possible interest payments from an investment in storage. To derive the condition for whether households continue lending, we must verify whether households are willing to invest even at arbitrarily high p_t. The term $\bar{p}/p_t$ in (29) vanishes and we can set φ_t to the limit φ. Expression (29) becomes

$$g(\varphi) = \frac{1}{\gamma+1}\left[(c+\beta_i)\frac{q\,\gamma}{\varphi^{1+\gamma}} + \beta_i - c\,\gamma\right]. \tag{30}$$

Note that this expression is the same for all households and does not depend on the actions of other households. We can therefore drop the index i and derive a condition that determines whether every or no household participates. The above value must exceed $\lambda Y l$. Solving for λ, a bubble can thus only evolve if

$$\lambda \leq \bar{\lambda} := \frac{1}{\gamma+1}\,\frac{1}{\beta}\,\left[(c+\beta)\frac{q\,\gamma}{\underline{\varphi}^{1+\gamma}} + \beta - c\,\gamma\right], \tag{31}$$

with $\underline{\varphi}$ representing the lowest solution of (26): $Y-\beta = q\,\varphi^{-\gamma}\,(\varphi-\beta)$.[24] If this condition fails to hold for $\beta = Yl$, households do not lend and $\beta = Y x^H = 0$. Bubbles are then not possible. To see this, insert $\beta=0$ in equation (26), which is violated for any $\varphi > 1$ in this case: traders do not invest in an overpriced asset with their own funds alone. The intermediation market, however, is in equilibrium: as in the steady-state case, traders are indifferent between borrowing or not and condition (8) holds. The asset market is in equilibrium as traders purchase the asset at the fundamental value with their own funds.

General Price Paths. In the proof of the following theorem, we show that condition (31), stemming from the households' participation constraint, holds not only for our example bubble with a trinomial price process, but also for bubbles in general. The reason is that out of all bubble processes, households suffer least from the trinomial process with the highest possible continuation probability. Therefore, if they are unwilling to lend in a trinomial process, they will not lend in any other bubble process.

THEOREM 2. *Assume that* (15) *and* (16) *hold, such that a* $\varphi > 1$ *exists with*

$$Y - \beta = q\,\varphi^{-\gamma-1}\,(\varphi - \beta),$$

i.e., the asset market allows for bubble stochastic processes in equilibrium. Then if

$$\lambda < \bar{\lambda} := \frac{1}{\gamma+1}\,\frac{1}{\beta}\,\left[(c+\beta)\frac{q\,\gamma}{\varphi^{1+\gamma}} + \beta - c\,\gamma\right], \tag{32}$$

households participate even in a bubble stochastic process, thus $x_i^H = l$ *for all* i. *If this condition does not hold, there are no bubble processes in equilibrium because households will stop lending to traders at some point. The set of parameters for both cases is non-empty. For an endogenous contract, set* $\beta = Yl$ *in the above equations.*

24. We focus on the lower intersection of the left- and the right-hand side of (26) (see Figure 3), as we are interested in the condition when bubbles can occur. The lower intersection corresponds to a slower price increase during a bubble, less risk, and hence an expected higher return to households. If households are not willing to lend with this flatter price path, they are not willing to lend at any other price path.

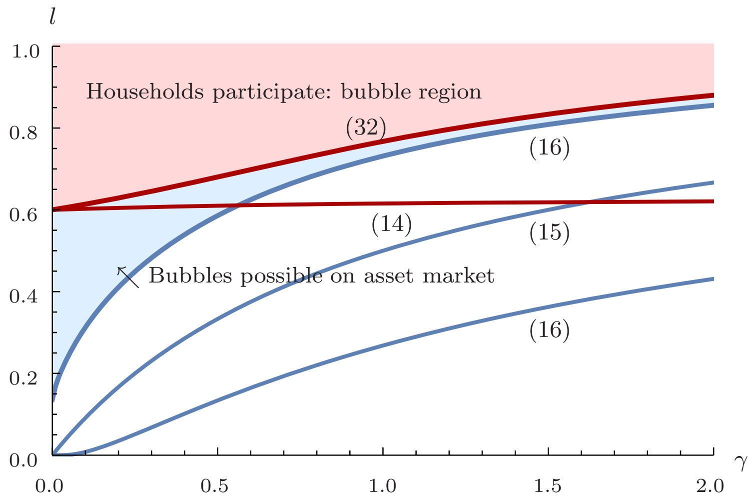

Figure 4. Parameter Range where Bubbles are Possible

The figure is based on a numerical example with $Y=1.1$ and $q=0.95$ (as in Figure 1) and additionally $\lambda=2/3$, $l=0.9$, and $c\to 0$. The blue curves are thus the same as in Figure 1. Condition (14) holds above the thin red curve (which nearly looks like a horizontal line). Here, households lend to traders in the absence of a bubble. Condition (32) holds above the thick red curve: households lend to traders even if they know there is an asset bubble. In both conditions, $\beta=Yl$ is inserted. In the red shaded region, bubbles are possible.

If condition (32) holds, we obtain $\beta=Yl$. This condition is depicted as a thick red curve in Figure 4. We hence get the following possible constellations: if (15) or (16) fail to hold, then (26) has no solution for φ and bubbles are impossible because traders do not invest in the asset. If both (15) and (16) hold, households participate according to (32) only if lending to traders beats the alternative investment. If this is not the case, $\beta=0$ and the only equilibrium is the steady state ($\varphi=1$). Condition (32) then reduces to (14), determining whether households lend at all. In sum, the households' participation constraint reduces the parameter space in which bubbles are possible but does not entirely prevent bubbles from existing.

How does the new condition (32) depend on exogenous parameters? Figure 4 shows one general property. Parameter constellations that tend to invalidate condition (16) also tend to invalidate (32). In particular, households are less willing to lend in a bubble when the underlying asset's expected lifetime is short (low q, not in the picture), when the trader's leverage ratio is low (low $\beta=Yl$, for a given Y and an active intermediation market), or when uncertainty about market depth N is low (high γ). The only difference to the comparative statics of Proposition 2 is the effect of the alternative yield Y; households are willing to lend to traders in particular when Y is high (for an endogenous $\beta = Yl$, not in the picture).

PROPOSITION 3. *For an endogenous contract, if the necessary condition* (32) *holds for given set of parameters (Y, q, γ, l, λ, c), then it also holds for any lower values of γ, λ, and c, and for any larger values of q, l and Y.*[25]

Two of these statements deserve additional explanations. First, why is a household reluctant to lend to a trader with a low leverage ratio? If leverage is sufficiently low, traders shun investment in an overpriced asset. If a bubble nevertheless exists, its expected price path must be steep enough to compensate traders (see equation (26) and Figure 3, in which a low l shifts the thin line upwards and hence the intersection to the right). This increases the risk of a burst and hence worsens the prospects for households.[26] Second, why is the effect of Y positive? Y has again a double role, as it appears in (26), determining φ, and via β in (32), which is based on a given φ. On the one hand, the effect of a higher Y is detrimental for households because with a more profitable safe investment, the bubble has to grow faster for traders to achieve market clearing, see (26). It is thus again more likely to burst. On the other hand, as in the case of l, we obtain an additional effect for $c > 0$ in (32). While a higher Y increases both the households' outside option and the return from investing through traders in case the bubble continues proportionally (as $\beta = Y l$), monitoring costs stay the same. They hence fall in relative terms, which makes lending to traders more attractive. This effect dominates.

As shown, for a given Y, the same parameter constellations that induce households to invest through traders induce traders to invest in bubbles. It is therefore likely that in situations in which households find it optimal to invest via traders, the resulting leverage creates the conditions for bubbles. The optimal investment strategy of households hence reduces the expected payoff to other households. Given that households do not take this externality on equilibrium prices into account, the expected payoff of all households is reduced. In the next section we show that this has negative welfare consequences.

25. If β represents an exogenous contract parameter and the condition (32) holds for given parameters Y and β, then it also holds for any lower values of Y and for any larger values of β.

26. Note that although a lower leverage level implies a lower β, that is a lower return in case the bubble does not burst, the rate of return, β/l, remains unchanged. Yet, since monitoring costs c do not depend on the invested amount, they become larger in relative terms. A lower l hence reduces the expected payoff from lending to traders in addition to the discussed increased riskiness, see (32).

5. Welfare

In this section we analyze the welfare effects of bubbles. In the baseline setup, welfare losses arise because the bubble crowds out productive investment in the safe asset. When evaluating welfare of the initial and future generations, we look at *realized* welfare of the initial generation and *expected* welfare of future generations.[27] Calculating an aggregate effect across generations would require additional assumptions on the discount rate between generations of households and traders. We therefore restrict ourselves to analyzing the inter- and intra-generational distributional consequences of bubbles, which can be derived without any further assumptions.

Proposition 4. *For given parameter values that allow for bubbles, utility of the initial owners of the asset is lowest in the steady-state equilibrium. For all future generations of traders, expected utility is the same on all price paths. For all future generations of households, expected utility is highest in the steady-state equilibrium.*

The initial owners of the asset gain unambiguously from a bubble, as expectations about future prices locate p_0 above the steady-state price. All following generations of traders must be indifferent between buying the asset and investing in storage, see condition (22). Traders are indifferent between these two options also in steady state. As the return to storage is independent from the existence of a bubble, it follows that, for traders of future generations, the expected return of the asset is the same with or without a bubble.

Expected utility of future generations of households, on the other hand, is affected negatively by a bubble. In fact, households cannot gain from a bubble: if the bubble continues, they obtain $\beta = Yl$. As soon as the price deviation from the steady state is large enough, they lose parts of their investment when the bubble bursts.[28] Absent a bubble, this risk does not exist. In addition, the

27. Specifically, we consider the following thought experiment: either parameters are such that bubbles are not possible and we remain in the steady state, or parameters are such that bubbles are possible when the model starts in $t = 0$. In this case, the possibility of higher future prices lets the initial price rise above $\bar{p}$, i.e., the bubble starts right away. We compare welfare in these two scenarios.

28. In case the intergenerational discount factor is equal to Y, the discounted sum of expected consumption $\sum_{t=0}^{\infty} Y^{-t} E(C_t^H + C_t^T)$ is independent of the existence of a bubble, neglecting monitoring costs. In this borderline case, the gain of the initial owners would offset the losses of later generations. Furthermore, possible compensations to the initial owners of the asset for letting the bubble burst would make someone else worse off. In fact, these transfers have a similar effect like a bubble, as they pull consumption forward from later generations to the current owners of the asset. Hence, even if later generations can be included in a transfer scheme, their expected welfare would not increase, again neglecting monitoring costs. Including these costs, however, increases welfare costs of bubbles, such

higher the current price, the more resources are absorbed by the bubble and the more likely traders are to invest in the bubbly asset. Hence, a larger fraction of households is at risk of loosing their investment because of bankruptcy of the underlying firm or because of a bursting bubble. This asymmetric distribution of gains and losses in expected welfare between traders and households is rooted in the limited liability of traders.

Limited liability shifts parts of the risk of a bursting bubble and bankruptcy to households. Traders' arbitrage condition (25) ensures that their expected *private* returns align across both assets. The risky asset's expected *social* return (i.e., under full liability), however, is below that to storage. Traders' behavior thus imposes a negative externality in the form of a lower expected return on households. Households recognize this, but as long as their own investment opportunities are sufficiently inferior relative to those of traders (condition (32)), they still expect a higher return from investing through traders. Limited liability hence leads to levels of investment in the risky assets above the social optimum, crowding out investment in the safe and productive asset.[29] If the bubble does not burst, the expected loss is passed on to the next generation, i.e., the bubble pulls expected consumption forward from future generations via a transfer from the young to the old.[30]

5.1. Heterogeneous households

If households have heterogeneous alternative investment opportunities, there can be additional welfare costs of bubbles due to a partial breakdown of the lending market. Since households know about the possibility that traders invest in the risky asset, those households with a good alternative investment opportunity stop lending to traders and make use of this opportunity instead if the bubble has grown large (and risky) enough. The lending market hence resembles a classic lemons market, where some households do not participate because lending of other households fuels the bubble. Since traders have access to a superior technology, this constitutes a welfare cost, comparable to Conlon

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that the mentioned sum is negatively affected by bubbles. Transfer schemes can thus have beneficial effects in this case.

29. Welfare is additionally reduced by the monitoring costs. These costs, which represent a deadweight loss, occur in steady state only if the underlying firm goes bankrupt. In a bubble they are also paid if a bubble bursts, such that their value increases in expectations.

30. Note the difference to models of rational bubbles in overlapping-generations models like in Tirole (1985). In such models, bubbles are possible if the economy features dynamic inefficiencies. Dynamic inefficiency implies that investing less in capital (and more in bubbly assets) increases current and future consumption, such that bubbles enhance welfare. In our model, welfare-reducing bubbles can exist without dynamic inefficiency because of unknown market depth and limited liability. In fact, we have assumed throughout that the rate of return on productive investments exceeds the growth rate of the economy (which is zero).

(2015). We now develop a version of the model with two types of households that differ in their alternative investment opportunity.

Changes to the baseline model. There are two types of households, indexed by $i \in \{L, H\}$. The types differ in their alternative investment disadvantages λ^i, where $0 < \lambda^L < \lambda^H < 1$. The share of households of type L in the economy is x, with $0 < x < 1$. We will construct an example where only these households will continue to lend to traders in a bubble. The variable x' denotes the share of households that currently lend. It is either 1 (all households lend), x (only type L lends) or zero (nobody lends). The relevant liquidity in the market for the risky asset is therefore $x'N$, as $x'N$ households each lend to $x'N$ traders.

The conditional probability that there is enough liquidity in the market to sustain p_{t+1}, given that p_t was already reached, is the same for $x'N$ as for N. If the share x' of households lends, traders know that $N \geq p_t/x'$. We hence get the same relevant distributions of N as for $x' = 1$, our baseline case, and the arbitrage condition for traders (26) remains unchanged. We thus concentrate on the modified participation constraint of households. The probability that the commissioned trader invests in the risky asset remains unaffected,

$$\int_{p_t/x'}^{\infty} \frac{p_t}{Nx'} f(N)\,\mathrm{d}N = \int_{p_t/x'}^{\infty} \frac{p_t}{Nx'} \frac{\gamma}{p_t/x'} \left(\frac{N}{p_t/x'}\right)^{-\gamma-1} \mathrm{d}N = \frac{\gamma}{\gamma+1}.$$

The expected payoff for households that lend to traders, equation (29), is therefore the same as in the baseline version. This payoff, however, needs to be above the individual outside option in order for the household to participate. Looking at the limiting case of $p_t \to \infty$ yields the equivalent to condition (32),

$$\lambda^i \leq \lambda^* = \frac{1}{\gamma+1} \frac{1}{\beta} \left[(c+\beta)\frac{q\,\gamma}{\varphi^{\gamma+1}} + \beta - c\,\gamma\right], \quad i \in \{L, H\}.$$

We assume that the parameter constellation is such that the low type lends, whereas the high type does not lend in a bubble, $\lambda^L < \lambda^* < \lambda^H$. Let us also assume that all households lend to traders in the steady state (to see when this is the case, insert λ^H into the steady-state participation constraint of households (14), instead of λ).

Welfare. To measure only those welfare loses that are due to bubbles, we assume that parameters are such that all households lend to traders in the absence of bubbles. Steady-state welfare is hence the same in the heterogeneous-household setup as in the baseline model.[31] We therefore derive those additional

31. We compare welfare for an identical bubble path. Due to the multiplicity of possible bubble paths, it is impossible to state which bubble paths will emerge in one or the other setting. We can therefore not make any statement about potential further differences due to differing bubble paths. Note, however, that the price of the asset only has an effect on

losses by comparing welfare in a bubble in the versions with heterogeneous and homogeneous households. In the heterogeneous-household setup with only one type of household lending to traders in a bubble, expected welfare (consumption) of a whole generation in period t amounts to

$$\mathrm{E}[W_t] = Q_t(p_{t+1} - \bar{p}) + q(\bar{p} + d) + \mathrm{E}\big[xN - p_t + (1-x)N(l\lambda^H + 1 - l)\big]Y - p_t(1 - Q_t)c,$$

taking into account that the fraction $1-x$ of households use their alternative investment opportunity with payoff $\lambda^H Y$ and the same fraction of traders invests their $1-l$ endowment goods in storage with a return of Y. Subtracting the expression for welfare of a generation in a bubble in the baseline version, i.e., the same equation with $x=1$, yields

$$-(1-\lambda^H)Yl\mathrm{E}[N](1-x) < 0.$$

We hence have an additional welfare loss from those households that stop investing through traders if a bubble emerges.[32] The intuition for this formula is straightforward. There are $N(1-x)$ households with a high outside investment opportunity, holding an endowment of l each. The expected (social) rate of return on their investment, if channeled through traders, is Y, and $\lambda^H Y$ for the alternative investment opportunity.[33]

In the next section, we discuss the effects of certain policy measures that have been discussed in the context of a prevention of asset-price bubbles. To keep the exposition short, we limit ourselves to the baseline case with homogeneous households.

6. Policy Measures

We examine whether a financial transaction (Tobin) tax or capital requirements can prevent bubbles. For either policy measure, we do two things. First, we check whether it prevents bubbles. Second, we analyze the measure's effect on the steady state.[34] As bubble processes by definition imply multiple equilibrium

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the degree of crowding-out of productive investment. As derived below, it does not affect the welfare loss due to the lemons problem, since the outside options of the households are unaffected by the asset's price.

32. In this setup, it might be possible that a generation of young households with a high outside option can compensate those old traders that hold the asset for letting the bubble burst. A discussion of the circumstances under which these private transfers can prevent bubbles is available upon request.

33. Since, for an identical price path, the bubble soaks up the same amount of resources with $x>0$ as with $x=0$, only investment into storage is reduced if a fraction of households stop lending.

34. To be precise, we compare two economies starting at $t=0$: one with the policy and one without. Thus, if we talk about, e.g., the initial generation, we calculate welfare of the

price paths, it is inherently impossible to analyze how a certain policy influences welfare if a bubble is not prevented. Based on Proposition 4, we can, however, make some statements about relative welfare in situations with or without a bubble.

6.1. Financial Transaction Tax

Financial transaction taxes (FTTs) have been discussed in the theoretical literature at least since Tobin (1978).[35] In our model, a period is interpreted as the investment duration of a trader. A tax on buying the asset, or on storage, can thus be interpreted as a FTT. Due to the setup, all assets are traded each period. The effects of a transaction tax in our model do therefore not work via a reduction in the volume of transactions but only via price effects. We call τ the tax rate on transactions of the safe asset and τ' the (potentially identical) tax rate on the risky asset. Tax revenues are redistributed as lump-sum transfers to households of the same generation that has paid the taxes, after investment decisions have taken place.

PROPOSITION 5. *If a financial transaction tax is levied on the risky asset only, the range of parameters in which bubbles are possible is reduced. If it is levied on the safe asset, this range increases or households might stop lending to traders. If it is levied with equal rates on both assets simultaneously, households might stop lending to traders.*

PROPOSITION 6. *A financial transaction tax on the risky asset that prohibits bubbles increases expected welfare of future generations but reduces welfare of the initial generation. A tax on the safe asset, or on both assets with the same rate, reduces expected welfare for future generations.*

We now derive the relevant version of the bubble condition equation (26). A leveraged trader has one unit of the endowment good to spend. The trader can hence afford to buy $1/(1+\tau)$ units of storage, yielding a surplus of $Y/(1+\tau)-\beta$. Given that the outside option of traders falls with a rising τ, the endogenous repayment to the households $\beta = Yl/(1+\tau)$ needs to fall as well. Regarding the risky asset, the trader can buy $1/\big(p_t\,(1+\tau')\big)$ units. Following the steps taken

generation that holds the asset when the model starts, with and without the policy. In the derivations, we assume a trinomial bubble process, which can be extended to general processes with the same arguments as before.

35. Constantinides (1986) finds limited price effects of a FTT. More recent contributions include Adam et al. (2015), who argue that while such a tax reduces the size and length of boom-bust cycles, it simultaneously increases the likelihood of these cycles.

in the derivation of equation (26) and using the new value for β, we find that the market-clearing condition changes to

$$\frac{Y(1-l)}{1+\tau} = q\,\varphi^{-\gamma-1}\left(\frac{\varphi}{1+\tau'} - \frac{Yl}{1+\tau}\right).$$

We can then derive the modified conditions (15) and (16) in Theorem 1 for the existence of bubbles as

$$(\gamma+1)\left(1 - \frac{1+\tau'}{1+\tau}Yx^H\right) < 1 \quad \text{and} \tag{33}$$

$$Y(1-l) \leq q\,\frac{\gamma^\gamma}{(Yl)^\gamma(\gamma+1)^{\gamma+1}}\left(\frac{1+\tau}{1+\tau'}\right)^{\gamma+1}. \tag{34}$$

As is visible in these equations, an increase in τ' has the same effect as a higher Y. We know from Proposition 2 that this tends to destroy the possibility of bubbles. Intuitively, a tax reduces the asset's return. If the tax is high enough, traders prefer storage instead. Yet, the way in which the tax is implemented is crucial. A tax on storage (τ) displays the exact opposite effects. It can therefore create the possibility of bubbles by reducing the relative expected return from storage. Additionally, a tax on the safe asset changes the households' participation constraint (32). Households obtain a lower outside option and hence require a proportionally reduced payment from traders. In case traders default, however, households have to pay the same monitoring costs as before. A high tax can thus deter households from lending to traders. In sum, we obtain the result that a tax on storage can either create the possibility of bubbles or destroy the market for intermediation.

In practice, identifying assets that correspond to the safe or the risky asset of the model is difficult. This can constitute a major obstacle to the implementation of a Tobin tax that aims at preventing bubbles. If the tax is levied on all financial assets, including storage ($\tau = \tau'$), it only impacts households' participation constraint. That is, a high enough tax rate can eliminate the possibility of bubbles, but again only at the cost of a breakdown of the intermediation market.

Welfare. The effects of a FTT on welfare are complicated by the fact that it does not only affect the possibility of bubbles, but changes steady-state prices as well. As shown in the proof of proposition 6, however, steady-state consumption actually increases with higher transaction taxes on the risky asset. The tax fulfills a function that households like but cannot carry out: it penalizes investment in the risky asset. Thus, investment in the risky asset becomes less attractive for traders and overall investment in storage rises. The opposite effect is obtained if the tax is levied on the safe asset: the return to storage falls and expected steady-state consumption falls. Steady-state consumption also falls if the tax is levied on both assets with the same rate.

We know from Proposition 4 that bubbles reduce expected welfare, except for the initial generation. Adding the result that a correctly implemented Tobin tax lowers the asset's price in steady-state, we conclude that such a tax reduces utility of the initial generation and raises expected welfare of future generations. A tax on the safe asset, in turn, reduces welfare for future generations for sure, as it decreases steady-state welfare and may create the possibility of bubbles or destroy the intermediation market. Due to an increase of the steady-state price of the risky asset, the initial generation gains from such a tax, as long as the intermediation market does not break down. Finally, given that tax revenues are not invested, steady-state consumption falls with a tax on both assets, additional to the possible breakdown of the intermediation market.

6.2. Capital Requirements

Another policy measure often discussed in connection to financial stability are capital requirements.[36] The analysis of capital requirements is straightforward in our setup. Capital regulation requires that for a given equity level of e, fixed at $1-l$, a trader can borrow up to $\overline{x}^T$ (previously equal to l). The balance sheet total is thus $e+\overline{x}^T$, and the equity ratio equals

$$\frac{e}{e+\overline{x}^T}.$$

The regulator can stipulate stricter (i.e., larger) capital requirements by setting a lower $\overline{x}^T$.

Proposition 7. *If capital requirements are increased, the range of parameters in which bubbles are possible is reduced.*

Based on Proposition 2, we know that lower leverage can destroy the possibility of bubbles. Thus, since capital requirements reduce leverage (traders have more “skin in the game”), they can eliminate potential bubbles.

Proposition 8. *Increases in capital requirements beyond the point where bubbles are eliminated reduce welfare for households and traders of all generations.*

Welfare. Regarding welfare consequences, we obtain several effects. *First,* if capital requirements prohibit a bubble, this effect raises welfare of future

36. Morrison and White (2005), Van den Heuvel (2008), and Harris et al. (2015), to name just a few recent contributions, discuss the effects of capital requirements in a variety of settings.

generations of households and reduces that of the initial generation. *Second*, higher capital controls reduce leverage and hence lower the steady-state price. Yet, as shown in the proof, the price falls more slowly than the available resources for the trader. The relative share of funds flowing into the risky asset, $\bar{p}/(e+\overline{x}^T)$, thus increases. Since the relative share of the safe asset falls and traders have absolutely less to spend, they actually invest less in the safe, productive (and in the risky) asset, which reduces welfare of future households. The lower steady-state price also hurts the initial generation. *Third*, related to the previous point, a decrease in $\overline{x}^T$ raises the amount that households must invest themselves. Because of the households' investment disadvantage λ, this is detrimental for welfare.

The overall effect of tightened capital controls thus depends on their initial level. If capital requirements are increased, but not enough to prevent bubbles, no welfare conclusions can be drawn. If capital requirements are raised sufficiently to prevent bubbles, further increases reduce welfare.[37] At some point along the path of increasing capital requirements households stop lending to traders, as the rising probability that traders invest in risky assets (second point above) makes investments through traders unattractive.

7. Conclusion

Our model endogenizes a specific reason why the price of an asset can exhibit large fluctuations despite unchanged fundamentals. If market depth is unknown, a trader might expect to sell the asset at an elevated price, increasing the amount she is willing to pay when buying. These price fluctuations are driven by expectations, typically involving unpredictable abnormal returns until the price falls back to its steady-state level. Such speculative bubbles can occur especially if traders are highly leveraged and if the information about market depth is imprecise. In addition, leverage also raises the steady-state price of a risky asset. This increase is not caused by expectations about rising prices, but by leveraged traders' risk-loving behavior. It does not raise the equilibrium price above the steady-state price and is therefore not a bubble, in our definition.

The policy measures differ in their impact on a bubble and on the price in steady state. A correctly implemented Tobin tax leads to welfare improvements

37. It is not possible to make definitive statements about welfare in a situation where capital requirements are just high enough to eliminate bubbles relative to the situation without any policy intervention. This is due to the effect of the policy intervention on steady-state welfare. Proposition 4 establishes that welfare in a bubble is lower than in the steady state with the same parameter values. Increases in capital requirements can eliminate the possibility of bubbles, but have at that point already reduced welfare in the alternative scenario of a steady state. Proposition 4 therefore does not apply and no definite statement about relative welfare effects can be made.

in steady state and can puncture a bubble. Capital requirements are detrimental for welfare in the steady state, but can puncture bubbles as well. By virtue of its relative simplicity, the model lends itself to discussions of related phenomena. For example, one could consider multiple assets and discuss whether the collapse of a bubble in one market can be contagious for other markets. Further applications include the introduction of the bubble mechanism into macro models, allowing for an investigation of its effects on business cycles and growth. Especially after the recent bursts of housing bubbles, applications seem both numerous and relevant.

Appendix

Proof of Lemma 1. In equilibrium, the risky asset is always bought by some trader, with or without bubbles. Hence, seen from the households' perspective, the return R_i which the commissioned trader will generate is inherently risky. The contract between trader and household contains the household's lending, a repayment function $z(R_i)$ and a function $B(z) \in \{0;1\}$ with $B(z)=1$ if the household verifies the true return R_i after some specific repayment z, and $B(z)=0$ if it does not. Then the optimal contract must fulfill incentive compatibility,

$$z(R_i) < z(R_i') \implies B(z(R_i)) = 1, \tag{A.A.1}$$

for all $R_i, R_i' \geq 0$. If the trader pays less in some state and more in another, she must be controlled in the state where she pays less. Otherwise, she would prefer to repay the lower amount also in the better state. Additionally, $z(R_i) \leq R_i x_i^H / x_i^T$ must hold for all $R_i \geq 0$, as the trader cannot repay more than she has earned. The trader's participation constraint for the individual contract is

$$\mathrm{E}[R_i x_i^H / x_i^T - z(R_i)] = \mathrm{E}[R_i x_i^H / x_i^T] - \mathrm{E}[z(R_i)] \geq (1-l)\,Y,$$

since the trader needs to repay $z(R_i)$ for an investment return of R_i. The profit of the trader needs to be larger than or equal to the return generated by investing her own funds in the safe asset, which is $(1-l)\,Y$. Since traders are in perfect competition, the above will be an equality, $\mathrm{E}[z(R_i)] = \mathrm{E}[R_i x_i^H / x_i^T] - (1-l)\,Y$. Finally, the household maximizes

$$\mathrm{E}[z(R_i)] - c\,\mathrm{E}[B(z(R_i))] = \mathrm{E}[R_i x_i^H / x_i^T] - (1-l)\,Y - c\,\mathrm{E}[B(z(R_i))].$$

This implies that the household wants to minimize the probability that it needs to verify. Because of (A.A.1), the verification states must be the ones with the lowest return R_i. In all other states, the repayment has to be the same (some value β_i set by household i). Also, in the verification states, the trader repays all she has proportionally to all households ($z(R_i) = R_i x_i^H / x_i^T$). Otherwise, a household would benefit from increasing the repayment in some states to R_i and compensating by reducing β_i, which also lowers the probability of verification. Putting the arguments together, we receive $z(R_i) = \min\{\beta_i; R_i x_i^H / x_i^T\}$. In our setting, in comparison to Gale and Hellwig (1985), there is one additional reason *not* to have a standard debt contract (SDC). With an SDC, the trader is incentivized to invest in the risky asset, with a positive probability of default and a negative externality on the household. A non-SDC, however, entails a verification cost of c with probability 1, whereas an SDC requires c only if the trader invests in the risky asset *and* the asset defaults. Therefore, if

$$\beta_i - c < \mathrm{E}[z(R_i)] - c\,\Pr[z(R_i) < \beta_i]$$

an SDC is the dominant financial contract, i.e., the results of Gale and Hellwig (1985) hold. We argue in footnote 23 that the probability of a bubble burst never approaches 100%, hence there is always a c such that total control is not optimal. Finally, we show that households have no incentive to deviate from setting $\beta_i = Y\, x_i^H$. The relevant question is whether households could have an incentive to increase β_i to economize on monitoring costs. In the absence of bubbles, the probability that an investment fails is $1 - q$. Expected monitoring costs are thus independent of β_i. We demonstrate that this result does not change in the presence of bubbles in the proof of Theorem 1. ■

Proof of Lemma 2. The lemma looks only at the optimal choice of y_i^T, so x_i^T is treated as a constant. We have $C_i^T = R_i - \min\{x_i^T Y; R_i\} = \max\{R_i - x_i^T Y; 0\}$, with $R_i = (1-l+x_i^T)[(1-y_i^T)Y + y_i^T Y^r]$. If we let $\underline{Y}^r$ denote the critical return at which $R_i = x_i^T Y$, we can rewrite the maximization problem (7) as

$$\max_{y_i^T} \int_0^\infty \max\{R_i - x_i^T Y; 0\} f(Y^r) dY^r = \max_{y_i^T} \int_{\underline{Y}^r}^\infty (R_i - x_i^T Y) f(Y^r) dY^r.$$

The second drivative of the argument with respect to y_i^T is

$$\frac{(1-l)^2 Y^2 f(\underline{Y}^r)}{(1-l+x_i^T)(y_i^T)^3},$$

which is positive. As a consequence, the argument is convex in y_i^T, and the optimal point must be a corner solution. ■

Proof of Proposition 1. We have constructed the steady-state process in the main text. The initial price is uniquely determined, hence the whole process is unique. ■

Proof of Theorem 1. In the exposition in the main text, we have already treated the case in which traders are not paid if a bubble bursts. Hence, we begin the proof of the theorem by providing a condition for this case and analyzing the alternative one. If a bubble bursts without the firm going bankrupt, the firm still pays the dividend. The return to the trader is then

$$\max\left\{\frac{d}{p_t} + \frac{\bar{p}}{p_t} - \beta; 0\right\} = \max\left\{\frac{d + \frac{d\,q}{Y - q - \beta\,(1-q)}}{p_t} - \beta; 0\right\}.$$

We have made the temporary assumption that $p_0 > (\bar{p}+d)/\beta$, such that, because $p_{t+1} > p_t$ and hence $p_t > p_0$, the trader loses everything if the bubble bursts. Now for the proof, we drop this assumption. If the price is only slightly above the steady-state price $\bar{p}$ (i.e., the bubble is small), the trader receives a payment even when the bubble bursts. The corresponding condition is

$$p_t < \check{p} := \left(d + \frac{d\,q}{Y - q - \beta\,(1-q)}\right)/\beta. \tag{A.A.2}$$

If $p_t < \check{p}$, such that (A.A.2) holds, a modified version of (22) applies. In asset-market equilibrium,

$$\begin{aligned} & Y-\beta = Q_t\left((p_{t+1}+d)/p_t-\beta\right)+(q-Q_t)\left((\bar{p}+d)/p_t-\beta\right) \\ \iff \quad & \frac{Y-\beta}{q} = \left(\frac{p_t}{p_{t+1}}\right)^{\gamma+1}\frac{p_{t+1}}{p_t}+\left(1-\left(\frac{p_t}{p_{t+1}}\right)^{\gamma+1}\right)\frac{\bar{p}}{p_t}+\frac{d}{p_t}-\beta. \quad \text{(A.A.3)} \end{aligned}$$

Again, beginning from p_t, we have an implicit equation for p_{t+1} in a rational-expectations equilibrium. Substituting $p_{t+1}=\varphi_t\, p_t$, we obtain

$$\frac{Y-\beta}{q} = \varphi_t^{-\gamma}+\left(1-\varphi_t^{-\gamma-1}\right)\frac{\bar{p}}{p_t}+\frac{d}{p_t}-\beta.$$

However, in a bubble, the price p_t increases over time and eventually exceeds the threshold $\check{p}$. Therefore, to determine whether bubbles are feasible it suffices to consider the case $p_t > \check{p}$, as done in the main text. Define the right-hand side of (22) as

$$\hat{h}(\varphi, p_t) = q\,\min(1, \varphi^{-\gamma-1})\,\max\left(0, \varphi+\frac{d}{p_t}-\beta\right)$$

For $\varphi > 1$ and $\varphi + d/p_t - \beta > 0$, this turns into

$$h(\varphi, p_t) = q\,\varphi^{-\gamma-1}\left(\varphi+\frac{d}{p_t}-\beta\right). \quad \text{(A.A.4)}$$

In Figure 3, $h(\varphi, \bar{p})$ is plotted as a dashed curve. When searching for intersections with the line $Y-\beta$, we only need to consider functions $h(\varphi, p_t)$ with $p_t > \bar{p}$ and $\varphi > 1$. Let p_l denote the lowest possible price that traders are willing to pay, given that no bankruptcy has happened so far. This price is reached for the lowest possible price expectations conditional on no bankruptcy. By definition, these are again given by p_l. Inserting $p_{t+1}=p_t=p_l$ into equation (24) yields the following arbitrage condition, with $\beta = Yx^H$ inserted

$$Y - Yx^H = q\left(1+\frac{d}{p_l}-Yx^H\right) \quad \iff \quad p_l = \bar{p}.$$

Prices below the steady-state price are hence not possible, except in the case of a bankruptcy. Furthermore, φ needs to be above one for an asset-market equilibrium to exist. For $\varphi < 1$, equation (23) applies, such that the curve has a positive slope, as long as the curve is above zero. Hence, if its slope is negative at $\varphi = 1$, the curve's maximum is at the same point and no equilibrium is reached. Formally, the slope of the curve at $\varphi = 1$ is $q\,(\beta - \gamma + \beta\,\gamma)$, which is positive for $(\gamma+1)\,(1-\beta) < 1$, the first condition (15) of Theorem 1. Intuitively, because of the min operator in equation (22) we calculate condition (16) based on equation (26). Condition (15) then guarantees that the min operator is nevertheless taken into account, ruling out $\varphi < 1$, such that we obtain solutions for equation (22).

The key question is now whether bubbles are possible for arbitrarily large prices. In a first step, we are therefore interested in $h(\varphi) := h(\varphi, \infty)$, where $d/p_t \to 0$. This gives the solid curve in Figure 3. The derivative of $h(\varphi)$ w. r. t. φ is

$$h'(\varphi) = q\,\varphi^{-\gamma-2}\,\big(\beta\,(\gamma+1) - \gamma\,\varphi\big).$$

The function $h(\varphi)$ has a maximum at $\varphi^* = \beta\,(1+1/\gamma)$. Thus, $\hat{h}(\varphi, \infty)$ also has a maximum at φ^* if $\varphi^* > 1$, and a maximum at one if $\varphi^* \leq 1$. If $\varphi^* \leq 1$, this maximum is $\hat{h}(1, \infty) = q\,(1-\beta)$, which is less than the return $Y - \beta$ on storage, see equation (23) and the above discussion.

Condition (22) can only have a solution if the maximum of its right-hand side, i.e., $\hat{h}(\varphi, p_t)$, is above its left-hand side. That is, the maximum expected gross return of the asset must weakly dominate storage. For any possible price process, this maximum return for $p \to \infty$ is bounded from above by

$$\mathrm{E}[\hat{h}(\varphi, \infty)] \leq \mathrm{E}[\hat{h}(\varphi^*, \infty)] = q\,\frac{\gamma^\gamma}{\beta^\gamma\,(\gamma+1)^{\gamma+1}},$$

where φ is now a random variable. Hence, traders will not invest in the risky asset if $Y - \beta$ is larger than the right-hand side of this inequality, giving condition (16). The above demonstrates that conditions (15) and (16) are necessary for a bubble to exist. The trinomial example then shows that these conditions are also sufficient for a bubble, assuming the distribution of N is Pareto-distributed. If the distribution is not Pareto, then, to get a sufficient condition, the weak inequality in (16) must be replaced by a strict inequality. Our figures are based on numerical examples, such that one set of parameters that enable bubbles is visible. To further see that the set of parameters that fulfill the conditions is non-empty, note that for $\gamma \to 0$, bubbles are possible whenever $q > Y - \beta = Y(1-l)$.

Finally, we show that $\beta = Yl$ holds also in a bubble. Assume there is a bubble in which φ is distributed such that some probability mass lies in a region below the negotiated repayment β. In that case, knowing this distribution, households would prefer to choose a contract with a lower β, such that they always obtain less from traders but economize on monitoring costs. The traders' participation would then not bind. Starting from such a distribution of φ, however, one can construct another distribution that also leads to market clearing but is strictly preferred by households. Specifically, if the probability mass from slightly below the default level is moved to the default level and some probability mass in the upper part of the distribution is moved to a lower level as compensation, the asset market still clears, i.e., $\mathrm{E}[h(\varphi)] = Y - \beta$. This distribution, together with the old β, is preferred by households, as it increases the probability to obtain the old β. We additionally know from the discussion in the proof of Theorem 2 that concentrating probability mass to the center of the distribution leads to a distribution that is preferred by the households. For the same reason, this distribution is itself dominated by the trinomial distribution. Summing up, even if bubble paths exist in which it is optimal for households not to push traders to their participation constraint, these bubbles are dominated by trinomial bubbles in which the participation constraint is binding. Hence, if households do not participate in trinominal bubbles, they do not participate in any other form of bubbles. Concentrating on the case of $\beta = Yl$ comes therefore without loss of generality. The parameter sets are non-empty because the left-hand side in (32) is a positive number, so λ can be smaller. ■

Proof of Proposition 2. There are two conditions that can become tighter or looser when changing a parameter: conditions (15) and (16). However, (15) is never binding. This can be seen in Figure 1, where the bubble region is bordered by condition (16) only. Mathematically, it can be shown by inserting (15), holding with equality, into (16) and making use of equation (10). We obtain $q/(1+\gamma) + d/\bar{p} \leq q/(1+\gamma)$, which is a contradiction. Without the additional condition (15), the area below the lower solution of (16) in Figure 1 would also be part of the bubble region. Hence, (15) determines the bubble region, but it does not touch it. Consequently, we only need to check whether (16) becomes tighter in the relevant region when changing a parameter. Condition (16) can be rewritten as

$$(Y - \beta)\,(\gamma + 1)^{\gamma+1} \left(\frac{\beta}{\gamma}\right)^{\gamma} - q \leq 0. \qquad \text{(A.A.5)}$$

The first comparative static is obvious. The derivative of the left-hand side w. r. t. q is negative, such that the condition is relaxed for higher q. The partial derivative w. r. t. Y is positive; hence, (16) gets tighter as Y increases. The derivative w. r. t. γ is

$$(Y - \beta)\,(\gamma + 1)^{\gamma+1} \left(\frac{\beta}{\gamma}\right)^{\gamma} \ln \frac{\beta\,(\gamma + 1)}{\gamma}.$$

Condition (15) implies that $\beta\,(\gamma{+}1) > \gamma$. The above logarithm and the complete derivative is thus positive. A larger γ tightens the condition for a bubble. Finally, we want to show that an increase in β relaxes condition (16). We first show that the left-hand side of (A.A.5) is concave in β. It is continuous and the derivative w. r. t. β is

$$(\gamma + 1)^{\gamma+1} \left(\frac{\beta}{\gamma}\right)^{\gamma} \frac{Y\gamma - \beta\,(\gamma + 1)}{\beta}. \qquad \text{(A.A.6)}$$

We are interested in the sign of this derivative especially near the border of the bubble region, i.e., where (A.A.5) holds with equality. Substituting (A.A.5) into (A.A.6) yields

$$\frac{q\,\gamma^{\gamma+1} - \beta^{\gamma+1}\,(\gamma + 1)^{\gamma+1}}{\beta\,\gamma^{\gamma}}.$$

The numerator is negative because $q\,\gamma^{\gamma+1} - \beta^{\gamma+1}\,(\gamma+1)^{\gamma+1} < 0$ if and only if $q < (\beta\,(\gamma+1)/\gamma)^{\gamma+1}$ which is always true, as the right-hand side exceeds unity as a consequence of (15). When considering $\beta = Y\,l$, equation (A.A.5) becomes

$$(1-l)Y^{\gamma+1}\,(\gamma+1)^{\gamma+1}\left(\frac{l}{\gamma}\right)^{\gamma} - q \leq 0,$$

where Y, as before, enters positively on the left-hand side. Leverage l has the same effect as calculated for β above. ■

Proof of Theorem 2. For trinomial bubbles, we have already derived the necessary and sufficient participation constraint (31). We want to show that the condition remains valid for non-trinomial bubbles. If (31) holds, households' participation constraint does not prevent the existence of multiple equilibria: the trinomial bubble is one example of an alternative price path. It remains to be shown that if (31) fails to hold, households' participation constraint is also violated for any other type of bubble. Hence, we need to show that of all possible bubble paths, the trinomial bubble is the most preferred one by households. Then, if for a certain parameter constellation trinomial bubbles do not exist because of households' participation constraint, households are even more reluctant to invest in a non-trinomial bubble.

As in the proof of Theorem 1, the function $h(\varphi) = q\varphi^{-\gamma-1}\,(\varphi - \beta)$ gives the expected asset return to the trader for high p, considering that the bubble might burst. The more advanced a bubble already is, the more reluctant households are to invest; hence, we can concentrate on large prices p. As defined in (30), $g(\varphi)$ gives the expected return to households in this case. These functions are plotted in Figure A.A.1. The lower, blue curve defines the market clearing condition. In a trinomial bubble, the market clears when the blue curve intersects with $Y - \beta$ (dashed line), i.e., traders are indifferent with respect to investing in the bubbly asset at this point. In a general bubble, the return can assume several values with different probabilities and $h(\varphi)$ defines an invariant for the probability distribution of φ. For the market to clear, $\mathrm{E}[h(\varphi)] = Y - \beta$ must hold. The households' expected return is then $\mathrm{E}[g(\varphi)]$. One possible solution is the trinomial bubble, where one single value of φ has 100% probability mass. Other solutions might have positive variance. We need to show that for all distributions of φ with strictly positive variance, households' expected return falls short of that in the trinomial bubble. We hence need to solve

$$\max \mathrm{E}[g(\varphi)] \quad \text{s.t.} \quad \mathrm{E}[h(\varphi)] = Y - \beta, \tag{A.A.7}$$

where the max operator is taken over all probability distributions of φ. In particular, we can restrict ourselves to distributions with support in the increasing part of $h(\varphi)$. If some distribution of φ has probability mass in the decreasing part of $h(\varphi)$, mirror that mass to the increasing part. This

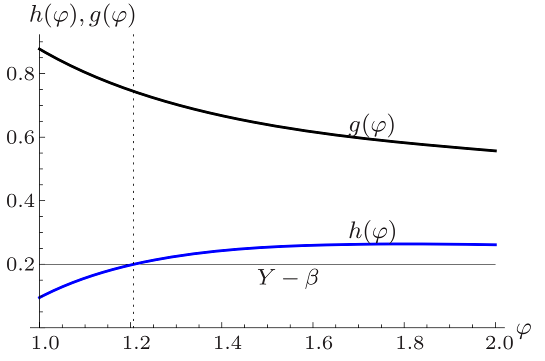

FIGURE A.A.1. Market clearing condition h and household participation constraint g

Parameters are $\gamma = 1$, $\beta = 0.9$, $q = 95\%$, $d = 1$, $Y = 1.1$, $\lambda = 2/3$, and $c \to 0$.

leaves traders indifferent, but is preferred by households. Hence, without loss of generality, we can assume that $h(\varphi)$ is an increasing function, and thus invertible. We now rescale the problem by distorting the φ-axis. We substitute $h(\varphi) \mapsto x$; thus, $\varphi \mapsto h^{-1}(x)$. Problem (A.A.7) becomes

$$\max \mathrm{E}[g(h^{-1}(x))] \quad \text{s.t.} \quad \mathrm{E}[x] = Y - \beta, \tag{A.A.8}$$

where the max operator is taken over all probability distributions of x. We want to show that $g(h^{-1}(x))$ is concave. The implicit function theorem yields

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} g(h^{-1}(x)) = \frac{h'(\varphi)\, g''(\varphi) - h''(\varphi)\, g'(\varphi)}{h'(\varphi)^3}.$$

Without loss of generality, we can concentrate on distributions of φ with support only in the increasing part of $h(\varphi)$. If there is probability mass on the decreasing part of h, we can move that mass to the increasing part that has the same level of h. This leaves traders indifferent, but improves the households' expected return since the risk of a bubble burst decreases. Hence $h'(\varphi) > 0$ and the denominator $h'(\varphi)^3$ is positive. The numerator is $-(q\,\gamma)^2\,(c+\beta)\,\varphi^{-2(\gamma+2)} < 0$. The entire fraction is thus negative, and $g(h^{-1}(x))$ is concave. A mean preserving spread of x deteriorates $\mathrm{E}[g(h^{-1}(x))]$, while $\mathrm{E}[x] = Y - \beta$ still holds. Problem (A.A.8) is thus solved by the degenerate one-point distribution. Hence, since $g(h^{-1}(x))$ is concave, households prefer trinomial bubbles above all other types. If households' participation constraint is violated within the class of trinomial bubbles, it is violated for any bubble. ■

Proof of Proposition 3. Consider condition (32). The effect of changes in λ is straightforward. When evaluating the effects of changes in the other parameters, however, we have to consider their effect on φ via the market clearing condition (26), $0=q\varphi^{-\gamma-1}(\varphi-\beta)-(Y-\beta)$. Implicitly differentiating this equality shows that φ depends positively on Y and γ as well as negatively on β and q. Recall from the proof of Theorem 2 that the right-hand side of (26) depends positively on φ in the relevant region. Furthermore, the right-hand side of condition (32) depends negatively on φ and γ as well as positively on q, which completes the proof.[38] ■

Proof of Proposition 4. Consider the initial generation at date $t=0$. The only difference for the owners of the asset, relative to the steady-state situation, is the higher price of the asset. Welfare of agents that do not own the asset does not change because of a bubble.

In the second and all following periods, the welfare difference between a bubbly situation and the steady state, denoted by variables with hats, $\hat{C}_1^T$ is 0 for traders, as already discussed in the main text. Households from the second period onward, however, are hurt by limited liability of the traders. Traders invested $\mathrm{E}[N]-p_0$ into storage, such that corresponding households get their payoff β for sure. The remaining funds with a value of p_0 have been invested in the bubble. If it continues (probability Q_0), households again obtain β, if it bursts (probability $q-Q_0$), traders repay what they have, up to β. Additionally, monitoring costs c are paid in this case and if the firm goes bankrupt,

$$\mathrm{E}[C_1^H]=(\mathrm{E}[N]-p_0)\,\beta+p_0Q_0\beta+p_0(q-Q_0)\left[\min\left(\frac{\bar{p}+d}{p_0};\beta\right)-c\right]-p_0(1-q)c. \tag{A.A.9}$$

Note that the term $-\beta p_0(1-Q_0)<0$, present in the above expression, corresponds to the expected loss due to the limited liability of traders for the households in a full-blown bubble. If such a bubble does not continue (probability $1-Q_0$), traders that have invested in the bubble default on households. As a mirror image, this expression equals the expected payoff gain due to limited liability for traders: with probability $1-Q_0$, they do not have to repay the contractual amount β. Households' welfare is hence reduced because a) the risk of default rises in a bubble ($Q_0<q$) and b) the amount p_0 of resources at risk, i.e., invested in the bubble, increases. Additionally, monitoring costs have to be paid in case of a bursting bubble. In steady state, expected consumption of the household sector is $q\bar{p}\beta+(\mathrm{E}[N]-\bar{p})\,\beta-\bar{p}(1-q)c$, such that the expected difference results as

$$\mathrm{E}[C_1^H]=(\bar{p}-p_0)\,\beta+(p_0Q_0-\bar{p}q)\beta+p_0(q-Q_0)\left[\min\left(\frac{\bar{p}+d}{p_0};\beta\right)-c\right]\ldots$$
$$-(p_0-\bar{p})(1-q)c<0.$$

38. When calculating the effect of γ, it is helpful to insert the transformed equality $\varphi^{-\gamma-1}=q^{-1}(Y-\beta)/(\varphi-\beta)$ into (32) to calculate the direct effect of γ and the indirect effect via φ on the resulting inequality. Observe that $\varphi>Y$ because of the market-clearing condition.

To determine the sign of this term, consider the best possible case for households, i.e., repayment of β in case of a bursting bubble—even then, the expression is negative. ■

Proof of Proposition 5. Defining $Y' = (1+\tau')/(1+\tau)$ immediately shows that Proposition 2 applies. Concerning the participation constraint of households, we find that it might start to bind if taxes are levied on storage. The constraint in steady state is

$$\int_{\bar{p}(1+\tau')}^{\infty} \left[\frac{\bar{p}(1+\tau')}{N}(q\,\beta-(1-q)\,c)+\left(1-\frac{\bar{p}(1+\tau')}{N}\right)\beta\right] f(N)dN \geq l\,\lambda\,\frac{Y}{1+\tau}.$$

Households take the transfer of tax receipts from the government as given; hence it does not appear in the participation constraint. Evaluating the above integral shows that the participation constraint is independent of τ', as the probability that the trader invests in the risky asset is unaffected by this tax. Intuitively, while the amount that flows in the asset (including taxes) is larger, the assumed minimum $N_0 = \bar{p}(1+\tau')$ adjusts proportionally (as before, we make this assumption to rule out cash-in-the-market pricing). This implies that if households participate in case no tax is implemented, they also to do so for a higher tax rates. Placing a tax on the safe asset, on the other hand, reduces the payoff from the household's alternative investment opportunity. At the same time, it decreases $\beta = Yl/(1+\tau)$. It is straightforward to verify that, for $c > 0$, the participation constraint of households might be violated for high enough values of τ. In a bubble, the expected payoff from lending to traders is lower than in steady state. Hence, if a tax on the safe asset violates the participation constraint for households in steady state, it will also do so in a bubble. Taken together, a tax on the safe asset can create or eliminate the possibility of bubbles. The latter, however, works only via destroying the intermediation market. ■

Proof of Proposition 6. Regarding welfare consequences, we know from the proof of Proposition 5 that increasing τ' can render bubbles impossible and vice versa for τ. We have to additionally check, however, if a higher τ' or τ will lead to worse outcomes outside a bubble, i.e., in steady state. Total expected consumption of future generations is

$$\mathrm{E}[C^n] = \frac{Y+\tau}{1+\tau}\big(\mathrm{E}[N] - \bar{p}^n(1+\tau')\big) + q(\bar{p}^n + d) + \tau'\bar{p}^n - \bar{p}^n(1-q)c,$$

where C^n stands for consumption when the tax is implemented, $\bar{p}^n$ denotes the steady-state price that is obtained with taxes, $N - \bar{p}^n(1+\tau')$ is the expected amount invested in the safe asset, $q\bar{p}^n$ is the expected reselling value, and $\tau'\bar{p}^n + \tau[N - \bar{p}^n(1+\tau')]/(1+\tau)$ is the tax return to the household. The steady-state price with taxes, observing $\beta = Yl$ in case of household participation, is

$$\bar{p}^n = \frac{dq}{\frac{1+\tau'}{1+\tau}Y[1-l(1-q)]-q}.$$

Its derivative w. r. t. τ' is negative and positive w. r. t. τ. The derivative of $\mathrm{E}[C^n]$ w. r. t. τ', setting $\tau = 0$ for simplicity and without loss of generality, is

$$\frac{\partial \mathrm{E}[C^n]}{\partial \tau'} = \bar{p}^n(1-Y) + [\tau'(1-Y) + q - Y - (1-q)c]\frac{\partial \bar{p}^n}{\partial \tau'}.$$

This expression is positive, which can be shown by inserting the value for $\bar{p}^n$, and using

$$\frac{(1-q)(1+c)}{1-Y} Y[1 - l(1-q)] < q.$$

Consumption in the steady state hence increases with higher transaction taxes on the risky asset. Knowing from Proposition 4 that welfare in steady state is higher if compared to a bubbly situation with the same parameter values (except for the initial generation), we can conclude that a policy measure that can prevent bubbles *and* increases steady-state welfare is unambiguously welfare enhancing for future generations, as long as the tax is high enough to prevent bubbles. The derivative of C^n w. r. t. τ, on the other hand, is

$$\begin{aligned}\frac{\partial C^n}{\partial \tau} = &\big(\mathrm{E}[N] - \bar{p}^n(1+\tau')\big)\left(1 - \frac{Y+\tau}{1+\tau}\right)/(1+\tau)\ldots \\ &+ \frac{\partial \bar{p}^n}{\partial \tau}\left(q - \frac{Y+\tau}{1+\tau} - \tau'\frac{Y-1}{1+\tau} - (1-q)c\right) < 0,\end{aligned}$$

demonstrating that a tax on the safe asset reduces steady-state welfare. Lastly, for $\tau' = \tau$,

$$\frac{\partial C^n}{\partial \tau} = \frac{\mathrm{E}[N]}{1+\tau}\left(1 - \frac{Y+\tau}{1+\tau}\right) < 0,$$

which shows that a tax on both assets reduces steady-state welfare as well. ■

Proof of Proposition 7. Assuming that each trader can borrow the maximum amount of $\overline{x}^T$, in combination with intermediation-market clearing ($x^H = \bar{x}^T$) and conditions (17), shows that we can directly use the result of Proposition 2 regarding l.

The participation constraint of traders with capital requirements changes to $eY \geq (e+\overline{x}^T)Y - \beta$, as traders can obtain eY by investing their own funds only, or $(e+\overline{x}^T)Y$ by additionally borrowing from households. Households will therefore again set $\beta = Y\overline{x}^T$. Introducing capital requirements by reducing $\overline{x}^T$ will hence destroy the possibility of bubbles, as condition (15) fails to hold for $\beta \to 0$. ■

Proof of Proposition 8. We compare two situations, the old steady state without capital requirements and a steady state with capital requirements. Expected aggregate period consumption in a given period with the capital requirement in place is

$$\mathrm{E}[C^n] = \lambda Y\, \mathrm{E}[N] + (1-\lambda)Y(e + \overline{x}^T)\mathrm{E}[N] + \bar{p}^n q + q\, d - \bar{p}^n\, Y - \bar{p}^n(1-q)\, c, \tag{A.A.10}$$

where $\lambda Y \mathrm{E}[N] + (1-\lambda) Y (e + \overline{x}^T) \mathrm{E}[N] - \bar{p}^n Y = Y\big(\mathrm{E}[N](e + \overline{x}^T) - \bar{p}^n\big) + \lambda Y \mathrm{E}[N](1 - e - \overline{x}^T)$ is the expected amount invested by the traders into the safe asset plus the investment of the household into the inferior investment technology. $\bar{p}^n$ is the steady-state price with the policy in place, $\bar{p}^o$ the one without. In the following, all $\bar{p}$ denote $\bar{p}^n$, except where explicitly mentioned. $\bar{p}q$ is the expected revenue from selling the asset, and qd the expected dividend. The derivative of the consumption difference between the new and the old steady state is

$$\frac{\partial \mathrm{E}[C^n] - \mathrm{E}[C^o]}{\partial \overline{x}^T} = (1-\lambda) Y \,\mathrm{E}[N] - (Y - q + (1-q)\,c) \partial \bar{p}^n / \partial \overline{x}^T.$$

The steady-state price (11) becomes

$$\bar{p} = \frac{dq}{Y - q - Y(1-q)\overline{x}^T/(e + \overline{x}^T)},$$

where $\beta = Yl$ is already inserted and $\overline{x}^T/(e + \overline{x}^T)$ represents the leverage ratio (equal to l in the baseline case). The derivative w. r. t. $\overline{x}^T$ is

$$\frac{\partial \bar{p}}{\partial \overline{x}^T} = \frac{Y(1-q)\bar{p}}{Y - q - Y(1-q)\overline{x}^T/(e + \overline{x}^T)} \frac{e}{(e + \overline{x}^T)^2} > 0.$$

Combining these equations yields

$$\frac{\partial \mathrm{E}[C^n] - \mathrm{E}[C^o]}{\partial \overline{x}^T} = \frac{Y(1-q)\bar{p}(q - Y - (1-q)c)}{Y - q - Y(1-q)\overline{x}^T/(e + \overline{x}^T)} \frac{e}{(e + \overline{x}^T)^2} + (1-\lambda) Y \mathrm{E}[N],$$

which is negative (capital requirements increase steady-state welfare) if

$$\frac{(1-q)\bar{p}(Y - q)}{Y - q - Y(1-q)\overline{x}^T/(e + \overline{x}^T)} \frac{e}{(e + \overline{x}^T)^2} > (1-\lambda)\mathrm{E}[N]. \tag{A.A.11}$$

This condition holds for certain parameter constellations; only if $q = 1$ there is no region in which this inequality is fulfilled (as $\lambda \leq 1$). In this case, the steady-state price does not depend on $\overline{x}^T$, since there is no limited liability due to the lack of risk. Lowering $\overline{x}^T$ is then unambiguously bad.

From the above it is not clear whether welfare increases or decreases relative to the baseline steady state with the introduction of capital requirements. On the one hand, they can destroy the possibility of bubbles. On the other hand they can decrease welfare because the household has to use the inferior investment technology. The lower steady-state price (once bubbles are prevented) is again beneficial for future generations because less is invested into the risky asset. We can make some statements about inequality (A.A.11) if we use the participation constraint of households. Expected payoff in steady state for a household is

$$\int_{N_0}^{\infty} \left[\frac{\bar{p}}{N(e + \overline{x}^T)} \left(q Y \overline{x}^T - (1-q)\,c \right) + \left(1 - \frac{\bar{p}}{N(e + \overline{x}^T)} \right) Y \overline{x}^T \right] f(N) dN \ldots$$
$$+ (1 - e - \overline{x}^T)\lambda Y, \tag{A.A.12}$$

where the probability that the trader invests into the asset $\bar{p}/\big(\mathrm{E}[N](e+\overline{x}^T)\big)$ is adjusted for the fact that the traders can now invest less resources. In this context, we assume $N_0 = \bar{p}/(e+\overline{x}^T)$ without loss of generality, as the total amount of funds that traders control (i.e., the maximum amount that all traders could have invested into the asset) is $N(e+\overline{x}^T)$ and agents know that $\bar{p}$ was invested into the asset. The above expression has to be larger than the alternative investment return to the household without lending to traders, which is $\overline{x}^T \lambda Y$. We therefore get

$$\int_{N_0}^{\infty} \frac{\bar{p}}{N(e+\overline{x}^T)}(q-1)(Y\,\overline{x}^T + c)f(N)dN + \int_{N_0}^{\infty} Y\overline{x}^T f(N)dN \geq \overline{x}^T \lambda Y$$
$$\implies \frac{\bar{p}}{e+\overline{x}^T}(1-q)\left(1+\frac{c}{Y\,\overline{x}^T}\right)\int_{N_0}^{\infty}\frac{1}{N}f(N)dN \leq 1-\lambda. \tag{A.A.13}$$

Taking the derivative of the left-hand side of this equation w. r. t. $\overline{x}^T$, we obtain

$$\left[\frac{\partial \bar{p}/(e+\overline{x}^T)}{\partial \overline{x}^T}\left(1+\frac{c}{Y\overline{x}^T}\right) - \frac{\bar{p}}{e+\overline{x}^T}\frac{c}{Y(\overline{x}^T)^2}\right](1-q)\int_{N_0}^{\infty}\frac{1}{N}f(N)dN < 0.$$

Decreasing $\overline{x}^T$ from a situation in which the household participates leads therefore at some point to a violation of the household participation constraint. A reduction in $\overline{x}^T$ results in an increase in the share of traders' funds flowing to the risky asset—the derivative of $\bar{p}/(e+\overline{x}^T)$ w. r. t. $\overline{x}^T$ is negative. The household hence needs to invest more in the inferior investment technology (involuntarily, as shown by revealed preferences in the situation without capital requirements) *and* traders' behavior becomes riskier.

Finally, we compare the condition required for steady-state welfare to depend negatively on $\overline{x}^T$, equation (A.A.11), with the participation constraint of households, equation (A.A.13). If both are fulfilled simultaneously, we can insert the latter into the former,

$$\frac{Y-q+(1-q)c}{Y-q-Y(1-q)\overline{x}^T/(e+\overline{x}^T)}\frac{e}{e+\overline{x}^T} > \left(1+\frac{c}{Y\overline{x}^T}\right)\int_{N_0}^{\infty}\frac{1}{N}f(N)dN\int_{N_0}^{\infty}Nf(N)dN. \tag{A.A.14}$$

Since

$$\int_{N_0}^{\infty}\frac{1}{N}f(N)dN\int_{N_0}^{\infty}Nf(N) > 1 \text{ and } \frac{(1-q)ce}{(Y-q)(e+\overline{x}^T)-Y(1-q)\overline{x}^T} < \frac{c}{Y\overline{x}^T}$$
$$\text{and } q\overline{x}^T(Y-1) > 0 \iff \frac{Y-q}{Y-q-Y(1-q)\overline{x}^T/(e+\overline{x}^T)}\frac{e}{e+\overline{x}^T} < 1,$$

we conclude that inequality (A.A.14) does not hold and hence conditions (A.A.11) and (A.A.13) cannot be fulfilled simultaneously. That is, if households participate, reducing $\overline{x}^T$ also reduces welfare. If households do not participate anymore, reducing $\overline{x}^T$ has no further effects. We thus get a negative effect of capital requirements on welfare in steady state. ■

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