Optimal Selling Mechanisms with Endogenous Proposal Rights

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Abstract

We study a model of optimal pricing where the right to propose a mechanism is determined endogenously: a privately informed buyer covertly invests to increase the probability of offering a mechanism. We show that higher types get to propose a mechanism more often, enabling the seller to learn from the trading process. In any equilibrium, the seller either offers the price he would have offered if he was always the one to make an offer or randomises over prices. Pure strategy equilibria may fail to exist, even when types are continuously distributed. A full characterization of equilibria is provided in the model with two types, where the seller’s profit is shown to be non-monotonic in the share of high-value buyers.

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1 Introduction

The question of how economic agents trade has spurred an immense literature with powerful insight. The main focus lies on the features of the optimal mechanism under informational asymmetries and how they are shaped by scarcity and competition. The latter two forces have long been recognised as driving forces of bargaining power. The literature has extensively explored how optimal mechanisms and the principal’s payoff depend on the number of buyers, as well as on the presence of competing sellers.\(^1\) For instance, it is well known that a seller can extract almost full surplus with a mechanism as simple as a second price auction if the number of participating buyers is large. Similar ideas have been discussed already by Adam Smith, who argued that masters have greater bargaining power than workers because they are fewer in numbers; Smith (1776).\(^2\) Little attention has, however, been dedicated to the question of how bargaining power is distributed in contracting problems when no side has a clear competitive advantage. In such situations, agents have incentives to find ways that improve their bargaining position: potential buyers search the internet for information on how to get a good deal from car salesmen, companies and individuals hire lawyers or other experts when selling or buying valuable assets, parties in a troubled marriage hire divorce lawyers to negotiate on their behalf, just to name a few. Although it is recognised that who determines the mechanism not only matters for the division of the surplus but also for many other features of the equilibrium allocation,\(^3\) by and large, the right to propose a mechanism has been treated as exogenous. In this paper, we consider the design of optimal mechanisms in a model with endogenous proposal rights.

We introduce a bilateral trade problem between a seller and a buyer with the novel feature that the proposer of the mechanism is selected endogenously. The buyer, who is privately informed about his valuation of the good, can make a covert investment in order to improve his bargaining position—e.g., the resources spent to find and hire the right lawyer (or other representatives), the amount of attention and time he devotes to specialised courses on ‘the art of negotiation’ (see Cialdini and Garde (1987), Fisher et al. (2011)), etc. Higher investment, naturally, translates into a higher probability of proposing a mechanism. Though investments are not observable, obtaining the right to make a proposal is a signal about the other party’s investment. Our focus lies on the

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\(^1\)For competition in mechanism design, see McAfee (1993), Peters and Severinov (1997) and for a recent review of the literature Pai (2010).

\(^2\)“It is not, however, difficult to foresee which of the two parties must, upon all ordinary occasions, have the advantage in the dispute, and force the other into a compliance with their terms. The masters, being fewer in number, can combine much more easily...”

\(^3\)For instance, Hagedorn and Manovskii (2008) showed that the distribution of bargaining power between workers and firms is an important determinant for the outcomes of search models regarding wages and unemployment volatility.
buyer’s investment, since he is the privately informed agent. The party with the proposal power determines the mechanism, which in the bilateral trade environment takes the form of a (distribution over) posted price(s). After the mechanism is executed the game ends; that is, the proposer commits to the mechanism.

Our environment is a reduced form model capturing two salient features. First, if a buyer invests, he is able to obtain a higher share of the surplus, captured by an increase in the probability that he gets to make an offer. Second, agents might be able to learn about their opponents during the trading process. Specifically, if a certain type of buyer invests less than others, then the seller is more likely to secure the chance to make an offer against that type. The revelation that the seller is the one making an offer, therefore, conveys information about the buyer’s willingness to pay. Our objective is to analyze how these two elements interplay, and what novel insights their interaction delivers.

The driving force of our paper is the observation that higher types of the buyer have stronger incentives to invest. A buyer with a higher valuation offers more surplus to be shared and therefore has more incentives to fight for it. Hence, the higher the buyer’s valuation, the more likely the buyer is to make an offer, establishing a novel source of bargaining power.

The effect of the buyer’s valuation on bargaining power has important implications on the seller’s optimal pricing policy. Taking into account that the seller is more likely to be able to make an offer against lower types, the seller revises his posterior (conditional on being able to make an offer) towards lower valuations. Despite the fact that this favours lower prices, our main result shows that in any equilibrium the ex-ante optimal price—the price the seller would charge if he were always the one to make an offer—belongs to the support of the seller’s price distribution. Moreover, it is the highest price in that support. As an immediate consequence, the only candidate pure-strategy equilibrium is the one where the seller offers the ex-ante optimal price. Due to the seller’s updating, however, such equilibrium may not exist. In that case, the seller randomises across a set of prices—a feature that is non-generic when bargaining power is exogenously fixed.

We show uniqueness of the equilibrium and derive further properties for the case where the buyer’s valuation is binary. A striking feature is that the seller’s expected payoff is non-monotonic in the prior probability of facing a high type. It is flat for low priors, decreasing in the intermediate region and increasing for high priors.

Finally, we discuss the implications of our findings on so-called probabilistic bargaining models. In such models the buyer and the seller have exogenously fixed probabilities of making an offer. Moreover, these probabilities tend to be interpreted as bargaining
powers; see for example Zingales (1995), Inderst (2001), Krasteva and Yildirim (2012) and Münnster and Reisinger (2015). Our results suggest that in some environments it would be more realistic to assume that bargaining power is increasing in the buyer’s valuation. We argue that taking the probability of making an offer exogenously fixed and increasing in the buyer’s type provides the researcher with a simpler model, yet captures some of the arresting features of our framework.

**Related Literature:** Myerson (1981) studied a mechanism design model in which the seller is always the one to choose a mechanism. Among other things, he shows that if the seller is selling to only one buyer, he can always restrict himself to a posted price. Moreover, the seller can extract nearly full surplus as the number of buyers grows. The seller’s bargaining power and, through that, the ability to extract surplus are put to the test in the models of competition in mechanism design; for an outline of the literature see Pai (2010). We explore an alternative way of reducing the seller’s bargaining power, namely by allowing the buyer to fight for his right to propose a mechanism himself.

To the best of our knowledge, we are the first to consider a model of endogenous proposal rights in the context of optimal pricing with commitment (mechanism design). Models where agents compete for proposal rights have been studied in the context of bargaining without commitment; see, for instance, Yildirim (2007), Yildirim (2010), Board and Zwiebel (2012) and Ali (2015). As opposed to our work, these papers analyse environments where the surplus is of commonly known size. Under the same assumption Karagozoglu and Rachmilevitch (2018) study a model where agents pay a cost to prepare for negotiations. In their framework, costly preparations do not affect the allocation of proposal rights, but are instead necessary to remain at the negotiation table.

Other papers analyse trading models with the feature that an agent partially learns from the interaction. In Lauermann and Wolinsky (2017), a privately informed seller gets to select a number of buyers that participate in a first price auction for an object. The fact that they have been selected reveals something about the common value of the object, similarly as being able to make an offer reveals something to the seller in our model. However, the objective is rather different from ours. While they study equilibria of a particular mechanism (the first price auction) and how they aggregate information, we study optimal pricing. Heinsalu (2017) analyses a model with endogenous entry into a competitive market with adverse selection. Entry is informative about the entrant’s private information and, therefore, reflected in the market price.

4A somewhat different, but related, literature–see Crawford (1979), Evans (1997) and Pérez-Castrillo and Wettstein (2001)–studies the effects of competition on coalitional bargaining games.
2 The Model

A buyer and a seller want to trade a good. The buyer’s valuation $v$ is drawn from a cumulative distribution function $F$ on $[0, 1]$. At this point we impose no additional assumptions on $F$; it can be continuous, as often assumed, or have atoms. The seller’s value is commonly known and normalised to zero. Upon observing his valuation, the buyer can make an investment to increase the probability with which he gets to make an offer.\(^5\) More precisely, the buyer chooses an investment $e \in \mathbb{R}_+$ at a cost $ke$, after which he gets to make an offer with probability $\rho(e)$, where $\rho : \mathbb{R}_+ \to [0, 1)$ is strictly increasing, strictly concave and differentiable on its support.\(^6\) The seller gets to make an offer with the complementary probability. After the offer is accepted/rejected the game ends. Since there is only one buyer, it is without loss of generality to restrict attention to price offers or randomizations over those.\(^7\)

Remark 1. An example to keep in mind is that of a Tullock contest in which agents’ chances to make an offer are proportional to their effort investments; Tullock (1980). In particular, $\rho(e) = e/(e + e_s)$, where $e_s > 0$ is the seller’s (fixed) effort.

The buyer’s pure strategy is a tuple of functions $(e, p_b, x_b)$, where $e : [0, 1] \to \mathbb{R}_+$ maps the buyer’s types into the investment choices, $p_b : [0, 1] \to \mathbb{R}_+$ specifies the price the buyer offers when he gets to make an offer (in the remainder of the text we suppress the qualifier ”when he gets to make an offer”), and $x_b : [0, 1] \times \mathbb{R}_+ \to \{0, 1\}$ specifies whether a type accepts (1) or rejects (0) a given price. The seller’s pure strategy is a tuple $(p_s, x_s)$, where $p_s \in \mathbb{R}_+$ is the price the seller offers and $x_s : \mathbb{R}_+ \to \{0, 1\}$ is the seller’s acceptance strategy.

The equilibrium concept we employ is sequential equilibrium; equilibrium for short. While a sequential equilibrium may not be well defined for games with a continuum of types and actions in general, such issues does not arise in our framework. The only possible out of equilibrium continuation games are the ones reached after unexpected prices.\(^8\)

Given the private values assumption, out of equilibrium beliefs in such information sets

\(^5\)In an earlier version of the paper, we considered the case where also the seller invests. Since the buyer knows the seller’s valuation and, hence, does not learn, the seller’s investment does not change the main insights of the analysis, so we abstract from the seller’s investment choice here.

\(^6\)By assuming that the range of $\rho$ is $[0, 1)$ we avoid rather uninteresting problems brought upon by the possibility of out of equilibrium beliefs. In particular, the assumption implies that the seller making an offer is always on the equilibrium path. The seller can not be surprised by receiving the opportunity to make an offer.

\(^7\)Any (direct) mechanism $(q, t)$, where $q$ specified the probability with which the buyer wins the good and $t$ the transfer can be implemented with randomization over prices. If revenue equivalence applies, the distribution is $H = q$, otherwise a bit more care is required.

\(^8\)There is one exception. Since $\rho$ can take value 0, under certain parameters an equilibrium can be constructed where no type of the buyer invests and therefore the buyer is never supposed to make an
are irrelevant: What the seller thinks about the buyer’s valuation after the buyer offers some price does not affect his decision whether or not to accept the price.

The outlined model is a simplified representation of reality. However, the model captures two important features that we would like to explore: 1.) A larger investment enables the agent to secure a higher share of the surplus. This is captured by the probability with which an agent gets to make an offer. The existing literature commonly uses the probability that an agent makes a take-it-or-leave-it offer as a proxy for bargaining power (see for example Zingales (1995) and Inderst (2001)). 2.) By observing a signal about the opponent’s investment, an agent may learn about the opponent’s type. The seller faced with an opportunity to make an offer can infer something about the buyer’s valuation due to different types of the buyer making (possibly) different investment decisions.

3 Equilibrium Analysis

It is useful to start with some preliminary observations. If the buyer gets to make an offer, he optimally offers price \( p_b = 0 \) and the seller accepts. The buyer’s strategy can therefore be reduced to his choice of investment \( e \) and his decision to accept/reject. The buyer’s optimal investment depends on the seller’s pricing strategy. The seller’s randomisation over prices can be described by a cumulative function \( H \), where \( H(p) \) is the probability that the seller offers a price weakly smaller than \( p \). Given \( H \), buyer type \( v \) solves the problem

\[
\max_{e \in [0, \infty)} \rho(e)v + (1 - \rho(e)) \int_0^v (v - p)dH(p) - ke.
\]  

(1)

After investing \( e \) the buyer gets to make an offer with probability \( \rho(e) \) and obtains a payoff \( v \). With the complementary probability the seller makes an offer, which the buyer accepts as long as his valuation is not below the price. The following lemma characterises the solution to the buyer’s problem and shows that the optimal investment is non-decreasing in the buyer’s valuation.

**Lemma 1.** If the buyer expects the seller to randomise across prices according to \( H \), his offer. In the off equilibrium continuation game where the buyer does make an offer, however, the seller’s beliefs over buyers valuations are irrelevant.
optimal investment is uniquely described by the function

\[ e^*(v; H) = (\rho')^{-1} \left( \frac{k}{v(1 - H(v)) + \int_0^v pdH(p)} \right), \tag{2} \]

for all \( v \) such that \( \rho'(0) \left( v(1 - H(v)) + \int_0^v tdH \right) \geq k \) and \( e^*(v; H) = 0 \) otherwise. For all \( H \), the function \( e^*(v; H) \) is non-decreasing in \( v \).

Proof. See Appendix for the proof.

The buyer’s investment function is equal to zero for types close to zero, increasing up to the lowest type who always trades when the seller makes an offer, and constant thereafter; see Figure 1.\(^9\) This is easy to see when the buyer expects the seller to offer a (deterministic) price \( p \). For types below the seller’s price \( p \), the loss of not being able to make an offer is equal to their type: if the buyer makes an offer, her surplus from trade is \( v \), whereas if the seller makes the offer \( p \), the buyer rejects and the surplus is 0. The loss is strictly increasing in \( v \).\(^10\) Types above \( p \), on the other hand, trade even when the seller makes an offer. The loss they incur due to the seller making an offer is the price the seller charges: when the seller makes the offer, their benefit from trade is \( v - p \), otherwise it is \( v \). This loss is constant for all types above the price, as are, therefore, the incentives to invest. The buyer’s expected payoff is strictly concave in \( e \), which means that each type has a unique best response. In equilibrium, the buyer thus plays a pure strategy.

\[ \begin{align*}
\text{Figure 1: Buyer’s optimal investment when expecting the seller to offer price } p. & \\
\end{align*} \]

\(^9\)In the situation we describe, \( k \) is sufficiently small such that some types want to invest. Instead, when \( k \) is large, the optimal investment is zero for all types.

\(^{10}\)For types close to zero, the gain from making an offer may not be sufficient to compensate the cost of investment, so they prefer not to invest.
The monotonicity of the buyer’s investment implies that higher types are more likely to make an offer and, conversely, that the seller is more likely to make an offer against lower types. Given the buyer’s investment strategy $e(v)$, the probability that type $v$ makes an offer is $\rho(e(v))$. Using this probability, the seller can compute the posterior distribution over the buyer’s valuations, conditionally on being able to make the offer himself:

$$G(v; e) = \frac{\int_0^v (1 - \rho(e(t)))dF(t)}{\int_0^1 (1 - \rho(e(t)))dF(t)}.$$  

(3)

**Lemma 2.** Let investment function $e(\cdot)$ be non-decreasing and non-constant on the support of $F$. Then $F$ first-order stochastically dominates $G$. 

Higher types of the buyer tend to invest more than lower types. The seller is, thus, more likely to be able to make an offer when facing a low type. As a consequence, the prior distribution of types, $F$, first-order stochastically dominates the seller’s posterior belief when given the chance to make an offer, $G$. While being able to make an offer is good news for the seller, he understands that he is more likely to win against lower types of the buyer. Winning, therefore, depresses his belief.

Given the buyer’s investment strategy $e$, the seller solves the problem

$$\max_{p \in \mathbb{R}_+} (1 - G(p; e))p.$$  

(4)

The revised beliefs dictate the optimal price, leading to a rather interesting fixed point problem. The buyer conjectures the price (or the distribution over prices) that the seller will charge and chooses his investment in response. The seller updates his beliefs and charges a price that is optimal with respect to his posterior. In equilibrium, the buyer’s conjecture about the price distribution must coincide with the seller’s optimal price distribution. The following lemma shows that this problem has a fixed point.

**Proposition 1.** An equilibrium of the game exists.

**Proof.** See Appendix for the proof. 

The main obstacle toward showing the existence is the discontinuity of the seller’s payoff in the price. If there is a mass point in the buyer’s distribution over valuations, the seller’s payoff jumps up as the price approaches the mass point from above. Here we rely on Reny (1999) and Carbonell-Nicolau and McLean (2017) results on the existence
of mixed strategy equilibria in discontinuous games with infinite strategy spaces. Before applying the result we simplify the game by using some of the above derived properties of equilibria. In particular, in any sequential equilibrium the buyer accepts a price offer if and only if the price is at least as high as his valuation and the seller if it is non-negative; we assume that the agent accepts an offer if indifferent. Moreover, knowing that the seller will accept any non-negative price, the buyer optimally offers price 0. The auxiliary game assumes such behavior. Leaving each type of the buyer to decide the investment and the seller to choose the price. After proving the existence of an equilibrium in the auxiliary game, we show that it embeds as a sequential equilibrium in the original game.

**Equilibrium pricing.** Having established the existence of an equilibrium, we turn attention to the question of the seller’s pricing strategy. If the seller were always the one to make an offer, he would offer the price that maximises \( p(1 - F(p)) \), denoted by \( p^* \). We abstract from the non-generic cases where \( p(1 - F(p)) \) has multiple maximisers. In what follows, we show that \( p^* \) is in the support of the seller’s randomisation over prices in every equilibrium. Moreover, it is the upper bound of the support.

**Proposition 2.** Fix an equilibrium. Then \( p^* \) is the maximum of the support of the seller’s distribution over prices. As a consequence, every equilibrium takes one of the following two forms:

- it is a pure-strategy equilibrium in which the seller offers price \( p^* \);
- it is a mixed-strategy equilibrium in which the seller randomises over a set of prices of which \( p^* \) is the highest.

Moreover, if two prices \( p', p \) are such that \( p < p' < p^* \) and \( p(1 - F(p)) > p'(1 - F(p')) \), then \( p' \) can not be in the support of the seller’s strategy in any equilibrium.

**Proof.** See Appendix for the proof.

Proposition 2 shows that in every equilibrium the ex-ante optimal price \( p^* \) is among the optimal prices for the seller and that the seller never offers a price higher than \( p^* \). One might be tempted to conclude that the latter property is a consequence of the fact that the seller’s posterior first-order stochastically dominates the prior distribution. This is, however, not enough to claim that the optimal price under the posterior is weakly lower than \( p^* \). A more direct argument is needed, as we show in the proof. Maybe more
surprising, the upper bound of the set of equilibrium prices cannot be smaller than \( p^* \).

This follows from the property that all types above the upper bound \( \bar{p} \) make the same investment (see Lemma 1). As a consequence, the relative posterior probabilities across these types are the same as under the prior and, by the same token, the maximizer over the prices in \([\bar{p}, 1]\) is the same before and after updating. Now if \( \bar{p} < p^* \), the fact that \( p^* \) is optimal ex-ante implies that it also yields the highest profit among the prices in \([\bar{p}, 1]\) after the revision of beliefs. Hence, \( p^* \) would yield a strictly higher profit than \( \bar{p} \), contradicting the assumption that \( \bar{p} \) belongs to the set of prices over which the seller randomises.

Given the above, the only possible pure-strategy equilibrium is the one where learning plays no role and the seller offers the ex-ante optimal price \( p^* \). Existence of such an equilibrium can be checked by computing the buyer’s best response to price \( p^* \), as specified in Lemma 1, and verifying that the seller has no incentives to deviate to a lower price after updating his beliefs. Simple sufficient conditions for the existence of the pure strategy equilibrium can be provided. For example, the pure strategy equilibrium exists if the cost of investment is sufficiently large so that when the buyer expects the sellers to offer \( p^* \), zero investment is optimal for all types. Alternatively, if the buyer’s valuation is distributed on some interval \([\theta, 1]\), where \( \theta > 0 \), and \( p^* = \theta \), then the pure strategy equilibrium is the unique equilibrium of the game. More interesting, however, is the possibility of mixed strategy equilibria.

While general conditions under which pure strategy equilibria do not exist remain elusive, the following intuition holds. A pure-strategy equilibrium does not exists when based on the prior there is a price \( p' \) sufficiently below \( p^* \) that achieves a profit sufficiently close to that generated by \( p^* \). In such a case, if the buyer expects \( p^* \) and invests accordingly, the seller’s posterior shifts sufficiently towards lower types in order to make the profit from offering \( p' \) larger than that from offering \( p^* \). The following example demonstrates that a pure-strategy equilibrium can fail to exist even in environments where \( F \) has a density on an interval.

**Example 1.** Let the function \( \rho(e) = e/(e + e_s) \) be a Tullock (1980) function with \( e_s > 0 \) and consider a probability distribution described by the CDF

\[
F(v) = \begin{cases} 
    av & \text{if } 0 \leq v \leq d \\
    c + bv & \text{if } v > d,
\end{cases}
\]

where \( b = (1 - ad)/(1 - d) \) and \( c = -(1-a)d/(1-d) \). We specify the following parameters: \( a = 1.5, d = 0.4, e_s = 0.5, k = 0.4 \), with the ex-ante optimal price being \( p^* = 0.5 \). When the buyer expects price 0.5 and invests optimally, the seller’s expected payoff evaluated at his posterior belief is no longer maximised at 0.5 but instead at \( \approx 0.326 \). Hence,
there is no pure-strategy equilibrium. However, there is an equilibrium where the seller randomises across two prices: the seller offers \( p \approx 0.332 \) with probability \( \approx 0.189 \) and the ex-ante optimal price \( p^* = 0.5 \) with the complementary probability. This is illustrated in Figure 2. The last part of the above proposition implies that there is no equilibrium where the seller randomises over the prices on the interval \([0.332, 0.5]\), since the prices just above 0.332 yield smaller payoffs than the price 0.332 when the profit is computed with respect to the prior. The only possibility for an equilibrium with randomization over a connected set of prices would be for the seller to randomize over prices in the interval \([\tilde{p}, 0.5]\) where \( \tilde{p} > 0.332 \) and the price \( \tilde{p} \) yields a higher payoff than 0.5 when computed with respect to the prior, i.e., randomization over prices just below 0.5. One can verify that such an equilibrium does not exist.

\[ \text{Figure 2: Seller’s expected payoff before updating (solid curve) and in equilibrium (dashed curve)} \]

The above example illustrates that even when the distribution of the buyer’s valuation has no atoms, in a mixed-strategy equilibrium the seller need not randomise over an interval of prices. To understand this better, we can consider a simpler discrete type example. Suppose there are three types \( v_1 < v_2 < v_3 \) and the probability distribution is such that with respect to the prior distribution the seller prefers charging a price equal to the high valuation \( v_3 \) over a price equal to the low valuation \( v_1 \) and the latter over a price equal to the intermediate valuation \( v_2 \). Suppose further that there is no equilibrium where the seller always offers the ex-ante optimal price \( v_3 \); this will be the case when the profits from offering \( v_3 \) and \( v_1 \) are sufficiently close with respect to the prior. By the last statement of Proposition 2 price \( v_2 \) cannot be in the support of the seller’s distribution. Therefore, in equilibrium the seller must randomise over prices \( v_1 \) and \( v_3 \).

Since we impose little restriction on the type distribution, we cannot exclude the possibility of multiple equilibria. Further insights can be gained by exploring the environment with binary types.
4 Binary Types

Suppose the buyer’s valuation belongs to the set \( \{v_L, v_H\} \), with \( 0 < v_L < v_H \), and let \( \mu \) denote the ex-ante probability that the buyer’s valuation is high. We assume that the marginal cost of investment is low enough for the low type to invest: \( k < \rho'(0)v_L \).

In equilibrium the seller offers either price \( v_L \) or price \( v_H \). Proposition 2 implies that when the ex-ante optimal price is \( v_L \)—equivalently, when \( \mu \leq v_L/v_H \)—there is a unique equilibrium in which the seller offers price \( v_L \). The more interesting case is the one where the ex-ante optimal price is \( v_H \), that is, \( \mu > v_L/v_H \). Letting \( \sigma \) be the probability that the buyer attaches to the seller charging price \( v_H \), the buyer’s optimal investment is:

\[
e^*(v; \sigma) = \begin{cases} (\rho')^{-1}(\frac{k}{\sigma v_H + (1-\sigma)v_L}) & \text{if } v = v_H \\ (\rho')^{-1}(\frac{k}{v_L}) & \text{if } v = v_L. \end{cases}
\]

Whenever the seller makes an offer, the low type’s payoff is 0—either the price is too high to trade or the seller extracts the full surplus. Consequently, the low type’s payoff does not depend on the probability of the seller charging the high price. The high type’s payoff, on the other hand, does. The larger is \( \sigma \), the higher is the expected price the high type has to pay when the seller makes the offer. Concavity of \( \rho \) implies that \( (\rho')^{-1} \) is a decreasing function and, hence, that the high type’s investment \( e^*(v_H; \sigma) \) is increasing in \( \sigma \). In other words, the higher is the expected price the seller charges, the larger is the buyer’s benefit of making the offer himself and, thus, the incentives to invest.

When \( \sigma \) is strictly positive, the high type invests more than the low type and the seller updates his beliefs accordingly.\(^{11}\) The seller’s posterior belief about the buyer’s type being high, conditional on the seller making an offer, is

\[
\hat{\mu} := \frac{\mu \left( 1 - \rho \left( (\rho')^{-1}(\frac{k}{\sigma v_H + (1-\sigma)v_L}) \right) \right)}{\mu \left( 1 - \rho \left( (\rho')^{-1}(\frac{k}{\sigma v_H + (1-\sigma)v_L}) \right) \right) + (1 - \mu) \left( 1 - \rho \left( (\rho')^{-1}(\frac{k}{v_L}) \right) \right)}.
\]

The seller’s posterior \( \hat{\mu} \) is strictly decreasing in \( \sigma \) and takes value \( \mu \) (equal to the prior) when \( \sigma = 0 \). Given the restriction \( \mu > v_L/v_H \), two cases need to be considered. If \( \hat{\mu} \) evaluated at \( \sigma = 1 \) is weakly greater than \( v_L/v_H \), the ex-ante optimal price \( p^* = v_H \) is optimal after updating for all values of \( \sigma \). In this case the only equilibrium is the pure-strategy equilibrium where the seller offers the price \( v_H \). On the other hand, if \( \hat{\mu} \)

\(^{11}\)When the seller offers the low price with certainty, both types trade with probability one. Their benefit from making an offer is the price they do not pay to the seller \( (v_L) \) which is the same for both. In consequence the two types invest the same amount.
evaluated at \( \sigma = 1 \) is smaller than \( v_L/v_H \), there is a unique value of \( \sigma \) at which the seller’s posterior \( \hat{\mu} \) coincides with \( v_L/v_H \), rendering him indifferent between the low and the high price. This value of \( \sigma \) is the randomisation strategy of the seller in the mixed-strategy equilibrium that obtains. Letting \( m \) be the value of \( \mu \) at which \( \hat{\mu} \) evaluated at \( \sigma = 1 \) is exactly \( v_L/v_H \), the discussion can be summarised as follows.

**Proposition 3.** In the environment with two types, the equilibrium is generically unique. The seller charges price \( v_H \) with probability

\[
\sigma = \begin{cases} 
0 & \text{if } \mu < \frac{v_L}{v_H} \\
\sigma^* & \text{if } \mu \in \left(\frac{v_L}{v_H}, m\right) \\
1 & \text{if } \mu \geq m,
\end{cases}
\]

where \( \sigma^* \) is the unique solution to

\[
1 = \frac{v_L}{v_H} \frac{1 - \rho((\rho')^{-1}(\frac{L}{H}))}{1 - \rho((\rho')^{-1}(\frac{v_L}{v_H} + (1 - \sigma)v_L))},
\]

and \( v_L \) with the remaining probability.

The equilibrium is most readily described as a function of the seller’s prior. When \( \mu \) is below \( v_L/v_H \) so that the ex-ante optimal price is \( v_L \), this price remains optimal after updating. Instead, when \( \mu \) is just above \( v_L/v_H \), the prior favors the high price only slightly. If the buyer expects the high price and chooses his investment accordingly, the seller’s posterior would drop below \( v_L/v_H \), making the low price optimal. However, if the buyer expects the seller to charge the low price, both types of the buyer would invest the same amount and the seller’s posterior would stay above \( v_L/v_H \), rendering the high price optimal. The seller, thus, randomises over the two prices. Finally, when \( \mu \) is higher than \( m \), the posterior belief \( \hat{\mu} \) is always above \( v_L/v_H \) and, therefore, the high price \( v_H \) is optimal.

In environments where the seller always makes the offer, the case where he is indifferent between the high and the low price is non-generic. Proposition 3 shows that when proposal rights are endogenous instead, there is always an interval of values of \( \mu \) where either of the two prices is optimal for the seller in equilibrium. Moreover, the seller’s equilibrium probability of offering the high price as a function of the prior \( \mu \) has two jumps, taking a value in \((0,1)\) in the intermediate region.

**Comparative Statics.** It is of interest to study how the seller’s payoff changes with
the proportion of the high types.

**Proposition 4.** The seller’s equilibrium payoff, \( u_s \), as a function of the prior \( \mu \) is non-monotonic:

\[
\begin{align*}
    u'_s(\mu) = 0 & \quad \text{if } \mu \leq v_L/v_H \\
    < 0 & \quad \text{if } \mu \in (v_L/v_H, m) \\
    > 0 & \quad \text{if } \mu \geq m.
\end{align*}
\]

*Proof.* See Appendix for the proof.

![Figure 3: Seller’s equilibrium expected payoff](image)

The striking property of the seller’s payoff is that it is non-monotonic in the probability of the buyer’s type being high; it is constant for low priors, decreasing for intermediate priors and increasing for high priors (see Figure 3).

In the first region, \( \mu \leq v_L/v_H \), the seller invariably offers the pooling price \( v_L \) and both types of the buyer make the same investment. His payoff in this region is therefore independent of the prior. In the intermediate region of priors, \( \mu \in (v_L/v_H, m) \), the seller is indifferent between the two price offers. His payoff conditional on making the offer is equal to \( v_L \)—the seller is randomising over the two prices and the low price is accepted with certainty—and as such constant in \( \mu \). As to the probability of the seller making an offer, it is useful to recall that in the mixed-strategy equilibrium, the seller’s posterior belief \( \hat{\mu} \) is fixed to \( v_L/v_H \). When the prior rises, this entails that the high-type buyer distinguishes himself from the low type by increasing his investment, thereby keeping the seller’s posterior belief constant. Hence, the seller’s probability of making an offer is falling for two reasons: as \( \mu \) increases, 1) the probability that the high type makes an
Figure 4: Seller’s probability of making an offer and posterior belief in equilibrium

offer increases, and 2) the seller faces the high type more often. This, together with the fact that the seller’s payoff conditional on making the offer is constant, implies that the seller’s expected payoff is decreasing in the prior probability of the buyer’s type being high. For high priors, $\mu \geq m$, the seller optimally offers the separating price $v_H$. The high type’s investment is constant in $\mu$, which means the probability of the seller making an offer against the high type is constant as well. Since the high price $v_H$ is only accepted by the high type, this implies that above $m$ the seller’s payoff is strictly increasing in $\mu$. The non-monotonicity is connected to the seller’s randomization over prices. The fact that randomization can occur in equilibrium even in environments where the buyer’s valuation is drawn from a distribution without atoms suggests that non-monotonicity may not be merely a consequence of discrete value environments.

Whether for high priors the seller’s payoff increases above the value it attains for low priors depends on the parameters of the problem. The seller’s expected payoff is smaller at $\mu = 1$ than at $\mu = 0$ when the difference in the buyer’s valuation is small relative to the difference in probabilities with which the seller gets to make the offer. This suggests that the case for the seller’s payoff being decreasing in the fraction of high types could have been made in an environment with perfect information. Our result is stronger: even when the seller’s payoff at $\mu = 1$ is larger than at $\mu = 0$, there is an intermediate region of priors under which the seller is strictly worse off than when he faces the low-value buyer with certainty.

The fact that the seller’s payoff can be decreasing in the probability of the high type has the following interesting application. If the seller could approach one of two potential buyers (or enter one of two markets)—the first consumer almost certainly to have a low valuation, the second more likely to have a high valuation—the seller might just prefer to trade with the agent who is likely to value the good less. There is less surplus to be split between the two parties, but the buyer will not fight aggressively for it either. The seller prefers, so to say, to pick the low hanging fruit.
5 A Probabilistic Bargaining Model

Our results have bearing on probabilistic bargaining models, which are a popular tool in many applications. In these models, a buyer and a seller get to make an offer with a probability, which stands as a proxy for their bargaining power. The probability of making an offer is exogenously fixed and independent of the agents’ valuations. The elegance and simplicity of such models make them a popular tool in economics and finance; see, for example, Zingales (1995) and Inderst (2001), though probabilistic bargaining goes back at least to Rubinstein and Wolinsky (1985). Our results show that if agents can increase their bargaining power at a cost, the probability of making an offer might depend on agents’ types. A natural question is whether probabilistic bargaining models can be generalised enough to capture the salient features of endogenously generated bargaining power, while preserving sufficient tractability to be interesting for wider applications.

We propose a modification of the probabilistic bargaining model, where the buyer’s probability of making an offer increases in his valuation. In particular, we assume that type \( v \) of the buyer makes an offer with probability \( \rho_0(v) \), where \( \rho_0(v) \) is a strictly increasing function. The seller makes an offer with the complementary probability. Under this assumption, the opportunity to make an offer carries information for the seller and thus affects his pricing decision.

For the case of binary types, we saw that the seller offers the high price \( v_H \) if his posterior belief conditional on making an offer is higher than \( v_L/v_H \) (see Section 4). That is:

\[
\frac{\mu(1 - \rho_0(v_H))}{\mu(1 - \rho_0(v_H)) + (1 - \mu)(1 - \rho_0(v_L))} \geq \frac{v_L}{v_H}.
\] (6)

Letting \( m_0 \) denote the value of \( \mu \) at which condition (6) is satisfied with equality, the seller optimally offers the high price \( v_H \) if and only if \( \mu \geq m_0 \). Given the assumption \( \rho_0(v_H) > \rho_0(v_L) \), the seller revises his beliefs downwards upon getting the right to propose. Therefore, he is willing to offer the high price only if his prior favors the high price sufficiently, implying that the threshold \( m_0 \) is strictly greater than \( v_L/v_H \). We can therefore write the seller’s expected payoff as a function of the prior \( \mu \).

\[
u_x = \begin{cases} 
[\mu(1 - \rho_0(v_H)) + (1 - \mu)(1 - \rho_0(v_L))]v_L & \text{if } \mu \leq m_0 \\
\mu(1 - \rho_0(v_H))v_H & \text{if } \mu > m_0.
\end{cases}
\]
For low priors, the seller’s payoff is decreasing in $\mu$. The seller offers the low price and both types accept it. The seller, however, suffers from the incidence of high types, as they decrease his probability of making an offer. In the region of high priors, the seller charges the high price, which is only accepted by the high type. Though the occurrence of high types decreases the seller’s chances of making an offer, the effect is overpowered by the fact that the seller’s price offer is accepted more often. The seller’s expected payoff is, hence, increasing in the probability of the buyer’s value being high.

The version of the probabilistic bargaining model with different probabilities approximates the model with endogenous proposal rights relatively well and captures the non-monotonicity of the seller’s payoff. Of course, it does not match the latter model exactly. The most notable difference is that in the model with endogenous proposal rights, both types make the same investment when they expect the seller to offer the low price. As a consequence, for priors below $v_L/v_H$, the seller’s payoff is constant rather than decreasing in $\mu$. More generally, in the model with investments, the difference between the probabilities of the high and the low type making an offer increases in the prior. This accounts, among other things, for the difference in the seller’s payoff below the threshold $v_L/v_H$. Despite this caveat, we believe that a researcher who wishes to use a more manageable tool will be well suited with the probabilistic model of bargaining proposed here.
6 Concluding Remarks

We introduce a model of bilateral trade in which the right to propose a mechanism is endogenous. The buyer, after learning his value for the object, makes an investment which determines the probability of choosing a mechanism; the seller selects the mechanism with the remaining probability.

We show that buyers with higher valuations invest (weakly) more resources. More precisely, the investment is flat for very low types, who find it too costly to invest, and very high types, who always trade given the seller’s equilibrium pricing strategy. These type have the same gain from obtaining the proposal right, namely, not paying the seller’s expected price. The fact that the investment is increasing in the buyer’s valuation prompts the seller to revise his beliefs towards lower types upon receiving the opportunity to select a mechanism. This leads to an interesting implication regarding the equilibria of the game: the only possible pure-strategy equilibrium is the one where the seller’s learning does not lead him to revise the price, that is, where the seller offers the ex-ante optimal price. If the latter does not constitute an equilibrium, the seller must randomise over prices in equilibrium, with the ex-ante optimal price being the maximum of the support of the seller’s randomisation.

We provide a full characterization of equilibria for the environment in which the buyer has two possible values. Interestingly, the seller’s payoff is non-monotonic in the share of high-value buyers. Therefore, if the seller were to choose between trading with a buyer that is almost certainly of low value and a buyer who is more likely to be of high value, he might choose the one with the smaller expected value. While high-value buyers offer a higher potential surplus, they also negotiate the terms of trade more aggressively.

Flexibility of our model offers several avenues for future research. In a separate work, we analyse the model where the investments are observable, and thus serve as a signalling device. As a consequence, high investments can become a double-edged sword for the buyer, on the one hand, enabling him to propose the mechanism more often, but on the other, making him seem too eager to trade, thereby revealing his valuation to the seller. Some further directions for research could be to study how the presence of two-sided private information or multiple agents affects the investment and the pricing problem.
A Appendix

Proof of Lemma 1. The first-order condition of (1) is given by

\[ \rho'(e) \left( v(1 - H(v)) + \int_0^v pdH(p) \right) \leq k. \] (7)

Let \( v^* \) be the value of \( v \in [0, 1] \) for which (7) holds as an equality at \( e = 0 \). The optimal investment function is described by (2) if \( v > v^* \) and by \( e^*(v; H) = 0 \) otherwise. Concavity of \( \rho \) implies that \( \rho' \) and its inverse are decreasing. Together with the fact that \( v(1 - H(v)) + \int_0^v pdH(p) \) is non-decreasing in \( v \), it follows that \( e^*(v; H) \) is non-decreasing in \( v \).

\[ \square \]

Proof of Lemma 2. We can write

\[ G(v; e) = \int_0^v \frac{1 - \rho(e(t))}{\int_0^1 (1 - \rho(e(s)))dF(s)} dF(t). \]

Since the investment function \( e(\cdot) \) is a non-decreasing function, \( \frac{1 - \rho(e(t))}{\int_0^1 (1 - \rho(e(s)))dF(s)} \) is non-increasing in \( t \). The fact that the investment function is not constant and that \( G \) is a distribution function further imply

\[ \frac{1 - \rho(e(0))}{\int_0^1 (1 - \rho(e(s)))dF(s)} > 1 > \frac{1 - \rho(e(1))}{\int_0^1 (1 - \rho(e(s)))dF(s)}. \]

Let \( t^* \) be such that \( \frac{1 - \rho(e(t))}{\int_0^1 (1 - \rho(e(s)))dF(s)} > 1 \) for all \( t \in [0, t^*) \) and \( \frac{1 - \rho(e(t))}{\int_0^1 (1 - \rho(e(s)))dF(s)} \leq 1 \) for \( t \geq t^* \). For \( v \in [0, t^*) \)

\[ G(v; e) = \int_0^v \frac{1 - \rho(e(t))}{\int_0^1 (1 - \rho(e(s)))dF(s)} dF(t) \]

\[ \geq \int_0^v \frac{1 - \rho(e(v))}{\int_0^1 (1 - \rho(e(s)))dF(s)} dF(t) \]

\[ = \frac{1 - \rho(e(v))}{\int_0^1 (1 - \rho(e(s)))dF(s)} F(v) \]

\[ > F(v), \]

where the first inequality follows from \( \rho \) and \( e \) being non-decreasing. On the other hand,
for $v \in [t^*, 1]$

$$1 - G(v; e) = \int_{v}^{1} \frac{1 - \rho(e(t))}{\int_{0}^{1} (1 - \rho(e(s)))dF(s)} dF(t)$$

$$\leq \int_{v}^{1} \frac{1 - \rho(e(v))}{\int_{0}^{1} (1 - \rho(e(s)))dF(s)} dF(t)$$

$$= \frac{1 - \rho(e(v))}{\int_{0}^{1} (1 - \rho(e(s)))dF(s)} (1 - F(v))$$

$$\leq 1 - F(v),$$

where the first inequality follows from $\rho$ and $e$ being non-decreasing and the second from

$$\frac{1 - \rho(e(t))}{\int_{0}^{1} (1 - \rho(e(s)))dF(s)} \leq 1$$

for $t \geq t^*$.

**Proof of Proposition 1.** The proof proceeds in two steps: first we argue existence of a Bayesian Nash equilibrium in an auxiliary game, then we show the existence of a sequential equilibrium in the original game.

In the auxiliary game we fix the buyer’s price offer to $p_b(v) = 0, \forall v \in [0, 1]$ and seller’s acceptance strategy to $x_s(p) = 1, \forall p \geq 0$ (accept). Similarly, we set the buyer’s acceptance strategy to $x_b(v, p) = 1$ if $p \leq v$ and $x_b(v, p) = 0$ otherwise. The seller’s strategy is then the price offer $p_s$ when he is asked to make an offer, and the buyer’s strategy is the investment choice $e$. Given a pair $(e, p_s)$ the buyer’s and seller’s payoffs are

$$\rho(e)v + (1 - \rho(e))1_{v \geq p_s}(v - p_s) - ke,$$

and

$$1 - \rho(e)1_{v \geq p_s}p_s,$$

respectively. In what follows we verify conditions needed for Theorem 1 in Carbonell-Nicolau and McLean (2017); which itself builds on existence results in Reny (1999) and Monteiro and Page Jr (2007). Uniform payoff security of the game is implied by the continuity of the seller’s payoff in $e$ and the buyer’s payoff in $p$. In addition, the sum of the two players’ payoffs, $\rho(e)v + (1 - \rho(e))1_{v \geq p_s}v - ke$, is upper semicontinuous for every $v$ in the profile of strategies $(e, p_s)$ due to the assumption that the buyer buys the good when indifferent. Finally, since only the buyer is privately informed, the absolute continuity of the distribution is automatically satisfied. The only remaining requirement is compactness of the strategy sets. The seller’s optimal price is never above 1, therefore we can restrict his set of prices to $[0, 1]$. Likewise, we confine the buyer’s investment
to the set \([0, 1/k]\); provided that prices are non-negative, the most the buyer can gain from making an offer himself is his highest possible valuation: 1. Restricting \(e\) to \([0, 1/k]\) and \(p_s\) to \([0, 1]\) renders the strategy sets compact. Theorem 1 of Carbonell-Nicolau and McLean (2017), then, delivers the existence of equilibrium.

Having established the existence of an equilibrium of the auxiliary game, we return to the original game. The seller’s strategy consists of the price he charges when given the opportunity and the acceptance decision for any price he may face. The buyer’s strategy consists of the investment, the price he charges if he gets to make an offer and the acceptance decision when the seller is offering the price. The seller’s assumed acceptance strategy in the continuation game where he faces the buyer’s price offer is clearly optimal. The seller’s strategy is, therefore, a best response to the buyer’s behavior. Likewise, one can argue that if the buyer had a profitable deviation, then he would have a profitable deviation where he plays the assumed acceptance strategy and offers price 0 when called upon. Such a profitable deviation, as we know, does not exist. 

**Proof of Proposition 2.** Let the seller’s equilibrium randomisation over prices be described by the cumulative distribution function \(H\) and let \(\bar{p}\) denote the maximum of the support of \(H\). We want to argue that \(\bar{p} = p^*\).

In equilibrium the probability with which buyer type \(v\) gets to make an offer is \(\rho(e^*(v; H))\). Given this probability, the seller’s posterior when making an offer is described by \(G(v; e^*(\cdot; H))\). Any price in the support of \(H\) must maximise \(p(1 - G(p; e^*(\cdot; H))) = p \frac{\int_0^1 (1 - \rho(e^*(t; H)))dF(t)}{\int_0^1 (1 - \rho(e^*(t; H)))dF(t)}\),

and therefore \(p \int_0^1 (1 - \rho(e^*(t; H)))dF(t)\). By Lemma 1, \(e^*(v; H)\) is non-decreasing in \(v\), which means that \(\rho(e^*(v; H))\) is non-decreasing in \(v\), and therefore that \(1 - \rho(e^*(v; H))\) is non-increasing in \(v\). Towards a contradiction, suppose now \(\bar{p} \neq p^*\). Then

\[
p^* \int_{p^*}^1 (1 - \rho(e^*(t; H)))dF(t) \geq p^* \int_{p^*}^1 (1 - \rho(e^*(\bar{p}; H)))dF(t),
\]

\[
= (1 - \rho(e^*(\bar{p}; H)))p^*(1 - F(p^*)),
\]

\[
> (1 - \rho(e^*(\bar{p}; H)))\bar{p}(1 - F(\bar{p})),
\]

\[
= \bar{p} \int_{\bar{p}}^1 (1 - \rho(e^*(t; H)))dF(t),
\]

where the first inequality is due to the fact that \(1 - \rho(e^*(v; H))\) is non-increasing in \(v\) and constant above \(\bar{p}\), and the second is due to the ex-ante optimality of \(p^*\). Given the
inequalities, the price \( \tilde{p} \) cannot maximise \( p(1 - G(p, e^*(\cdot; H))) \) and, hence, cannot belong to the support of \( H \).

This proves that \( \tilde{p} = p^* \) holds. Since \( p^* \) always belongs to the support of \( H \), the only possible pure-strategy equilibrium is the one where the seller offers \( p^* \).

To prove the last part of the statement we show that in any equilibrium for \( p < p' < p^* \), \[
\frac{p(1 - G(p))}{p(1 - F(p))} > \frac{p'(1 - G(p'))}{p'(1 - F(p'))},
\] or in words, the seller’s profit with respect to the posterior as the fraction of the profit with respect to the prior declines for the prices below \( p^* \). If in addition \( p(1 - F(p)) > p'(1 - F(p')) \), that would mean that \( p(1 - G(p)) > p'(1 - G(p')) \) and, therefore, \( p' \) is not optimal for the seller. Formally,

\[
\frac{p(1 - G(p))}{p(1 - F(p))} - \frac{p'(1 - G(p'))}{p'(1 - F(p'))} = c[(1 - G(p))(1 - F(p')) - (1 - G(p'))(1 - F(p))],
\]

where \( c \) is a strictly positive number. Now

\[
(1 - G(p))(1 - F(p')) - (1 - G(p'))(1 - F(p))
\]

\[
= \int_p^1 (1 - \tilde{p}(x))dF(x) \int_{p'}^1 dF(x) - \int_{p'}^1 (1 - \tilde{p}(x))dF(x) \int_p^1 dF(x)
\]

\[
= \int_p^{p'} (1 - \tilde{p}(x))dF(x) \int_{p'}^1 dF(x) - \int_p^1 (1 - \tilde{p}(x))dF(x) \int_p^{p'} dF(x)
\]

\[
\geq (1 - \tilde{p}(p')) \int_p^{p'} dF(x) \int_{p'}^1 dF(x) - \int_p^1 (1 - \tilde{p}(x))dF(x) \int_p^{p'} dF(x)
\]

\[
\geq (1 - \tilde{p}(p')) \int_p^{p'} dF(x) \int_{p'}^1 dF(x) - (1 - \tilde{p}(p')) \int_p^1 dF(x) \int_p^{p'} dF(x)
\]

\[
= 0,
\]

where \( \tilde{p}(x) \) is an abbreviation for \( \rho(e^*(x; H)) \), and where both the inequalities are due to \( (1 - \tilde{p}(x)) \) being non-increasing. The inequality is strict if \( F \) assigns probability mass above \( p \), which is the case here, as can be inferred from \( p(1 - F(p)) > p'(1 - F(p')) \).

**Proof of Proposition 3.** The assumption \( k < \rho'(0)v_L \) assures that \( e^*(v_i; \sigma) > 0, i = L, H \) for all \( \sigma \in [0, 1] \). We distinguish three cases:

- \( \sigma = 0 \): suppose in equilibrium the seller offers price \( v_L \) with probability one. In this case the investment choice of both buyers is the same \( (e^*(v_L; 0) = e^*(v_H; 0)) \), so the seller does not learn in equilibrium. Offering price \( v_L \) is optimal for the seller if and only if it is optimal according to the prior, that is: \( \mu \leq v_L/v_H \).
• $\sigma = 1$: suppose that in equilibrium the seller offers price $v_H$ with probability one. The high-type buyer invests strictly more than the low-type buyer and the seller’s posterior is

$$\hat{\mu} = \frac{\mu(1 - \rho(e^*(v_H; 1)))}{\mu(1 - \rho(e^*(v_H; 1))) + (1 - \mu)(1 - \rho(e^*(v_L; 1)))}.$$  

This posterior is strictly increasing in $\mu$ and equal to $v_L/v_H$ at $\mu = m$. The equilibrium exists if and only if price $v_H$ is optimal after updating. This is satisfied if $\hat{\mu}$ is greater than $v_L/v_H$ or, equivalently, $\mu \geq m$.

• $\sigma \in (0, 1)$: suppose the seller randomises across prices in equilibrium. This requires that the high-type buyer’s investment, $e^*(v_H; \sigma)$, is such that after updating the seller is indifferent between both prices:

$$\frac{\mu(1 - \rho(e^*(v_H; \sigma)))}{\mu(1 - \rho(e^*(v_H; \sigma))) + (1 - \mu)(1 - \rho(e^*(v_L; \sigma)))} = v_L/v_H \quad (8)$$

Since $e^*(v_H; \sigma)$ is continuous and strictly increasing on $[0, 1]$ and $e^*(v_L; \sigma)$ is constant, the term on the left-hand side of (8) is continuous and strictly decreasing on $[0, 1]$. At $\sigma = 0$, the left hand side is equal to $\mu$. Substituting for $e^*(v_i; \sigma), i = L, H$, it follows that the unique solution of (8) is implicitly defined by (5) if and only if $\mu \in (v_L/v_H, m)$. Hence, there exists an equilibrium where the seller randomises over both prices if and only if $\mu \in (v_L/v_H, m)$. Substituting for $e^*(v_i; \sigma), i = L, H$ in (8) yields (5), which implicitly defines the seller’s equilibrium randomisation, $\sigma^*$.

Proof of Proposition 4. Given the buyer’s equilibrium investment strategy, as described in Proposition 3, the seller’s payoff is given by the product of the probability with which he makes an offer and his conditional payoff in the event he does. His expected payoff as a function of the prior belief $\mu$ can thus be written as

$$u_s(\mu) = \begin{cases} 
(1 - \rho(e_L^*)v_L) & \text{if } \mu < \frac{v_L}{v_H} \\
[\mu(1 - \rho(e_H^*(\sigma^*))) + (1 - \mu)(1 - \rho(e_L^*))v_L] & \text{if } \mu \in \left(\frac{v_L}{v_H}, m\right) \\
\mu(1 - \rho(e_H^*(1)))v_H & \text{if } \mu \geq m.
\end{cases}$$

where $\sigma^*$ is defined by (5).

• For $\mu \leq v_L/v_H$, the payoff function $u_s$ does not depend on $\mu$ and hence we have $u'_s(\mu) = 0$. 

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• For $\mu \in (v_L/v_H, m)$, taking the first derivative of the seller’s payoff with respect to $\mu$ yields

$$u'_s(\mu) = \left[ -(\rho(e^*_H(\sigma^*))) - \rho(e^*_L(\sigma^*)) - \mu \rho'(e^*_H(\sigma^*)) e^*_H(\sigma^*) \frac{d\sigma^*}{d\mu} \right] v_L.$$ 

The first term in the square bracket is strictly negative. Since $\rho$ and $e^*_H$ are increasing functions, the second term is negative if $d\sigma^*/d\mu$ is positive. Given that the seller’s posterior on the left-hand side of (5) is increasing in $\mu$ and decreasing in $\sigma$, $d\sigma^*/d\mu > 0$ is indeed satisfied. Hence, the second term in the square bracket is negative as well and the seller’s expected payoff is strictly decreasing in $\mu$ in the parameter region $\mu \in (v_L/v_H, \hat{m})$.

• Finally, for $\mu \geq m$ we have $u'_s(\mu) = (1 - \rho(e^*_H(1)))v_H$, which is strictly positive.

References


