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Optimal Voting Mechanisms on Generalized Single-Peaked Domains

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Abstract

This paper studies the design of voting mechanisms in a setting with more than two alternatives and arbitrarily many voters who have generalized single-peaked preferences derived from median spaces as introduced in [Nehring and Puppe, 2007b]. This class of preferences is considerably larger than the well-known class of preferences that are single-peaked on a line. I characterize the voting rules that maximize ex-ante utilitarian welfare among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity. The optimal mechanism takes the form of voting by properties, that is, the social choice is determined through a collection of binary votes on subsets of alternatives involving qualified majority requirements that reflect the characteristics of these subsets of alternatives. I illustrate my general optimality result by means of applications including, for instance, collective choice when preferences are single-peaked with respect to a tree. Finally, I emphasize the importance of my characterization result for the analysis of stable constitutions.

Keywords: Voting; Generalized Single-Peaked Preferences; Mechanism Design
JEL Classification: D71, D72, D82

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1 Introduction

In this paper, I characterize the optimal utilitarian voting mechanisms, meaning, the voting rules that maximize ex-ante utilitarian welfare, among all mechanisms satisfying strategy-proofness, anonymity, and surjectivity in a setting with more than two alternatives and arbitrarily many voters who have generalized single-peaked preferences derived from median spaces as introduced in [Nehring and Puppe, 2007b]. This class of preferences is much larger than the well-known class of preferences that are single-peaked on a line and it includes, for instance, the class of preferences that are single-peaked with respect to a tree introduced in [Demange, 1982]. [Nehring and Puppe, 2007b] extend previous work in strategy-proof social choice like the seminal contribution of [Moulin, 1980], who considers single-peaked preferences on a line, to generalized single-peaked domains. I build on [Nehring and Puppe, 2007b]'s characterization of strategy-proof social choice functions. [Gershkov et al., 2017] study the stated optimality question for preferences which are single-peaked on a line.[1] For these preferences, they derive the utilitarian mechanism, and they show that, in this case, the optimal voting rule takes the form of a successive procedure with weakly decreasing thresholds that depend on the intensities of preferences. The present paper extends the work of [Gershkov et al., 2017] to a considerably larger class of preferences.

In the outlined setting, the utilitarian mechanism takes the form of voting by properties, that is, the social choice is determined through a collection of binary votes on subsets of alternatives and the involved qualified majority requirements reflect the characteristics of these subsets of alternatives. Let me discuss the optimal mechanism more in detail for an example giving rise to single-peaked preferences on a tree. Suppose that the set of alternatives amounts to $\{1, 2, 3, 4\}$, i.e., there are four alternatives, and assume that voters have preferences that are single-peaked with respect to the tree shown in Figure 1. This means that any voter's preference has to meet the following restriction: There exists an alternative p, which is the most preferred or peak alternative, such that for all alternatives b and c with $b \neq c$ it holds that whenever b lies on a shortest path in the tree depicted in Figure 1 connecting p and c, the voter prefers b over c. For instance, if

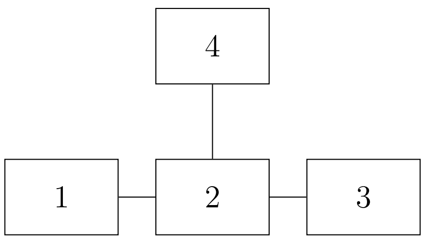

Figure 1: Tree example

some voter's most preferred alternative is 1, generalized single-peakedness requires here

[1] To be more precise, they assume that preferences are single-crossing and single-peaked on a line.

that alternative 2 is preferred over 3 and 4, but it does not impose whether 3 is preferred to 4 or the other way around. In this example, and, more generally, if preferences are single-peaked with respect to an arbitrary tree, the optimal mechanism can be intuitively described as "voting by edges". Take any edge of the tree shown in Figure 1, and cut this edge yielding two subsets of alternatives or, more precisely, two connected components of the tree. Then, decide according to qualified majority voting which connected component is winning. This means that agents vote sincerely for one of the two connected components, i.e., every agent votes for the connected component containing his or her peak alternative, and the qualified majority decision yields one winning connected component, that is, the social choice must be contained in the set of nodes associated with this connected component. These qualified majority decisions are performed for all edges yielding a collection of connected components that are winning. Eventually, the final outcome is given by the intersection of the sets of nodes linked to the connected components that are winning.[2] The expressions for the welfare-maximizing quotas related to the discussed qualified majority decisions are determined by the voters' preference intensities. Again, consider some edge in the tree depicted in Figure 1, say $\{a, b\}$, and denote by A' and A'' the two sets of nodes associated with the two connected components that are generated when cutting the edge $\{a, b\}$. Focus on the utilitarian majority requirement linked to the set A', describing the minimal number of peaks from the set A' that are needed to ensure that the social choice is contained in A' as well. Let q denote this optimal majority quota.[3] The utilitarian designer compares the welfare of the two neighbors in the tree that are directly connected through the edge $\{a, b\}$, i.e., the designer compares the welfare induced by a and b. In particular, the welfare generated by any other alternative does not matter for the determination of q. Moreover, the welfare-maximizing quota linked to the set A' is calibrated such that the designer is indifferent between implementing alternatives a and b conditional on the event that the number of voters who have peaks from the set A' coincides with the optimal majority requirement q. For instance, consider the edge $\{1, 2\}$. Then, the optimal majority quota representing the minimal number of voters with peak on 1 that are needed to make sure that 1 is the final outcome is determined in a way such that the designer is indifferent between implementing alternatives 1 and 2 conditional on the event that the number of voters with peak on 1 equals the discussed quota. These features of the welfare-maximizing quotas are not specific to the tree shown in Figure 1, but they also hold for preferences that are single-peaked with respect to any other tree.

Why study optimal voting mechanisms on generalized single-peaked domains while deviating from single-peaked preferences on a line? Consider collective decision-making in

[2]Under the assumptions I impose in the characterization of the utilitarian mechanisms, this intersection always amounts to one single alternative.

[3]The majority quota associated with the other set A'' is simply given by the number of voters plus one minus q.

parliaments and suppose that the deputies face several different bills related to the same policy issue. [Benoit and Laver, 2006] analyze the dimensionality of policy spaces arising in reality and find that, in many countries, the diversity in the positions of political parties with regard to various policy issues cannot be accurately captured by a single dimension of political conflict (see chapter 5 of their book).[4] In these countries, models featuring preferences that are single-peaked on a line do not apply whenever the policy proposals deputies are facing relate to more than one dimension of political conflict. Therefore, the empirical evidence from [Benoit and Laver, 2006] suggests that there is need to deviate from single-peaked preferences on a line in order to better understand according to which voting rules parliaments should collectively decide about different bills or policy proposals. Considering generalized single-peaked preferences makes it possible to account for the multidimensionality of politics in a number of ways.[5] Again, one way to incorporate the multidimensionality of politics that is covered in my analysis is to consider preferences which are single-peaked with respect to a tree instead of a line as introduced in [Demange, 1982]. Interestingly, [Kleiner and Moldovanu, 2020] argue that in some real-world voting problems from the German as well as the British parliament preferences were single-peaked on a tree. Consequently, from a practical point of view, my findings in this paper could provide some guidance for parliaments when it comes to the design of voting mechanisms for collective choice problems that give rise to single-peaked preferences on a tree or, more generally, that do not feature a unidimensional conflict structure. Moreover, from a theoretical perspective, it seems to be natural to move away from single-peaked preferences on a line and to study optimal voting mechanisms for larger classes of preferences that, nevertheless, admit well-behaved strategy-proof social choice functions.

The structure of this paper is as follows: In the following section 2, I discuss the related literature. The subsequent section 3 introduces the model. Next, in section 4, I review the characterization of strategy-proof social choice functions from [Nehring and Puppe, 2007b]. In section 5, I present my general optimality finding, constituting the main result of this paper. Then, in section 6, I illustrate my general optimality result by means of applications including collective choice when preferences are single-peaked with respect to an arbitrary tree. I also provide an application to voting over multiple public goods under constraints. The final section 7 concludes while emphasizing the importance of my characterization result for the analysis of stable constitutions. The proofs are contained in the Appendix.

[4]Their comprehensive empirical analysis covering 47 countries is based on expert surveys that were mostly conducted in 2003. In terms of methods, they employ statistical techniques of data reduction.

[5]In the following section on the related literature, I will be more concrete as to which kind of preferences are captured by my analysis.

2 Literature

The present paper relates to work on social choice and mechanism design and it contributes, specifically, to the literature on the evaluation of the utilitarian efficiency of voting rules. This literature starts with [Rae, 1969], who focuses on *binary decisions*. More recent contributions that also consider binary decisions include [Schmitz and Tröger, 2012], [Azrieli and Kim, 2014] and [Drexl and Kleiner, 2018], tackling each somewhat different research questions.

When moving away from the binary setting and allowing for more than two alternatives, the *Gibbard-Satterthwaite-Theorem* ([Gibbard, 1973], [Satterthwaite, 1975]) implies that restrictions on the preference domain have to be imposed, since otherwise, dictatorship arises.[6]

[Nehring and Puppe, 2007b] offer a characterization of strategy-proof social choice functions for all rich generalized single-peaked domains. Among many other preference structures, the unrestricted domain as well as domains that give rise to median spaces constitute generalized single-peaked domains.[7] Their results reveal that the latter preference domain admits a large class of well-behaved strategy-proof social choice functions, circumventing the Gibbard-Satterthwaite-Theorem. This is the reason why I consider generalized single-peaked domains derived from median spaces. Again, I will make use of [Nehring and Puppe, 2007b]'s characterization of strategy-proof social choice functions for these preference structures. Furthermore, [Nehring and Puppe, 2005] as well as [Nehring and Puppe, 2007a] also treat strategy-proof social choice on generalized single-peaked domains, albeit having each a somewhat different emphasis. The difference between all these contributions and my work is that these authors are concerned about incentive compatibility, whereas I optimize over incentive-compatible mechanisms.

The preference domains from the literature discussed below are all instances of generalized single-peaked domains derived from median spaces (see [Nehring and Puppe, 2007b]).[8] Therefore, subject to my assumptions on the utility function and the type distribution, my optimality analysis covers all these preference domains. In that way, I unify and generalize previous results in the mechanism design literature. To the best of my knowledge, the optimization over strategy-proof mechanisms on generalized single-peaked domains giving rise to median spaces while relying on the utilitarian principle is novel. This is the main contribution of this paper.

One strand of the literature investigates *hypercubes* or coupled binary decisions, meaning, voters face a collection of binary decisions. In terms of strategy-proof social choice,

[6] In a setting with more than two alternatives, [Apesteguia et al., 2011] evaluate voting rules according to different normative standards. Since these authors assume voters to be non-strategic, they do not need to restrict the set of preferences.

[7] [Nehring and Puppe, 2007b] generalize previous work by [Barberà et al., 1997].

[8] Related contributions that also allow for more than two alternatives, but do not fit in the subsequent classification include [Börgers and Postl, 2009] and [Kim, 2017].

[Barberà et al., 1991] provide a characterization of strategy-proof and onto mechanisms when preferences are separable across the binary issues. When it comes to mechanism design, [Jackson and Sonnenschein, 2007] offer a mechanism that is based on the idea of budgeting. For sufficiently many decisions, their mechanism is approximately Bayesian incentive-compatible as well as nearly ex-ante Pareto efficient. Other voting rules in the context of Bayesian mechanism design where voters report cardinal utility information include qualitative voting studied in [Hortala-Vallve, 2012] as well as storable votes due to [Casella, 2005]. In contrast to [Jackson and Sonnenschein, 2007], [Hortala-Vallve, 2010] considers finitely many decision problems as well as strategy-proof mechanisms. Allowing for random mechanisms, he finds that ex-ante Pareto efficiency cannot be attained and, moreover, in the presence of strategy-proofness, there is no unanimous mechanism which is sensitive to preference intensities.

Another branch of the literature considers preferences which are *single-peaked on a line*. [Moulin, 1980] characterizes in his seminal contribution peaks-only and strategy-proof social choice functions for the full domain of preferences which are single-peaked on a line. His elegant characterization involves min-max rules or generalized median mechanisms when restricting attention to anonymous social choice functions. Again, [Gershkov et al., 2017] characterize the utilitarian mechanism when preferences are single-crossing and single-peaked on a line. In contrast to their work, I allow for a much larger class of preferences, going beyond single-peaked preferences on a line. In terms of the proof argument for my optimality result, I expand on their insights, but the much larger class of preferences requires additional proof arguments as well as different assumptions. Further, similar to [Gershkov et al., 2017], [Gersbach, 2017] also emphasizes the importance of flexible majority rules. Moreover, [Kleiner and Moldovanu, 2017] analyze dynamic, binary, and sequential voting procedures. They identify conditions on the voting procedures under which the induced dynamic games possess an ex-post perfect equilibrium in which voters behave sincerely. Moreover, they illustrate their theoretical findings by means of several empirical case studies involving collective decisions from different parliaments.

Building on preferences which are single-peaked on a line, *products of lines*, the coupling of unidimensional decisions or, as [Barberà et al., 1993] put it, multidimensional single-peaked preferences have also received attention in the literature. Removing the peaks-only assumption in [Moulin, 1980], [Border and Jordan, 1983] as well as [Barberà et al., 1993] provide characterizations of strategy-proof social choice functions for the stated class of voting problems. Despite considering each somewhat different preferences, the main conclusion following from these contributions is that any strategy-proof social choice function is peaks-only and it can be decomposed into unidimensional functions such that each dimension is treated in a separate way. In other words, any strategy-proof social choice function is composed of a collection of the mechanisms that [Moulin, 1980] identified for the unidimensional case. Finally, regarding mechanism design, [Gershkov et al., 2019] consider a spatial voting environment, but they keep the voting procedure fixed in the sense that, essentially, the collective choice in each coordinate of the multidimensional setting is determined via simple majority voting. They show that the redefinition of the involved issues or, in other words, the rotation of the initial coordinate axes leads, generally, to improvements in terms of welfare.

While extending single-peaked preferences on a line in a somewhat different direction compared to products of lines, but maintaining the general idea of single-peakedness, [Demange, 1982] investigates preferences which are *single-peaked with respect to a tree*. She establishes that these domains ensure the existence of a Condorcet winner. However, when it comes to aggregation theory instead of voting, the majority relation need not be transitive. Moreover, [Kleiner and Moldovanu, 2020] study dynamic, binary, and sequential voting procedures in the context of single-peaked preferences on a tree. They derive conditions on the voting procedures such that voting sincerely constitutes an ex-post perfect equilibrium and the Condorcet winner is implemented in this equilibrium. Also, again, they apply their theoretical findings to real-world voting problems from the German and the British parliament.

3 Model

There is a finite set of voters $N = \{1, ..., n\}$ with $n \geq 2$ and a finite set of alternatives A with $|A| \geq 2$. Following [Nehring and Puppe, 2007b], the set of alternatives is endowed with a property space structure. Elements of A are distinguished by *properties* which are described by $\mathcal{H} \subseteq \mathcal{P}(A)$ where $\mathcal{H} \neq \emptyset$ and $\mathcal{P}(A)$ denotes the power set of A. Each $H \in \mathcal{H}$ captures the property shared by all elements in $H \subseteq A$, but violated by all alternatives in $H^c := A \setminus H$. In other words, properties are subsets of the set of alternatives A. The set of properties $\mathcal{H}$ satisfies the regularity conditions

$$H \in \mathcal{H} \Rightarrow H \neq \emptyset \text{ (non-triviality)},$$
$$H \in \mathcal{H} \Rightarrow H^c \in \mathcal{H} \text{ (closedness under negation), and}$$
$$\forall a, b \in A, a \neq b : \exists H \in \mathcal{H} : a \in H \wedge b \notin H \text{ (separation)}.$$

Each pair (H, H^c) involving some property and its complement forms an *issue*. The tuple $(A, \mathcal{H})$ is called *property space*. The property space $(A, \mathcal{H})$ induces some ternary relation on A, denoted by $B_{\mathcal{H}}$, in the following way: For all $(a, b, c) \in A \times A \times A$,

$$(a, b, c) \in B_{\mathcal{H}} :\Leftrightarrow [\forall H \in \mathcal{H} : \{a, c\} \subseteq H \Rightarrow b \in H].$$

The relation $B_{\mathcal{H}}$ is called *betweenness relation*. This means that some alternative b is between the alternatives a and c if and only if all properties that are jointly shared by a and c are also shared by b.

Moreover, I suppose that any property space constitutes a *median space* as introduced in [Nehring and Puppe, 2007b].[9] This requires that that the betweenness relation $B_{\mathcal{H}}$ satisfies the following constraint: For any $a, b, c \in A$, there exists some alternative $m = m(a, b, c) \in A$, called the median, such that

$$\{(a, m, b), (a, m, c), (b, m, c)\} \subseteq B_{\mathcal{H}}.$$

Take any set that is composed of three alternatives. The restriction of being a median space demands that there must be some alternative having the feature that it is between all pairs of alternatives that can be formed from the given set of three alternatives.

Given some alternative $a \in A$, let $\mathcal{H}_a$ be the set of all properties shared by alternative a, meaning, formally, define $\mathcal{H}_a := \{H \in \mathcal{H} : a \in H\}$. Due to separation, it holds that $\cap_{H \in \mathcal{H}_a} H = \{a\}$. Hence, there is a one-to-one relationship between alternatives and collections of properties characterizing them. However, not all collections of properties are feasible in the sense that the intersection of them amounts to some alternative.

Based on these concepts, I introduce preferences. There is a set of types T which is given by

$$T := [\times_{H \in \mathcal{H}} \tilde{T}_H] \times [\times_{H \in \mathcal{H}} V_H].$$

For any property $H \in \mathcal{H}$, the random variable $\tilde{T}_H$ takes on values from the set $\{H, H^c\}$. Moreover, assume that some collection of sets $\times_{H \in \mathcal{H}} \{S_H\}$ with $S_H \in \{H, H^c\}$ is part of the support of $\times_{H \in \mathcal{H}} \tilde{T}_H$ if and only if there exists some alternative $a \in A$ such that $\cap_{H \in \mathcal{H}} S_H = \{a\}$. Let π denote the pmf corresponding to $\times_{H \in \mathcal{H}} \tilde{T}_H$ and let S be the support of π. In other words, a collection of properties is part of the support of π if and only if these properties characterize some alternative.[10] Given some exogenous $\overline{v} > 0$, the distribution of $\times_{H \in \mathcal{H}} V_H$ has full support on $[0, \overline{v}]^{|\mathcal{H}|}$. Let G denote the joint cdf of $\times_{H \in \mathcal{H}} V_H$, and, for any property $H \in \mathcal{H}$, G_H describes the marginal cdf corresponding to property H.[11] Further, I suppose that types are distributed iid across voters.

Assumption 1. *The type vectors T are distributed independently and identically across*

[9] This assumption will ensure that there is a rich class of non-degenerate incentive-compatible social choice functions. I will discuss it in more detail in section 4.

[10] Alternatively, I could have directly introduced some random variable that is supported on the set of alternatives. However, the formulation in terms of properties will be convenient for the ensuing analysis.

[11] The support assumptions related to the distributions of the random variables $\times_{H \in \mathcal{H}} \tilde{T}_H$ and $\times_{H \in \mathcal{H}} V_H$ make sure that the set of ordinal preferences generated by the utility function below is sufficiently rich such that the characterization of incentive-compatible social choice functions from [Nehring and Puppe, 2007b] applies.

voters.

Some voter having type realization $t = [\times_{H \in \mathcal{H}} \tilde{t}_H] \times [\times_{H \in \mathcal{H}} v_H]$ derives utility

$$u^a(t) := \sum_{H \in \mathcal{H}: \tilde{t}_H = H \wedge a \in H} v_H$$

from alternative $a \in A$. This utility function is taken from [Nehring and Puppe, 2007b]. The intuition behind it is as follows. The realization $\times_{H \in \mathcal{H}} \tilde{t}_H$ of $\times_{H \in \mathcal{H}} \tilde{T}_H$ induces the collection of properties $\cup_{H \in \mathcal{H}} \tilde{t}_H$ determining the properties characterizing the peak of the preference relation. Based on this collection of peak properties, utility levels are determined by the properties that are jointly shared by the peak and the implemented alternative. If some alternative possesses more peak properties than some other alternative, the derived utility is higher when the former compared to the latter alternative is implemented. Regarding a more concrete interpretation of the random variable $\times_{H \in \mathcal{H}} V_H$, consider two alternatives $a, b \in A$ that are separated by exactly one property. In formal terms, this means that there exists some property $K \in \mathcal{H}$ such that

$$\{H \in \mathcal{H} : a \in H, b \notin H\} = \{K\}.$$

Whenever the peak of some preference relation shares this property K, the realization v_K of V_K captures the gain in utility terms when moving from alternative b that does not share property K to alternative a that does share property K. Furthermore, the additive separability across properties imposes that this utility gain is the same across all pairs of alternatives that are separated only by property K.

Moreover, following [Nehring and Puppe, 2007b], note that any type realization $t \in S \times [0, \overline{v}]^{|\mathcal{H}|}$ together with the utility function above induces an ordinal preference relation $\succ_t$ that satisfies the following condition: There exists some alternative $p \in A$ such that, for all $b, c \in A$ with $b \neq c$, it holds

$$(p, b, c) \in B_{\mathcal{H}} \Rightarrow b \succ_t c.$$

These ordinal preference relations are said to be *generalized single-peaked* with respect to the underlying betweenness relation $B_{\mathcal{H}}$. Intuitively, any generalized single-peaked preference relation is characterized by two main ingredients. On the one hand, the alternative p describes the peak of that preference relation. On the other hand, the constraint formalizing the generalized notion of single-peakedness requires that any alternative which is between the peak p and some alternative c according to the betweenness relation $B_{\mathcal{H}}$ must be preferred to c.

Finally, again, following [Nehring and Puppe, 2007b], observe that this notion of generalized single-peakedness reduces to the well-known concept of single-peakedness on a line in the following way: Suppose that the set of alternatives amounts to $A = \{1, ..., m\}$ with $m \geq 2$ and assume that the properties are given by

$$H_{\leq a} := \{a' \in A : a' \leq a\} \; \forall a \in A : a < m \text{ as well as}$$
$$H_{\geq a} := \{a' \in A : a' \geq a\} \; \forall a \in A : a > 1.$$

In this case, preferences are single-peaked on a line and, more precisely, they are single-peaked with respect to the natural ordering $1 < 2 < ... < m - 1 < m$.

4 Incentive Compatibility

In this section, for completeness, I review the characterization of strategy-proof, anonymous, and surjective social choice functions for generalized single-peaked domains giving rise to median spaces due to [Nehring and Puppe, 2007b].

First of all, a *social choice function* or, equivalently, a direct mechanism f is a mapping assigning to each type profile an alternative from the set A. In formal terms, this mapping amounts to $f : \{S \times [0, \overline{v}]^{|\mathcal{H}|}\}^n \rightarrow A$.[12] Due to the revelation principle, without loss of generality, I restrict attention to such direct mechanisms. Subsequently, I recall some well-known properties of social choice functions.

Definition 1. *A social choice function f is strategy-proof if it holds, for all $i \in N$ and for all $t_i, t_i' \in S \times [0, \overline{v}]^{|\mathcal{H}|}$ and $t_{-i} \in \{S \times [0, \overline{u}]^{|\mathcal{H}|}\}^{n-1}$, that*

$$u^{f(t_i, t_{-i})}(t_i) \geq u^{f(t_i', t_{-i})}(t_i).$$

In words, *strategy-proofness* requires that all voters have a weakly dominant strategy to truthfully reveal their types. Further, observe that strategy-proofness implies that social choice functions do not condition on preference intensities.

Definition 2. *A social choice function f is anonymous if it holds, for all $(t_1, ..., t_n) \in \{S \times [0, \overline{u}]^{|\mathcal{H}|}\}^n$, that $f(t_1, ..., t_n) = f(t_{\sigma(1)}, ..., t_{\sigma(n)})$ with σ being an arbitrary permutation of the set of voters N.*

Intuitively, *anonymity* imposes that mechanisms treat all voters equally. To put it differently, anonymity ensures that mechanisms respect the democratic principle of "one person, one vote".

Definition 3. *A social choice function f is surjective if, for all $a \in A$, there exists $(t_1, ..., t_n) \in \{S \times [0, \overline{u}]^{|\mathcal{H}|}\}^n$ such that $f(t_1, ..., t_n) = a$.*

[12] In particular, I focus on deterministic mechanisms.

The requirement that social choice functions are *surjective* represents a mild condition ensuring that no alternative is a priori excluded from the set of outcomes.
[Nehring and Puppe, 2007b] show that strategy-proof and surjective social choice functions must be *peaks-only*, meaning, the outcome of any strategy-proof and surjective social choice function depends only on the voters' most preferred alternatives. In the following, with abuse of notation, social choice functions are simply mappings $f : S^n \to A$ assigning to every profile of peak alternatives or, equivalently, to each profile of peak properties $(\times_{H \in \mathcal{H}} P_{H,1}, ..., \times_{H \in \mathcal{H}} P_{H,n}) = (P_1, ..., P_n) \in S^n$ some winning alternative from the set A.
In order to be able to state [Nehring and Puppe, 2007b]'s characterization result, I need the following supplementary definitions from their paper. To begin with, introduce the notion of a *family of quotas* relative to some property space $(A, \mathcal{H})$.

Definition 4. *[Nehring and Puppe, 2007b]*
Given some property space $(A, \mathcal{H})$, a family of quotas $\{q_H : H \in \mathcal{H}\}$ is a function that assigns an integer-valued quota $1 \leq q_H \leq n$ to each property $H \in \mathcal{H}$ such that, for all $H \in \mathcal{H}$, the associated quotas satisfy $q_H + q_{H^c} = n + 1$.

Take any property $H \in \mathcal{H}$. The associated absolute quota, threshold or majority requirement q_H describes the minimal number of votes that are needed in order to ensure that some alternative sharing property H is winning. Furthermore, the condition $q_H + q_{H^c} = n + 1$ reflects that whenever the quota q_H linked to property H is reached, the quota associated with the complementary property H^c cannot be attained, and vice versa. Moreover, exactly one of these two quotas is always achieved.
On the basis of the definition of families of quotas, consider the following class of functions which is termed anonymous *voting by properties*. These functions will be central for the ensuing characterization result.

Definition 5. *[Nehring and Puppe, 2007b]*
Given some property space $(A, \mathcal{H})$ and associated family of quotas $\{q_H : H \in \mathcal{H}\}$, voting by properties is the function $f_{\{q_H : H \in \mathcal{H}\}} : S^n \to \mathcal{P}(A)$ such that, for all profiles of peak properties $P = (P_1, ..., P_n) \in S^n$, it holds that

$$a \in f_{\{q_H : H \in \mathcal{H}\}}(P) :\Leftrightarrow [\forall H \in \mathcal{H}_a : |\{i \in N : P_{H,i} = H\}| \geq q_H].$$

Intuitively, under voting by properties, the social choice is determined through a collection of binary votes on subsets of alternatives involving qualified majority requirements. In more detail, it works as follows. Take some family of quotas $\{q_H : H \in \mathcal{H}\}$. For any issue (H, H^c), it is collectively decided according to the quotas q_H and q_{H^c} whether the winning alternative is supposed to share property H or its complement H^c. These binary decisions yield a collection of properties that the winning alternative is supposed to share. However, it has to be ensured that this set of, loosely speaking, winning properties is consistent in the sense that the intersection of these properties is not empty, but it contains exactly one alternative which, then, constitutes the winning alternative. Thus, in general, the considered mapping need not represent a proper social choice function. However, as the following result reveals, under some conditions on the family of quotas, the stated mapping forms a social choice function.

I state [Nehring and Puppe, 2007b]'s characterization of strategy-proof, anonymous, and surjective social choice functions.

Theorem 1. *[Nehring and Puppe, 2007b]*
A social choice function f is strategy-proof, anonymous, and surjective if and only if it is voting by properties $f_{\{q_H:H\in\mathcal{H}\}} : S^n \to A$ with family of quotas $\{q_H : H \in \mathcal{H}\}$ such that, for all properties $K, H \in \mathcal{H}$, it holds

$$K \subseteq H \Rightarrow q_K \geq q_H.$$

In particular, if the number of voters n is odd, this characterization shows that voting by properties with simple majority quotas, i.e., $q_H = \frac{n+1}{2}$ for all $H \in \mathcal{H}$, is feasible in median spaces. In other words, in median spaces, there exists a social choice function that is strategy-proof, anonymous, surjective, and neutral.[13] In fact, [Nehring and Puppe, 2007b] have more results: First, they establish that strategy-proof, anonymous, and surjective social choice functions take the form of consistent voting by properties on generalized single-peaked domains derived from any property space. Second, they show that there exist strategy-proof, anonymous, surjective, and neutral social choice functions on generalized single-peaked domain giving rise to any property space if and only if the property space is a median space and the number of voters n is odd. [Nehring and Puppe, 2007b] describe the intuition behind the proof of the necessity part of the second finding while employing the first result as follows: For simplicity, suppose that there are $n = 3$ voters. Towards a contradiction, assume that voting by properties with simple majority quotas is feasible, but the property space does not form a median space. Then, there exists some triple of alternatives that does not admit a median. Now, suppose that the three voters have peak properties $P_1, P_2, P_3 \in S$ characterizing the alternatives from the discussed triple. Hence, all properties that are in the set of peak properties for at least two voters must be winning. However, this means that the social choice must be between any pair alternatives from the discussed triple. In other words, this triple must admit a median which is the desired contradiction. This reasoning uncovers the intuition behind the median space assumption. I focus on median spaces because they admit a rich class of non-degenerate incentive-compatible mechanisms.[14]

[13] Voting by properties satisfies *neutrality* if and only if the involved quotas are constant across properties, that is, there exists $1 \leq q \leq n$ such that $q = q_H$ for all $H \in \mathcal{H}$ and the number of voters n is odd.

[14] However, note that I do not impose neutrality and, as shown in [Nehring and Puppe, 2005], strategy-

Combing back to the optimality question I am addressing, Theorem 1 implies that, when searching for the optimal mechanism among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity, it is sufficient to optimize over the set of quotas $\{q_H : H \in \mathcal{H}\}$ related to voting by properties while respecting the collection of inequalities stated in Theorem 1. I tackle this problem in the subsequent part.

5 Optimal Mechanism

In this section, I characterize the welfare-maximizing mechanism among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity, constituting the main result of this paper.

By Theorem 1, it is sufficient to find the optimal quotas related to voting by properties. Also, the existence of a solution is ensured since a bounded function is maximized over a finite set of elements. The structure of the proof of the main theorem below is as follows: First, consider some optimal mechanism and derive necessary conditions for optimality by means of studying the implications of alterations of this optimal mechanism. Second, argue that these necessary conditions are also sufficient for optimality and conclude that they determine a unique optimal mechanism. Again, the proof builds on [Gershkov et al., 2017], but the much larger class of preferences requires additional proof arguments as well as different assumptions.

When deriving the discussed necessary conditions for optimality, it turns out that I have to compare the welfare induced by the following two sets of alternatives: For every property $H \in \mathcal{H}$, define the sets of alternatives[15]

$$A_H := \begin{cases} H, & \nexists S \in \mathcal{H} : H \subset S \textit{ and } \nexists M \in \mathcal{H} : M \subset H \\ H \cap [\cap_{\{S:H\subset S\}} S], & \exists S \in \mathcal{H} : H \subset S \textit{ and } \nexists M \in \mathcal{H} : M \subset H \\ H \cap [\cap_{\{M:M\subset H\}} M^c], & \nexists S \in \mathcal{H} : H \subset S \textit{ and } \exists M \in \mathcal{H} : M \subset H \\ H \cap [\cap_{\{S:H\subset S\}} S] \cap [\cap_{\{M:M\subset H\}} M^c], & \exists S \in \mathcal{H} : H \subset S \textit{ and } \exists M \in \mathcal{H} : M \subset H \end{cases}$$

and

$$A_{H^c} := \begin{cases} H^c, & \nexists S \in \mathcal{H} : H \subset S \textit{ and } \nexists M \in \mathcal{H} : M \subset H \\ H^c \cap [\cap_{\{S:H\subset S\}} S], & \exists S \in \mathcal{H} : H \subset S \textit{ and } \nexists M \in \mathcal{H} : M \subset H \\ H^c \cap [\cap_{\{M:M\subset H\}} M^c], & \nexists S \in \mathcal{H} : H \subset S \textit{ and } \exists M \in \mathcal{H} : M \subset H \\ H^c \cap [\cap_{\{S:H\subset S\}} S] \cap [\cap_{\{M:M\subset H\}} M^c], & \exists S \in \mathcal{H} : H \subset S \textit{ and } \exists M \in \mathcal{H} : M \subset H \end{cases}.$$

proofness, anonymity and surjectivity without imposing neutrality would be also compatible in some property spaces distinct from median spaces. Nevertheless, for tractability reasons, I restrict attention to median spaces.

[15] The subsequent expressions can be simplified. It is on purpose that I do not simplify terms here.

Any alternative contained in the set A_H shares property H as well as all properties which are more moderate compared to H in the sense that they constitute supersets of H, but, at the same time, these alternatives violate all properties which are more extreme than H in the sense that they are subsets of H. Likewise, any alternative from the set A_{H^c} satisfies property H^c as well as all properties which are more moderate relative to H^c, but, at the same time, properties that are more extreme compared to H^c are violated.[16] In Lemma 1, I establish that the sets A_H and A_{H^c} have a particular tuple structure.

Lemma 1.
Consider any property $H \in \mathcal{H}$. The sets A_H and A_{H^c} satisfy $A_H \neq \emptyset$ and $A_{H^c} \neq \emptyset$.[17] *Moreover, all elements in both sets can be uniquely matched into tuples having the form (j, l) with $j \in A_{H^c}$ and $l \in A_H$ such that j and l are separated only by property H, meaning $\{H^c\} = \{K \in \mathcal{H} : j \in K \wedge l \notin K\}$.*

The proof of Lemma 1 employs a characterization of median spaces in terms of the involved properties instead of relying on the induced betweenness relation due to [Nehring and Puppe, 2007b]. Let Z_H denote the set of tuples generated in the way just described. It is clear that $|A_H| = |A_{H^c}|$, but, in general, it does not hold that $|A_H| = |A_{H^c}| = 1$. However, the tuple structure established in Lemma 1 implies that the comparison of the welfare generated by the sets A_H and A_{H^c} reduces to contrasting a collection of pairs of alternatives such that the elements within each pair are separated by one property only.

Furthermore, in order to characterize the optimal quotas in a separable way, I have to make sure that the welfare gains and losses involved in the welfare comparison within the discussed tuples do not depend on the tuple under consideration, but that they are the same across all tuples. The purpose of Assumption 2 together with Lemma 2 below is to ensure exactly that. For all alternatives in the sets A_{H^c} and A_H, the following sets of properties collect the properties that do not appear in the construction of the sets A_{H^c} and A_H. For any property $H \in \mathcal{H}$ and alternative $l \in A_H$, consider the collection of properties

$$\mathcal{C}_l^H := \mathcal{H}_l \setminus [\{H\} \cup (\cup_{\{M: M \subset H\}} M^c) \cup (\cup_{\{S: H \subset S\}} S)]$$

and, for any property $H \in \mathcal{H}$ and alternative $j \in A_{H^c}$, introduce the family of properties

$$\mathcal{C}_j^{H^c} := \mathcal{H}_j \setminus [\{H^c\} \cup (\cup_{\{M: M \subset H\}} M^c) \cup (\cup_{\{S: H \subset S\}} S)].$$

Next, I introduce the notion of *independent properties* from [Nehring and Puppe, 2007a] that builds the basis for Assumption 2 below.

[16] Also, note that $A_H \cap A_{H^c} = \emptyset$.

[17] The non-emptiness of these sets can be inferred directly from Theorem 1. The proof in the Appendix contains a different argument establishing this feature.

Definition 6. *[Nehring and Puppe, 2007a]*
The properties $K, H \in \mathcal{H}$ are independent, denoted as $K \perp H$, if

$$H \cap K \neq \emptyset, H \cap K^c \neq \emptyset, H^c \cap K \neq \emptyset \text{ and } H^c \cap K^c \neq \emptyset.$$

Two properties $K, H \in \mathcal{H}$ are independent if any two properties taken from the set $\{K, K^c, H, H^c\}$ that do not form an issue are compatible in the sense that their intersection is not empty. In other words, two properties are independent if the consistency of the properties does not imply to resolve one of the associated issues in a particular way, as soon as the other issue is resolved. Now, fix some property $H \in \mathcal{H}$. It turns out that, on the one hand, all properties from the sets $\mathcal{C}_l^H$ and $\mathcal{C}_j^{H^c}$ and, on the other hand, the properties H and H^c are independent. This feature is shown in Lemma 2.

Lemma 2.
(i) Consider any property $H \in \mathcal{H}$ and alternative $j \in A_{H^c}$. For all $K \in \mathcal{C}_j^{H^c}$, it holds that $K \perp H$ and $K \perp H^c$.
(ii) Take any property $H \in \mathcal{H}$ and alternative $l \in A_H$. For all $K \in \mathcal{C}_l^H$, it holds that $K \perp H$ and $K \perp H^c$.

Assumption 2 achieves that the welfare comparisons within the tuples identified in Lemma 1 are identical across these tuples.

Assumption 2.
For all $K, H \in \mathcal{H}$ such that $K \perp H$, the random variables $\tilde{T}_K$ and V_H are independent.

Take some property $H \in \mathcal{H}$ and consider any two alternatives (j, l) with $j \in A_{H^c}$ and $l \in A_H$ separated only by property H. By construction, it holds that $\mathcal{C}_l^H = \mathcal{C}_j^{H^c}$. The welfare comparison between j and l involves specific conditional expectations of the random variables V_H and V_{H^c}. Now, in general, these conditional expectations depend on the tuple under consideration via the type components capturing the properties related to properties that do not appear in the construction of the sets A_{H^c} and A_H, that is, the type components $\tilde{T}_K$ with $K \in \mathcal{C}_l^H = \mathcal{C}_j^{H^c}$. However, by Lemma 2, these properties K and the properties H and H^c are independent of each other. Consequently, Assumption 2 imposes that the discussed conditional expectations are not sensitive to these type components $\tilde{T}_K$, implying that the welfare comparison within the mentioned tuples is the same across all tuples.
Finally, Assumption 3 takes care of the fact that the alterations of some optimal mechanism that build the basis for the derivation of necessary conditions for optimality need not be feasible due to the constraints on the family of quotas appearing in Theorem 1. Essentially, it implies that the discussed necessary conditions remain valid even if the considered alterations are not feasible.

Assumption 3.

Consider two arbitrary properties $K, L \in \mathcal{H}$ satisfying $K \subseteq L$.
The following inequality holds:

$$\delta_K := \frac{\mathbb{E}[V_{K^c}|\tilde{T}_K = K^c]}{\mathbb{E}[V_{K^c}|\tilde{T}_K = K^c] + \mathbb{E}[V_K|\tilde{T}_K = K]} \geq \frac{\mathbb{E}[V_{L^c}|\tilde{T}_L = L^c]}{\mathbb{E}[V_{L^c}|\tilde{T}_L = L^c] + \mathbb{E}[V_L|\tilde{T}_L = L]} =: \delta_L.$$

If two properties $K, L \in \mathcal{H}$ satisfy $K \subseteq L$, property L might be interpreted as being more moderate than property K. Then, in intuitive terms, Assumption 3 requires that voters care in expectation more about more moderate properties. To put it differently, following a related discussion in [Nehring and Puppe, 2007b], Assumption 3 can be interpreted as a specific concavity restriction on the average voter, suggesting that it constitutes a rather mild condition.
Having presented the required assumptions as well as some preliminary steps for the analysis, I state the main result of this paper, that is, I provide a characterization of the welfare-maximizing mechanism among all strategy-proof, anonymous, and surjective social choice functions.

Theorem 2.

Suppose that Assumptions 1, 2 and 3 hold.
The optimal mechanism among all strategy-proof, anonymous, and surjective social choice functions takes the form of voting by properties with quotas

$$q_H^* = \lceil n\delta_H \rceil \text{ for all } H \in \mathcal{H}.$$

While taking into account that mechanisms have to be dominant-strategy incentive-compatible, Theorem 2 characterizes the optimal utilitarian mechanism for generalized single-peaked domains derived from median spaces. In particular, Theorem 2 provides closed-form expressions for the welfare-maximizing quotas related to voting by properties. What is the intuition behind the optimal quotas $q_H^* = \lceil n\delta_H \rceil$? First of all, observe that the quota q_H^* is shaped by the ratio of preference intensities

$$\frac{\mathbb{E}[V_H|\tilde{T}_H = H]}{\mathbb{E}[V_{H^c}|\tilde{T}_H = H^c]},$$

reflecting the utilitarian objective of the designer. Also, regarding comparative statics, the quota q_H^* decreases in the discussed ratio of preference intensities. For the purpose of a more detailed understanding, ignore the aspect that quotas must be integer-valued. Plugging in the expression for δ_H and rearranging yields

$$q_H^* \cdot \mathbb{E}[V_H|\tilde{T}_H = H] = [n - q_H^*] \cdot \mathbb{E}[V_{H^c}|\tilde{T}_H = H^c].$$

Consider any tuple of alternatives $a, b \in A$ separated only by property H, and, without loss of generality, suppose that $a \in H$ and $b \in H^c$. Intuitively, the trade-off the designer is facing constitutes in comparing the following two expected utility differences. On the one hand, the term $\mathbb{E}[V_H|\tilde{T}_H = H]$ captures the gain in utility terms for voters with peaks sharing H when implementing a instead of b. On the other hand, the expression $\mathbb{E}[V_{H^c}|\tilde{T}_H = H^c]$ describes the increase in terms of utility for voters with peaks sharing H^c when selecting b instead of a. The optimal quota q^*_H is calibrated in a way such that the utilitarian designer is indifferent between implementing alternatives a and b whenever the number of voters having peaks that share property H amounts to q^*_H. Moreover, it depends only on the two discussed conditional expectations capturing the outlined expected utility differences, but it is not sensitive to other aspects of the preference distribution.

Let me discuss the proof of Theorem 2. To begin with, by Theorem 1, it is sufficient to optimize over the set of quotas related to voting by properties.[18] Furthermore, again, due to Theorem 1, for all $H', H \in \mathcal{H}$, the optimal quotas must satisfy

$$H' \subseteq H \Rightarrow q^*_{H'} \geq q^*_H.$$

Consider some property $H \in \mathcal{H}$ as well as the associated quota q^*_H which is supposed to be part of an optimal mechanism. To simplify the exposition, I divide the proof of Theorem 2 into two lemmata.

Lemma 3.
Suppose that Assumptions 1 and 2 hold. Consider any property $H \in \mathcal{H}$.
(i) If $H' \subset H \Rightarrow q^*_{H'} > q^*_H$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$, the inequality

$$q^*_H \geq n \cdot \delta_H$$

constitutes a necessary condition for optimality.
(ii) If $H \subset H' \Rightarrow q^*_H > q^*_{H'}$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H \subset H'' \subset H'$, any optimal mechanism meets the inequality

$$q^*_H \leq n \cdot \delta_H + 1.$$

Suppose that increasing q^*_H by 1 is feasible, meaning, this alteration does not violate the inequalities from Theorem 1. This change matters only if there are q^*_H voters having some peak from the set H and $n - q^*_H$ voters with peaks from the set H^c. In this case, since $q^*_S \leq q^*_H$ for all $H \subset S$, the properties $\{S : H \subset S\}$ are accepted whenever there are such

[18] Again, since a bounded function is maximized over a finite set of elements, the existence of a solution is ensured.

properties. Additionally, since increasing q_H^* by 1 is feasible, I must have that $q_M^* > q_H^*$ for all $M \subset H$. Thus, the properties $\{M : M \subset H\}$ are rejected or, equivalently, the properties $\{M^c : M \subset H\}$ are winning whenever there are such properties. Putting these aspects together and using the introduced notation, if the quota is q_H^*, some element of the set $A_H \neq \emptyset$ is the winning alternative. However, if the quota amounts to $q_H^* + 1$, some element of the set $A_{H^c} \neq \emptyset$ is selected. Since q_H^* is part of an optimal mechanism, the modification of this quota should weakly decrease welfare. In other words, the expected welfare induced by alternatives from the set A_H must be weakly higher compared to the welfare generated by alternatives from the set A_{H^c}. This observation translates into a condition which is necessary for optimality whenever the considered change in the optimal quota q_H^* is feasible. Exploiting the tuple structure derived in Lemma 1, the comparison of the expected welfare induced by the two sets of alternatives reduces to contrasting a collection of tuples of alternatives such that the elements within each tuple are separated only by property H. Now, imposing Assumption 2 while employing Lemma 2 implies, as discussed above, that these within-tuple welfare comparisons are not sensitive to the tuple under consideration. This aspect simplifies the involved expressions and leads to the inequality appearing in part (i) of Lemma 3.
Studying the effect of a decrease of q_H^* by 1 yields via an analogous argument the inequality appearing in part (ii) of Lemma 3. This inequality is necessary for optimality as long as the considered decrease in the optimal quota q_H^* is feasible.
The second step of the proof of Theorem 2 is summarized in Lemma 4.

Lemma 4.
Suppose that Assumptions 1, 2 and 3 hold. Consider any properties $H', H \in \mathcal{H}$ such that $H' \subset H$ and $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$.
If $q_{H'}^* = q_H^*$, any optimal mechanism nevertheless satisfies

$$q_H^* \geq n \cdot \delta_H$$

as well as

$$q_{H'}^* \leq n \cdot \delta_{H'} + 1.$$

The two alterations of the quota q_H^* that is part of an optimal mechanism considered above might not be feasible. Lemma 4 addresses this issue. Making use of Assumption 3, I show that the two inequalities derived in Lemma 3 still hold even if these alterations are not feasible.
Finally, it turns out that these inequalities are not only necessary, but also sufficient for optimality and they determine the generically unique optimal mechanism featuring the quotas appearing in Theorem 2.

To conclude this section, let me emphasize a major difference between my analysis and the work of [Gershkov et al., 2017]. In their paper, because preferences are single-peaked on a line, the sets A_H and A_{H^c} discussed above are singletons.[19] Therefore, the additional complications arising from the fact that in my setting these sets are, in general, not singletons are completely absent in their analysis. This difference has substantial implications requiring supplementary arguments (in particular, Lemma 1 and Lemma 2) as well as different assumptions on the preference distribution (in particular, Assumption 2) in order to characterize the welfare-maximizing mechanism.
In the following section, I apply this characterization to specific classes of median spaces.

6 Applications

The purpose of this section is to apply the general characterization of welfare-maximizing mechanism developed in Theorem 2 to more specific collective decision-making problems while providing more concrete sufficient conditions for Assumption 2. The first part deals with products of trees, encompassing as a special case collective choice problems featuring preferences that are single-peaked with respect to an arbitrary tree. The second part discusses voting problems that go beyond products of trees. I also provide an application to voting over multiple public goods under constraints falling in the latter class of preferences.

6.1 Products of Trees

In this part, I focus on the specific class of median spaces representing products of trees. The intuitive idea behind products of trees can be described as follows: Suppose that the alternatives are multidimensional objects in the sense that each alternative is composed of several attributes. Further, the structure of the utility function will imply that preferences are additively separable across these attributes. Finally, the preferences over the multidimensional alternatives are restricted such that, for each attribute or dimension, the preferences over this attribute are single-peaked with respect to some possibly dimension-specific tree.
To start, identify each alternative $a \in A$ with a set of attributes where each attribute is supposed to correspond to some tree. Let d be the number of involved trees. More formally, there are sets A_m with $m \in D := \{1, ..., d\}$ such that $A = \times_{m \in D} A_m$.
Furthermore, for each $m \in D$, there is some tree (A_m, E_m), meaning (A_m, E_m) constitutes some undirected graph that is connected and acyclic. The set A_m describes the set of nodes and the set E_m captures the set of edges corresponding to that tree. In particular,

[19] This can be seen from the definition of the property space inducing preferences that are single-peaked on a line contained in section 3.

$E_m \subseteq \{V \in \mathcal{P}(A_m) : |V| = 2\}$.
Following [Nehring and Puppe, 2007a], for any $m \in D$ as well as for any corresponding edge $V_m = \{b_m, c_m\} \in E_m$, define the two properties

$$H^m_{V_m,b_m} := \{a = (a_1, ..., a_d) \in A : \text{“}a_m \text{ lies in direction of } b_m\text{”}\} \text{ and}$$
$$H^m_{V_m,c_m} := \{a = (a_1, ..., a_d) \in A : \text{“}a_m \text{ lies in direction of } c_m\text{”}\}.$$ [20]

The properties of the form $(H^m_{V_m,b_m}, H^m_{V_m,c_m})$ constitute an issue. Let $\mathcal{H}_{Product\,of\,Trees}$ denote the collection of all these properties. Further, for any $m \in D$, let $\mathcal{H}^m_{Product\,of\,Trees}$ be the set of properties that belongs to the tree (A_m, E_m), meaning, loosely speaking, $\mathcal{H}^m_{Product\,of\,Trees}$ collects all properties having superscript m.
To illustrate the class of property spaces giving rise to products of trees, consider the example discussed in the introduction. Recall that, in this example, preferences are single-peaked with respect to the tree shown in Figure 1. For convenience, I reproduce this tree again here in Figure 2.[21] In this case, there is only one tree, i.e., $d = 1$, and the

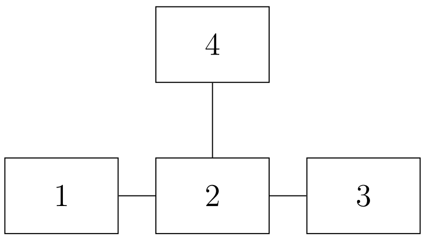

Figure 2: Tree example

collection of properties $\mathcal{H}_{Product\,of\,Trees}$ amounts to

$$H^1_{\{2,4\},2}, H^1_{\{2,4\},4} \text{ and}$$
$$H^1_{\{a,a+1\},a}, H^1_{\{a,a+1\},a+1} \; \forall a \in \{1, 2\}.$$

Therefore, in this example, and, more generally, if preferences are single-peaked on a tree, corresponding to the special case of products of trees where $d = 1$, every property coincides with the set of nodes associated with a connected component of the underlying tree.
Consider again an arbitrary property space $(\times_{m \in D} A_m, \mathcal{H}_{Product\,of\,Trees})$ giving rise to a

[20]More formally, following [Nehring and Puppe, 2007a], the property $H^m_{V_m,b_m}$ is composed of all alternatives $a = (a_1, ..., a_d) \in A$ such that b_m lies on the shortest path from a_m to c_m in the tree (A_m, E_m). Similarly, $H^m_{V_m,c_m}$ comprises all alternatives $a \in A$ such that c_m lies on the shortest path from a_m to b_m in the tree (A_m, E_m).
[21]Note that this is the simplest tree structure that is not a line.

product of trees and rewrite the type vector T as follows:

$$T = [\times_{H \in \mathcal{H}_{Product\,of\,Trees}} \tilde{T}_H] \times [\times_{H \in \mathcal{H}_{Product\,of\,Trees}} V_H]$$
$$= \times_{m \in D}[(\times_{H \in \mathcal{H}^m_{Product\,of\,Trees}} \tilde{T}_H) \times (\times_{H \in \mathcal{H}^m_{Product\,of\,Trees}} V_H)].$$

Equipped with this alternative expression, I offer a sufficient condition for Assumption 2 that applies to any product of trees. Essentially, it requires that preferences are distributed independently across trees and, therefore, it represents a sufficient condition that appears to be intuitive and easily interpretable.[22]

Assumption 4. *The random variables* $(\times_{H \in \mathcal{H}^m_{Product\,of\,Trees}} \tilde{T}_H) \times (\times_{H \in \mathcal{H}^m_{Product\,of\,Trees}} V_H)$ *with* $m \in D$ *are distributed independently across trees, that is, for all* $m', m'' \in D$ *with* $m' \neq m''$*, the random variables*

$$(\times_{H \in \mathcal{H}^{m'}_{Product\,of\,Trees}} \tilde{T}_H) \times (\times_{H \in \mathcal{H}^{m'}_{Product\,of\,Trees}} V_H) \text{ and}$$
$$(\times_{H \in \mathcal{H}^{m''}_{Product\,of\,Trees}} \tilde{T}_H) \times (\times_{H \in \mathcal{H}^{m''}_{Product\,of\,Trees}} V_H)$$

are independent.

I claim that Assumption 4 implies Assumption 2. First, observe that any two properties $K, H \in \mathcal{H}_{Product\,of\,Trees}$ are dependent if and only if there exists some $m \in D$ such that $K, H \in \mathcal{H}^m$ (see [Nehring and Puppe, 2007a]). Now, take any two properties $K, H \in \mathcal{H}_{Product\,of\,Trees}$ such that $K \perp H$. Since K and H are independent, there exist $m', m'' \in D$ with $m' \neq m''$ such that $K \in \mathcal{H}^{m'}_{Product\,of\,Trees}$ and $H \in \mathcal{H}^{m''}_{Product\,of\,Trees}$. Then, Assumption 4 implies that the random variables $\tilde{T}_K \times V_K$ and $\tilde{T}_H \times V_H$ are independent. However, this means that $\tilde{T}_K$ and V_H must be independent as well. Thus, Assumption 2 is met.

Consequently, I obtain the following corollary of the general optimality result captured in Theorem 2.

Corollary 1.
Consider the median space $(\times_{m \in D} A_m, \mathcal{H}_{Product\,of\,Trees})$ *and suppose that Assumptions 1, 4 and 3 are satisfied.*
The optimal mechanism among all strategy-proof, anonymous, and surjective social choice functions takes the form of voting by properties with quotas

$$q^*_H = \lceil n\delta_H \rceil \text{ for all } H \in \mathcal{H}_{Product\,of\,Trees}.$$

The optimal quotas derived in Corollary 1 can be interpreted as follows. Consider some $m \in D$ and an arbitrary issue $(H^m_{V_m,b_m}, H^m_{V_m,c_m})$ with $V_m \in E_m$ and $b_m, c_m \in A_m$. Now,

[22] Again, note that the structure of the utility function implies that preferences are additively separable across trees.

suppose that the designer is pivotal, that is, there are exactly $q^*_{H^m_{V_m,b_m}}$ voters with peaks sharing $H^m_{V_m,b_m}$. In this case, the designer is indifferent between any tuple of alternatives separated only by property $H^m_{V_m,b_m}$. In the context of products of trees, this means that these two alternatives differ only in the m-th component and the m-th coordinates of these two alternatives are b_m and c_m which are neighbors in the tree (A_m, E_m).

Moreover, note that Assumption 4 is vacuously met if there is only one tree, i.e., $d = 1$. For simplicity, let the set of alternatives, the underlying tree and the set of properties be A, (A, E), and $\mathcal{H}_{Tree}$ respectively. I immediately have the subsequent corollary of Theorem 2.

Corollary 2.

Consider the median space $(A, \mathcal{H}_{Tree})$ and suppose that Assumptions 1 and 3 are satisfied. The optimal mechanism among all strategy-proof, anonymous, and surjective social choice functions takes the form of voting by properties with quotas

$$q^*_H = \lceil n\delta_H \rceil \text{ for all } H \in \mathcal{H}_{Tree}.$$

As discussed in the introduction, if preferences are single-peaked on a tree, the optimal mechanism can be intuitively described as "voting by edges". Take any edge $V = \{b, c\} \in E$ with $b, c \in A$. Now, cutting this edge yields two connected components of the tree (A, E). Then, decide according to qualified majority involving the quotas $q^*_{H_{V,b}}$ and $q^*_{H_{V,c}}$ which connected component is winning. This means that agents vote sincerely for one of the two connected components, i.e., every agent votes for the connected component containing his or her peak alternative and the qualified majority decision yields one winning connected component, that is, the social choice must be contained in the set of nodes associated with this connected component. These qualified majority decisions are performed for all edges yielding a collection of winning connected components. The final outcome is given by the intersection of the sets of nodes linked to the connected components that are winning. Moreover, the expressions for the optimal quotas involve the comparison of the welfare generated by two graph neighbors and the welfare-maximizing quotas are calibrated such that, conditional on being pivotal, the designer is indifferent between implementing two graph neighbors.

Finally, let me discuss how Corollary 2 extends the main result in [Gershkov et al., 2017]. First of all, it can be verified that [Gershkov et al., 2017]'s model can be incorporated into my model in the following sense. Fix any utility function and take any type realization from their model. Then, there is a type realization in my model such that this type realization together with the utility function I employ induces the same utility levels. Further, it turns out that Assumption 3 reduces exactly to assumption B in their paper and Assumption 1 is assumption A in their work. However, it can be verified that the set of ordinal preferences induced by [Gershkov et al., 2017]'s utility representation does, in general, not satisfy the richness condition of [Nehring and Puppe, 2007b] and, thus, Theorem 1 need not apply. Consequently, the application of Corollary 2 to [Gershkov et al., 2017]'s model yields, in general, only the welfare-maximizing voting rule within the class of voting by properties mechanisms.[23] The case in which $|A| \leq 3$ constitutes an exception. Here, [Gershkov et al., 2017]'s preference domain amounts to the full domain of single-peaked preferences on a line and, thus, Corollary 2 formally reduces to the corresponding result from [Gershkov et al., 2017].

6.2 Beyond Products of Trees

Having studied the case of products of trees in detail, I emphasize that there are several relevant median spaces which do not give rise to products of trees. Applications falling in the class of median spaces, but going beyond products of trees include, for instance, voting over multiple public goods under constraints (see [Barberà et al., 1997], [Nehring and Puppe, 2005] and [Nehring and Puppe, 2007a]). More generally, collective choice problems giving rise to voting on a distributive lattice are covered here (see [Nehring and Puppe, 2007b]).

The discussion of the independence notion for properties in [Nehring and Puppe, 2007a] reveals that this concept is inherently connected to products of trees. Therefore, it seems to be difficult to provide sufficient sufficient conditions for Assumption 2 that are tailored towards different subclasses of property spaces that do not give rise to products of trees. However, albeit being rather strong, the following restriction provides a simple sufficient condition for Assumption 2 that applies to all median spaces.

Assumption 5. *The random variable $\times_{H\in\mathcal{H}}\tilde{T}_H$ is independent of $\times_{H\in\mathcal{H}}V_H$.*

It is clear that Assumption 5 implies Assumption 2. Hence, I obtain the following corollary of Theorem 2.

Corollary 3.
Consider some median space $(A, \mathcal{H})$ and suppose that Assumptions 1, 5 and 3 are satisfied. The optimal mechanism among all strategy-proof, anonymous, and surjective social choice functions takes the form of voting by properties with quotas

$$q^*_H = \lceil n\delta_H \rceil \text{ for all } H \in \mathcal{H}.$$

Furthermore, note that, in the presence of Assumption 5, Assumption 3 reduces to

[23] However, when combining results from [Moulin, 1980], [Nehring and Puppe, 2007b], [Saporiti, 2009] and [Gershkov et al., 2017], it can be inferred that any strategy-proof, anonymous, and surjective social choice functions for the preference domain considered in [Gershkov et al., 2017] can be represented by some voting by properties mechanism.

the following constraint: For all $K, L \in \mathcal{H}$ such that $K \subseteq L$, it holds that

$$\delta_K = \frac{\mathbb{E}[V_{K^c}]}{\mathbb{E}[V_{K^c}] + \mathbb{E}[V_K]} \geq \frac{\mathbb{E}[V_{L^c}]}{\mathbb{E}[V_{L^c}] + \mathbb{E}[V_L]} = \delta_L.$$

Finally, let me discuss a concrete application illustrating Corollary 3. I derive the optimal voting mechanisms for the provision of two ex-ante identical public goods subject to the constraint that the expenditures for the first good must be weakly higher than the investments in the second good. The outlined constraint on the set of feasible allocations might be imposed by law or there might be a consensus among all voters that allocations have to meet that condition and this consensus is common knowledge.[24]
Suppose that there are two public goods α and β that can be supplied in non-negative integer quantities ranging from 0 to some exogenous upper bound $M \in \mathbb{N}$ with $M \geq 2$.[25] Therefore, an allocation is described by a pair $(l_\alpha, l_\beta) \in \mathbb{N}_0 \times \mathbb{N}_0$ where the first component corresponds to the provided level of public good α and the second coordinate refers to the expenditures for public good β. Imposing that the expenditures for α must be weakly higher than the investment in β, the set of alternatives A amounts to

$$A = \{(l_\alpha, l_\beta) \in \mathbb{N}_0 \times \mathbb{N}_0 : l_\alpha \leq M, l_\beta \leq M, l_\alpha \geq l_\beta\}.$$

Following [Nehring and Puppe, 2007a], define the properties

$$H^\alpha_{\leq l} := \{(l'_\alpha, l'_\beta) \in A : l'_\alpha \leq l\} \; \forall l \in \mathbb{N}_0 : l < M \text{ and}$$
$$H^\alpha_{\geq l} := \{(l'_\alpha, l'_\beta) \in A : l'_\alpha \geq l\} \; \forall l \in \mathbb{N}_0 : M \geq l > 0$$

as well as

$$H^\beta_{\leq l} := \{(l'_\alpha, l'_\beta) \in A : l'_\beta \leq l\} \; \forall l \in \mathbb{N}_0 : l < M \text{ and}$$
$$H^\beta_{\geq l} := \{(l'_\alpha, l'_\beta) \in A : l'_\beta \geq l\} \; \forall l \in \mathbb{N}_0 : M \geq l > 0.$$

Denote by $\mathcal{H}_{Public\,Good}$ the collection of these properties. Figure 3 shows the set of feasible allocations for the case of $M = 2$, meaning, every node corresponds to some feasible allocation. For instance, the node labeled as $(2, 1)$ describes the allocation where 2 units of money are invested in public good α and 1 unit of money is spent for good β. Further, following [Nehring and Puppe, 2007b], the graph depicted in Figure 3 visualizes the betweenness relation $B_{\mathcal{H}_{Public\,Good}}$ in the sense that $(a, b, c) \in B_{\mathcal{H}_{Public\,Good}}$ if and only if b lies

[24] Again, examples which are similar to the one I am studying here appear in [Barberà et al., 1997], [Nehring and Puppe, 2005], [Nehring and Puppe, 2007a], [Block, 2010] and [Block de Priego, 2014]. As far as the property space is concerned, the present example is identical to an example that can be found in [Nehring and Puppe, 2005] and [Nehring and Puppe, 2007a]. However, the interpretation I am offering is somewhat different.

[25] If $M = 1$, preferences are single-peaked on a line. This is the reason why this case is ruled out.

on a shortest path connecting a and c in the graph shown in Figure 3. For instance, suppose that some voter's peak alternative is the allocation $(1, 1)$. Then, single-peakedness requires, among other things, that this voter must prefer $(1, 0)$ and $(2, 1)$ over $(2, 0)$, but it does not impose whether $(1, 0)$ is preferred to $(2, 1)$ or the other way around. Note that

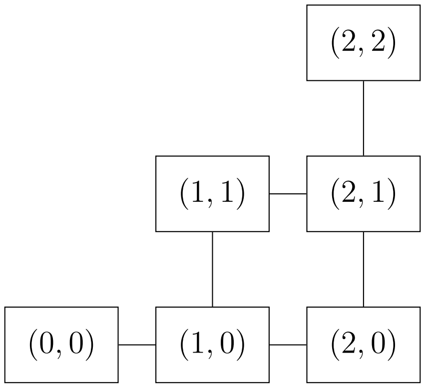

Figure 3: Graphic representation of $B_{\mathcal{H}_{Public\,Good}}$ $(M = 2)$

the corresponding property space $(A, \mathcal{H}_{Public\,Good})$ does not have the structure of a product of trees, but it gives rise to a distributive lattice (see [Nehring and Puppe, 2007b]). In particular, it constitutes a median space (see [Nehring and Puppe, 2007a]).
For simplicity, suppose that the distribution G of $\times_{H \in \mathcal{H}} V_H$ is a product measure. Moreover, the requirement that the two public goods are ex-ante identical amounts to assuming that the marginal distributions of G satisfy $G_{H_{\leq l}} := G_{H^{\alpha}_{\leq l}} = G_{H^{\beta}_{\leq l}}$ as well as $G_{H_{\geq l}} := G_{H^{\alpha}_{\geq l}} = G_{H^{\beta}_{\geq l}}$.
What is the interpretation of these random variables? For concreteness, consider the random variable $V_{H^{\alpha}_{\leq l}}$ and suppose that some voter's peak alternative stipulates to invest at most l in good α. The random variable $V_{H^{\alpha}_{\leq l}}$ measures the loss in utility terms when moving from an arbitrary allocation with $l_\alpha = l$ to another allocation satisfying $l_\alpha = l + 1$ such that both allocations or alternatives are separated only by property $H^{\alpha}_{\leq l}$. To put this differently, it measures this voter's willingness to pay for one unit of expenditures less in good α if the initial investment is $l + 1$ while holding the investment in β constant.
Subsequently, I offer sufficient conditions for Assumption 3. This allows me to apply Corollary 3 to the outlined public goods problem.

Corollary 4. *Consider the median space $(A, \mathcal{H}_{Public\,Good})$ and suppose that Assumptions 1 and 5 are satisfied. If $\mathbb{E}[V_{H^{\alpha}_{\leq l}}] = \mathbb{E}[V_{H^{\beta}_{\leq l}}]$ is increasing in l and $\mathbb{E}[V_{H^{\alpha}_{\geq l}}] = \mathbb{E}[V_{H^{\beta}_{\geq l}}]$ is decreasing in l, the optimal mechanism among all strategy-proof, anonymous, and surjective* *social choice functions takes the form of voting by properties with quotas*

$$\begin{aligned} q^*_{H^\alpha_{\leq l}} &= q^*_{H^\beta_{\leq l}} \\ &= \lceil n \frac{\mathbb{E}[V_{H^\alpha_{\geq l+1}}]}{\mathbb{E}[V_{H^\alpha_{\geq l+1}}] + \mathbb{E}[V_{H^\alpha_{\leq l}}]} \rceil \\ &= \lceil n \frac{\mathbb{E}[V_{H^\beta_{\geq l+1}}]}{\mathbb{E}[V_{H^\beta_{\geq l+1}}] + \mathbb{E}[V_{H^\beta_{\leq l}}]} \rceil \end{aligned}$$

Consequently, the application to voting over multiple public goods under constraints demonstrates that Corollary 3 can be applied to relevant voting problems going beyond products of trees and the sufficient conditions for Assumption 3 reduce to rather mild and transparent restrictions.

7 Conclusion

In this paper, I offered a welfare analysis of voting rules. Specifically, I derived the optimal utilitarian mechanism among all strategy-proof, anonymous, and surjective social choice functions for generalized single-peaked domains giving rise to median spaces. The optimal mechanism takes the form of voting by properties, meaning, the social choice is determined through a collection of binary votes on subsets of alternatives involving qualified majority requirements that incorporate the characteristics of these subsets of alternatives. Consequently, on a qualitative level, my results emphasize the importance of flexible and qualified majority requirements for utilitarian welfare in voting on a broad scale. Furthermore, my optimality analysis suggests that products of trees are fundamentally different from other median spaces like distributive lattices when it comes to the optimization over strategy-proof, anonymous, and surjective mechanisms while relying on the utilitarian principle.

Finally, I emphasize the importance of the characterization of utilitarian mechanisms for generalized single-peaked domains derived from median spaces, which is, again, the main result of this paper, for the analysis of stable constitutions. [Barbera and Jackson, 2004] study stable constitutions in a setting with two alternatives only. One approach to extend their modeling of constitutions as well as their notion of stability to my setting is as follows: A constitution is composed of a tuple of voting mechanisms (F, f) where F is a standard binary qualified majority rule with q_F representing the majority requirement for "change", and f is some strategy-proof, anonymous, and surjective social choice function.[26] The first voting rule is employed to decide about amendments of the voting mechanism f, whereas the second voting rule determines which policy alternative is im-

[26]The majority requirement q_F is assumed to satisfy $1 \leq q_F \leq n$.

plemented. More concretely, take some constitution (F, f), and consider the following two period game: In the first period, voters decide in a binary vote whether to amend the mechanism f to some other strategy-proof, anonymous, and surjective social choice function f' without knowing their types. If there are at least q_F voters who prefer f' over f, f' is applied to determine the policy alternative that is implemented, and, otherwise, f prevails. In the second period, voters learn their types, and, depending on the first period outcome, some policy alternative is selected according to f or f' and voters play the dominant-strategy equilibrium of the respective direct mechanism. Then, define a constitution (F, f) to be stable if for any strategy-proof, anonymous, and surjective social choice function f' with f' being distinct from f, there are strictly less than q_F voters who prefer f' over f. Similar to [Barbera and Jackson, 2004], the idea is that stable constitutions are expected to persist in the long run or, in other words, once a stable constitution is reached, it is expected that this constitution is not amended. Since in my model the voters' preferences over policy alternatives are distributed independently and identically across voters, all voters have the same preference over strategy-proof, anonymous, and surjective social choice functions prior to learning their types. Consequently, in this setting, a constitution (F, f) is stable if and only if f maximizes ex-ante utilitarian welfare among all strategy-proof, anonymous, and surjective social choice functions. This observation demonstrates the importance of characterizing the utilitarian mechanism for generalized single-peaked domains derived from median spaces.

Appendix

The proof of Lemma 1 employs a result from [Nehring and Puppe, 2007b] that is stated as Lemma 5 below. In order to present this result, I need to introduce the notion of *critical families of properties* from their paper. These sets are collections of properties having the following characteristic.

Definition 7. *[Nehring and Puppe, 2007b]*
A set of properties $\mathcal{F} \subseteq \mathcal{H}$ is a critical family of properties if

$$\cap_{\bar{F} \in \mathcal{F}} \bar{F} = \emptyset \text{ and}$$
$$\forall F \in \mathcal{F} : \cap_{\bar{F} \in \mathcal{F}: \bar{F} \neq F} \bar{F} \neq \emptyset.$$

In words, a collection of properties constitutes a critical family of properties if the intersection of all involved properties is empty, but these properties have a non-empty intersection whenever an arbitrary single property of the collection is removed. Also, note that any critical family of properties involves at least two elements. Based on this definition, [Nehring and Puppe, 2007b] obtain the following result about the size of critical families of properties in median spaces.

Lemma 5. *[Nehring and Puppe, 2007b]*
If $(A, \mathcal{H})$ constitutes a median space, all critical families of properties have length two.

Lemma 5 says that median spaces share the characteristic that there are no critical families of properties involving more than two properties.[27]

Proof of Lemma 1.
Take any property $H \in \mathcal{H}$ and consider the related sets A_H and A_{H^c} as defined in the main text.
Concerning the first aspect, consider the set A_H. The argument for the set A_{H^c} is analogous. Towards a contradiction, suppose that $A_H = \emptyset$. In the first two cases, I have that $H \subseteq A_H$. Since $H \neq \emptyset$, it follows that $A_H \neq \emptyset$. Concerning the remaining two cases, $A_H = \emptyset$ implies $H \cap (\cap_{M \subset H} M^c) = \emptyset$. In other words, the collection of properties $\{H\} \cup \{M^c : M \subset H\}$ is not consistent. However, this means that there must be some subset of the set of these properties which constitutes a critical family of properties. If this critical family involves at least three elements, the desired contradiction is derived since, due to Lemma 5, all critical families have length two in a median space. In case this critical family involves only two properties, there are two possibilities. On the one hand, if H is part of this critical family, the other element must be some single property M^c such that $M \subset H$. However, the collection of these two properties cannot be inconsistent

[27] In fact, [Nehring and Puppe, 2007b] show that this feature even characterizes median spaces.

and, hence, not critical since the intersection of H and M^c must be non-empty. On the other hand, if H is not part of the critical family, this family must be composed of two properties from the set $\{M^c : M \subset H\}$, but both of them are by definition supersets of H^c which means that they are consistent and, thus, not critical since $H^c \neq \emptyset$. Therefore, in all possible cases, I derived the desired contradiction.

Regarding the second aspect, consider any $j \in A_{H^c}$ and assume to the contrary that there is no corresponding alternative $l \in A_H$ such that j and l are separated only by property H. This means that the collection of properties $(\mathcal{H}_j \setminus \{H^c\}) \cup \{H\}$ is not consistent. To begin with, if $\mathcal{H}_j \setminus \{H^c\} = \emptyset$, the set of properties $(\mathcal{H}_j \setminus \{H^c\}) \cup \{H\}$ must be consistent since $H \neq \emptyset$. Thus, subsequently, assume that $\mathcal{H}_j \setminus \{H^c\}$ contains at least one property. $(\mathcal{H}_j \setminus \{H^c\}) \cup \{H\}$ being inconsistent implies that there must be some subset of these properties which constitutes a critical family of properties. Since all property spaces are median spaces, due to Lemma 5, this critical family must involve exactly two properties. Again, there are two possibilities. On the one hand, if H is part of this critical family, the other element must be some single property $K \in \mathcal{H}_j \setminus \{H^c\}$ satisfying $K \subset H^c$. In particular, it must hold that $j \in K$. However, by definition of A_{H^c}, because of $K \subset H^c$ or, equivalently, $H \subset K^c$, I have $j \in K^c$. This contradicts $j \in K$. On the other hand, if H is not part of the critical family, this family must be composed of two properties from the set $\mathcal{H}_j \setminus \{H^c\}$, but, by construction, the alternative j shares both of them which means that they are consistent and, thus, not critical. Therefore, in all possible cases, I obtain the desired contradiction. Thus, I infer that there exists some $l \in A_H$ such that j and l are separated only by property H. Moreover, there cannot be another alternative $l' \in A_H$ with $l \neq l'$ such that j and l' are also separated only by property H since this would contradict separation. The argument for the other direction, meaning, starting with some $l \in A_H$ and showing that there is some unique $j \in A_{H^c}$ such that both alternatives are separated only by property H works in the same way. This establishes the claimed unique tuple structure. □

Proof of Lemma 2.

To begin with, I establish part (i). Fix some property $H \in \mathcal{H}$ and alternative $j \in A_{H^c}$. Consider any property $K \in \mathcal{C}_j^{H^c}$ and assume to the contrary that $K \not\perp H$ or $K \not\perp H^c$. First, by definition, $K \not\perp H$ means that

$$H \cap K = \emptyset \vee H \cap K^c = \emptyset \vee H^c \cap K = \emptyset \vee H^c \cap K^c = \emptyset.$$

If $H \cap K = \emptyset$, I have that $H \subset K^c$, implying that $j \in K^c$ because $j \in A_{H^c}$. However, $K \in \mathcal{C}_j^{H^c} \subseteq \mathcal{H}_j$ yields $j \in K$ contradicting $j \in K^c$. If $H \cap K^c = \emptyset$, I obtain that $H \subset K$. Thus, by definition of $\mathcal{C}_j^{H^c}$, I have $K \notin \mathcal{C}_j^{H^c}$ contradicting $K \in \mathcal{C}_j^{H^c}$. If $H^c \cap K = \emptyset$, I infer that $K \subset H$. Hence, $j \in A_{H^c}$ yields $j \in K^c$ which contradicts $j \in K$. If $H^c \cap K^c = \emptyset$, I conclude that $K^c \subset H$. Therefore, by definition of $\mathcal{C}_j^{H^c}$, I have $K \notin \mathcal{C}_j^{H^c}$ contradicting

$K \in \mathcal{C}_j^{H^c}$. Consequently, $K \not\subset H$ cannot hold. By the same arguments, $K \not\subset H^c$ yields a contradiction as well. This establishes part (i). Since the proof of part (ii) is analogous to the proof of part (i), it is omitted here. □

Proof of Lemma 3.

Take any property $H \in \mathcal{H}$.

Assume that $H' \subset H \Rightarrow q^*_{H'} > q^*_H$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$.

Consider the quota q^*_H being part of an optimal mechanism and imagine that it is increased by 1, i.e. the quota linked to property H moves to q^*_H+1. In particular, as long as $q^*_H \neq n$, the modified quota $q^*_H + 1$ is still feasible because $q^*_{H'} \geq q^*_H + 1 > q^*_H$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$.

This alteration matters only if there are q^*_H voters having some peak from the set H and $n - q^*_H$ voters with peaks from the set H^c or, equivalently, there are q^*_H voters with type components $\tilde{t}_H = H$ and $n - q^*_H$ voters having type components $\tilde{t}_H = H^c$.

In this case, since $q^*_S \leq q^*_H$ for all $H \subset S$, the properties $\{S : H \subset S\}$ are accepted whenever there are such properties.

Additionally, $q^*_{H'} > q^*_H$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$ implies that $q^*_M > q^*_H$ for all $M \subset H$. Thus, the properties $\{M : M \subset H\}$ are rejected or, equivalently, the properties $\{M^c : M \subset H\}$ are winning whenever there are such properties. Putting these aspects together and using the notation introduced in the main text, if the quota is q^*_H, some element of the set $A_H \neq \emptyset$ is the winning alternative. However, if the quota amounts to $q^*_H + 1$, some element of the set $A_{H^c} \neq \emptyset$ is selected.

Further, as introduced in the main text, for any $l \in A_H$, consider the collection of properties

$$\mathcal{C}_l^H = \mathcal{H}_l \setminus [\{H\} \cup (\cup_{\{M:M\subset H\}} M^c) \cup (\cup_{\{S:H\subset S\}} S)],$$

and, for any $j \in A_{H^c}$, focus on the set of properties

$$\mathcal{C}_j^{H^c} = \mathcal{H}_j \setminus [\{H^c\} \cup (\cup_{\{M:M\subset H\}} M^c) \cup (\cup_{\{S:H\subset S\}} S)].$$

Hence, for both quotas, employing Assumption 1, the expected welfare conditional on the event where the alteration of q^*_H matters called "piv_H" can be expressed in the following way.

If the quota is q^*_H, the resulting welfare amounts to

$$\sum_{l\in A_H} Pr(\mathcal{C}_l^H \ win|piv_H) \cdot \{n \cdot \mathbb{E}[u^l(T)|piv_H \wedge \mathcal{C}_l^H \ win]\}.$$

In contrast, if the quota is $q_H^* + 1$, the induced welfare satisfies

$$\sum_{j \in A_{H^c}} Pr(\mathcal{C}_j^{H^c}\ win | piv_H) \cdot \{n \cdot \mathbb{E}[u^j(T) | piv_H \wedge \mathcal{C}_j^{H^c}\ win]\}.$$

Because q_H^* is part of an optimal mechanism, it must be that the former expression is weakly higher than the latter term. This necessary condition for optimality translates into the inequality

$$\sum_{l \in A_H} Pr(\mathcal{C}_l^H\ win \wedge piv_H) \mathbb{E}[u^l(T) | piv_H \wedge \mathcal{C}_l^H\ win] \geq$$
$$\sum_{j \in A_{H^c}} Pr(\mathcal{C}_j^{H^c}\ win \wedge piv_H) \mathbb{E}[u^j(T) | piv_H \wedge \mathcal{C}_j^{H^c}\ win].$$

Consider the tuple structure derived in Lemma 1 and, with abuse of notation, suppose that (j, l) constitutes such a tuple. Again, this means that $j \in A_{H^c}$ and $l \in A_H$ satisfy the restriction that they are separated only by property H.
This means that the events "$\mathcal{C}_l^H\ win \wedge piv_H$" and "$\mathcal{C}_j^{H^c}\ win \wedge piv_H$" must coincide, meaning, they refer to the same set of type realizations. This is true because, by construction, it holds that $\mathcal{C}_l^H = \mathcal{C}_j^{H^c}$. Call this very same event "$\mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H$". In particular, I have

$$Pr(\mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H) = Pr(\mathcal{C}_l^H\ win \wedge piv_H) = Pr(\mathcal{C}_j^{H^c}\ win \wedge piv_H).$$

Hence, the inequality can be rewritten as follows:

$$\sum_{(j,l) \in Z_H} Pr(\mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H) \{\mathbb{E}[u^l(T) - u^j(T) | \mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H]\} \geq 0.$$

By construction of the tuple (j, l), for any type realization $t = [\times_{H \in \mathcal{H}} \tilde{t}_H] \times [\times_{H \in \mathcal{H}} v_H] \in S \times [0, \overline{v}]^{|\mathcal{H}|}$, it holds that

$$u^l(t) - u^j(t) = \begin{cases} v_H, & \tilde{t}_H = H \\ -v_{H^c}, & \tilde{t}_H = H^c \end{cases}$$

This feature allows me to rewrite the conditional expectation involved in the inequality as follows:

$$\begin{aligned} &\mathbb{E}[u^l(T) - u^j(T) | \mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H] \\ =& Pr(\tilde{T}_H = H | \mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H) \mathbb{E}[V_H | \tilde{T}_H = H \wedge \mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H] \\ +& Pr(\tilde{T}_H = H^c | \mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H) \mathbb{E}[-V_{H^c} | \tilde{T}_H = H^c \wedge \mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H]. \end{aligned}$$

Now, Assumption 1 implies

$$\mathbb{E}[V_H|\tilde{T}_H = H \wedge \mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H] = \mathbb{E}[V_H|\tilde{T}_H = H \wedge \mathcal{C}_{(j,l)}^{(H^c,H)}\ win]$$

and

$$\mathbb{E}[-V_{H^c}|\tilde{T}_H = H^c \wedge \mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H] = \mathbb{E}[-V_{H^c}|\tilde{T}_H = H^c \wedge \mathcal{C}_{(j,l)}^{(H^c,H)}\ win].$$

By construction of $\mathcal{C}_{(j,l)}^{(H^c,H)}$ and because voting by properties is peaks-only, the event "$\mathcal{C}_{(j,l)}^{(H^c,H)}\ win$" provides only information about the frequency of the type components $\tilde{T}_K$ with $K \in \mathcal{C}_{(j,l)}^{(H^c,H)}$ in society, but it does not contain any information about other type coordinates. Moreover, by Lemma 2, all properties contained in the set $\mathcal{C}_l^H = \mathcal{C}_j^{H^c}$ are independent of the properties H^c and H, i.e., for all $K \in \mathcal{C}_l^H = \mathcal{C}_j^{H^c}$, it holds $K \perp H$ and $K \perp H^c$. Hence, Assumption 2 implies that V_H and V_{H^c} are independent of $\tilde{T}_K$ with $K \in \mathcal{C}_{(j,l)}^{(H^c,H)}$. Consequently, the conditional expectations above satisfy

$$\mathbb{E}[V_H|\tilde{T}_H = H \wedge \mathcal{C}_{(j,l)}^{(H^c,H)}\ win] = \mathbb{E}[V_H|\tilde{T}_H = H]$$

as well as

$$\mathbb{E}[-V_{H^c}|\tilde{T}_H = H^c \wedge \mathcal{C}_{(j,l)}^{(H^c,H)}\ win] = \mathbb{E}[-V_{H^c}|\tilde{T}_H = H^c].$$

In particular, $\mathbb{E}[V_H|\tilde{T}_H = H]$ and $\mathbb{E}[-V_{H^c}|\tilde{T}_H = H^c]$ do not depend on the concrete pair (j,l) from the set Z_H that is under consideration. Hence, these terms can be put outside the sums appearing in the inequality. Consequently, the inequality reduces to

$$\mathbb{E}[V_H|\tilde{T}_H = H]\{\sum_{(j,l)\in Z_H} Pr(\mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H)Pr(\tilde{T}_H = H|\mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H)\} +$$
$$\mathbb{E}[-V_{H^c}|\tilde{T}_H = H^c]\{\sum_{(j,l)\in Z_H} Pr(\mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H)Pr(\tilde{T}_H = H^c|\mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H)\} \geq 0.$$

Since

$$\begin{aligned}
&Pr(\mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H)Pr(\tilde{T}_H = H|\mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H)\\
&= Pr(\tilde{T}_H = H \wedge \mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H) \text{ and}\\
&Pr(\mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H)Pr(\tilde{T}_H = H^c|\mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H)\\
&= Pr(\tilde{T}_H = H^c \wedge \mathcal{C}_{(j,l)}^{(H^c,H)}\ win \wedge piv_H),
\end{aligned}$$

the inequality becomes

$$\mathbb{E}[V_H|\tilde{T}_H = H]\{\sum_{(j,l)\in Z_H} Pr(\tilde{T}_H = H \wedge \mathcal{C}^{(H^c,H)}_{(j,l)}\ win \wedge piv_H)\} +$$
$$\mathbb{E}[-V_{H^c}|\tilde{T}_H = H^c]\{\sum_{(j,l)\in Z_H} Pr(\tilde{T}_H = H^c \wedge \mathcal{C}^{(H^c,H)}_{(j,l)}\ win \wedge piv_H)\} \geq 0.$$

Moreover, because the events "$\tilde{T}_H = H \wedge \mathcal{C}^{(H^c,H)}_{(j,l)}\ win \wedge piv_H$" as well as "$\tilde{T}_H = H^c \wedge \mathcal{C}^{(H^c,H)}_{(j,l)}\ win \wedge piv_H$" are disjoint across tuples (j,l) from the set Z_H, I infer that

$$\sum_{(j,l)\in Z_H} Pr(\tilde{T}_H = H \wedge \mathcal{C}^{(H^c,H)}_{(j,l)}\ win \wedge piv_H) = Pr(\tilde{T}_H = H \wedge piv_H) \text{ and}$$
$$\sum_{(j,l)\in Z_H} Pr(\tilde{T}_H = H^c|\mathcal{C}^{(H^c,H)}_{(j,l)}\ win \wedge piv_H) = Pr(\tilde{T}_H = H^c \wedge piv_H).$$

Thus, the inequality can be restated as

$$\mathbb{E}[V_H|\tilde{T}_H = H]Pr(\tilde{T}_H = H|piv_H) + \mathbb{E}[-V_{H^c}|\tilde{T}_H = H^c]Pr(\tilde{T}_H = H^c|piv_H) \geq 0.$$

Further, by definition of the event "piv_H" and because of Assumption 1, the probabilities involved in the inequality satisfy

$$Pr(\tilde{T}_H = H|piv_H) = \frac{q^*_H}{n} \text{ and}$$
$$Pr(\tilde{T}_H = H^c|piv_H) = \frac{n - q^*_H}{n}.$$

Plugging these expressions into the inequality, I conclude that

$$\frac{q^*_H}{n}\mathbb{E}[V_H|\tilde{T}_H = H] + \frac{n - q^*_H}{n}\mathbb{E}[-V_{H^c}|\tilde{T}_H = H^c] \geq 0.$$

Hence, rearranging yields

$$q^*_H \geq n \cdot \delta_H$$

while I use the notation introduced in the main text. In addition, if $q^*_H = n$, the derived inequality still holds since $\delta_H \in (0,1)$. This establishes the first claim of the lemma.
Turning to the second point of the lemma, suppose that $H \subset H' \Rightarrow q^*_H > q^*_{H'}$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H \subset H'' \subset H'$.
Consider again the quota q^*_H related to an optimal mechanism and imagine that it is decreased by 1, i.e. the quota q^*_H moves to $q^*_H - 1$. In particular, the altered quota is still feasible as long as $q^*_H \neq 1$. This change matters only if there are $q^*_H - 1$ voters with type components $\tilde{t}_H = H$ and $n - q^*_H + 1$ voters having type components $\tilde{t}_H = H^c$.

Following the steps employed in the reasoning above in an analogous way, it can be verified that the inequality

$$q_H^* \leq n \cdot \delta_H + 1$$

constitutes a necessary condition for optimality. Additionally, observe that the derived inequality also holds if $q_H^* = 1$ since $n \cdot \delta_H > 0$. $\square$

Proof of Lemma 4.
Assume that there are properties $H', H \in \mathcal{H}$ with $H' \subset H$ as well as $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$ and the quotas related to an optimal mechanism satisfy $q_{H'}^* = q_H^*$.
Define

$$\begin{aligned}\mathcal{Q} :=& \{K \in \mathcal{H} : [(K \subseteq H' \vee H \subseteq K) \wedge q_K^* = n] \textit{ and} \\ & \nexists K' \in \mathcal{H} : [K \subset K' \wedge (K' \subseteq H' \vee H \subseteq K') \wedge q_{K'}^* = n]\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{R} :=& \{K \in \mathcal{H} : [(K \subseteq H' \vee H \subseteq K) \wedge q_K^* = 1] \textit{ and} \\ & \nexists K' \in \mathcal{H} : [K' \subset K \wedge (K' \subseteq H' \vee H \subseteq K') \wedge q_{K'}^* = 1]\}.\end{aligned}$$

In the following, I perform a case distinction:
1) Suppose that $\mathcal{Q} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$.
1a) $\exists \bar{Q} \in \mathcal{Q} : H \subseteq \bar{Q}$
By definition of $\mathcal{Q}$, it holds that $q_{\bar{Q}}^* = n$ and, because $H \subseteq \bar{Q}$, it follows that $q_H^* = n$. Thus, the inequality $q_H^* \geq n \cdot \delta_H$ is met.
Moreover, I obtain $H \subseteq Q$ for all $Q \in \mathcal{Q}$ since otherwise, $Q \subseteq H'$ or, equivalently, $Q \subset H$ which would imply $Q \notin \mathcal{Q}$ because $q_H^* = n$.
Take some $Q' \in \mathcal{Q}$ and consider the related set

$$\mathcal{S} := \{K \in \mathcal{H} : Q' \subset K \textit{ and } \nexists K' \in \mathcal{H} : Q' \subset K' \subset K\}$$

of properties.
If $\mathcal{S} = \emptyset$, this means that there are no properties $Q'' \in \mathcal{H}$ such that $Q' \subset Q''$. Consequently, decreasing the quota $q_{Q'}^* = n$ by 1 is feasible and, thus, Lemma 3 implies that the inequality $q_{Q'}^* \leq n \cdot \delta_{Q'} + 1$ holds.
If $\mathcal{S} \neq \emptyset$, it must be that $q_S^* < q_{Q'}^*$ for all $S \in \mathcal{S}$. Suppose not, meaning, there exists some $S \in \mathcal{S}$ such that $q_S^* \geq q_{Q'}^*$. Since, by construction $Q' \subset S$, I obtain $q_S^* = q_{Q'}^* = n$. But, then, it holds that $H \subseteq Q' \subset S$ and $q_S^* = n$ and, thus, it follows that $Q' \notin \mathcal{Q}$ which is the desired contradiction.

Now, the aspect $q_S^* < q_{Q'}^*$ for all $S \in \mathcal{S}$ implies that decreasing the quota $q_{Q'}^*$ by 1 is feasible and, therefore, by Lemma 3, the inequality $q_{Q'}^* \leq n \cdot \delta_{Q'} + 1$ is met.
Hence, in both possible cases, the inequality $q_{Q'}^* \leq n \cdot \delta_{Q'} + 1$ is true. Furthermore,

$$q_{H'}^* = q_H^* = n = q_{Q'}^* \leq n \cdot \delta_{Q'} + 1 \leq n \cdot \delta_{H'} + 1$$

by Assumption 3 since $H' \subset H \subseteq Q'$. Thus, the inequality $q_{H'}^* \leq n \cdot \delta_{H'} + 1$ holds.

1b) $\exists \bar{R} \in \mathcal{R} : \bar{R} \subseteq H'$
By definition of $\mathcal{R}$, it holds $q_{\bar{R}}^* = 1$ and, since $\bar{R} \subseteq H'$, I obtain $q_{H'}^* = 1$. Therefore, the second inequality $q_{H'}^* \leq n \cdot \delta_{H'} + 1$ is true.
Furthermore, I obtain $R \subseteq H'$ for all $R \in \mathcal{R}$ because $H' \subset H \subseteq R$ would imply $R \notin \mathcal{R}$ since $q_{H'}^* = 1$.
Take some $R' \in \mathcal{R}$ and consider the related set

$$\mathcal{J} := \{K \in \mathcal{H} : K \subset R' \; and \; \nexists K' \in \mathcal{H} : K \subset K' \subset R'\}$$

of properties.
If $\mathcal{J} = \emptyset$, this means that there are no properties $R'' \in \mathcal{H}$ satisfying $R'' \subset R'$. Consequently, increasing the quota $q_{R'}^* = 1$ by 1 must be feasible yielding the inequality $q_{R'}^* \geq n \cdot \delta_{R'}$ because of Lemma 3.
If $\mathcal{J} \neq \emptyset$, it must be that $q_J^* > q_{R'}^*$ for all $J \in \mathcal{J}$. To see this point, suppose that the contrary is true, meaning, there exists some $J \in \mathcal{J}$ such that $q_J^* \leq q_{R'}^*$. Thus, because of $J \subset R'$, I obtain $q_J^* = q_{R'}^* = 1$. However, since $J \subset R' \subseteq H'$ and $q_J^* = 1$, the property R' cannot be part of the set $\mathcal{R}$ which contradicts $R' \in \mathcal{R}$.
Employing the aspect $q_J^* > q_{R'}^*$ for all $J \in \mathcal{J}$, I observe that increasing the quota $q_{R'}^*$ by 1 is feasible and, therefore, by Lemma 3, the inequality $q_{R'}^* \geq n \cdot \delta_{R'}$ holds.
Hence, in both possible scenarios, I obtain that the inequality $q_{R'}^* \geq n \cdot \delta_{R'}$ is satisfied. Consequently, since $R' \subseteq H' \subset H$, Assumption 3 implies

$$1 = q_H^* = q_{H'}^* = q_{R'}^* \geq n \cdot \delta_{R'} \geq n \cdot \delta_H.$$

Therefore, the inequality $q_H^* \geq n \cdot \delta_H$ is also true.

1c) $\forall \bar{Q} \in \mathcal{Q} : \bar{Q} \subseteq H'$ and $\forall \bar{R} \in \mathcal{R} : H \subseteq \bar{R}$
Define

$$\mathcal{O} = \{K \in \mathcal{H} : [K \subseteq H' \wedge q_K^* > q_H^*] \; and \; \nexists K' \in \mathcal{H} : [K \subset K' \subseteq H' \wedge q_{K'}^* > q_H^*]\}$$

and

$$\mathcal{P} = \{K \in \mathcal{H} : [H \subseteq K \wedge q_K^* < q_H^*] \, and \, \nexists K' \in \mathcal{H} : [H \subseteq K' \subset K \wedge q_{K'}^* < q_H^*]\}.$$

In particular, $\mathcal{O} \neq \emptyset$ as well as $\mathcal{P} \neq \emptyset$ since $\mathcal{Q} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$.
Take some $O \in \mathcal{O}$. By construction, I have $q_L^* = q_H^*$ for all $L \in \mathcal{H}$ such that $O \subset L \subseteq H'$. Also, since $\mathcal{Q} \neq \emptyset$ and $\bar{Q} \subseteq H'$ for all $\bar{Q} \in \mathcal{Q}$, it must be that $q_H^* \neq n$.
Moreover, there exists some $L' \in \mathcal{H}$ such that $\nexists L'' \in \mathcal{H} : O \subset L'' \subset L' \subseteq H'$. Consider the set

$$\mathcal{I} := \{K \in \mathcal{H} : K \subset L' \, and \, \nexists K' \in \mathcal{H} : K \subset K' \subset L'\}$$

of properties. In particular, I have $\mathcal{I} \neq \emptyset$ because, by construction, $O \in \mathcal{I}$.
If $q_I^* > q_{L'}^*$ for all $I \in \mathcal{I}$, increasing $q_{L'}$ by 1 is feasible and, therefore, by Lemma 3, the inequality $q_{L'}^* \geq n \cdot \delta_{L'}$ holds.
If there exists $I' \in \mathcal{I}$ such that $q_{I'}^* \leq q_{L'}^*$, it follows that $q_{I'}^* = q_{L'}^* = q_H^*$ because $I' \subset L' \subseteq H'$. Now, employ the reasoning that I used to tackle L' and apply it to I'. Again, there are two possibilities: Either increasing $q_{I'}^*$ is feasible or there must be some property $I'' \in \mathcal{H}$ such that $I'' \subset I' \subseteq H'$ satisfying $q_{I''}^* = q_{L'}^* = q_H^*$. If necessary, since there are finitely many properties, repeat this argument for a finite number of times. This yields that there exist either some property $I''' \in \mathcal{H}$ with $I''' \subseteq H'$ such that increasing $q_{I'''}^*$ satisfying $q_{I'''}^* = q_{L'}^* = q_H^*$ is feasible or, otherwise, there must be some property $I'''' \in \mathcal{H}$ with $I'''' \subseteq H'$, $q_{I''''}^* = q_{L'}^* = q_H^*$ and $\nexists I''''' \in \mathcal{H} : I''''' \subset I''''$. However, concerning the latter case, increasing $q_{I''''}^*$ by 1 is feasible.
Therefore, in any scenario, there must be some $\tilde{I} \in \mathcal{H}$ with $\tilde{I} \subseteq L' \subseteq H' \subset H$ such that increasing $q_{\tilde{I}}$ by 1 is feasible and $q_{\tilde{I}}$ satisfies $q_{\tilde{I}}^* = q_{L'}^* = q_H^*$. Employing Lemma 3, this means that the inequality $q_{\tilde{I}}^* \geq n \cdot \delta_{\tilde{I}}$ is met. But, then, since $\tilde{I} \subset H$, Assumption 3 implies

$$q_H^* = q_{\tilde{I}}^* \geq n \cdot \delta_{\tilde{I}} \geq n \cdot \delta_H$$

and, thus, the inequality $q_H^* \geq n \cdot \delta_H$ is met.
Consider some arbitrary $P \in \mathcal{P}$. By construction, I have $q_M^* = q_{H'}^*$ for all $M \in \mathcal{H}$ such that $H \subseteq M \subset P$. Further, since $\mathcal{R} \neq \emptyset$ and $H \subseteq \bar{R}$ for all $\bar{R} \in \mathcal{R}$, it must be that $q_{H'}^* \neq 1$.
Additionally, there exists some $M' \in \mathcal{H}$ such that $\nexists M'' \in \mathcal{H} : H \subseteq M' \subset M'' \subset P$. Focus on the set

$$\mathcal{C} := \{K \in \mathcal{H} : M' \subset K \, and \, \nexists K' \in \mathcal{H} : M' \subset K' \subset K\}$$

of properties. In particular, I have $\mathcal{C} \neq \emptyset$ because, by construction, $P \in \mathcal{C}$.
If $q_C^* < q_{M'}^*$ for all $C \in \mathcal{C}$, decreasing $q_{M'}$ by 1 is feasible and, therefore, due to Lemma 3, the inequality $q_{M'}^* \leq n \cdot \delta_{M'} + 1$ holds.
If there exists $C' \in \mathcal{C}$ such that $q_{C'}^* \geq q_{M'}^*$, it follows that $q_{C'}^* = q_{M'}^* = q_{H'}^*$ because $H \subseteq M' \subset C'$. Now, employ the reasoning that I used to tackle M' and apply it to C'. Again, there are two possibilities: Either decreasing $q_{C'}^*$ is feasible or there must be some property $C'' \in \mathcal{H}$ such that $H \subseteq C' \subset C''$ satisfying $q_{C''}^* = q_{M'}^* = q_{H'}^*$. If necessary, since there are finitely many properties, repeat this argument for a finite number of times. This yields that there exist either some property $C''' \in \mathcal{H}$ with $H \subseteq C'''$ such that increasing $q_{C'''}^*$ satisfying $q_{C'''}^* = q_{M'}^* = q_{H'}^*$ is feasible or, otherwise, there must be some property $C'''' \in \mathcal{H}$ with $H \subseteq C''''$, $q_{C''''}^* = q_{M'}^* = q_{H'}^*$ and $\nexists C''''' \in \mathcal{H} : C'''' \subset C'''''$. However, concerning the latter case, decreasing $q_{C''''}^*$ by 1 is feasible.
Therefore, in any scenario, there must be some $\tilde{C} \in \mathcal{H}$ with $H' \subset H \subseteq M' \subseteq \tilde{C}$ such that decreasing $q_{\tilde{C}}$ by 1 is feasible and $q_{\tilde{C}}$ satisfies $q_{\tilde{C}}^* = q_{M'}^* = q_{H'}^*$. Invoking Lemma 3, this means that the inequality $q_{\tilde{C}}^* \leq n \cdot \delta_{\tilde{C}} + 1$ is met. But, then, since $H' \subset \tilde{C}$, Assumption 3 implies

$$q_{H'}^* = q_{\tilde{C}}^* \leq n \cdot \delta_{\tilde{C}} + 1 \leq n \cdot \delta_{H'} + 1$$

and, thus, the inequality $q_{H'}^* \leq n \cdot \delta_{H'} + 1$ is met.
In conclusion, as desired, despite $q_{H'}^* = q_H^*$, both relevant inequalities are met at q_H^*.
2) If $\mathcal{Q} = \emptyset$ and $\mathcal{R} = \emptyset$, the argument from case 1c applies.
3) Suppose that $\mathcal{Q} \neq \emptyset$, but $\mathcal{R} = \emptyset$.
If $\exists \bar{Q} \in \mathcal{Q} : H \subseteq \bar{Q}$, the reasoning in case 1a yields the desired conclusion; in case $Q \subset H$ for all $Q \in \mathcal{Q}$, take the argument from case 1c.
4) Suppose that $\mathcal{R} \neq \emptyset$, but $\mathcal{Q} = \emptyset$.
In case $H \subset R$ for all $R \in \mathcal{R}$, replicate the steps in case 1c; if $\exists \bar{R} \in \mathcal{Q} : \bar{R} \subseteq H$, the argument from case 1b applies.
Taking all four cases together, this shows that the two relevant inequalities

$$q_H^* \geq n \cdot \delta_H \text{ and}$$
$$q_{H'}^* \leq n \cdot \delta_{H'} + 1$$

determining q_H^* as well as $q_{H'}^*$ hold despite $q_{H'}^* = q_H^*$. Therefore, overall, the claim in the lemma follows. □

Proof of Theorem 2.
It is sufficient to find the quotas related to voting by issues that are part of an optimal mechanism. The existence of a solution is ensured since a bounded function is optimized over a finite set of elements.

Recall, by Theorem 1, the optimal quotas must satisfy

$$K' \subseteq K \Rightarrow q^*_{K'} \geq q^*_K$$

for all $K', K \in \mathcal{H}$.

Consider some arbitrary property $H \in \mathcal{H}$ and the associated quota q^*_H being part of an optimal mechanism. Subsequently, I perform case distinctions.

1a) If $\forall H' \in \mathcal{H}$ with $H' \subset H$ and $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$, it holds that $q^*_{H'} > q^*_H$, part (i) of Lemma 3 yields that the inequality $q^*_H \geq n \cdot \delta_H$ is met.

1b) If there is some $H' \in \mathcal{H}$ with $H' \subset H$ and $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$ such that $q^*_{H'} = q^*_H$, Lemma 4 implies that the inequality $q^*_H \geq n \cdot \delta_H$ holds.

Therefore, no matter the shape of the optimal mechanism, the inequality $q^*_H \geq n \cdot \delta_H$ constitutes a necessary condition for optimality.

2a) If $\forall H' \in \mathcal{H}$ with $H \subset H'$ and $\nexists H'' \in \mathcal{H} : H \subset H'' \subset H'$, it holds $q^*_H > q^*_{H'}$, part (ii) of Lemma 3 yields that the inequality $q^*_H \leq n \cdot \delta_H + 1$ is true.

2b) If there is some $H' \in \mathcal{H}$ with $H \subset H'$ and $\nexists H'' \in \mathcal{H} : H \subset H'' \subset H'$ such that $q^*_H = q^*_{H'}$, Lemma 4 implies that the inequality $q^*_H \leq n \cdot \delta_H + 1$ is satisfied.

Thus, no matter the shape of the optimal mechanism, the inequality $q^*_H \leq n \cdot \delta_H + 1$ is necessary for optimality.

Taking both inequalities together, since the quotas are integer-valued, the quotas q^*_H satisfying these inequalities are, generically, unique and they amount to $q^*_H = \lceil n\delta_H \rceil$ with $H \in \mathcal{H}$. Consequently, it remains to be verified that these quotas are feasible in the sense that they constitute a family of quotas and that they meet the inequalities from Theorem 1. First, for all $H \in \mathcal{H}$, since $0 < \delta_H < 1$, I have that $1 \leq q^*_H = \lceil n\delta_H \rceil \leq n$. Second, observe that, for any $H \in \mathcal{H}$, it holds that $q^*_H + q^*_{H^c} = n+1$. Thus, the discussed quotas constitute a family of quotas. Finally, it is immediate from Assumption 3 that the inequalities from Theorem 1 are met. Consequently, the theorem follows. □

Proof of Corollary 4.

To establish the claim, I show that Assumption 3 is implied by the restrictions appearing in the statement.

Since $\mathbb{E}[V_{H^{\alpha}_{\leq l}}]$ is increasing in l and because $\mathbb{E}[V_{H^{\alpha}_{\geq l}}]$ is decreasing in l, the ratio $\frac{\mathbb{E}[V_{H^{\alpha}_{\leq l}}]}{\mathbb{E}[V_{H^{\alpha}_{\geq l+1}}]}$ is increasing in l. Hence, $\delta_{H^{\alpha}_{\leq l}}$ is decreasing in l. By a similar argument, $\delta_{H^{\beta}_{\leq l}}$ is also decreasing in l. Therefore, for any l, I have $\delta_{H^{\alpha}_{\leq l}} \geq \delta_{H^{\alpha}_{\leq l+1}}$ as well as $\delta_{H^{\beta}_{\leq l}} \geq \delta_{H^{\beta}_{\leq l+1}}$. In words, the part of Assumption 3 that concerns only properties corresponding to the same good is met.

However, properties are also interrelated across goods because any feasible allocation must satisfy $l_\alpha \geq l_\beta$. Specifically, for any l, the relevant interdependence is

$$H^{\alpha}_{\leq l} \subset H^{\beta}_{\leq l}$$

or, equivalently, $H^{\beta}_{\geq l+1} \subset H^{\alpha}_{\geq l+1}$. Since $\mathbb{E}[V_{H^{\alpha}_{\leq l}}] = \mathbb{E}[V_{H^{\beta}_{\leq l}}]$ and $\mathbb{E}[V_{H^{\alpha}_{\geq l}}] = \mathbb{E}[V_{H^{\beta}_{\geq l}}]$, it is immediate that

$$\frac{\mathbb{E}[V_{H^{\alpha}_{\leq l}}]}{\mathbb{E}[V_{H^{\alpha}_{\geq l+1}}]} = \frac{\mathbb{E}[V_{H^{\beta}_{\leq l}}]}{\mathbb{E}[V_{H^{\beta}_{\geq l+1}}]}.$$

Thus, I directly obtain that $\delta_{H^{\alpha}_{\leq l}} = \delta_{H^{\beta}_{\leq l}}$. Therefore, in particular, it holds that $\delta_{H^{\alpha}_{\leq l}} \geq \delta_{H^{\beta}_{\leq l}}$. Overall, I infer that Assumption 3 is met and, hence, Corollary 3 applies. This yields the claim. □

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