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# Information Aggregation in Poisson-Elections

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# Information Aggregation in Poisson-Elections[*]

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**Abstract**

The modern Condorcet jury theorem states that under weak conditions, when voters have common interests, large elections will aggregate information, in any responsive and symmetric equilibrium. Here, we study the performance of large elections with population uncertainty. We find that the modern Condorcet jury theorem holds if and only if the expected number of voters is independent of the state. If the expected number of voters depends on the state, then additional equilibria exist in which information is not aggregated. The main driving force is that, everything else remaining equal, voters are more likely to be pivotal if the population is small. We provide conditions under which the additional equilibria are stable. In addition, we show that the Condorcet jury theorem also fails if abstention is allowed and characterize equilibrium with binary signals. Finally, a state-dependent population can provide additional information to voters, and we characterize how voters can take advantage of this information.

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# Contents

# 1 Introduction

Elections are said to be effective in aggregating information that is dispersed among citizens, for example, about uncertainty regarding future economic prospects, costs and benefits of a public good, or the political ramifications of a trade deal. This belief has been justified by the so-called Condorcet jury theorem (see Ladha (1992)), which asserts that large electorates choose correct outcomes, and its modern form by Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1997, 1998), Wit (1998), Duggan and Martinelli (2001), and others. Precisely, the modern Condorcet jury theorem states that under weak conditions, in a large voting game with common values, *all* responsive and symmetric Nash equilibria aggregate information. The Condorcet jury theorem is one motivation for using elections to make collective choices. In its modern form, it provides "*a rational choice foundation for the claim that majorities invariably 'do better' than individuals*" (at least, for large electorates); see Austen-Smith and Banks (1996).

Most of these earlier contributions assume that the number of voters is deterministic and known. Myerson (1998a) observed that the size of the electorate is often uncertain. Importantly, this uncertainty may not be independent of the underlying state of the world.

In fact, there are plenty of reasons why the state of the world may be correlated with the expected number of voters. For example, in the case of local elections or referenda, awareness about the election taking place may depend on the perceived economic prospects. Similarly, the awareness of elections may depend on the competency or the motivation of the current office holders because of its effect on news coverage and general political engagement. Finally, election turnouts are often subject to manipulation by interested parties who may choose to influence turnout strategically and differently across states; see Ekmekci and Lauermann (2019).

This paper studies whether the modern Condorcet Jury Theorem is robust to population uncertainty. To do so, we use the model by Myerson (1998a): Voters have to choose between two alternatives (two policies). They share common values that depend on an unknown binary state of nature. The number of voters is Poisson-distributed and the mean of the distribution may be state-dependent. Each voter observes a private, conditionally independent signal.

To start, note that any asymmetry in the expected number of voters itself contains additional information about the state of the world; hence, there is one more source of information—in addition to the private signals of the voters—that the electorate could use to aggregate information. However, as we argue below, even though there is more information that could be used, large electorates may fail to aggregate any information—that is, the modern Condorcet Jury Theorem is not robust to population uncertainty.

Because our environment is a common-value environment, we follow McLennan (1998), who showed that in common interest games, every symmetric strategy profile that maximizes social welfare is also a symmetric Nash equilibrium. Our first order of business is to extend McLennan (1998)'s theorem for a deterministic population size to our environment, where participation is Poisson-distributed and the expected number of voters is state-dependent. One notable observation we make is that when the population size is state-dependent, the voting game from the viewpoint of the actual participants fails to be a common interest game. An extension of McLennan (1998)'s result can be obtained only when the social welfare function considered in this program is selected to include the payoffs of the nonparticipating voters in the welfare calculation.

We use our extension of McLennan (1998) to re-derive the main result from Myerson (1998a): If voters have noisy but informative signals about the state of the world, then large electorates in which the population size is state-dependent admit at least one symmetric Nash equilibrium that aggregates information—that is, that selects the correct outcome with a probability close to one.[1] Thus, as is known from Myerson (1998a), one part of the Condorcet Jury theorem survives: large electorates are able to aggregate information.

However, we show that the second part of the Condorcet Jury theorem fails: there are plausible equilibria that fail to aggregate information when the population is state-dependent. In such equilibria, the majority of voters vote as if the state is the one in which there are fewer voters. Therefore, the implemented policy is the one that is preferred in the state in which there are fewer voters. Such equilibria are responsive, and when sufficiently informative signals are possible, these equilibria are stable.

Thus, our main finding is that the modern Condorcet Jury theorem holds with population uncertainty if and only if this uncertainty is independent from the state. Otherwise, if the population is statistically state-dependent, additional responsive

[1] In Myerson (1998a), this result is proven directly, not using the common interest structure.

equilibria exist that fail to aggregate information.

The key force that helps sustain such equilibria is a "**participation curse.**" A vote is more likely to change the outcome of the election, that is, to be *pivotal*, when there are fewer voters, all else being equal. Therefore, a majority of voters—but not all voters—vote as if the state is the one with fewer voters.

We then explore whether strategic abstention can help eliminate such "bad" equilibria. Krishna and Morgan (2012) showed that voluntary voting improves on compulsory voting and induces sincere voting outcomes when there are binary signals. In Feddersen and Pesendorfer (1997), abstention allows uninformed players to participate at a rate that cancels out the effect of partisans who cast their votes in one direction independently of their signals. Hence, one may hope that strategic abstention would help the electorate "undo" the asymmetry in the population size induced by exogenous factors. However, for the binary signal setting by Krishna and Morgan (2012), we show that allowing abstention does not eliminate responsive equilibria that fail to aggregate information.

Somewhat surprisingly, if the asymmetry in the population size is not too large, then in the equilibria that fail to aggregate information, voters with a certain signal are mixing between all three options; they vote with positive probability for both policies and abstain. This behavior is due to a swing voter's *blessing*.[2]

We also explore abstention with state-dependent participation rates when signals are continuous. In this case, if there is no bound on the informativeness of the signals, then there are always equilibria that fail to aggregate information.

Finally, we explore the idea that state-dependent participation provides additional information to voters by considering a setting in which signals are uninformative and thus this is the *only* source of information. For this setting, we show how voters can utilize the additional information and how much additional information can be maximally provided by state-dependent participation.

Information aggregation fails in our setting because, whenever the expected number of voters depends on the state in a non-trivial manner, the probability of being pivotal is different across states (even when voters use constant strategies). We also explore this general topic in a companion paper, Ekmekci and Lauermann (2019), where we study a setting in which the number of voters is state-dependent but de-

[2] To the best of our knowledge, this is the first paper to identify an equilibrium with a swing voter's blessing, rather than a swing voter's curse, in the sense of Feddersen and Pesendorfer (1997).

terministic. The companion paper focuses on how the number of voters may endogenously emerge to give rise to equilibria in which information aggregation fails. Here, we consider a different setting in which the number of voters is not deterministic but Poisson-distributed in each state. For this case, the present paper provides a comprehensive equilibrium analysis. We allow for abstention and characterize equilibrium for voluntary voting with a state-dependent population size.

Further, the literature has identified other circumstances in which information may fail to aggregate. Feddersen and Pesendorfer (1997) show such a failure in an extension (Section 6 of their paper) when the aggregate distribution of preferences remains uncertain conditional on the realized state. Mandler (2012) demonstrates a similar failure if the aggregate distribution of signals remains uncertain. In these settings, the effective state is multi-dimensional. Intuitively, this implies an invertibility problem from the relevant order statistic of the vote shares to payoff-relevant states. A similar problem is identified by Bhattacharya (2013), who observes the necessity of preference monotonicity for information aggregation; see also Bhattacharya (2018) and Ali, Mihm, and Siga (2017). Barelli, Bhattacharya, and Siga (2018) study which conditions on the joint distributions of states and voters' signals make information aggregation feasible. Gul and Pesendorfer (2009) show that information aggregation fails when there is policy uncertainty. In our setting, conditional on the state, there is no aggregate uncertainty (in the sense that the mean of the Poisson distribution is known), preferences over policies are monotone in the state, and there is no policy uncertainty.

## 2 Model

The model setup follows Myerson (1998a). Voters have to decide between two policies, $A$ and $B$. There are two states, $\alpha$ and $\beta$, with prior probability

$$\pi = \Pr\{\alpha\},$$

where $0 < \pi < 1$ and $\Pr\{\beta\} = 1 - \pi$. Voters have common values: Each voter receives a payoff of 1 if the policy matches the state, and a payoff of 0 otherwise.

However, voters do not know the realized state. Instead, voters observe noisy signals $x \in [\underline{x}, \bar{x}]$.[3] Conditional on the state, the signals are independent and identically distributed. The c.d.f. of the signal distribution is $G(\cdot|\omega)$. The distribution is atomless and admits a density. Without loss of generality, signals are ordered so the weak monotone likelihood ratio property (MLRP) holds,

$$\frac{g(x|\alpha)}{g(x|\beta)} \text{ is weakly decreasing in } x.$$

In addition, $g(x|\omega) > 0$ for all $x \in (\underline{x}, \bar{x})$. This, together with $G$ being atomless, rules out that voters receive perfectly revealing signals with positive probability. Finally, for technical convenience, we assume right continuity of $\frac{g(\cdot|\alpha)}{g(\cdot|\beta)}$ on $(\underline{x}, \bar{x})$, defining $\lim_{x \to \underline{x}} \frac{g(x|\alpha)}{g(x|\beta)} =: \frac{g(\underline{x}|\alpha)}{g(\underline{x}|\beta)} \in \mathbb{R} \cup \{\infty\}$ and $\lim_{x \to \bar{x}} \frac{g(x|\alpha)}{g(x|\beta)} =: \frac{g(\bar{x}|\alpha)}{g(\bar{x}|\beta)} \in \mathbb{R}$.[4] Signals contain some information, meaning, $\frac{g(\underline{x}|\alpha)}{g(\underline{x}|\beta)} > 1 > \frac{g(\bar{x}|\alpha)}{g(\bar{x}|\beta)}$. Two important special cases are boundedly informative signals,

$$\infty > \frac{g(\underline{x}|\alpha)}{g(\underline{x}|\beta)} > 1 > \frac{g(\bar{x}|\alpha)}{g(\bar{x}|\beta)} > 0,$$

and unboundedly informative signals,

$$\infty = \frac{g(\underline{x}|\alpha)}{g(\underline{x}|\beta)} \text{ and } \frac{g(\bar{x}|\alpha)}{g(\bar{x}|\beta)} = 0.$$

The number of voters is Poisson-distributed in each state, with an expected number of $n_\alpha = n$ and $n_\beta = \theta n$; so, the probability that there are $t$ voters in state $\omega$ is

$$\Pr\{t|\omega\} = \frac{(n_\omega)^t e^{-n_\omega}}{t!}.$$

The policy is decided by simple majority rule among submitted votes. If there is a tie, then a fair coin flip decides. Abstention is not possible for now.

We consider symmetric and pure voting strategies. Given the Poisson setup, symmetry is without loss of generality; see Myerson (1998a). A voting strategy is a function $a : [\underline{x}, \bar{x}] \to [0, 1]$, with $a(x)$ being the probability to vote for $A$.

Let $U(x, W; a, n)$ be the expected utility for a voter having signal $x$ who votes for $W \in \{A, B\}$, given that all other voters use strategy $a$ and the expected number of voters is $(n, \theta n)$ in states $\alpha$ and $\beta$, respectively. We often omit $a$ and $n$.

[3] We allow $\underline{x} = -\infty$ and $\bar{x} = +\infty$.

[4] Because we allow for a discontinuous density, discrete signals are a special case of our model.

We study voting strategies that form a (Bayesian) Nash equilibrium. A voting strategy $a$ is a Nash equilibrium if and only if $U(x, A; a, n) > U(x, B; a, n)$ implies $a(x) = 1$ and $U(x, A; a, n) < U(x, B; a, n)$ implies $a(x) = 0$.

To characterize equilibrium, the following is useful. The likelihood ratio of the two states conditional on having signal $x$ and participating is[5]

$$\frac{\Pr(\alpha|x)}{\Pr(\beta|x)} = \frac{\pi}{1-\pi}\frac{n}{\theta n}\frac{g(x|\alpha)}{g(x|\beta)}. \tag{1}$$

Let $T$ denote the event in which the number of $A$ and $B$ votes is the same, $T-1$ the event in which there is one less $A$ vote than $B$ votes, and $T+1$ the event in which there is one more $A$ vote. Then, the difference $U(x, A; a, n) - U(x, B; a, n)$ is equal to

$$\begin{aligned} &\Pr(\alpha|x)\left(\Pr[T-1|\alpha]\frac{1}{2} + \Pr[T|\alpha] + \Pr[T+1|\alpha]\frac{1}{2}\right) \\ &-\Pr(\beta|x)\left(\Pr[T-1|\beta]\frac{1}{2} + \Pr[T|\beta] + \Pr[T+1|\beta]\frac{1}{2}\right). \end{aligned} \tag{2}$$

Voting $A$ versus voting $B$ changes the payoffs only in the events $T-1$, $T$, and $T+1$. In the first event, voting $A$ rather than $B$ increases the probability of $A$ winning from 0 to 1/2; in the second event, it increases the probability from 0 to 1; and in the third event, it increases the probability from 1/2 to 1.

The probability that the decision to vote $A$ versus $B$ turns out to be *pivotal* is[6]

$$\Pr(Piv_0|\omega) = \frac{1}{2}\Pr[T-1|\omega] + \Pr[T|\omega] + \frac{1}{2}\Pr[T+1|\omega].$$

Then, it is evident from (1) and (2) that voting for $A$ is a best response for a voter having signal $x$ if

$$\gamma(x; a, n) := \frac{\pi}{1-\pi}\frac{1}{\theta}\frac{g(x|\alpha)}{g(x|\beta)}\frac{\Pr(Piv_0|\alpha)}{\Pr(Piv_0|\beta)} \geq 1,$$

where $\gamma$ denotes the *critical likelihood ratio*.

A strategy $a$ is a cutoff strategy if for some $\hat{x}$, $a(x) = 1$ if $x > \hat{x}$ and $a(x) = 0$

[5] See also Milchtaich (2004) for a discussion of updating in Poisson games.

[6] We use the subscript 0 to distinguish the pivotal event with compulsory voting from the pivotal event with voluntary voting that is introduced later.

if $x < \hat{x}$. We state without proof that cutoff strategies are without loss of generality. This is immediate from $\gamma$ being nonincreasing in $x$.

**Lemma 1.** *If a strategy forms a Nash equilibrium, it is equivalent to a cutoff strategy.*[7]

Our generic notation is $\hat{x}$ for the strategy: "Vote for $A$ if $x \in (\underline{x}, \hat{x})$, and vote for $B$ if $x \in (\hat{x}, \bar{x})$." Abusing notation, let $\gamma(x; \hat{x}, n)$ be the critical likelihood ratio given cutoff $\hat{x}$. If $\gamma(t; t, n)$ is continuous in $t$, then $\hat{x} \in (\underline{x}, \bar{x})$ is an interior Nash equilibrium if and only if

$$\gamma(\hat{x}; \hat{x}, n) = 1.$$

# 3 Welfare Maximization and Nash Equilibria

McLennan (1998) observed that for common interest games, welfare maximizing strategy profiles are also Nash equilibria. This result extends almost immediately to Poisson games with an infinite population. However, we first need to clarify the welfare function to be maximized, since there are two natural candidates.

Let $u(i, \hat{x}|\omega)$ be the expected payoff of a voter conditional on state $\omega$ and $i$ voters being present who vote according to $\hat{x}$. The *expected surplus* of the participating voters given a voting profile $\hat{x}$ is

$$E^{sur}[u; \hat{x}] = \sum_{\omega \in \{\alpha, \beta\}} \Pr\{\omega\} \sum_{i=0}^{\infty} i \Pr\{\tilde{n} = i|\omega\} u(i, \hat{x}|\omega).$$

The *expected payoff of a representative agent* given $\hat{x}$ is

$$E^{rep}[u; \hat{x}] = \sum_{\omega \in \{\alpha, \beta\}} \Pr\{\omega\} \sum_{i=0}^{\infty} \Pr\{\tilde{n} = i|\omega\} u(i, \hat{x}|\omega).$$

The expected surplus *weighs* the payoffs in the two states by the population size; however, the expected payoff of a representative agent does not. The first welfare criterion is maybe more appropriate if those who do not participate receive payoff 0 (and/or do not exist). The second welfare criterion is maybe more appropriate if the

[7] If the likelihood ratio is constant on some interval, every voting strategy is equivalent to a voting strategy in cutoffs (because we can reorder votes on that interval). If the likelihood ratio is strictly increasing, every voting strategy is in cutoffs.

non-participating agents are present and receive the same payoff $u(i, \hat{x}|\omega)$, but just do not vote.

Of these two criteria, only the second one corresponds to a game of common interest.

**Lemma 2.** *Any voting strategy $x^*$ that maximizes the expected payoff of a representative agent $E^{rep}[u; \cdot]$ is a Nash equilibrium.*

One may have expected Nash equilibrium to be tilted towards the state in which more voters are present, since the participating voters consider this state to be relatively more likely.

# 4 Compulsory Voting

Lemma 2 implies that, for all $\theta$, equilibria exist that aggregate information. Myerson (1998a) provides a direct proof for this result.

**Theorem 1.** [Myerson (1998a)] *For all $\theta \in (0, \infty)$, there exists a sequence of equilibria $\{\hat{x}^n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} E^{rep}[u|\hat{x}^n] = 1$.*

For this and subsequent proofs, it will be useful to define the median signals

$$x_\alpha : G(x_\alpha|\alpha) = 1/2 \text{ and } x_\beta : G(x_\beta|\beta) = 1/2.$$

Since signals contain information, $x_\alpha < x_\beta$.

**Proof.** Take any $x' \in (x_\alpha, x_\beta)$. Let $\hat{x}^n$ maximize the welfare of the representative agent given $n$. Then, the proposition follows from

$$1 = \lim_{n\to\infty} E^{rep}[u; x'] \leq \lim_{n\to\infty} E^{rep}[u; \hat{x}^n] \leq 1.$$

The first equality follows from the weak law of large numbers, the first inequality from the choice of $\hat{x}^n$, and the last inequality from the definition of $E^{rep}$ and $u(i, \hat{x}|\omega) \in [0, 1]$. □

Now, we study whether all equilibria aggregate information in large elections: How reliably do elections enable voters to make good choices? Is the Condorcet jury theorem robust to population uncertainty? If not, what drives the difference?

## 4.1 Auxiliary Results

We use the following simple extension of the intermediate value theorem. This lemma deals with the problem that the signal likelihood ratio $g(x|\alpha)/g(x|\beta)$ is monotone decreasing but may be discontinuous, implying that the critical likelihood ratio $\gamma(t;t,n)$ may be discontinuous in $t$ as well.

**Lemma 3.** *Suppose there are two points $a$ and $b$, such that $\gamma(a;a,n) \leq 1 \leq \gamma(b;b,n)$ (either $a < b$ or $a > b$). Then, there exists a Nash equilibrium with cutoff $\hat{x}$ with $\min\{a,b\} \leq \hat{x} \leq \max\{a,b\}$. If $a < b$, then there exists a Nash equilibrium cutoff with $\gamma(\hat{x};\hat{x},n) = 1$.*

We also use the following approximation of the critical likelihood ratio.

**Lemma 4.** *If $\lim_{n\to\infty} \hat{x}^n \in (\underline{x}, \bar{x})$, then*

$$\lim_{n\to\infty} \frac{\Pr[Piv_0|\alpha;\hat{x}^n,n]}{\Pr[Piv_0|\beta;\hat{x}^n,n]}$$
$$= \begin{cases} \infty & \textit{if} \ \lim\limits_{n\to\infty} 2\sqrt{G(\hat{x}^n|\alpha)(1-G(\hat{x}^n|\alpha))} - 1 > \lim\limits_{n\to\infty} \theta\left(2\sqrt{G(\hat{x}^n|\beta)(1-G(\hat{x}^n|\beta))} - 1\right), \\ 0 & \textit{if} \ \lim\limits_{n\to\infty} 2\sqrt{G(\hat{x}^n|\alpha)(1-G(\hat{x}^n|\alpha))} - 1 < \lim\limits_{n\to\infty} \theta\left(2\sqrt{G(\hat{x}^n|\beta)(1-G(\hat{x}^n|\beta))} - 1\right). \end{cases}$$

*For $\theta = 1$, this simplifies to*

$$\lim_{n\to\infty} \frac{\Pr[Piv_0|\alpha;\hat{x}^n,n]}{\Pr[Piv_0|\beta;\hat{x}^n,n]} = \begin{cases} \infty & \textit{if} \ \lim\limits_{n\to\infty} \left|G(\hat{x}^n|\alpha) - \frac{1}{2}\right| < \left|G(\hat{x}^n|\beta) - \frac{1}{2}\right|, \\ 0 & \textit{if} \ \lim\limits_{n\to\infty} \left|G(\hat{x}^n|\alpha) - \frac{1}{2}\right| > \left|G(\hat{x}^n|\beta) - \frac{1}{2}\right|. \end{cases}$$

This lemma follows from standard approximations to pivot probabilities; see Krishna and Morgan (2012). The proof of the lemma is provided in a separate section in the appendix, where we re-state these general approximations and this and other lemmas for our purposes. The case $\theta = 1$ is particularly intuitive. Roughly, the state in which the election is closer to being tied in expectation becomes arbitrarily more likely conditional on the election being actually tied.

## 4.2 The Modern Condorcet Jury Theorem

The modern Condorcet theorem states that in large elections, all "reasonable" (symmetric and responsive) equilibria aggregate information. For a deterministic number of voters, this result has been proven by Feddersen and Pesendorfer (1998) and Wit (1998) for binary signals and by Duggan and Martinelli (2001) for a continuum of signals. Krishna and Morgan (2012) prove it for Poisson elections when the expected number of voters is independent of the state, $\theta = 1$, and signals are binary. Here, we extend this result to a continuum of signals.

As in Duggan and Martinelli (2001) and Krishna and Morgan (2012), we assume that

$$\frac{\pi}{1-\pi} \frac{g\left(\underline{x}|\alpha\right)}{g\left(\underline{x}|\beta\right)} > 1 > \frac{\pi}{1-\pi} \frac{g\left(\bar{x}|\alpha\right)}{g\left(\bar{x}|\beta\right)}. \tag{3}$$

With this assumption, based on their own signal alone, a voter with the strongest signal for $\alpha$ would prefer policy $A$ and a voter with the strongest signal for $\beta$ would prefer policy $B$. The assumption holds if the prior is uniform. The assumption also holds if signals are sufficiently informative.

**Theorem 2.** Condorcet Jury Theorem. *All equilibria aggregate information if $\theta = 1$ (there is no imbalance) and (3) holds (signals are sufficiently informative): For every sequence of Nash equilibrium cutoffs $\{\hat{x}^n\}_{n=1}^{\infty}$ with $\underline{x} \leq \hat{x}^n \leq \bar{x}$ for all $n$, $\lim_{n\to\infty} E^{rep}\left[u|\hat{x}^n\right] = 1$.*

The proof of the theorem is by contradiction. The main step is to show that for any sequence of Nash equilibrium cutoffs we have

$$x_\alpha < \lim_{n\to\infty} \hat{x}^n < x_\beta, \tag{4}$$

which then implies that information aggregates by the weak law of large numbers. To show (4), we verify that for any sequence of cutoffs $\hat{x}^n$,

$$\lim_{n\to\infty} \frac{\Pr\left[Piv_0|\alpha;\hat{x}^n,n\right]}{\Pr\left[Piv_0|\beta;\hat{x}^n,n\right]} = \begin{cases} \infty & \text{if} \quad \underline{x} < \lim_{n\to\infty}\hat{x} \leq x_\alpha, \\ 0 & \text{if} \quad x_\beta \leq \lim_{n\to\infty}\hat{x}^n < \bar{x}. \end{cases} \tag{5}$$

This rules out that such sequences are Nash equilibria, of course. A somewhat different argument deals with the remaining cases $\hat{x}^n \to \underline{x}$ or $\hat{x}^n \to \bar{x}$.

Equation (5) follows from the second part of Lemma 4. In particular, if $\underline{x} < \lim_{n\to\infty}\hat{x}^n \leq x_\alpha$, then with $x^0 = \lim_{n\to\infty}\hat{x}^n$, the MLRP implies that $0 < G\left(x^0|\beta\right) < G\left(x^0|\alpha\right) \leq 1/2$. Therefore, the election is closer to being tied in state $\alpha$—and conditional on being pivotal, a voter becomes almost certain that it is this state. Conversely, if $\lim_{n\to\infty}\hat{x}^n = x^0$ is interior, then Lemma 4 requires that the election is equally close to being tied in both states, meaning, $\left|G\left(x^0|\alpha\right)-\frac{1}{2}\right| = \left|G\left(x^0|\beta\right)-\frac{1}{2}\right|$, as otherwise the relative likelihood of being pivotal explodes. However, for the election to be equally close to being tied, it is easy to see that it must be that $x_\alpha < x^0 < x_\beta$; hence, information is aggregated.

For deterministic elections, the analogous result holds only for symmetric and responsive equilibria. For Poisson elections, the symmetry assumption is without loss of generality, and there are no nonresponsive equilibria if (3) holds; hence, the result is stronger. If $\theta = 1$, *all* Nash equilibria aggregate information.[8]

## 4.3 The Failure of the Modern Condorcet Jury Theorem

We now show that the modern Condorcet theorem fails if $\theta \neq 1$.

**Theorem 3.** *Consider a sequence of voting games in which the expected number of participants is $(n, \theta n)$ in states $\alpha$ and $\beta$, respectively, and $n \to \infty$.*

- *If $\theta < 1$, then there is a sequence of interior Nash equilibrium cutoffs $\{\hat{x}^n\}_{n=1}^\infty$ with $\hat{x}^n \in (\underline{x}, x_\alpha)$ for large $n$, such that $B$ wins in both states with probability converging to one.*
- *If $\theta > 1$, then there is a sequence of interior Nash equilibrium cutoffs $\{\hat{x}^n\}_{n=1}^\infty$ with $\hat{x}^n \in (x_\beta, \bar{x})$ for large $n$, such that $A$ wins in both states with probability converging to one.*

The proof is provided in the appendix. The main observations are that

$$\lim_{n\to\infty} \gamma\left(x_\alpha; x_\alpha, n\right) = \infty, \tag{6}$$

and that for any $x_R$ sufficiently small, with $\underline{x} < x_R < x_\alpha$,

$$\lim_{n\to\infty} \gamma\left(x_R; x_R, n\right) = 0. \tag{7}$$

[8] We emphasize the importance of (3) for these strong conclusions. If the condition fails, then it can be easily seen that nonresponsive equilibria exist. Moreover, one can also show that in this case there are responsive equilibrium sequences that do not aggregate information.

Hence, the intermediate value theorem from Lemma 3 implies that, for all $n$ large enough, there exists some $\hat{x}^n \in (x_R, x_\alpha)$ such that $\gamma(\hat{x}^n; \hat{x}^n, n) = 1$. We verify that $\lim_{n\to\infty} \hat{x}^n < x_\alpha$, and, hence, $B$ wins with probability converging to 1 in both states.

Again, the critical observations (6) and (7) follow from Lemma 4. For (6), note that for a cutoff $\hat{x} = x_\alpha$, the election is tied in state $\alpha$, while $B$ wins with certainty in state $\beta$. Thus, it is intuitive that conditional on being pivotal, a voter becomes certain that the state is $\alpha$. Now, consider some $x_R$ close to $\underline{x}$. Then, with $x_R$ small enough, $B$ will win in both states. Moreover, because $\theta < 1$, the number of $B$ votes is actually larger in state $\alpha$. Now, the expected vote difference in state $\alpha$ is

$$n\left(\left(1 - G\left(x_R|\alpha\right)\right) - G\left(x_R|\alpha\right)\right),$$

while the expected vote difference in state $\beta$ is

$$n\theta\left(\left(1 - G\left(x_R|\beta\right)\right) - G\left(x_R|\beta\right)\right).$$

For $x_R$ close enough to $\underline{x}$, the expected vote difference is larger in state $\alpha$. Because of the smaller margin of victory, it is therefore intuitive that $B$ is less likely to win in state $\beta$ than in state $\alpha$. Thus, conditional on being pivotal, the state is almost certainly $\beta$ for $n$ large, explaining (7).

The fact that there are fewer voters in state $\beta$ when $\theta < 1$ is at the heart of the aggregation failure. Because the number is smaller, a voter is more likely to be pivotal in that state, and, given that sophisticated voters condition on being pivotal, they tend to support $B$, even if their signals are strongly in favor of $A$.

## 4.4 Stability

If signals are unboundedly informative and $\frac{g(\underline{x}|\alpha)}{g(\underline{x}|\beta)} = \infty$, it is immediate that for any fixed $n$, there exists $x_L^n$ with $\underline{x} < x_L^n < x_R$ such that[9]

$$\gamma\left(x_L^n; x_L^n, n\right) > 1. \tag{8}$$

[9]This holds because, for any fixed $n$, the ratio $\frac{\Pr[Piv_0|\alpha; x, n]}{\Pr[Piv_0|\beta; x, n]}$ is bounded.

Together with (7) and Lemma 3, (8) implies that there exists some equilibrium with $\hat{x}_s^n \in (x_L^n, x_R)$ for all $n$ large enough. Thus, when signals are unboundedly informative, there are at least two interior equilibria in which information aggregation fails, this one and the previous one with $\hat{x}^n \in (x_R, x_\alpha)$.

This argument also implies that when signals are unboundedly informative, there is one "stable" equilibrium that fails to aggregate information by the following argument. From (7) and (8), for all $n$ large enough, $\tilde{\gamma}(x) = \gamma(x; x, n)$ cuts 1 from above, at least once at some point $\hat{x}_s < x_R$. If $\hat{x}_s$ is the only equilibrium cutoff in some neighborhood, then $\hat{x}_s$ is an equilibrium cutoff that is responsive, and this equilibrium is an expectationally stable equilibrium in the sense of Fey (1997). Intuitively, this equilibrium has the property that if this is the outcome of a dynamic best-response iteration and if the process starts in a neighborhood of the equilibrium cutoff, then the process will eventually converge to the cutoff.[10]

**Unstable Equilibria.** Applying the same argument to the case with boundedly informative signals, it follows from (6) and (7) that there exists at least one point $\hat{x}_r \in (x_L, x_\alpha)$ at which $\gamma$ crosses 1 from below. Thus, when signals are boundedly informative, there exists at least one equilibrium cutoff $\hat{x}_r$ that is unstable.[11]

**Nonresponsive Equilibria.** As observed by Myerson (1998a), for all $\theta \neq 1$, if signals are boundedly informative, then there are also nonresponsive equilibria when $n$ is large enough. Consider $\theta < 1$ and suppose $\hat{x} = \underline{x}$, so that all voters support $B$. In this case, the relative probability of being pivotal is

$$\frac{\Pr\left[Piv_0|\alpha; \underline{x}, n\right]}{\Pr\left[Piv_0|\beta; \underline{x}, n\right]} = \frac{e^{-n}(1+n)}{e^{-\theta n}(1+\theta n)} \approx e^{-n(1-\theta)}\frac{1}{\theta} \rightarrow_{n\rightarrow\infty} 0.$$

Thus, given that signals are boundedly informative, it is a best response for a voter to vote for $B$ independently of her signal, for $n$ large enough. Nonresponsive equilibria are stable.

[10] It may be that for some $\varepsilon > 0$ and $\tilde{x}_0 < \tilde{x}_1$, we have $\tilde{\gamma}(x) > 1$ for $x \in (\tilde{x}_0 - \varepsilon, \tilde{x}_0)$, $\tilde{\gamma}(x) = 1$ for $x \in [\tilde{x}_0, \tilde{x}_1]$ and $\tilde{\gamma}(x) < 1$ for $x \in (\tilde{x}_1, \tilde{x}_1 + \varepsilon)$. We may call such a set of equilibria "pseudo-stable," with singletons being special cases.

[11] On the other hand, even though there must be some equilibrium that is not stable, we cannot rule out that there are stable equilibria. In fact, our previous discussion implies that it is easily possible to construct stable equilibria for some boundedly informative signals.

# 5 Voluntary Voting (Abstention)

We now consider the possibility of abstention or "voluntary voting." Feddersen and Pesendorfer (1996) noted that voters may have a strict incentive to abstain because of the "swing voter's curse." Moreover, the possibility of abstention necessarily increases the expected payoff of a representative agent in the best equilibrium relative to compulsory voting. This is an immediate implication of Lemma 2; see also Krishna and Morgan (2012).

In addition, Krishna and Morgan (2012) observe that with abstention and a binary signal, there is no longer a conflict between voting strategically and voting sincerely, which is often present with compulsory voting even when $\theta = 1$.[12] Thus, abstention may help eliminate the equilibria that we identified earlier since these equilibria relied on voters with a strong signal towards state $\alpha$ to nevertheless vote $B$. Thus, we now ask whether abstention may help eliminate the bad equilibria.

With abstention, our generic notation is $(y, z)$ for the following strategy: "Vote for $A$ if $x \in (\underline{x}, y)$, abstain if $x \in (y, z)$, and vote for $B$ if $x \in (z, \bar{x})$." We call a voting strategy $(y, z)$ *nonresponsive* if either $z = \underline{x}$ or $y = \bar{x}$, so, either all participants vote $B$ or all of them vote $A$. Otherwise, an equilibrium is *responsive*.

Again, Lemma 2 implies that information aggregation is possible in some equilibria. We state this without proof.

**Theorem 4.** *Suppose voting is voluntary. Consider a sequence of voting games in which the expected number of participants is $(n, \theta n)$ in states $\alpha$ and $\beta$, respectively, and $n \to \infty$. For all $\theta > 0$, there exists a sequence of Nash equilibria that aggregates information.*

## 5.1 Voluntary Voting: Unboundedly Informative Signals

First, we study the case with unboundedly informative signals. As before, we study whether every sequence of equilibria aggregate information.

**Theorem 5.** [Unboundedly Informative Signals.] *Suppose voting is voluntary and signals are unboundedly informative. Consider a sequence of voting games in which*

[12] Moreover, if voting is costly, abstention enables a reduction in overall voting costs.

*the expected number of participants is $(n, \theta n)$ in states $\alpha$ and $\beta$, respectively, and $n \to \infty$.*

1. *If $\theta < 1$, then there is a sequence of responsive Nash equilibria such that $B$ wins in both states with probability converging to* $1$*.*
2. *If $\theta > 1$, then there is a sequence of responsive Nash equilibria such that $A$ wins in both states with probability converging to* $1$*.*

The proof is provided in the appendix. The basic idea is this: Consider an auxiliary game $\Gamma(x_R, n)$ in which voters with signals $x \geq x_R > \underline{x}$ must vote for $B$, but which otherwise remains unchanged. By a standard argument, this game has an equilibrium. Then, for $n$ large enough, this equilibrium is also shown to be an equilibrium of the original game if $x_R$ is small enough, in particular, if $x_R < x_\alpha$. The critical argument for this proof is that for $x_R$ small enough, given any admissible strategy profile with $y \leq z \leq x_R$, the probability of state $\beta$ conditional on being pivotal converges to one.[13] Thus, voters with signals around $x_R > \underline{x}$ will optimally vote for $B$; hence, this restriction does not bind. Thus, this is an equilibrium of the original game. Moreover, the equilibrium is responsive: since signals are unboundedly informative, $\frac{g(\underline{x}|\alpha)}{g(\underline{x}|\beta)} = \infty$. Therefore, for every given $n$, voters will optimally vote $A$ for some sufficiently small signal.

## 5.2 Voluntary Voting: Binary Signals

When signals are boundedly informative, the construction via an auxiliary game does not work. If we simply search for an equilibrium with cutoffs $y \leq z \leq x_R$, that equilibrium may turn out to be a nonresponsive equilibrium, with $y = z = \underline{x}$.[14] We also cannot use arguments using the intermediate value theorem, as in the compulsory voting case, since we are now looking for a two-dimensional strategy profile. Instead, in the Appendix in Section B, we introduce a new result, Lemma 10, which generalizes the intermediate value theorem in a certain sense to two dimensions. With this result, we can study the case with binary signals, which was also analyzed in Krishna and Morgan (2012). Signals are binary if there is some $x_B$, such that the signal likelihood

[13] Note that this argument is very similar to the one used in (7) for compulsory voting.

[14] Relatedly, we cannot use a fixed point theorem on a restricted strategy space because we cannot ensure that points on the boundary of the space are mapped into the interior.

ratios are constant below and above that cutoff, with $\frac{g(x|\alpha)}{g(x|\beta)} = c_\alpha \in (1, \infty)$ for all $x \in [\underline{x}, x_B)$ and $\frac{g(x|\alpha)}{g(x|\beta)} = c_\beta \in (0, 1)$ for all $x \in (x_B, \bar{x}]$. Following Krishna and Morgan (2012), we also assume $\pi = 1/2$, so that (3) holds.

**Theorem 6.** [Binary Signals.] *Suppose voting is voluntary, signals are binary, and $\pi = 1/2$. Consider a sequence of voting games in which the expected number of participants is $(n, \theta n)$ in states $\alpha$ and $\beta$, respectively, and $n \to \infty$.*

1. [Krishna and Morgan, 2012]. *If $\theta = 1$, then all sequences of Nash equilibria aggregate information.*

2. *If $\theta < 1$ and $\frac{1-G(x_B|\alpha)}{1-G(x_B|\beta)} \neq \theta$, there is a sequence of responsive Nash equilibria such that $B$ wins in both states with probability converging to* 1.

3. *If $\theta > 1$ and $\frac{G(x_B|\alpha)}{G(x_B|\beta)} \neq \theta$, there is a sequence of responsive Nash equilibria such that $A$ wins in both states with probability converging to* 1.

In the theorem, $\frac{1-G(x_B|\alpha)}{1-G(x_B|\beta)} \neq \theta$ is a genericity requirement.[15] To the best of our knowledge, the equilibrium construction via the new Lemma 10 is new. This construction may be useful in other contexts as well and may constitute an independent technical contribution of the paper.

**Mixed Equilibrium.** With binary signals, the equilibrium turns out to be typically mixed. Consider the case when $\theta < 1$ but not too small such that $\frac{1-G(x_B|\alpha)}{1-G(x_B|\beta)} < \theta$. In this case, equilibrium may have somewhat surprising properties. In particular, it will be the case that a voter with a signal $x < x_B$ (which is a signal indicative of state $\alpha$) is mixing between voting for $A$, voting for $B$, and abstaining, choosing each of the three actions with a strictly positive probability. A necessary condition for this is that there is no swing voters curse. Note that, in contrast to Krishna and Morgan (2012), despite voting being voluntary, the voting strategy is not sincere,[16] because voters with signal $x < x_B$ are voting against their signal.

[15] This condition implies that a voter's posterior conditional on participation and a high signal is not exactly equal to the prior.

[16] We say that voting is *sincere* if $x \leq y$ implies $\frac{\pi}{1-\pi}\frac{1}{\theta}\frac{g(x|\alpha)}{g(x|\beta)} \geq 1$ and $x \geq z$ implies $\frac{\pi}{1-\pi}\frac{1}{\theta}\frac{g(x|\alpha)}{g(x|\beta)} \leq 1$ (if the strategy profile requires voting for either $A$ or $B$ for some signal, then given that signal, voting for $A$ and $B$ is also myopically optimal, that is, if a voter were to be the sole voter and used only the information contained in her signal and participating.)

**General Signal.** We conjecture that Theorem 6 can be extended to all boundedly informative signals. We have verified this for the case in which the signal likelihood ratio $\frac{g(x|\alpha)}{g(x|\beta)}$ is continuous. However, the proof becomes much more involved and we need to strengthen Lemma 10. To move even further to general discontinuous signals would require a result analogous to Lemma 10 for certain sets of functions that are discontinuous but monotone along one of the two dimensions. While we believe this to be true, we have not been able to establish it.

**Compulsory versus Voluntary Voting.** In the beginning of this section, we noted that abstention increases the expected payoff of a representative agent in the best equilibrium relative to compulsory voting by Lemma 2; see also Krishna and Morgan (2012). However, this is not necessarily the case relative to expected surplus. While we have not found a specific example of this, we believe that there may be parameters for which the expected surplus with compulsory voting is higher than that with voluntary voting. Further exploring the costs and benefits of voluntary voting will be an interesting avenue for further research.

# 6 Learning from the Population-Size

As mentioned in the introduction, any asymmetry in the expected number of voters provides information about the state of the world that can be utilized by the electorate. Here, we focus on this aspect by considering the special case in which the voters have no further private signals at all and, therefore, the population asymmetry is the only source of information.

Nevertheless, a substantial amount of information remains: If the expected population size is large and state-dependent, an outsider who observes the realized population size would know the state with arbitrarily high precision. We show that even though voters do not observe the realized population size, the voters can still use the information contained in the population size to increase the probability of the correct choice.

Consider the following example in which the number of voters is deterministic in each state (rather than Poisson-distributed). Specifically, in state $\alpha$, there are $n$ voters, and in state $\beta$ there is only 1 voter. Moreover, assume a uniform prior, that is, $\pi = \frac{1}{2}$. In such an environment, a voter updates her belief about the state conditional on participating and her posterior belief that the state is $\alpha$ is $\frac{n}{n+1}$. Then, the naive (sincere) voting strategy would be to vote for $A$; hence, $a$ would be selected regardless of the state. Therefore, from an ex-ante perspective, the correct outcome is selected with probability 1/2.

Now, consider the following alternative strategy profile in which a voter votes $A$ with a probability $\frac{1}{2}+\varepsilon$, for some $\varepsilon>0$. If $n$ is sufficiently large, then, by the law of large numbers, $A$ wins in state $\alpha$ with a probability close to 1. On the other hand, in state $\beta$, because there is only 1 voter, $B$ wins with probability $\frac{1}{2}-\varepsilon$. Therefore, this strategy profile selects the correct choice with a probability close to 3/4 for $\varepsilon$ small and $n$ large. Note that, without an asymmetric population and $n$ voters in *both* states, *any* voting strategy leads to a correct choice with probability 1/2.

The key feature of the strategy profile that improves the probability of the correct choice given the state-dependent participation is that the variance of the outcome is larger in state $\beta$ than in state $\alpha$. Therefore, voters can use the asymmetry in the variance of the outcome across the states to increase the probability of the correct choice.

We now find the maximum probability with which the electorate can choose the correct outcome in equilibrium when $n$ is large. A symmetric strategy profile is a pair of probabilities that we denote by $q=(q_a,q_b)$, where $q_a+q_b=1$ and $q_a$ is the probability that a voter votes for $A$ (and $q_b$ the probability to vote for $B$). Given a symmetric strategy profile, the expected payoff of a representative agent is

$$E_n^{rep}\left[u;q\right]=\pi Pr\left\{\text{a wins}|\alpha,q\right\}+(1-\pi)Pr\left\{\text{b wins}|\beta,q\right\}.$$

The strategy profile $q_*^n$ that maximizes $E_n^{rep}\left[u;\cdot\right]$ is a Nash equilibrium by Lemma 2, following McLennan (1998). We are interested in the limit of the sequence $\{q_*^n\}_{n=1}^{\infty}$.

The first lemma derives the limiting outcome for a given sequence of strategy profiles $\{q^n\}$, utilizing the central limit theorem to approximate the Poisson-distributed vote totals for each alternative. What matters for the outcome is the expected margin of victory of each alternative relative to the standard deviation of the vote total. In state $\alpha$, the expected margin of victory is $n\left(q_a^n-1/2\right)$ and the standard deviation of the vote total is $\sqrt{n}$. We denote the limit of their ratio as $k:=\lim_{n\to\infty}\frac{n(q_a^n-1/2)}{\sqrt{n}}$. As is intuitive, if $k$ is large (i.e., the margin of victory is positive and large relative to the standard deviation), then $A$ wins with a high probability; if $k$ is positive but close to zero, each alternative is almost equally likely to win; and if $k$ is negative, then $B$ is more likely to win. The lemma verifies this intuition. The proof for this is provided in the appendix. Recall that $\Phi$ denotes the c.d.f. of the standard normal distribution.

**Lemma 5.** *Consider a sequence of voting games in which the expected number of participants is $(n, \theta n)$ in states $\alpha$ and $\beta$, respectively, and $n \to \infty$. Take a sequence of strategy profiles $\{q^n\}$ with $\lim \sqrt{n}\,(q_a^n - 1/2) = k \in [-\infty, \infty]$. Then,*

1. $\lim_{n\to\infty} Pr\{a \text{ wins}|\alpha, q_n\} = \Phi(2k)$,
2. $\lim_{n\to\infty} Pr\{b \text{ wins}|\beta, q_n\} = 1 - \Phi\left(2\sqrt{\theta}k\right)$,

   *where $\Phi(-\infty) = 0$, and $\Phi(\infty) = 1$.*

Utilizing the lemma, the expected payoff from a given sequence of strategy profiles is

$$\lim_{n\to\infty} E_n^{rep}[u; q^n] = \max_{k\in[-\infty,\infty]} \pi\Phi(k) + (1-\pi)\left(1 - \Phi\left(\sqrt{\theta}k\right)\right).$$

Maximizing the expected payoff with respect to $k$ yields the optimal strategy profile and, hence, the limit of the sequence of the welfare maximizing equilibrium profiles, $\{q_*^n\}_{n=1}^{\infty}$.

**Theorem 7.** *Consider a sequence of voting games in which the expected number of participants is $(n, \theta n)$ in states $\alpha$ and $\beta$, respectively, and $n \to \infty$. Suppose that $\theta < 1$ and $\pi \geq 1/2$. Let $k^* := \sqrt{\frac{2}{1-\theta}\ln\left(\frac{\pi}{(1-\pi)\sqrt{\theta}}\right)}$. Then, the sequence of Nash equilibria $\{q_*^n\}$ that maximizes $E_n^{rep}[u;\cdot]$ is such that*

1. $\lim_{n\to\infty} E_n^{rep}[u; q_*^n] = \pi\Phi(k^*) + (1-\pi)\left(1 - \Phi\left(\sqrt{\theta}k^*\right)\right)$, *and*
2. $\lim_{\theta\to 0}\lim_{n\to\infty} E_n^{rep}[u; q_*^n] = \pi + \frac{1}{2}(1-\pi)$.

The proof of the theorem is given in the appendix. There, we also characterize the best equilibria for the remaining parameters $(\theta, \pi)$.

# 7 Conclusion

In this paper, we study the set of equilibria of Poisson elections when the expected number of voters is potentially state-dependent. We show that large Poisson elections robustly aggregate information—in the sense that *all* equilibria imply the correct choice with probability converging to one—if and only if the expected number of voters is constant across states. If the expected number of voters is different, then there are additional responsive equilibria that fail to aggregate information. The basic reason for this is that voters are more likely to be pivotal when the electorate is smaller. This leads to equilibria in which voters systematically vote for the policy that is optimal in the state with fewer expected voters. Moreover, when signals are sufficiently informative, these equilibria can be chosen to be stable. Abstention does not eliminate additional equilibria. For the case with abstention and binary signals, we use a certain "two-dimensional extension" of the intermediate value theorem to construct an equilibrium with novel properties (voters mix between all three options and there is a swing-voters blessing). We end on a positive note: Although the state-dependent population introduces a coordination problem, it also injects additional information that the voters can utilize when coordinating on the best equilibrium.

Our results relate to three contributions: Ekmekci and Lauermann (2019) consider a setting with compulsory voting and a deterministic number of voters in each state and show that information aggregation may fail. Myerson (1998a) introduces a model of Poisson elections in which the expected number of voters may be state-dependent and voting is compulsory. He shows that there exist equilibria that aggregate information. Krishna and Morgan (2012) study a model of Poisson elections in which the expected number of voters is independent of the state and abstention is allowed, showing that for the case with binary signals, all equilibria aggregate information.[17] Relative to Ekmekci and Lauermann (2019), we consider Poisson elections and allow abstention; relative to Myerson (1998a), we show the existence of additional equilibria and allow abstention; and relative to Krishna and Morgan (2012) we allow continuous signals and show that there are additional equilibria when the expected number of voters depends on the state.

# A Appendix

**Notation.** In the appendix, we denote the expected number of $A$ and $B$ votes in state $\alpha$ as $\sigma_A = nG(\hat{x}|\alpha)$ and $\sigma_B(\hat{x}) = n(1 - G(\hat{x}|\alpha))$. Similarly, for state $\beta$, $\tau_A(\hat{x}) = \theta n G(\hat{x}|\beta)$, $\tau_B(\hat{x}) = \theta n (1 - G(\hat{x}|\beta))$. We often drop the arguments from

[17] Their paper considers costly voting, which is the main focus of their analysis.

$\sigma_\omega$ and $\tau_\omega$.

Sequences and Limits. When taking limits, we mean with respect to subsequences for which a limit exists (in the extended reals). In the context of our proofs, such subsequences can always be found and proving statements for all converging subsequences will be sufficient for the desired claims. This convention saves us from introducing notation for layers of subsequences. We will not always repeat this qualifier.

## A.1 Proofs for Section 3 (Welfare Maximization)

**Proof of Lemma 2.**

We apply the same argument as McLennan (1998), with a slight modification to account for random number of voters . Let $u(i, a|\omega)$ denote the expected payoff of a voter when there are $i$ voters, the state is $\omega$, and each voter is following the strategy $a$, and let $u(i, a'|\omega, a)$ denote the expected payoff of a voter in state $\omega$ if she follows the strategy $a'$ while all other voters are using the strategy $a$, and when there are $i+1$ voters in total. Finally, let $u(a, i, a', j|\omega)$ be the expected payoff of a voter when $i$ voters are using the strategy $a$, and $j$ voters are using the strategy $a'$. The representative voter's payoff is

$$E^{rep}\left[u|a\right] = \sum_{\omega} \Pr\{\omega\} \sum_{i=0}^{\infty} \Pr\left\{\tilde{n} = i|\omega\right\} u\left(i, a|\omega\right).$$

Let $(1-\epsilon)a + \epsilon a'$ denote mixing over the two strategies $a$ and $a'$. We characterize the derivative of $E^{rep}\left[u|a\right]$ in the direction of $a'$:

**Claim.**

$$\lim_{\epsilon\to 0} \frac{E^{rep}\left[u|a\right] - E^{rep}[u|(1-\epsilon)a + \epsilon a']}{\epsilon} = \sum_{\omega} \Pr(\omega) n_\omega \sum_{i=0}^{\infty} \Pr(\tilde{n} = i|\omega)(u(i+1, a|\omega) - u(i, a'|\omega, a)).$$

Note that the right-hand side is proportional to the increase of an individual voter's expected utility when changing her strategy from $a'$ to $a$, provided all other voters follow strategy $a$.

**Proof of the claim:** Let $x$ be the realized number of participants who play $a$ and $y$ the realized number of participants who play $a'$, and note that $x + y = \tilde{n}$. Notice that $x$ has a Poisson distribution with mean $n_\omega(1-\epsilon)$, and $y$ has a Poisson distribution with mean $n_\omega \epsilon$. Then,

$$\begin{aligned}
E^{rep}[u|a] - E^{rep}[u|(1-\epsilon)a + \epsilon a'] = \sum_\omega \Pr(\omega)(\sum_{i=0}^{\infty} \Pr(\tilde{n} = i|\omega) u(i, a|\omega) \\
- \Pr(y = 0|\omega) \sum_{i=0}^{\infty} \Pr(x = i|\omega) u(i, a|\omega) \\
- \Pr(y = 1|\omega) \sum_{i=0}^{\infty} \Pr(x = i|\omega) u(i, a'|\omega, a) \\
- \sum_{j>1} \Pr(y = j|\omega) \sum_{i=0}^{\infty} \Pr(x = i|\omega) u(a, i, a', j|\omega)),
\end{aligned}$$

and so

$$\begin{aligned}
&E^{rep}[u|a] - E^{rep}[u|(1-\epsilon)a + \epsilon a'] \\
&= \sum_\omega \Pr(\omega)( \\
&\Pr(y = 1|\omega) \sum_{i=0}^{\infty} \Pr(x = i|\omega)(u(i+1, a|\omega) - u(i, a'|\omega, a)) \\
&+ \sum_{j>1} \Pr(y = j|\omega) \sum_{i=0}^{\infty} \Pr(x = i|\omega)(u(i+j, a|\omega) - u(a, i, a', j|\omega))).
\end{aligned}$$

Consider the term

$$\frac{1}{\epsilon} \sum_\omega \Pr(\omega) \Pr(y = 1|\omega) \sum_{i=0}^{\infty} \Pr(x = i|\omega)(u(i+1, a|\omega) - u(i, a'|\omega, a)).$$

Because

$$\lim_{\epsilon \to 0} \frac{\Pr(y = 1|\omega)}{\epsilon} = n_\omega,$$

and because

$$\lim_{\epsilon \to 0} \sum_{i=0}^{\infty} \Pr(x = i|\omega)(u(i+1, a|\omega) - u(i, a'|\omega, a)) = \sum_{i=0}^{\infty} \Pr(\tilde{n} = i|\omega)(u(i+1, a|\omega) - u(i, a'|\omega, a)),$$

we have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\sum_{\omega}\Pr(\omega)\Pr(y=1|\omega)\sum_{i=0}^{\infty}\Pr(x=i|\omega)(u(i+1,a|\omega)-u(i,a'|\omega,a)) =$$
$$\sum_{\omega}\Pr(\omega)n_{\omega}\sum_{i=0}^{\infty}\Pr(\tilde{n}=i|\omega)(u(i+1,a|\omega)-u(i,a'|\omega,a)).$$

Consider the term

$$\frac{1}{\epsilon}\sum_{j>1}\Pr(y=j|\omega)\sum_{i=0}^{\infty}\Pr(x=i|\omega)(u(i+j,a|\omega)-u(a,i,a',j|\omega).$$

Because $\sum_{j>1}\Pr(y=j|\omega)$ is at the order of $\epsilon^2$, and because payoffs are bounded, we have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\sum_{j>1}\Pr(y=j|\omega)\sum_{i=0}^{\infty}\Pr(x=i|\omega)(u(i+j,a|\omega)-u(a,i,a',j|\omega)=0.$$

Therefore,

$$\lim_{\epsilon\to 0}\frac{E^{rep}[u|a]-E^{rep}[u|(1-\epsilon)a+\epsilon a']}{\epsilon}=\sum_{\omega}\Pr(\omega)n_{\omega}\sum_{i=0}^{\infty}\Pr(\tilde{n}=i|\omega)(u(i+1,a|\omega)-u(i,a'|\omega,a)),$$

proving the claim.

Suppose now to the contrary of the claim of the Lemma that the strategy $a$ maximizes $E^{rep}[u|a]$, and $a$ is not a symmetric Nash equilibrium of the voting game. Then, there is a strategy $a'$ such that

$$\sum_{\omega}\Pr(\omega)n_{\omega}\sum_{i=0}^{\infty}\Pr(\tilde{n}=i|\omega)(u(i+1,a|\omega)-u(i,a'|\omega,a))<0,$$

because $\sum_{\omega}\Pr(\omega)n_{\omega}\sum_{i=0}^{\infty}\Pr(\tilde{n}=i|\omega)u(i,a'|\omega,a)$ is proportional to the payoff to a voter from following the strategy $a'$ when other voters follow strategy $a$. However, then it follows from the claim there is some $\epsilon>0$ such that $E^{rep}[u|(1-\epsilon)a+\epsilon a']>E^{rep}[u|a]$, which contradicts that $a$ maximizes $E^{rep}[u|a]$. $\square$

## A.2 Proofs for Section 4.1 (Auxiliary Results)

**Proof of Lemma 3. (The Generalized Intermediate Value Theorem.)**

We are done if either $\gamma(a; a, n) = 1$ or $1 = \gamma(b; b, n)$: If $\gamma(a; a, n) = 1$, then the weak MLRP implies $\gamma(x; a, n) \geq 1$ for all $x \leq a$ (and hence voting $A$ is optimal) and $\gamma(x; a, n) \leq 1$ for all $x \geq a$ (and hence voting $B$ is optimal).

So, suppose $\gamma(a; a, n) < 1 < \gamma(b; b, n)$.

<u>Case $a > b$.</u> Let $\hat{x} = \sup\{x|\gamma(x; x, n) > 1\}$. By the continuity of $\Pr(Piv_0|\alpha) / \Pr(Piv_0|\beta)$ on $[\underline{x}, \bar{x}]$ and the right-continuity of $g(x|\alpha)/g(x|\beta)$, $\gamma(\hat{x}; \hat{x}, n) = \lim_{\varepsilon\to 0} \gamma(\hat{x} + \varepsilon; \hat{x} + \varepsilon, n) \leq 1$. Also, $\lim_{\varepsilon\to 0} \gamma(\hat{x} - \varepsilon; \hat{x} - \varepsilon, n) \geq 1$. By the MLRP, $\gamma(x; \hat{x}, n) \geq 1$ for all $x < \hat{x}$ and $\gamma(x; \hat{x}, n) \leq 1$ for all $x \geq \hat{x}$. Hence, $\hat{x}$ is a Nash equilibrium cutoff.

<u>Case $a < b$.</u> Let $\hat{x} = \sup\{x|\gamma(x; x, n) < 1\}$. As before, by the continuity of $\Pr(Piv_0|\alpha) / \Pr(Piv_0|\beta)$ and the right-continuity of $g(x|\alpha)/g(x|\beta)$, $\gamma(\hat{x}; \hat{x}, n) = \lim_{\varepsilon\to 0} \gamma(\hat{x} + \varepsilon; \hat{x} + \varepsilon, n) \geq 1$. By the continuity of $\Pr(Piv_0|\alpha) / \Pr(Piv_0|\beta)$ and the MLRP, $\lim_{\varepsilon\to 0} \gamma(\hat{x} - \varepsilon; \hat{x} - \varepsilon, n) \geq \gamma(\hat{x}; \hat{x}, n)$. By the definition of $\hat{x}$, $\lim_{\varepsilon\to 0} \gamma(\hat{x} - \varepsilon; \hat{x} - \varepsilon, n) \leq 1$. Hence, $\gamma(\hat{x}; \hat{x}, n) \leq 1$. Together with the previous bound, $\gamma(\hat{x}; \hat{x}, n) = 1$. $\square$

Recall that $\sigma_A = nG(\hat{x}|\alpha)$ and $\sigma_B(\hat{x}) = n(1 - G(\hat{x}|\alpha))$ and similarly $\tau_A(\hat{x}) = \theta nG(\hat{x}|\beta)$, $\tau_B(\hat{x}) = \theta n(1 - G(\hat{x}|\beta))$. We approximate the pivotal probabilities. Here and in the following, given any sequence $\{x^k\}_{k=1}^{\infty}$, we say $f \approx g$ for two functions if $\lim_{k\to\infty} \frac{f(x^k)}{g(x^k)} = 1$. To improve readability, we suppress the sequence index $k$ in the following statement.

**Lemma 6.** *If $\sigma_A\sigma_B \to \infty$, then*

$$\begin{aligned}
\Pr[T|\alpha] &\approx e^{-\sigma_A-\sigma_B}\frac{e^{2\sqrt{\sigma_A\sigma_B}}}{\sqrt{2\pi 2\sqrt{\sigma_A\sigma_B}}}, \\
\Pr[T \pm 1|\beta] &\approx e^{-\sigma_A-\sigma_B}\frac{e^{2\sqrt{\sigma_A\sigma_B}}}{\sqrt{2\pi 2\sqrt{\sigma_A\sigma_B}}}\left(\frac{\sigma_A}{\sigma_B}\right)^{\pm 1/2}.
\end{aligned}$$

*If* $\sigma_A\sigma_B \to k \in (0,\infty)$*, then*

$$\begin{aligned}
\Pr[T-1|\alpha] &\approx e^{-\sigma_A-\sigma_B}\sigma_B\frac{I_1\left(2\sqrt{k}\right)}{\sqrt{k}},\\
\Pr[T|\alpha] &\approx e^{-\sigma_A-\sigma_B}I_0\left(2\sqrt{k}\right),\\
\Pr[T+1|\alpha] &\approx e^{-\sigma_A-\sigma_B}\sigma_A\frac{I_1\left(2\sqrt{k}\right)}{\sqrt{k}},
\end{aligned}$$

*with* $I_0 : \mathbb{R}_+ \to \mathbb{R}_+$ *a continuous, strictly positive function with* $\lim_{z\to\infty} I_0(z) = I_1(z) = \infty$*,* $I_0(0) = 1$ *and* $I_1 : \mathbb{R}_+ \to \mathbb{R}_+$ *a continuous function that is strictly positive on* $(0,\infty)$ *but* $\lim_{z\to 0}\frac{I_1(z)}{z} = 1/2$*.*

*If* $\sigma_A\sigma_B \to 0$*, then*

$$\begin{aligned}
\Pr[T-1|\alpha] &\approx e^{-\sigma_A-\sigma_B}\sigma_B,\\
\Pr[T|\alpha] &\approx e^{-\sigma_A-\sigma_B},\\
\Pr[T+1|\alpha] &\approx e^{-\sigma_A-\sigma_B}\sigma_A.
\end{aligned}$$

Of course, all analogous approximations hold for state $\beta$, after substituting $\tau_W$ for $\sigma_W$.

**Proof of Lemma 6.**

The Lemma follows immediately from observations from Krishna and Morgan (2012), equations (4) and (5), namely,

$$\Pr[T|\alpha] = e^{-\sigma_A-\sigma_B}I_0\left(2\sqrt{\sigma_A\sigma_B}\right), \tag{9}$$

$$\Pr[T\pm 1|\alpha] = e^{-\sigma_A-\sigma_B}\left(\frac{\sigma_A}{\sigma_B}\right)^{\pm 1/2} I_1\left(2\sqrt{\sigma_A\sigma_B}\right), \tag{10}$$

where $I_0$ and $I_1$ are the so-called "modified Bessel functions." The approximations then use properties of the modified Bessel functions, namely, that

$$\lim_{z\to\infty}\frac{\frac{e^z}{\sqrt{2\pi z}}}{I_0(z)} = \lim_{z\to\infty}\frac{\frac{e^z}{\sqrt{2\pi z}}}{I_1(z)} = 1,$$

and that

$$\lim_{z\to 0}\frac{I_1(z)}{z}=\frac{1}{2}\Rightarrow\frac{\left(\frac{\sigma_B}{\sigma_A}\right)^{1/2}I_1\left(2\sqrt{\sigma_A\sigma_B}\right)}{\sigma_B}=\frac{I_1\left(2\sqrt{\sigma_A\sigma_B}\right)}{\sqrt{\sigma_A\sigma_B}}\to 1.$$

Now, the approximations follow. $\square$

**Proof of Lemma 4. (Approximation for Compulsory Voting.)**

Recall that

$$\Pr\left(Piv_0|\omega\right)=\frac{1}{2}\Pr\left[T-1|\omega\right]+\Pr\left[T|\omega\right]+\frac{1}{2}\Pr\left[T+1|\omega\right].$$

From $\lim\hat{x}^n\in(\underline{x},\bar{x})$, we have $\sigma_A\sigma_B\to\infty$ and $\tau_A\tau_B\to\infty$. So, Lemma 6 implies that

$$\begin{aligned}\frac{1}{2}\Pr\left[T-1|\alpha\right]+\Pr\left[T|\alpha\right]+\frac{1}{2}\Pr\left[T+1|\alpha\right] &\approx e^{-\sigma_A-\sigma_B}\frac{e^{2\sqrt{\sigma_A\sigma_B}}}{\sqrt{2\pi 2\sqrt{\sigma_A\sigma_B}}}\frac{1}{2}\left(2+\left(\frac{\sigma_A}{\sigma_B}\right)^{+1/2}+\left(\frac{\sigma_A}{\sigma_B}\right)^{-1/2}\right)\\ &= \frac{e^{-n+2\sqrt{\sigma_A\sigma_B}}}{\sqrt{2\pi 2\sqrt{\sigma_A\sigma_B}}}\frac{1}{2}\left(2+\left(\frac{\sigma_A}{\sigma_B}\right)^{+1/2}+\left(\frac{\sigma_A}{\sigma_B}\right)^{-1/2}\right).\end{aligned}$$

Furthermore, $\lim\hat{x}^n\in(\underline{x},\bar{x})$ implies that

$$0<\lim\frac{\sqrt[4]{\tau_A\tau_B}}{\sqrt[4]{\sigma_A\sigma_B}}\frac{2+\sqrt{\frac{\sigma_B}{\sigma_A}}+\sqrt{\frac{\sigma_A}{\sigma_B}}}{2+\sqrt{\frac{\tau_B}{\tau_A}}+\sqrt{\frac{\tau_A}{\tau_B}}}=:K<\infty.$$

This is because with $\hat{x}=\lim\hat{x}^n$,

$$\lim\frac{\sqrt[4]{\tau_A\tau_B}}{\sqrt[4]{\sigma_A\sigma_B}}=\lim\frac{\sqrt[4]{\theta^2n^2G\left(\hat{x}^n|\beta\right)\left(1-G\left(\hat{x}^n|\beta\right)\right)}}{\sqrt[4]{n^2G\left(\hat{x}^n|\alpha\right)\left(1-G\left(\hat{x}^n|\alpha\right)\right)}}=\frac{\sqrt{\theta}\sqrt[4]{G\left(\hat{x}|\beta\right)\left(1-G\left(\hat{x}|\beta\right)\right)}}{\sqrt[4]{G\left(\hat{x}|\alpha\right)\left(1-G\left(\hat{x}|\alpha\right)\right)}},$$

and

$$\lim\frac{2+\sqrt{\frac{\sigma_B}{\sigma_A}}+\sqrt{\frac{\sigma_A}{\sigma_B}}}{2+\sqrt{\frac{\tau_B}{\tau_A}}+\sqrt{\frac{\tau_A}{\tau_B}}}=\frac{2+\sqrt{\frac{(1-G(\hat{x}|\alpha))}{G(\hat{x}|\alpha)}}+\sqrt{\frac{G(\hat{x}|\alpha)}{(1-G(\hat{x}|\alpha))}}}{2+\sqrt{\frac{(1-G(\hat{x}|\beta))}{G(\hat{x}|\beta)}}+\sqrt{\frac{G(\hat{x}|\beta)}{(1-G(\hat{x}|\beta))}}}.$$

So,

$$\lim \frac{\Pr\left[Piv_0|\alpha\right]}{\Pr\left[Piv_0|\beta\right]} = \lim \frac{e^{-n+2\sqrt{\sigma_A\sigma_B}}}{e^{-\theta n+2\sqrt{\tau_A\tau_B}}}\frac{\sqrt[4]{\tau_A\tau_B}}{\sqrt[4]{\sigma_A\sigma_B}}\frac{2+\sqrt{\frac{\sigma_B}{\sigma_A}}+\sqrt{\frac{\sigma_A}{\sigma_B}}}{2+\sqrt{\frac{\tau_B}{\tau_A}}+\sqrt{\frac{\tau_A}{\tau_B}}}$$
$$= K \lim e^{n\left(2\sqrt{G(\hat{x}^n|\alpha)(1-G(\hat{x}^n|\alpha))}-1-\theta\left(2\sqrt{G(\hat{x}^n|\beta)(1-G(\hat{x}^n|\beta))}-1\right)\right)}.$$

and the lemma now follows. □

### A.3 Proofs for Sections 4.2 and 4.3 (Compulsory Voting)

**Proof of Theorem 2. (If $\theta = 1$, all equilibria aggregate information.)**

We show first that for any sequence of cutoffs $x^n$,

$$\lim_{n\to\infty} \frac{\Pr\left[Piv_0|\alpha; x^n, n\right]}{\Pr\left[Piv_0|\beta; x^n, n\right]} = \begin{cases} \infty & \text{if } \ \underline{x} < \lim_{n\to\infty} x^n \leq x_\alpha, \\ 0 & \text{if } \ x_\beta \leq \lim_{n\to\infty} x^n < \bar{x}. \end{cases}$$

This rules out that such sequences are Nash equilibria, of course. Consider $x_\beta \leq \lim_{n\to\infty} x^n < \bar{x}$. Then, from the MLRP and the fact that signals contain information, $1/2 \leq \lim G\left(\hat{x}^n|\beta\right) < \lim G\left(\hat{x}^n|\alpha\right) < 1$. Now, the claim follows from Lemma 4.

We now rule out equilibria in which $x^n$ is close to $\bar{x}$. By assumption (3), there is some $x_r > x_\beta$ such that for $x > x_r$

$$1 > \frac{\pi}{1-\pi}\frac{g\left(x|\alpha\right)}{g\left(x|\beta\right)}.$$

Now, if $x \in (x_r, \bar{x})$, then $1/2 < \lim G\left(\hat{x}^n|\beta\right) < \lim G\left(\hat{x}^n|\alpha\right) < 1$ implies that the probability $\Pr\left[Piv_0|\beta; x, n\right] > \Pr\left[Piv_0|\alpha; x, n\right]$. To see this, consider a fixed voter and suppose the realized number of other voters is $m$ and each of the $m$ other voters supports $A$ with i.i.d. probability $G\left(x|\omega\right)$. If $m = 0$, then the voter is pivotal in both states with equal likelihood. If $m > 0$ is even, then the fixed voter affects the election if and only if exactly $\frac{m}{2}$ other voters support $A$ and $B$. The probability that exactly $\frac{m}{2}$ voters support each policy is strictly larger in state $B$ since $G\left(x|\beta\right)\left(1 - G\left(x|\beta\right)\right) > G\left(x|\alpha\right)\left(1 - G\left(x|\alpha\right)\right)$ by $1/2 < \lim G\left(\hat{x}^n|\beta\right) < \lim G\left(\hat{x}^n|\alpha\right) < 1$. If $m$ is odd, then the fixed voter affects the election if and only if she votes $A$ and $\frac{m-1}{2}$ other voters support $A$ and $\frac{m+1}{2}$ support $B$ and vote for $B$ changes the outcome if $\frac{m+1}{2}$ support $A$ and $\frac{m-1}{2}$

support $B$. With $\frac{m-1}{2} =: r$ and $q_\omega := G(x|\omega)$, the sum of these two probabilities is

$$\begin{aligned}
& \binom{2r+1}{r} (q_\omega)^r (1-q_\omega)^{r+1} + \binom{2r+1}{r+1} (q_\omega)^{r+1} (1-q_\omega)^r \\
= & \binom{2r+1}{r} (q_\omega)^r (1-q_\omega)^r (q_\omega + (1-q_\omega)) = \binom{2r+1}{r} (q_\omega)^r (1-q_\omega)^r .
\end{aligned}$$

Again, $G(x|\beta)(1-G(x|\beta)) > G(x|\alpha)(1-G(x|\alpha))$ implies that this probability is higher in state $\beta$. Thus, conditional on any realization of the number of other voters (either even or odd), the probability to affect the election is higher in state $\beta$. Hence, for all $\bar{x} > x > x_r$,

$$\frac{\Pr[Piv_0|\alpha; x, n]}{\Pr[Piv_0|\beta; x, n]} < 1.$$

Thus, for all $n$ and $x > x_r$,

$$\frac{\pi}{1-\pi} \frac{g(x|\alpha)}{g(x|\beta)} \frac{n}{n\theta} \frac{\Pr[Piv_0|\alpha; x, n]}{\Pr[Piv_0|\beta; x, n]} < \frac{\pi}{1-\pi} \frac{g(x|\alpha)}{g(x|\beta)} < 1.$$

There can be no equilibrium with a cutoff $x \in (x_r, \bar{x})$ for any $n$. A symmetric argument rules out equilibria with cutoffs $x$ close to $\underline{x}$.

Finally, from $\theta = 1$, we have for a cutoff $\bar{x}$ that

$$\begin{aligned}
\frac{\Pr[Piv_0|\alpha; \bar{x}, n]}{\Pr[Piv_0|\beta; \bar{x}, n]} &= \frac{\frac{1}{2}\Pr[T-1|\alpha] + \Pr[T|\alpha] + \frac{1}{2}\Pr[T+1|\alpha]}{\frac{1}{2}\Pr[T-1|\beta] + \Pr[T|\beta] + \frac{1}{2}\Pr[T+1|\beta]} \\
&= \frac{0 + e^{-n} + \frac{1}{2}e^{-n}n}{0 + e^{-n} + \frac{1}{2}e^{-n}n} = 1,
\end{aligned}$$

which follows because $A$ cannot be behind if the cutoff is $\bar{x}$ (all vote $A$), a tie occurs only if no voter participates, and $A$ is one ahead if there is exactly one voter. Thus,

$$\begin{aligned}
\gamma(\bar{x}; \bar{x}, n) &= \frac{\pi}{1-\pi} \frac{g(\bar{x}|\alpha)}{g(\bar{x}|\beta)} \frac{n}{n\theta} \frac{\Pr[Piv_0|\alpha; \bar{x}, n]}{\Pr[Piv_0|\beta; \bar{x}, n]} \\
&= \frac{\pi}{1-\pi} \frac{g(\bar{x}|\alpha)}{g(\bar{x}|\beta)} < 1,
\end{aligned}$$

and so by the left-continuity of $g(\cdot|\alpha)/g(\cdot|\beta)$ at $\bar{x}$, $\gamma(x; \bar{x}, n) < 1$ for all $x < \bar{x}$. There can be no equilibrium with cutoff $\bar{x}$ for any $n$. Similarly, there can be no equilibrium with cutoff $\underline{x}$ for any $n$. $\square$

**Proof of Theorem 3. (If $\theta \neq 1$ then there are interior equilibria that do not aggregate information.)**

We prove the theorem for $\theta < 1$. The argument for $\theta > 1$ is analogous and omitted.

Suppose signals are boundedly informative, that is, $\frac{g(\underline{x}|\alpha)}{g(\underline{x}|\beta)} < \infty$.

We verify that

$$\lim_{n\to\infty} \gamma\left(x_\alpha; x_\alpha, n\right) = \infty, \tag{11}$$

$$\lim_{n\to\infty} \gamma\left(\underline{x}; \underline{x}, n\right) = 0. \tag{12}$$

From the MLRP and the fact that signals contain information, $1/2 = G\left(x_\alpha|\alpha\right) > G\left(x_\alpha|\beta\right) > 0$. Now, (11) follows from Lemma 4.

For $x = \underline{x}$, we have

$$\begin{aligned} \frac{\Pr\left[Piv_0|\alpha; \underline{x}, n\right]}{\Pr\left[Piv_0|\beta; \underline{x}, n\right]} &= \frac{\frac{1}{2}\Pr\left[T-1|\alpha\right] + \Pr\left[T|\alpha\right] + \frac{1}{2}\Pr\left[T+1|\alpha\right]}{\frac{1}{2}\Pr\left[T-1|\beta\right] + \Pr\left[T|\beta\right] + \frac{1}{2}\Pr\left[T+1|\beta\right]} \\ &= \frac{\frac{1}{2}e^{-n}n + e^{-n} + 0}{\frac{1}{2}e^{-\theta n}\theta n + e^{-\theta n} + 0} \to_{n\to\infty} 0, \end{aligned}$$

where the second equality follows because $A$ is one behind if there is exactly one voter, a tie occurs only if no voter participates, and $A$ cannot be ahead if the cutoff is $\underline{x}$ (all vote $B$). The limit follows from $\theta < 1$. This implies (12).

Given (11) and (12), existence of an interior Nash equilibrium $\hat{x}^n$ with $\gamma\left(\hat{x}^n; \hat{x}^n, n\right) = 1$ for all $n$ large enough follows from the generalized intermediate value theorem, Lemma 3. By Lemma 4, the conclusion of (11) also holds if $\hat{x}^n \to x_\alpha$. Thus, $\lim \hat{x}^n < x_\alpha$, and so $B$ wins with probability converging to one.

Suppose signals are unboundedly informative, that is, $\frac{g(\underline{x}|\alpha)}{g(\underline{x}|\beta)} = \infty$.

Since $\theta < 1$, there exists some $x_R \in \left(\underline{x}, x_\alpha\right)$ small enough such that

$$2\sqrt{G\left(x_R|\alpha\right)\left(1 - G\left(x_R|\alpha\right)\right)} - 1 < \theta\left(2\sqrt{G\left(x_R|\beta\right)\left(1 - G\left(x_R|\beta\right)\right)} - 1\right),$$

noting that the left-hand side approaches $-1$ for $x_R \to \underline{x}$ and the right-hand side approaches $-\theta$. Lemma 4 implies that

$$\lim_{n\to\infty} \gamma\left(x_R; x_R, n\right) = 0. \tag{13}$$

Since signals are unboundedly informative, it is immediate that for any given $n$, there exists $x_L^n$ with $\underline{x} < x_L^n < x_R$ such that

$$\gamma\left(x_L^n; x_L^n, n\right) > 1. \tag{14}$$

To check: For a fixed $n$, for $\hat{x} \to \underline{x}$, we have $\lim_{\hat{x}\to\underline{x}} \frac{\Pr[Piv_0|\alpha;\hat{x},n]}{\Pr[Piv_0|\beta;\hat{x},n]} = \frac{\frac{1}{2}e^{-n}n+e^{-n}+0}{\frac{1}{2}e^{-\theta n}\theta n+e^{-\theta n}+0} \in (0,1)$. Hence, from signals being unboundedly informative, $\lim_{\hat{x}\to\underline{x}} \gamma\left(\hat{x}; \hat{x}, n\right) = \infty$ for all $n$.

Given (13) and (14), existence of a Nash equilibrium $\hat{x}^n$ for all $n$ large enough follows again from Lemma 3. □

## A.4 Proofs for Section 5 (Abstention)

**Notation.** In the following, we denote the critical likelihood ratio at the cutoff types as

$$\gamma_A\left(y,z,n\right) = \frac{\pi}{1-\pi}\frac{1}{\theta}\frac{g\left(y|\alpha\right)}{g\left(y|\beta\right)}\frac{\Pr\left[PivA|\alpha;y,z,n\right]}{\Pr\left[PivA|\beta;y,z,n\right]},$$

and

$$\gamma_B\left(y,z,n\right) = \frac{\pi}{1-\pi}\frac{1}{\theta}\frac{g\left(z|\alpha\right)}{g\left(z|\beta\right)}\frac{\Pr\left[PivB|\alpha;y,z,n\right]}{\Pr\left[PivB|\beta;y,z,n\right]}.$$

A strategy profile $(y,z)$ is an interior Nash equilibrium if $\underline{x} < y \le z < \bar{x}$ and

$$1 = \gamma_A\left(y,z,n\right) = \gamma_B\left(y,z,n\right).$$

### A.4.1 Auxiliary Approximation Results

**Lemma 7.** *Suppose abstention is possible and suppose signals are boundedly informative. Pick a sequence of equilibrium cutoffs $(y^n, z^n)$.*

*If $\sigma_A\sigma_B \to \infty$ (or, equivalently, $\tau_A\tau_B \to \infty$), then there is some $K_\alpha\left(\sigma,\tau\right)$ such that $\lim K_\alpha\left(\sigma,\tau\right) \in (0,\infty)$ and*

$$\lim \frac{\Pr\left[PivA|\alpha\right]}{\Pr\left[PivA|\beta\right]} = \lim \frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}} K_\alpha\left(\sigma,\tau\right),$$

*and similarly there is some* $K_\beta(\sigma,\tau)$ *such that* $\lim K_\beta(\sigma,\tau)\in(0,\infty)$ *and*

$$\lim \frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]} = \lim \frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}} K_\beta(\sigma,\tau).$$

*It follows that under these conditions that for* $W\in\{A,B\}$,

$$\lim_{n\to\infty}\frac{\Pr[PivW|\alpha]}{\Pr[PivW|\beta]} = \begin{cases} \infty & \textit{if} \quad \lim_{n\to\infty}\left|\sqrt{\sigma_A}-\sqrt{\sigma_B}\right| < \lim_{n\to\infty}\left|\sqrt{\tau_A}-\sqrt{\tau_B}\right|, \\ 0 & \textit{if} \quad \lim_{n\to\infty}\left|\sqrt{\sigma_A}-\sqrt{\sigma_B}\right| > \lim_{n\to\infty}\left|\sqrt{\tau_A}-\sqrt{\tau_B}\right|, \end{cases}$$

**Remark.** Observe that for any $z<x_\alpha$

$$\sqrt{\sigma_A}-\sqrt{\sigma_B}<0 \text{ and } \sqrt{\tau_A}-\sqrt{\tau_B}<0.$$

Hence,

$$\begin{aligned} & -\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2+\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2 \\ = & -n\left(\left(\sqrt{G(y|\alpha)}-\sqrt{1-G(z|\alpha)}\right)^2+\theta\left(\sqrt{G(y|\beta)}-\sqrt{1-G(z|\beta)}\right)^2\right)\to-\infty, \end{aligned}$$

if

$$\sqrt{1-G(z|\alpha)}-\sqrt{G(y|\alpha)}>\sqrt{\theta}\left(\sqrt{1-G(z|\beta)}-\sqrt{G(y|\beta)}\right).$$

Thus, it follows from the lemma that

$$\begin{aligned} & \lim_{n\to\infty}\frac{\Pr[PivW|\alpha]}{\Pr[PivW|\beta]} \qquad (15) \\ = & \begin{cases} \infty & \text{if} \quad \lim_{n\to\infty}\sqrt{1-G(z|\alpha)}-\sqrt{G(y|\alpha)}<\lim_{n\to\infty}\sqrt{\theta}\left(\sqrt{1-G(z|\beta)}-\sqrt{G(y|\beta)}\right), \\ 0 & \text{if} \quad \lim_{n\to\infty}\sqrt{1-G(z|\alpha)}-\sqrt{G(y|\alpha)}>\lim_{n\to\infty}\sqrt{\theta}\left(\sqrt{1-G(z|\beta)}-\sqrt{G(y|\beta)}\right). \end{cases} \end{aligned}$$

**Proof.** We consider $W=A$. With abstention, we have

$$\Pr[PivA|\alpha]=\frac{1}{2}\Pr[T|\alpha]+\frac{1}{2}\Pr[T-1|\alpha].$$

Suppose signals are boundedly informative. So, from Lemma 6,

$$\begin{aligned} 1 &= \frac{\Pr\left[PivA|\alpha\right]}{\frac{1}{2}\Pr\left[T|\alpha\right]+\frac{1}{2}\Pr\left[T-1|\alpha\right]} \\ &\underset{n\to\infty}{\approx} \frac{\frac{1}{2}e^{-\sigma_A-\sigma_B}\frac{e^{2\sqrt{\sigma_A\sigma_B}}}{\sqrt{2\pi 2\sqrt{\sigma_A\sigma_B}}}\left(1+\frac{\sqrt{\sigma_B}}{\sqrt{\sigma_A}}\right)}{\frac{1}{2}\Pr\left[T|\alpha\right]+\frac{1}{2}\Pr\left[T-1|\alpha\right]} \\ &= \frac{\frac{1}{2}\frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{\sqrt{2\pi 2\sqrt{\sigma_A\sigma_B}}}\left(1+\frac{\sqrt{\sigma_B}}{\sqrt{\sigma_A}}\right)}{\frac{1}{2}\Pr\left[T|\alpha\right]+\frac{1}{2}\Pr\left[T-1|\alpha\right]}. \end{aligned}$$

Let

$$K_\alpha\left(\sigma,\tau\right)=\frac{\sqrt{2\pi 2\sqrt{\tau_A\tau_B}}}{\sqrt{2\pi 2\sqrt{\sigma_A\sigma_B}}}\frac{1+\frac{\sqrt{\sigma_B}}{\sqrt{\sigma_A}}}{1+\frac{\sqrt{\tau_B}}{\sqrt{\tau_A}}}.$$

Then, arguing analogously for state $\beta$,

$$\lim_{n\to\infty}\frac{\Pr\left[PivA|\alpha\right]}{\Pr\left[PivA|\beta\right]}=\lim_{n\to\infty}\frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}}K_\alpha\left(\sigma,\tau\right).$$

Given the hypothesis that signals are boundedly informative, it is immediate that

$$\lim_{n\to\infty}K_\alpha\left(\sigma,\tau\right)\in\left(0,\infty\right).$$

The claim follows. □

### A.4.2 Proof for the Unboundedly Informative Signal Case

Throughout this section, we consider the case with

$$\theta<1,$$

and signals are unboundedly informative, in particular,

$$\frac{g\left(\underline{x}|\alpha\right)}{g\left(\underline{x}|\beta\right)}=\infty.$$

Let $\Gamma\left(x_R,n\right)$ be an auxiliary game in which voters with signals $x\geq x_R$ must vote $B$, while voters below $x_R$ can choose between voting $A$, $B$, or abstaining as before. We will show that $\Gamma\left(x_R,n\right)$ has an equilibrium that satisfies the properties of the theorem and for which the constraint at $x_R$ does not bind.

Note that $\Gamma(x_R, n)$ has an equilibrium $(y, z)$ by standard arguments for all $x_R$; see Myerson (1998b).

We use the following Lemma, proven at the end of this section.

**Lemma 8.** *Suppose $\theta < 1$, abstention is possible, and signals are unboundedly informative. There exists some $x_R \in (\underline{x}, x_\alpha)$ such that for any sequence $(y^n, z^n)$ with $y^n \leq z^n \leq x_R$,*

$$\lim_{n\to\infty} \frac{\Pr[PivA|\alpha; y^n, z^n, n]}{\Pr[PivA|\beta; y^n, z^n, n]} = \lim_{n\to\infty} \frac{\Pr[PivB|\alpha; y^n, z^n, n]}{\Pr[PivB|\beta; y^n, z^n, n]} = 0.$$

Now, suppose that $(y^n, z^n)$ is an equilibrium sequence of $\Gamma(x_R, n)$, for $x_R$ chosen to satisfy Lemma 8. It cannot be that $z^n \to x_R$. Suppose otherwise. If $z^n \to x_R$, then

$$\begin{aligned}
\lim_{n\to\infty} \gamma_B(y^n, z^n, n) &= \lim_{n\to\infty} \frac{\pi}{1-\pi} \frac{g(z^n|\alpha)}{g(z^n|\beta)} \frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]} \\
&= \frac{\pi}{1-\pi} \frac{g(x_R|\alpha)}{g(x_R|\beta)} \lim_{n\to\infty} \frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]} \\
&= 0.
\end{aligned}$$

Thus, for any $x'$, any voter having a signal in $(x', x_R)$ would have a strict preference to vote for $B$. Thus, it must be that $\lim z^n < x_R$. But this implies that all voters with signals in $(\lim z^n, x_R)$ prefer voting $B$ to voting $A$ or abstaining. By the MLRP, this implies that in particular all voters with signals $x \geq x_R$ prefer voting $B$. Hence, the initial restriction of $\Gamma(x_R, n)$ relative to the original game does not bind. Therefore, for large $n$, $(y^n, z^n)$ is also an equilibrium of the original game. Clearly, from $\lim z^n < x_R < x_\alpha$, policy $B$ is chosen with probability converging to one. This proves the claim of the theorem.

**Proof of Lemma 8.**

There exists a signal $x_R \in (\underline{x}, x_\alpha)$ such that for all $y \leq z \leq x_R$,

$$\sqrt{1-G(z|\alpha)} - \sqrt{G(y|\alpha)} > \sqrt{\theta}\left(\sqrt{1-G(z|\beta)} - \sqrt{G(y|\beta)}\right).$$

Hence, it follows from the remark after Lemma 7 that for all $y^n \leq z^n \leq x_R$,

$$\lim_{n\to\infty} \frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}} = 0. \tag{16}$$

Moreover,

$$\lim \frac{e^{-\sigma_A-\sigma_B}}{e^{-\tau_A-\tau_B}} = 0.$$

This follows from

$$\begin{aligned} -\sigma_A - \sigma_B + \tau_A + \tau_B &= \tau_A - \sigma_A + \tau_B - \sigma_B \\ &= n\left(\theta G\left(y|\beta\right) - G\left(y|\alpha\right) + \theta\left(1 - G\left(z|\beta\right)\right) - \left(1 - G\left(z|\alpha\right)\right)\right), \end{aligned}$$

and

$$\left(\theta G\left(y|\beta\right) - G\left(y|\alpha\right) + \theta\left(1 - G\left(z|\beta\right)\right) - \left(1 - G\left(z|\alpha\right)\right)\right) < 0,$$

from $\theta G(y|\beta) < G(y|\alpha)$ (by the MLRP) and $\theta(1 - G(z|\beta)) < (1 - G(z|\alpha))$ (which is necessary by $z \leq x_R$ for our choice of $x_R$).

Case 1. Suppose $\tau_A\tau_B \to \infty$. From MLRP, $\sigma_A\sigma_B \to \infty$. Then, from Lemma 6,

$$\lim \frac{\Pr\left[PivA|\alpha\right]}{\Pr\left[PivA|\beta\right]} = \lim \frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}} \frac{\sqrt{2\pi 2\sqrt{\tau_A\tau_B}}}{\sqrt{2\pi 2\sqrt{\sigma_A\sigma_B}}} \frac{1+\frac{\sqrt{\sigma_B}}{\sqrt{\sigma_A}}}{1+\frac{\sqrt{\tau_B}}{\sqrt{\tau_A}}}.$$

If $\lim y^n > \underline{x}$, then we are done because of (16) and the last fractions are bounded. Suppose $\lim y^n = \underline{x}$. Then,

$$\lim \frac{\sqrt{2\pi 2\sqrt{\tau_A\tau_B}}}{\sqrt{2\pi 2\sqrt{\sigma_A\sigma_B}}} \frac{1+\frac{\sqrt{\sigma_B}}{\sqrt{\sigma_A}}}{1+\frac{\sqrt{\tau_B}}{\sqrt{\tau_A}}} < \infty,$$

since $\lim \frac{\tau_B}{\sigma_B} \in (0,\infty)$ and $\frac{\tau_A}{\sigma_A} \leq 1$.

Similarly,

$$\lim \frac{\Pr\left[PivB|\alpha\right]}{\Pr\left[PivB|\beta\right]} = \lim \frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}} \frac{\sqrt{2\pi 2\sqrt{\tau_A\tau_B}}}{\sqrt{2\pi 2\sqrt{\sigma_A\sigma_B}}} \frac{1+\frac{\sqrt{\sigma_A}}{\sqrt{\sigma_B}}}{1+\frac{\sqrt{\tau_A}}{\sqrt{\tau_B}}},$$

and

$$\lim \frac{\sqrt{2\pi 2\sqrt{\tau_A\tau_B}}}{\sqrt{2\pi 2\sqrt{\sigma_A\sigma_B}}}\frac{1+\frac{\sqrt{\sigma_A}}{\sqrt{\sigma_B}}}{1+\frac{\sqrt{\tau_A}}{\sqrt{\tau_B}}}<\infty,$$

follows from $\lim \frac{\sqrt{\sigma_A}}{\sqrt{\sigma_B}}<\infty$, $\lim \frac{\tau_B}{\sigma_B}\in(0,\infty)$, and $\frac{\tau_A}{\sigma_A}\leq 1$.

Case 2a. Suppose $\tau_A\tau_B\to k<\infty$ and $z=\lim \sigma_A\sigma_B<\infty$. This requires $y^n\to \underline{x}$. Then, from Lemma 6,

$$\begin{aligned}
\Pr[PivA|\beta] &\approx e^{-\tau_A-\tau_B}\left(I_0\left(2\sqrt{k}\right)+\tau_B\frac{I_1\left(2\sqrt{k}\right)}{\sqrt{k}}\right),\\
\Pr[PivB|\beta] &\approx e^{-\tau_A-\tau_B}\left(I_0\left(2\sqrt{k}\right)+\tau_A\frac{I_1\left(2\sqrt{k}\right)}{\sqrt{k}}\right),
\end{aligned}$$

with $\frac{I_1\left(2\sqrt{k}\right)}{\sqrt{k}}=1$ if $k=0$. Similarly, from $z=\lim \sigma_A\sigma_B<\infty$, we have

$$\begin{aligned}
\Pr[PivA|\alpha] &\approx e^{-\sigma_A-\sigma_B}\left(I_0\left(2\sqrt{z}\right)+\sigma_B\frac{I_1\left(2\sqrt{z}\right)}{\sqrt{z}}\right),\\
\Pr[PivB|\alpha] &\approx e^{-\sigma_A-\sigma_B}\left(I_0\left(2\sqrt{z}\right)+\sigma_A\frac{I_1\left(2\sqrt{z}\right)}{\sqrt{z}}\right).
\end{aligned}$$

So, if $\lim \sigma_A\sigma_B<\infty$ then

$$\lim \frac{\Pr[PivA|\alpha]}{\Pr[PivA|\beta]}=\lim \frac{e^{-\sigma_A-\sigma_B}}{e^{-\tau_A-\tau_B}}\frac{I_0\left(2\sqrt{z}\right)+\sigma_B\frac{I_1\left(2\sqrt{z}\right)}{\sqrt{z}}}{I_0\left(2\sqrt{k}\right)+\tau_B\frac{I_1\left(2\sqrt{k}\right)}{\sqrt{k}}}\to 0,$$

since $\frac{e^{-\sigma_A-\sigma_B}}{e^{-\tau_A-\tau_B}}\to 0$ by (16), $I_0\left(2\sqrt{k}\right)>0$, $\frac{I_1\left(2\sqrt{k}\right)}{\sqrt{k}}>0$ and $\lim \frac{\sigma_B}{\tau_B}<\infty$. Analogously,

$$\lim \frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]}=\lim \frac{e^{-\sigma_A-\sigma_B}}{e^{-\tau_A-\tau_B}}\frac{I_0\left(2\sqrt{z}\right)+\sigma_A\frac{I_1\left(2\sqrt{z}\right)}{\sqrt{z}}}{I_0\left(2\sqrt{k}\right)+\tau_A\frac{I_1\left(2\sqrt{k}\right)}{\sqrt{k}}}\to 0,$$

since $\lim \sigma_A=\lim \tau_A=0$.

Case 2b. Suppose $\tau_A \tau_B \to k < \infty$ and $\lim \sigma_A \sigma_B = \infty$. Then, from Lemma 6,

$$\lim \frac{\Pr[PivA|\alpha]}{\Pr[PivA|\beta]} = \lim \frac{e^{-\sigma_A - \sigma_B}}{e^{-\tau_A - \tau_B}} \frac{e^{2\sqrt{\sigma_A \sigma_B}} \left( \frac{\left(1 + \frac{\sqrt{\sigma_B}}{\sqrt{\sigma_A}}\right)}{\sqrt{2\pi 2\sqrt{\sigma_A \sigma_B}}} \right)}{I_0\left(2\sqrt{k}\right) + \tau_B \frac{I_1\left(2\sqrt{k}\right)}{\sqrt{k}}}.$$

Now, observe that

$$\begin{aligned} & \lim e^{2\sqrt{\tau_A \tau_B}} \frac{e^{-\sigma_A - \sigma_B}}{e^{-\tau_A - \tau_B}} \frac{e^{2\sqrt{\sigma_A \sigma_B}}}{e^{2\sqrt{\tau_A \tau_B}}} \\ = \; & e^{2\sqrt{k}} \lim \frac{e^{-\left(\sqrt{\sigma_A} - \sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A} - \sqrt{\tau_B}\right)^2}} \\ = \; & 0. \end{aligned}$$

Moreover, from $I_0\left(2\sqrt{k}\right) \in (0, \infty)$, $\frac{I_1\left(2\sqrt{k}\right)}{\sqrt{k}} \in (0, \infty)$, $\tau_B \to \infty$, and $\sigma_A \sigma_B \to \infty$

$$\lim \frac{\left( \frac{\left(1 + \frac{\sqrt{\sigma_B}}{\sqrt{\sigma_A}}\right)}{\sqrt{2\pi 2\sqrt{\sigma_A \sigma_B}}} \right)}{I_0\left(2\sqrt{k}\right) + \tau_B \frac{I_1\left(2\sqrt{k}\right)}{\sqrt{k}}} \leq \lim \frac{\frac{\sqrt{\sigma_B}}{\sqrt{\sigma_A}}}{\tau_B} \leq \lim \frac{\frac{\sqrt{\sigma_B}}{\sqrt{\sigma_A}}}{\sigma_B} = \frac{1}{\sqrt{\sigma_A \sigma_B}} \to 0.$$

This proves the result. □

### A.4.3 Proof for the Binary Case

Suppose signals are binary and suppose

$$\theta < 1$$

throughout the section.

We sometimes refer to a signal $x < x_B$ as an "$a$ signal" and a signal $x \geq x_B$ as a "$b$ signal," analogously to the notation from existing work on binary signals by Krishna and Morgan (2012) and others. We use the following notation for the signals,

$$\begin{aligned} r &= G\left(x_B|\alpha\right) \text{ and } 1 - r = 1 - G\left(x_B|\alpha\right) \\ s &= 1 - G\left(x_B|\beta\right) \text{ and } 1 - s = G\left(x_B|\beta\right), \end{aligned}$$

so that $\frac{r}{1-r} > 1 > \frac{1-s}{s}$. We only consider strategies $(y, z)$ with $y \le x_B$. Let

$$\begin{aligned} p &= \frac{G\left(\min\left\{y, x_B\right\}|\alpha\right)}{G\left(x_B|\alpha\right)} = \Pr\left(\text{Vote } A|x \le x_B\right) \\ q &= \frac{G\left(x_B|\alpha\right) - G\left(\min\left\{z, x_B\right\}|\alpha\right)}{G\left(x_B|\alpha\right)} = \Pr\left(\text{Vote } B|x \le x_B\right) \\ \gamma &= \frac{1 - G\left(\max\left\{z, x_B\right\}|\alpha\right)}{1 - G\left(x_B|\alpha\right)} = \Pr\left(\text{Vote } B|x \ge x_B\right). \end{aligned}$$

Noting that the definitions above also hold with $\alpha$ replaced by $\beta$ since $\frac{G(\min\{y,x_B\}|\alpha)}{G(x_B|\alpha)} = \frac{G(\min\{y,x_B\}|\beta)}{G(x_B|\beta)}$, etc...

Step 1. Suppose

$$\frac{1-r}{s} > \theta.$$

Then, there exist equilibria such that $\lim np = \infty$ and $\lim pr < \lim \gamma(1-r)$. In these equilibria, $\gamma = 1 > p > 0 = q$ for all $n$ large.

Proof of Step 1. We want to prove the existence of an equilibrium with $p \in (0,1)$, $\gamma = 1$, and $q = 0$. Such a strategy profile is an equilibrium if

$$\begin{aligned} \frac{r}{1-s}\frac{1}{\theta}\frac{\Pr\left[PivA|\alpha\right]}{\Pr\left[PivA|\beta\right]} &= 1 \text{ and } \frac{r}{1-s}\frac{1}{\theta}\frac{\Pr\left[PivB|\alpha\right]}{\Pr\left[PivB|\beta\right]} \ge 1, \\ \frac{1-r}{s}\frac{1}{\theta}\frac{\Pr\left[PivA|\alpha\right]}{\Pr\left[PivA|\beta\right]} &\le 1 \text{ and } \frac{1-r}{s}\frac{1}{\theta}\frac{\Pr\left[PivB|\alpha\right]}{\Pr\left[PivB|\beta\right]} \le 1. \end{aligned}$$

Given $\gamma = 1$, and $q = 0$, (15) becomes

$$\begin{aligned} &\lim_{n\to\infty} \frac{\Pr\left[PivW|\alpha\right]}{\Pr\left[PivW|\beta\right]} \qquad (17) \\ &= \begin{cases} \infty & \text{if } \lim_{n\to\infty}\sqrt{1-r} - \sqrt{pr} < \lim_{n\to\infty}\sqrt{\theta}\left(\sqrt{s} - \sqrt{p(1-s)}\right), \\ 0 & \text{if } \lim_{n\to\infty}\sqrt{1-r} - \sqrt{pr} > \lim_{n\to\infty}\sqrt{\theta}\left(\sqrt{s} - \sqrt{p(1-s)}\right). \end{cases} \end{aligned}$$

Fix some $\theta < \frac{1-r}{s}$. Let $p^*$ be the $p$ solution to

$$\sqrt{(1-r)} - \sqrt{pr} = \sqrt{\theta}\left(\sqrt{s} - \sqrt{p(1-s)}\right). \qquad (18)$$

A solution exists by the intermediate value theorem because at $p = 0$, the left-hand side$>$ right-hand side by $1 - r > s\theta$, and for $p = 1$, the left-hand side $< 0 <$ right-hand side. The solution $p^*$ satisfies $0 < p^* < 1$. Moreover, the solution also satisfies

$$p^* r < 1 - r,$$

for otherwise the left-hand side would be non-positive while the right-hand side$>$ 0. The solution is unique: For this, note that both sides are linear in $\sqrt{p}$ with coefficients $\sqrt{r} > \sqrt{\theta}\sqrt{1-s}$. Thus,

$$p < p^* \Rightarrow \sqrt{(1-r)} - \sqrt{pr} > \sqrt{\theta}\left(\sqrt{s} - \sqrt{p(1-s)}\right) \tag{19}$$

and vice versa for $p > p^*$. Pick $\varepsilon > 0$ such that $0 < p^* - \varepsilon < p^* + \varepsilon < 1$ and $(p^* + \varepsilon) r < (1 - r)$.

Notice that for all $p \in [p^* - \varepsilon, p^* + \varepsilon]$, $\sigma_A \sigma_B = (npr)(n(1-r)) \to \infty$, validating the use of the previous approximations. With $p$ as a free variable, define

$$P(p, n) := \frac{\Pr[PivA|\alpha; p]}{\Pr[PivA|\beta; p]}.$$

From (17) and (19),

$$\lim P(p^* - \varepsilon, n) = 0, \text{ and } \lim P(p^* + \varepsilon, n) = \infty. \tag{20}$$

(A rough intuition may be this: $B$ wins in both states. The margin of victory is necessarily larger in state $\beta$ and the election closer to being tied in state $\alpha$. Increasing $p$ brings the election even closer to being tied in state $\alpha$. Thus, the probability of state $\alpha$ increases if we increase $p$.)

Notice that for fixed $n$, $P(p, n)$ is a continuous function. This and (20) implies that for all large $n$, there is a $\bar{p}^n \in (p^* - \varepsilon, p^* + \varepsilon)$ such that

$$\frac{r}{1-s}\frac{1}{\theta} P(\bar{p}^n, n) = 1 \text{ for all } n. \tag{21}$$

From (17) and from the definition and uniqueness of $p^*$, we have $\bar{p}^n \to p^*$.

Now, observe that

$$\frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]} = \frac{\Pr[T|\alpha]}{\Pr[T|\beta]}\frac{1+\frac{\sqrt{\sigma_A}}{\sqrt{\sigma_B}}}{1+\frac{\sqrt{\tau_A}}{\sqrt{\tau_B}}} = \frac{\Pr[T|\alpha]}{\Pr[T|\beta]}\frac{1+\frac{\sqrt{\sigma_B}}{\sqrt{\sigma_A}}}{1+\frac{\sqrt{\tau_B}}{\sqrt{\tau_A}}}\frac{\frac{\sqrt{\sigma_A}}{\sqrt{\sigma_B}}}{\frac{\sqrt{\tau_A}}{\sqrt{\tau_B}}} = \frac{\Pr[PivA|\alpha]}{\Pr[PivA|\beta]}\frac{\frac{\sqrt{\sigma_A}}{\sqrt{\sigma_B}}}{\frac{\sqrt{\tau_A}}{\sqrt{\tau_B}}}, \quad (22)$$

using (9) and (10). From,

$$\frac{\frac{\sigma_A}{\sigma_B}}{\frac{\tau_A}{\tau_B}} = \frac{\frac{r}{(1-r)}}{\frac{(1-s)}{s}} = \frac{r}{(1-r)}\frac{s}{(1-s)} > 1, \quad (23)$$

we have that the second optimality condition for voters with an $a$ signal holds since,

$$\frac{r}{1-s}\frac{1}{\theta}\frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]} > 1.$$

Now, consider voters with a $b$ signal. From the indifference condition of the $a$ voter,

$$\lim \frac{\Pr[T|\alpha]}{\Pr[T|\beta]}\frac{1+\frac{\sqrt{\sigma_B}}{\sqrt{\sigma_A}}}{1+\frac{\sqrt{\tau_B}}{\sqrt{\tau_A}}} = \theta\frac{1-s}{r}.$$

Hence, from (22) and (23)

$$\lim \frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]} = \theta\frac{(1-s)}{r}\sqrt{\frac{r}{(1-r)}\frac{s}{(1-s)}} = \theta\sqrt{\frac{(1-s)s}{(1-r)r}}.$$

Hence,

$$\lim \frac{1-r}{s}\frac{1}{\theta}\frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]} = \frac{1-r}{s}\frac{1}{\theta}\theta\sqrt{\frac{(1-s)s}{(1-r)r}} = \sqrt{\frac{1-s}{s}\frac{1-r}{r}} < 1.$$

So, the $b$ voter strictly prefers voting for $B$ to abstaining. Moreover, from MLRP,

$$\frac{1-r}{s}\frac{1}{\theta}\frac{\Pr[PivA|\alpha]}{\Pr[PivA|\beta]} < \frac{r}{1-s}\frac{1}{\theta}\frac{\Pr[PivA|\alpha]}{\Pr[PivA|\beta]} = 1.$$

Therefore, $\gamma = 1$ is a best response to $(\bar{p}^n, q = 0, \gamma = 1)$ for $n$ large enough.

Thus, for large $n$, $(\bar{p}^n, q = 0, \gamma = 1)$ is an equilibrium profile. Finally, from $\bar{p}^n \to p^*$ and $p^*$ solving (18), it must be that $\bar{p}^n r < \gamma(1-r) = (1-r)$. This proves the first step.

Step 2. Suppose

$$1 > \theta > \frac{1-r}{s}. \tag{24}$$

For all $n \geq \bar{n}$, there exists an equilibrium profile $(p^*, q^*, \gamma^*)$ with $p^* \in (0,1)$, $q^* \in (0,1)$, and $\gamma^* = 1$ such that for $n \to \infty$

$$\lim rp^* < \lim rq^* + (1-r).$$

A strategy profile $(p, q, \gamma)$ with $p \in (0,1)$, $q \in (0,1)$, and $\gamma = 1$ is an equilibrium if

$$\begin{aligned}
\frac{r}{1-s}\frac{1}{\theta}\frac{\Pr[PivA|\alpha]}{\Pr[PivA|\beta]} &= 1 \text{ and } \frac{r}{1-s}\frac{1}{\theta}\frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]} = 1, \\
\frac{1-r}{s}\frac{1}{\theta}\frac{\Pr[PivA|\alpha]}{\Pr[PivA|\beta]} &\leq 1 \text{ and } \frac{1-r}{s}\frac{1}{\theta}\frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]} \leq 1.
\end{aligned}$$

Note that the inequalities are implied by the equalities.

Define

$$\begin{aligned}
g(p,q) &: = \frac{\Pr(PivA|\alpha; p, q)}{\Pr(PivA|\beta; p, q)}, \\
f(p,q) &: = \frac{\Pr(PivB|\alpha; p, q)}{\Pr(PivB|\beta; p, q)},
\end{aligned}$$

for $(p,q) \in [0,1]^2$, $p + q \leq 1$.

The proof uses two lemmas.

**Lemma 9.** *There exist $\bar{n}$, and $\bar{p}, \bar{q}, q_0 \in [0,1]^3$, with $\bar{q} > q_0$, $\bar{p} > 0$, $\bar{q} + \bar{p} \leq 1$, and $rq_0 + (1-r) > r\bar{p}$ such that for all $n \geq \bar{n}$ and $p \in [0, \bar{p}]$ ,*

$$\frac{r}{1-s}\frac{1}{\theta} f(p, q_0) > 1, \tag{25}$$

*and*

$$\frac{r}{1-s}\frac{1}{\theta} f(p, \bar{q}) < 1, \tag{26}$$

*So, by continuity of $f$, for all $p \in [0, \bar{p}]$ there exists a solution $\tilde{q}(p) \in [q_0, \bar{q}]$ to*

$$\frac{r}{1-s}\frac{1}{\theta}f(p,q) = 1.$$

*Any such solution satisfies*

$$g(0, \tilde{q}(0)) > f(0, \tilde{q}(0)), \tag{27}$$

*and*

$$g(\bar{p}, \tilde{q}(\bar{p})) < f(\bar{p}, \tilde{q}(\bar{p})). \tag{28}$$

**Proof:** Pick some $q_0 \in (0,1)$ with

$$q_0 r + (1-r) < \theta(q_0(1-s) + s),$$

which exists by (24) and pick some $\bar{q} \in (q_0, 1)$ with

$$\bar{q}r + (1-r) > \theta(\bar{q}(1-s) + s),$$

which exists by $\theta < 1$ from (24). Finally, pick $\bar{p} \in (0, 1-\bar{q})$ small enough such that

$$\sqrt{r\bar{q} + (1-r)} - \sqrt{r\bar{p}} > \sqrt{\theta((1-s)\bar{q} + s)} - \sqrt{\theta(1-s)\bar{p}}. \tag{29}$$

Such $\bar{p}$ exists since the inequality holds strictly for $\bar{p} = 0$. Since $\theta((1-s)\bar{q} + s) > \theta(1-s)p$,

$$rq_0 + (1-r) > r\bar{p}.$$

**Remark:** The choices imply that for all $(p,q) \in [0,\bar{p}] \times [q_0, \bar{q}]$,

$$\sqrt{\sigma_B} > \sqrt{\sigma_A},$$

and hence $\sqrt{\tau_B} > \sqrt{\tau_A}$. Furthermore,

$$\sqrt{\tau_B} - \sqrt{\tau_A} > \sqrt{\sigma_B} - \sqrt{\sigma_A} \text{ for } q = q_0 \text{ and any } p \in [0, \bar{p}],$$

and

$$\sqrt{\tau_B} - \sqrt{\tau_A} < \sqrt{\sigma_B} - \sqrt{\sigma_A} \text{ for } q = \bar{q} \text{ and any } p \in [0, \bar{p}].$$

**Proof of Equation (25),**

$$\frac{r}{1-s}\frac{1}{\theta}f\left(p,q_0\right)>1 \text{ for all } 0\leq p\leq \bar{p}.$$

The proof shows that for any sequence of $(p,n)$, for $n\to\infty$, $f\left(p,q_0\right)\to\infty$.
Case 1: $\sigma_A\sigma_B\to\infty$. Then,

$$\lim\frac{\Pr\left[PivB|\alpha\right]}{\Pr\left[PivB|\beta\right]}=\lim\frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}}K_\beta\left(\sigma,\tau\right),$$

with $\lim K_\beta\left(\sigma,\tau\right)\in(0,\infty)$. We show that

$$\lim\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2=\infty,$$

that is, by choice of $p\leq\bar{p}$ (which implies $\sqrt{\sigma_B}>\sqrt{\sigma_A}$),

$$\theta\left(\sqrt{(1-s)\,q_0+s}-\sqrt{(1-s)\,p}\right)^2-\left(\sqrt{rq_0+(1-r)}-\sqrt{rp}\right)^2>0.$$

By choice of $q_0$,

$$\theta\left(\left(1-s\right)q_0+s\right)>rq_0+\left(1-r\right),$$

and from $\theta<1$ and $(1-s)<r$, for all $p$,

$$\theta\left(1-s\right)p<rp.$$

Case 2: $\sigma_A\sigma_B\to k<\infty$. Suppose $k=0$ (the case $k\in(0,\infty)$ is analogous). Then, $\tau_A\tau_B\to 0$ as well. Also $\sigma_A\to 0$ from $\sigma_B\geq n\left(1-r\right)\to\infty$. Similarly, $\tau_A\to 0$. From Lemma 6,

$$\lim\frac{\Pr\left[PivB|\alpha\right]}{\Pr\left[PivB|\beta\right]}=\lim\frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}}\frac{\sigma_B+1}{\tau_B+1}=\infty,$$

since $\lim\frac{\sigma_B+1}{\tau_B+1}>0$ and $\lim\frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}}=\infty$ (as before).

**Proof of Equation (26),**

$$\frac{r}{1-s}\frac{1}{\theta}f\left(p,\bar{q}\right)<1.$$

Case 1: $\sigma_A\sigma_B\to\infty$. Then, from Lemma 7

$$\lim \frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]} = \lim \frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}} K_\beta(\sigma,\tau),$$

with $\lim K_\beta(\sigma,\tau) \in (0,\infty)$. We show that

$$\lim \left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2 - \left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2 = -\infty$$

that is,

$$\theta\left(\sqrt{(1-s)\bar{q}+s}-\sqrt{(1-s)\,p}\right)^2 - \left(\sqrt{r\bar{q}+(1-r)}-\sqrt{rp}\right)^2 < 0$$

but this holds by choice of $\bar{p}$ in (29).

Case 2: $\sigma_A\sigma_B \to k$. This is just as the proof of Case 2 in the proof of Equation (25) and therefore omitted.

**Proof of Equation (27):** Any solution $\tilde{q}(p)$ with

$$\frac{r}{1-s}\frac{1}{\theta} f(p,\tilde{q}(p)) = 1.$$

satisfies

$$g(0,\tilde{q}(0)) > f(0,\tilde{q}(0)). \tag{30}$$

So, suppose $p = 0$ in the following. A $B$ vote is only pivotal if there is no other vote. An $A$ vote is pivotal is there is either no other vote or one other vote (which would be a B vote). Let $m$ count the number of other $B$ votes. The probability that there is either no voter or 1 voter is

$$\begin{aligned}
\Pr(m=0|\alpha) &= e^{-n(qr+(1-r))} \\
\Pr(m=1|\alpha) &= e^{-n(qr+(1-r))} n(qr+(1-r))
\end{aligned}$$

and

$$\begin{aligned}
\Pr(m=0|\beta) &= e^{-\theta n(q(1-s)+s)} \\
\Pr(m=1|\beta) &= e^{-\theta n(q(1-s)+s)} \theta n(q(1-s)+s)
\end{aligned}$$

and so a $B$ vote is pivotal implies

$$\frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]} = \frac{e^{-n(qr+(1-r))}}{e^{-\theta n(q(1-s)+s)}} = e^{n(\theta(q(1-s)+s)-(qr+(1-r)))}$$

and

$$\begin{aligned}\frac{\Pr[PivA|\alpha]}{\Pr[PivA|\beta]} &= \frac{e^{-n(qr+(1-r))}+e^{-n(qr+(1-r))}n\left(qr+(1-r)\right)}{e^{-\theta n(q(1-s)+s)}+e^{-\theta n(q(1-s)+s)}\theta n\left(q\left(1-s\right)+s\right)} \\ &= \frac{e^{-n(qr+(1-r))}}{e^{-\theta n(q(1-s)+s)}}\frac{1+n\left(qr+(1-r)\right)}{1+\theta n\left(q\left(1-s\right)+s\right)}\end{aligned}$$

Let $f(q,n) \equiv \frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]}$ (recall that we set $p \equiv 0$ for this part). Let $q^* \in (q_0, \bar{q})$ solve

$$\theta\left(q\left(1-s\right)+s\right) = \left(qr+(1-r)\right)$$

Such a solution exists because by our choices at $q = q_0$, we have $\theta s > q_0 r + (1-r)$ and for $q = \bar{q}$ we have $\theta s < \bar{q}r + (1-r)$.

Then,

$$\lim f\left(q_0, n\right) = \infty$$

since at $q = q_0$,

$$\lim n\left(\theta\left(q\left(1-s\right)+s\right)-\left(qr+(1-r)\right)\right) = \infty,$$

and

$$\lim f\left(\bar{q}, n\right) = 0,$$

since at $q = \bar{q}$

$$\lim n\left(\theta\left(q\left(1-s\right)+s\right)-\left(qr+(1-r)\right)\right) = -\infty.$$

Thus, for all large $n$, we can find a $\tilde{q}^n$ such that

$$\frac{r}{1-s}\frac{1}{\theta}f\left(\tilde{q}^n, n\right) = 1.$$

Finally, from

$$f\left(\tilde{q}^n, n\right) < 1,$$

and
$$\frac{\Pr\left[PivB|\alpha\right]}{\Pr\left[PivB|\beta\right]}=\frac{e^{-n(qr+(1-r))}}{e^{-\theta n(q(1-s)+s)}}=e^{n(\theta(q(1-s)+s)-(qr+(1-r)))},$$
we get
$$\theta\left(\tilde{q}^{n}\left(1-s\right)+s\right)<\left(\tilde{q}^{n}r+\left(1-r\right)\right),$$
and so with
$$\frac{\Pr\left[PivA|\alpha\right]}{\Pr\left[PivA|\beta\right]}=\frac{e^{-n(qr+(1-r))}}{e^{-\theta n(q(1-s)+s)}}\frac{1+n\left(qr+\left(1-r\right)\right)}{1+\theta n\left(q\left(1-s\right)+s\right)},$$
we have
$$\frac{1+n\left(\tilde{q}^{n}r+\left(1-r\right)\right)}{1+\theta n\left(\tilde{q}^{n}\left(1-s\right)+s\right)}>1.$$
This and
$$\frac{\Pr\left[PivA|\alpha\right]}{\Pr\left[PivA|\beta\right]}=\frac{\Pr\left[PivB|\alpha\right]}{\Pr\left[PivB|\beta\right]}\frac{1+n\left(\tilde{q}^{n}r+\left(1-r\right)\right)}{1+\theta n\left(\tilde{q}^{n}\left(1-s\right)+s\right)},$$
implies
$$\frac{\Pr\left[PivA|\alpha\right]}{\Pr\left[PivA|\beta\right]}>\frac{\Pr\left[PivB|\alpha\right]}{\Pr\left[PivB|\beta\right]},$$
proving what we wanted to show.

**Remark.** This step fails with $\pi\neq 1$. If $\pi\neq 1$, then there may be equilibria in which $p=0$ and $q\in(0,1)$, supported by the fact that $\frac{\Pr[PivA|\alpha]}{\Pr[PivA|\beta]}\leq\frac{\Pr[PivB|\alpha]}{\Pr[PivB|\beta]}$. This is because when $\pi\neq 1$, then it does not need to be the case that $f\left(\tilde{q}^{n},n\right)<1$.

**Proof Equation (28):** Any solution $\tilde{q}\left(p\right)$ with
$$\frac{r}{1-s}\frac{1}{\theta}f\left(p,\tilde{q}\left(p\right)\right)=1.$$
satisfies
$$g\left(\bar{p},\tilde{q}\left(\bar{p}\right)\right)<f\left(\bar{p},\tilde{q}\left(\bar{p}\right)\right).$$

By choice of the boundaries, $\tilde{q}\left(\bar{p}\right)\in\left[q_{0},\bar{q}\right]\subset(0,1)$ and $\bar{p}\in(0,1)$, implying that $\sigma_{A}\sigma_{B}\rightarrow\infty$. So,
$$\frac{\Pr\left[PivB|\alpha\right]}{\Pr\left[PivB|\beta\right]}\approx\frac{\Pr\left[PivA|\alpha\right]}{\Pr\left[PivA|\beta\right]}\frac{\frac{\sqrt{\sigma_{A}}}{\sqrt{\sigma_{B}}}}{\frac{\sqrt{\tau_{A}}}{\sqrt{\tau_{B}}}},$$

and inspection shows

$$\frac{\sigma_A}{\sigma_B}\frac{\tau_B}{\tau_A} = \frac{rp}{(1-s)\,p}\frac{(1-s)\,q+s}{rq+(1-r)} = \frac{q+\frac{s}{1-s}}{q+\frac{1-r}{r}} > 1,$$

and hence,

$$\frac{\Pr\left[PivB|\alpha\right]}{\Pr\left[PivB|\beta\right]} > \frac{\Pr\left[PivA|\alpha\right]}{\Pr\left[PivA|\beta\right]},$$

as claimed. □

**Lemma 10.** *Continuous Selection. Suppose $f : [0,1] \times [0,1] \to [0,1]$ is a function such that*

- *$f(0,r) < 1$ for all $r$*
- *$f(1,r) > 1$ for all $r$*
- *$f$ is continuous in $(x,r)$*

*Then, there exist continuous functions $\hat{x} : [0,1] \to [0,1]$ and $\hat{r} : [0,1] \to [0,1]$ such that $\hat{r}(0) = 0$, $\hat{r}(1) = 1$ and*

$$f\left(\hat{x}(t), \hat{r}(t)\right) = 1 \text{ for all } t.$$

**Proof:** See Section B. □

**Proof of Step 2:**

For all $n \geq \bar{n}$, there exists an equilibrium profile $(p^*, q^*, \gamma^*)$ with $p^* \in (0,1)$, $q^* \in (0,1)$, and $\gamma^* = 1$ such that for $n \to \infty$

$$\lim rp^* < \lim rq^* + (1-r).$$

We are looking for $(p^*, q^*) \in [0, \bar{p}] \times [q_0, \bar{q}]$ such that

$$g\left(p^*, q^*\right) = f\left(p^*, q^*\right) = \theta\frac{1-s}{r}.$$

Any such solution will be an equilibrium. Since $q^* \geq q_0$ and $p \leq \bar{p}$, information fails to aggregate.

Note that $f$ is a continuous function with

$$f(p, q_0) < \theta \frac{1-s}{r} < f(p, \bar{q}),$$

for all $p \in [0, \bar{p}]$ and $n \geq \bar{n}$, by (25) and (26) from Lemma 9.

Thus, by the continuous selection lemma, there are continuous functions $\hat{p} : [0,1] \to [0, \bar{p}]$ and $\hat{q} : [0,1] \to [q_0, \bar{q}]$ such that $\hat{p}(0) = 0$ and $\hat{p}(1) = \bar{p}$ and

$$f(\hat{p}(t), \hat{q}(t)) = \theta \frac{1-s}{r} \text{ for all } t.$$

Now,

$$g(\hat{p}(0), \hat{q}(0)) - f(\hat{p}(0), \hat{q}(0)) = g(0, \hat{q}(0)) - f(0, \hat{q}(0)) > 0,$$

and

$$g(\hat{p}(1), \hat{q}(1)) - f(\hat{p}(1), \hat{q}(1)) = g(1, \hat{q}(1)) - f(1, \hat{q}(1)) < 0,$$

by (27) and (28).

By construction, the difference $g(p,q) - f(p,q)$ is continuous in $(p,q)$. Hence, $g(\hat{p}(t), \hat{q}(t)) - f(\hat{p}(t), \hat{q}(t))$ is continuous in $t$. Therefore, by the intermediate value theorem, there exists some $t^*$ such that

$$g(\hat{p}(t^*), \hat{q}(t^*)) - f(\hat{p}(t^*), \hat{q}(t^*)) = 0.$$

Thus, $(p^*, q^*) = (\hat{p}(t^*), \hat{q}(t^*))$ solves the equilibrium conditions. Since $\hat{p}(t^*), \hat{q}(t^*) \in [0, \bar{p}] \times [q_0, \bar{q}]$, we have $\lim rp^* < \lim rq^* + (1-r)$. This proves Step 2.

## A.5 Proofs for Section 6 (Learning from Participation)

**Proof of Lemma 5.**

Suppose $k \in (-\infty, \infty)$. Let $t_a^n, t_b^n$ be the number of $A$ and $B$ votes in state $\alpha$. $t_a^n$ is Poisson-distributed with mean $nq_a^n$, and $t_b^n$ is Poisson-distributed with mean $nq_n^b$, and they are independent from each other. Let

$$z_a^n := \frac{t_a^n - nq_a^n}{\sqrt{nq_a^n}},$$
$$z_b^n := \frac{t_b^n - nq_b^n}{\sqrt{nq_b^n}}.$$

Because $k \in (-\infty, \infty)$, $nq_a^n, nq_b^n \to \infty$. Hence, the central limit theorem (e.g. Greene, 2003, p. 912) implies that

$$(z_a^n, z_b^n) \to \mathcal{N}\left((0,0), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right). \tag{31}$$

Because $\lim Pr\{t_a^n = t_b^n | \alpha\} = 0$, we obtain

$$\begin{aligned} \lim Pr\{\text{a wins}|\alpha, q_n\} &= \lim Pr\{t_a^n > t_b^n\} \\ &= \lim Pr\left\{z_a^n \sqrt{nq_a^n} + nq_a^n > z_b^n \sqrt{nq_b^n} + nq_b^n\right\} \\ &= \lim Pr\left\{z_a^n \sqrt{nq_a^n} - z_b^n \sqrt{nq_b^n} > n\left(q_b^n - q_a^n\right)\right\} \\ &= \lim Pr\left\{z_a^n \sqrt{q_a^n} - z_b^n \sqrt{q_b^n} > \sqrt{n}\left(q_b^n - q_a^n\right)\right\}. \end{aligned}$$

Equation 31 together with $q_a^n + q_b^n = 1$ implies that

$$z_a^n \sqrt{q_a^n} - z_b^n \sqrt{q_b^n} \to^D \mathcal{N}(0,1).$$

Hence,

$$\begin{aligned} \lim Pr\left\{z_a^n \sqrt{q_a^n} - z_b^n \sqrt{q_b^n} > \sqrt{n}\left(q_b^n - q_a^n\right)\right\} &= 1 - \Phi\left(\lim \sqrt{n}\left(q_b^n - q_a^n\right)\right) \\ &= 1 - \Phi(-2k) \\ &= \Phi(2k). \end{aligned}$$

If $k = \infty$, then $\lim Pr\{t_a^n > n/2\} = 1$, and $\lim Pr\{t_b^n < n/2\} = 1$, hence $\lim Pr\{\text{a wins}|\alpha, q_n\} = 1 = \Phi(\infty)$. An analogous argument establishes that if $k = -\infty$, $\lim Pr\{\text{a wins}|\alpha, q_n\} = 0 = \Phi(-\infty)$.

For state $\beta$, $\lim \sqrt{n}\left(q_a^n - 1/2\right) = k$ implies that $\lim \sqrt{\theta n}\left(q_a^n - 1/2\right) = \sqrt{\theta}k$. Hence, the same analysis for state $\alpha$ applies here, and

$$\lim Pr\{\text{a wins}|\beta, q_n\} = \Phi\left(2\sqrt{\theta}k\right).$$

□

**Proof of Theorem 7.**

Item 1:

Let $g(k) := \pi\Phi(k) + (1-\pi)\left(1 - \Phi\left(\sqrt{\theta}k\right)\right)$. The maximum of $g(k)$ for $k \in [-\infty, \infty]$ is attained at one of its critical points. Taking the first order condition, we find that

$$g'(k) = \pi\phi(k) - (1-\pi)\sqrt{\theta}\phi(\sqrt{\theta}k). \tag{32}$$

$g'(k) < 0$ if $\pi < (1-\pi)\sqrt{\theta}$. Hence, in this case, the maximum is attained at $k = -\infty$. Because we assume $\theta < 1$, and $\pi \geq 1/2$, $\pi > (1-\pi)\sqrt{\theta}$. In this case, we obtain that $g'(k) = 0$ if and only if $k \in \{-k^*, k^*\}$. Hence, the maximum is attained at $k \in \{-\infty, -k^*, k^*, \infty\}$. Because $\pi \geq 1/2$, $k = -\infty$ is never the unique optimal solution (it is never optimal if $\pi > 1/2$). Taking the second derivative of $g$ delivers that $g''(-k^*) > 0$. Hence, the only candidates that attain the maximum are $k \in \{k^*, \infty\}$.

We now verify that $g''(k^*) < 0$. Taking the derivative of equation 32, we obtain that

$$sign(g''(k)) = sign\left(-k\pi\phi(k) + k(1-\pi)\theta\phi(\sqrt{\theta}k)\right).$$

Because $k^* > 0$, we obtain that

$$sign(g''(-k^*)) = sign\left(\pi\phi(k^*) - (1-\pi)\theta\phi(\sqrt{\theta}k^*)\right).$$

Because $g'(-k^*) = \pi\phi(-k^*) - (1-\pi)\sqrt{\theta}\phi(-\sqrt{\theta}k^*) = 0$, $\sqrt{\theta} < 1$, and because $\phi(-\sqrt{\theta}k^*) > 0$, we obtain that

$$\pi\phi(k^*) - (1-\pi)\theta\phi(\sqrt{\theta}k^*) > \pi\phi(-k^*) - (1-\pi)\sqrt{\theta}\phi(-\sqrt{\theta}k^*) = 0.$$

Therefore,

$$g''(-k^*) > 0.$$

A similar argument establishes that $g''(k^*) < 0$.

Observe that by L'Hopital's rule,

$$\lim_k \frac{1 - \Phi(\sqrt{\theta}k)}{1 - \Phi(k)} = \lim_k \frac{\sqrt{\theta}\phi\left(\sqrt{\theta}k\right)}{\phi(k)} = \infty.$$

Therefore, there exists some $\bar{k} > 0$ such that $g(k) > \pi$ for all $k > \bar{k}$. Therefore, the maximum cannot be attained at $k = \infty$. Hence, the maximum is attained at $k^*$.

<u>Item 2:</u>

Denote $k^*(\theta) := \sqrt{\frac{2}{1-\theta}\ln\left(\frac{\pi}{(1-\pi)\sqrt{\theta}}\right)}$. Observe that $\lim_{\theta\to 0} k^*(\theta) = \infty$, therefore, in state $\alpha$, $A$ wins with probability approaching to 1. We will now show that

$$\lim_{\theta\to 0} \sqrt{\theta}k^*(\theta) = 0.$$

Note that $\sqrt{\theta}k^*(\theta) = \sqrt{\frac{2\theta}{1-\theta}\ln\left(\frac{\pi}{(1-\pi)\sqrt{\theta}}\right)}$. Note also that, by L'Hopital's rule we have

$$\lim_{\theta\to 0} \theta \ln\theta = 0.$$

Hence,

$$\lim_{\theta\to 0} \frac{2\theta}{1-\theta}\ln\left(\frac{\pi}{(1-\pi)\sqrt{\theta}}\right) = \lim_{\theta\to 0} -\theta\ln(\theta) = 0.$$

Therefore, $\lim_{\theta\to 0}\sqrt{\theta}k^*(\theta) = 0$. Hence, $\lim_{\theta\to 0}\lim E_n^{rep}[u; q_*^n] = \pi + \frac{1}{2}(1-\pi)$. □

We now complete the remaining case for Theorem 7.

**Theorem 8.** *Suppose* $0 < \theta < 1$*. There exists a* $\rho^*(\theta) \in \left(\max\{1/2, \sqrt{\theta}\}, 1\right)$ *such that:*

1. *If* $\frac{\pi}{1-\pi} \leq \rho^*(\theta)$*, then* $\lim E_n^{rep}[u; q_*^n] = (1-\pi)$*.*
2. *If* $\frac{\pi}{1-\pi} > \rho^*(\theta)$*, then* $\lim E_n^{rep}[u; q_*^n] = \pi\Phi(k^*) + (1-\pi)\left(1 - \Phi\left(\sqrt{\theta}k^*\right)\right)$*.*
3. $\rho^*$ *is increasing in* $\theta$ *for* $\theta \in (0,1)$*.*

**Proof.** Recall that $g(k) := \pi\Phi(k) + (1-\pi)\left(1-\Phi\left(\sqrt{\theta}k\right)\right)$, and $g'(k) = \pi\phi(k) - (1-\pi)\sqrt{\theta}\phi(\sqrt{\theta}k)$.

If $\pi \leq \sqrt{\theta}(1-\pi)$, then $g'(k) < 0$ for all $k \neq 0$, because $\phi(k) < \phi(\sqrt{\theta}k)$ for $k \neq 0$ (equality is attained only if $k = 0$, and if $\pi = \sqrt{\theta}(1-\pi)$). Hence, $g$ is decreasing, and the optimum is attained at $k = -\infty$.

If $\pi \leq \frac{1}{2}(1-\pi)$, then for all $k > 0$, $g(k) < 1-\pi$, because $\Phi\left(\sqrt{\theta}k\right) > 1/2$. Hence, $g(-\infty) > g(k^*)$ if $k^* > 0$. Note that $k^* = 0$ if and only if $\pi = \sqrt{\theta}(1-\pi)$. In that case, the previous item delivers that $g(-\infty) > g(k^*)$.

If $\pi \geq 1-\pi$, then Theorem 7 established that $\lim E_n^{rep}[u; q_*^n] = \pi\Phi(k^*) + (1-\pi)\left(1-\Phi\left(\sqrt{\theta}k^*\right)\right)$.

Hence, if there exists a cutoff $\rho^*(\theta)$, then it is in the interval we claimed. Now we show the existence of the cutoff $\rho^*(\theta)$.

When $\pi < 1/2$, the optimal is attained either at $k = k^*$, or at $k = -\infty$.

Viewing $g$ also dependent on $\pi$, observe that,

$$g(k;\pi) - (1-\pi) = \pi\Phi(k) - (1-\pi)\Phi\left(\sqrt{\theta}k\right).$$

$g(k;\pi) - (1-\pi)$ is increasing in $\pi$, for every $k \neq 0$, and is constant for $k = 0$. Viewing $k^*$ as a function of $\pi$, we have that if for some $\pi_1$, $g(k^*(\pi_1);\pi_1) - (1 - \pi_1) \geq 0$, then, for all $\pi > \pi_1$, we have that $g(k^*(\pi_1);\pi_2) - (1-\pi_2) > 0$. Moreover, $g(k^*(\pi_1);\pi_2) < g(k^*(\pi_2);\pi_2)$. This is because, as shown in the proof of Theorem 7, $g'(k) = 0$ has a unique positive solution (when $\pi$ is in the range we are considering), and $g''(k^*) < 0$. Hence, $g(\cdot)$ is maximized at $k = k^*$ in the range $k \geq 0$. Hence, $g(k^*(\pi_2);\pi_2) - (1-\pi_2) > 0$. Moreover, $g(k^*(1/2);1/2) - (1/2) > 0$, and by continuity, the property holds for an open interval around 1/2. This shows the existence of the cutoff $\rho^*(\theta)$.

We now show that $\rho^*$ is increasing in $\theta$ for $\theta \in (0,1)$. Observe that $g$ viewed as a function of $\theta$, $g(k;\theta)$ is decreasing in $\theta$ for $k > 0$. Hence, if for $\pi > 0$, and $k > 0$, $\pi\Phi(k) - (1-\pi)\Phi\left(\sqrt{\theta}k\right) \geq 0$, then for all $\theta' < \theta$, we have $\pi\Phi(k) - (1-\pi)\Phi\left(\sqrt{\theta'}k\right) > 0$. Hence, the same reasoning in the previous paragraph delivers that, if $k = -\infty$ is not uniquely optimal for some $(\pi, \theta)$, then it is suboptimal for $(\pi, \theta')$ for all $\theta' \in (0, \theta)$. Hence, $\rho^*$ is increasing in $\theta$. $\square$

# B The Generalized Intermediate Value Theorem (Lemma 10)

**Lemma 10 (Restatement).** *Continuous Selection. Suppose* $f : [0,1] \times [0,1] \to [-1,1]$ *is a function such that*

- $f(0,r) < 0$ *for all* $r$
- $f(1,r) > 0$ *for all* $r$
- $f$ *is continuous in* $(x,r)$

*Then, there exist continuous functions $\hat{x} : [0,1] \to [0,1]$ and $\hat{r} : [0,1] \to [0,1]$ such that $\hat{r}(0) = 0$, $\hat{r}(1) = 1$ and*

$$f(\hat{x}(t), \hat{r}(t)) = 0 \text{ for all } t.$$

For intuition, consider the following related statement. Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with $g(x_0) = g_0 > 0$ for some $x_0 \in (0,1)^2$ and $g(x) = 0$ for all $x \notin [0,1]^2$. Then, for every $b \in (0, g_0)$ there exists a loop in $[0,1]^2$ that goes around $x_0$ such that $g(x) \equiv b$ for all $x$ from this loop. (A loop is a continuous map from the unit circle to $\mathbb{R}^2$.) Put differently, consider the iso-height lines of some mountain on a map. Then, for every elevation level that is between the peak of the mountain and the surrounding plains, there is at least one unbroken iso-height line that goes around the peak of the mountain. The lemma makes an analogous statement for the segment of a mountain ridge.

The lemma extends the intermediate value theorem in the following sense to two-dimensions. Suppose there is another function $q : [0,1] \times [0,1] \to [-1,1]$ that is continuous and which has the property that $q(x,0) < 0$ if $f(x,0) = 0$ and $q(x,1) > 0$ if $f(x,1) = 0$. Then, the lemma implies that there exists a point $(x_0, r_0)$ such that $f(x_0, r_0) = q(x_0, r_0) = 0$. This is because the lemma implies that $q(\hat{x}(\cdot), \hat{r}(\cdot))$ is a continuous function with $q(\hat{x}(0), \hat{r}(0)) < 0$ and $q(\hat{x}(1), \hat{r}(1)) > 0$.

**Proof:**[18] Let $F$ be the upper contour set of $f$:

$$F = \{(x,r) | f(x,r) > 0\}.$$

Notice that $F$ is an open subset of $X = [0,1] \times [0,1]$ in induced topology, since $f$ is continuous.

Let $E$ be the component of $F$ containing the right edge, $Y_R = \{[1,y] \mid y \in [0,1]\}$. Let $E' = E - \partial X$. Then, $E'$ is an open planar surface. As we can restrict ourselves only to the end of $E'$ containing $Y_R$, without loss of generality, we can assume the open surface $E'$ has finite topology, i.e. it has finite number of ends (it has already genus 0 as it is planar). Therefore, by the classification of surfaces, there exists a

[18] We are thankful for help with the proof of this result to Baris Coskunuzer, Mathematics Department at Koc University.

compact surface with boundary, which we call $S$, such that $E'$ is homeomorphic to the interior of $S$, $int(S)$. Let $g : int(S) \to E'$ be the homeomorphism (continuous map with continuous inverse). By inducing the path metric of $E'$, we define a metric on $int(S)$ via $g$.

Let $\overline{E}$ be the closure of $E'$ in $X$. Let $\widehat{g} : S \to \overline{E}$ be the continuous extension of $g$. In particular, for any $p \in \partial S$, let $\{p_n\}$ be a sequence in $int(S)$ with $p_n \to p$. Then $q_n = g(p_n)$ defines a Cauchy sequence in $E'$ as $S$ has the induced metric. Since $\overline{E}$ is compact, $\{q_n\}$ is convergent sequence in $\overline{E}$, say $q_n \to q$. Then, we naturally define $\widehat{g}(p) = q$ which is continuous by construction. Notice that the restriction of $\widehat{g}$ to $\partial S$ defines a continuous and onto map from collection of loops $(\partial S)$ to $\overline{E} - E'$, i.e. $\widehat{g} : \partial S \to (\overline{E} - E')$

Let $x_T : \inf\{x \in [0,1] \mid (x,1) \in E\}$, and $x_B : \inf\{x \in [0,1] \mid (x,0) \in E\}$. Note that $x_B, x_T \in (0,1)$, because of the intermediate value theorem. Also, because $f$ is continuous, $f(x_T,1) = f(x_B,0) = 0$, and also for any $(x,y) \in (\overline{E} - E)$, $f(x,y) = 0$.

Let $\gamma$ be the component of $\partial S$ where $\widehat{g}(\gamma) \supset Y_R$. Since $x_T$ and $x_B$ are connected to $Y_R$ in $\overline{E} - E'$ and $\widehat{g} : \partial S \to (\overline{E} - E')$ is onto, then $x_T, x_B \in \widehat{g}(\gamma)$. Then, let $p_T, p_B \in \gamma$ with $\widehat{g}(p_T) = x_T$ and $\widehat{g}(p_B) = x_B$. As $\gamma$ is a loop, there are two closed arcs $\tau^+$ and $\tau^-$ in $\gamma$ with endpoints $\{p_T, p_B\}$, i.e. $\tau^+ \cup \tau^- = \gamma$ and $\partial\tau^\pm = \{p_T, p_B\}$. Let $\tau^+$ be the arc where $\widehat{g}(\tau^+)$ is not containing $Y_R$. Then, $\alpha = \widehat{g}(\tau^+)$ is a continuous arc in $\overline{E} - E$ with $\partial\alpha = \{x_T, x_B\}$. As the arc $\tau^+$ is homeomorphic to an interval $I$, $\widehat{g}$ provides a continuous parametrization of $\alpha$. As $\alpha \subset \overline{E} - E$ and $f = 0$ along $\overline{E} - E$, the proof follows. $\square$

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