Discussion Paper Series – CRC TR 224

Discussion Paper No. 081
Project B 04

Informal Elections with Dispersed Information

Mehmet Ekmekci*
Stephan Lauermann**

April 2019

*Boston College, Department of Economics. Email: ekmekci@bc.edu
**University of Bonn, Department of Economics. Email: s.lauermann@uni-bonn.de

Funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 is gratefully acknowledged.

Informal Elections with Dispersed Information*

Mehmet Ekmekci[†] and Stephan Lauermann[‡]

April 3, 2019

Abstract

We study a model of information transmission through an informal election. Partially informed senders send binary messages to a receiver, and the receiver chooses a policy after observing the number of messages sent. Our leading example is protests in which the citizens' participation choices are their messages, and there may be positive costs or benefits of participation. A policy maker infers information from the aggregate turnout. However, the presence of activists who obtain direct benefits from participation adds noise to turnout. We show that the interplay between noise and costs leads to strategic substitution and strategic complementarity effects in the participation decisions, and we characterize their implications for the informativeness of protests. When there is no noise, information aggregates and the outcome is efficient. Our findings contrast with existing work, which shows that for many informal election scenarios with costless participation, a bias of the policy maker may prohibit any information transmission.

*We thank Daron Acemoglu, Nageeb Ali, Attila Ambrus (discussant), Sourav Bhattacharya, Francesc Dilme, Hulya Eraslan, Navin Kartik, Nenad Kos, Vijay Krishna, Shengwu Li, Elliot Lipnowski, Qingmin Liu, Nadya Malenko, Andrey Malenko, Benny Moldovanu, Roger Myerson, Alex Smolin, Alex Wolitzky, and Huseyin Yildirim for their comments. Jan Knöpfle and Carl Heese provided excellent research assistance. This work was supported by a grant from the European Research Council (ERC 638115) and the German Science Foundation (CRC TR 224). Ekmekci thanks the MIT Economics Department for their hospitality. We are thankful for comments by seminar audiences at Western Ontario, Utah Eccles, Cowles conference, Boston College, ASSA meetings, CETC, Queen Mary, Chicago, Northwestern, Bocconi, Math Econ Conference at Austin, Rice, PSE, Tel-Aviv, Hebrew, and Toulouse.

[†]Boston College, Department of Economics. Email: ekmekci@bc.edu.

[‡]University of Bonn, Department of Economics. Email: s.lauermann@uni-bonn.de.

Contents

1 Introduction

We present and study a model of certain political processes that we call *informal elections*. Examples are protests, petitions, polls, surveys, and nonbinding shareholder voting. These scenarios share some qualitative features with elections. First, the citizens (or experts, shareholders, etc.) take a coarse action that is often effectively binary: participate in a protest or not, sign a petition or not, etc. Second, what matters is the aggregate outcome: how many citizens participate in a protest, how many people sign a petition, etc., with the protest or petition being more effective at convincing an audience the more citizens participate. However, there is an important difference: In a formal election, there is a prespecified rule that determines the policy as a function of the vote count. This is not the case in these scenarios. Instead, the main effect of the informal election is due to the information that the audience infers from it, where the audience could be a particular policy maker but also the general electorate. So, informal elections are primarily about communication.

The Condorcet Jury Theorem and its modern versions have shown that formal elections effectively aggregate dispersed opinions of the citizens under quite general conditions (Feddersen and Pesendorfer, 1997; Myerson, 1998a,b).[1] We explore whether informal elections share these information aggregation properties. Examples of informal election scenarios include the following:

1. Nonbinding Elections. The board of a firm holds a nonbinding vote among the shareholders to decide whether to approve an executive compensation package or a shareholder-submitted proposal on corporate governance (Levit and Malenko, 2011).[2]

2. Petitions and Protests. A policy maker decides whether to change a policy based on the petitions signed by the citizens or based on the turnout in a protest. For example, the Gezi Park movements informed citizens living in rural parts of Turkey about the government's plans to replace a park with a shopping mall and influenced the citizens' opinions. Many other examples are provided in Battaglini (2017).

3. Polls. A manager holds a survey among the employees to learn about the prospects of a new product or the effectiveness of a marketing strategy. A king asks his generals for advice in war (Wolinsky, 2002). A policy maker organizes polls to elicit citizens' information regarding current policies (former US President Nixon did this regularly, as does the current Turkish President, Erdogan; for more examples, see Morgan and Stocken, 2008).

These scenarios have been studied before, as we discuss below. However, there are two features of informal political processes that have been somewhat neglected but that we believe to be inherent to these settings: There are costs of participation, and there can be

[1] However, see Feddersen and Pesendorfer (1997), Razin (2003), Mandler (2012), Bhattacharya (2013), Acharya (2016), Ekmekci and Lauermann (Forthcoming), and Ali et al. (2017) for related models of elections that perform poorly in aggregating information.

[2] The Five Star Movement, a political organization in Italy, holds nonbinding votes on various decisions including the organization's policy stance, the votes of its members in the parliament, and its candidates in local and general elections.

significant noise in the turnout. Participation costs can arise, for example, due to the time commitment or possible repercussions of participating in a protest or a petition. Noise stems from the policy maker's inability to distinguish the motives of citizens in a protest, counting errors in the number of messages, or the presence of citizens who answer poll questions randomly.

More generally, the meaning of turnout in protests—and, therefore, its ability to influence policy or public opinion—depends on the motivations of its participants. If the participants are thought to be participating based on their information, then the turnout is informative. This is further strengthened if participation is costly because this makes the participation decision an even stronger signal. On the other hand, if the protesters are thought to be motivated by non-informational motives, such as monetary incentives for participation via astroturf politics,[3] entertainment value,[4] or extremist tendencies, then the turnout is uninformative.

Indeed, it is no surprise that politicians frequently try to undermine the meaning of protests by arguing that the protesters have motivations that are orthogonal to informational motivations, such as the protesters being paid or having ulterior motives.[5] Similarly, senators are worried about "bot-calling" when making an inference from calls to their office. There are, of course, many other sources of noise, including the possibility that turnout itself is only imperfectly observable.

In this paper, we consider the following protest model, based on Battaglini (2017), to study informal elections with costly participation and noise: A policy maker needs to choose one of two policies, A or B.[6] He prefers the policy to match the unknown state α or β, respectively. There is a pool of citizens who are privately and imperfectly informed about the state of the world. The citizens, like the policy maker, prefer policy A in state α and policy B in state β. (However, when there is uncertainty, the policy maker may have a bias for policy B in the form of a higher "threshold of doubt".) Each citizen draws a participation cost c from a distribution F with support $[\underline{c}, \bar{c}]$ with $\underline{c} < 0 < \bar{c}$. Each citizen also observes a private binary signal, a or b, that is indicative of the state. After observing her participation cost and signal, each citizen chooses whether to participate in the protest. Then, the policy maker observes the total turnout and chooses the policy.[7]

In this model, citizens communicate their information to the policy maker via their participation decision. A priori, the protest may be in favor of A or B. The meaning of the protest is determined in equilibrium, and we look at equilibria in which protests are in favor

[3]There are plenty of examples of astroturfing: Tobacco industry, McDonald's, Walmart. See https://en.wikipedia.org/w/index.php?title=Astroturfing&oldid=887263113 for more details and examples.

[4]There is anecdotal evidence that many people participated in anti-war protests during the Vietnam war era simply to meet with friends and to socialize.

[5]US President Trump sent a tweet in the aftermath of the Supreme Court nomination of Brett Kavanaugh stating that the protesters against Kavanaugh's nomination were paid to participate in the protests. The Hungarian Prime Minister Orban, and many other politicians, often claim that left-wing protests or petitions across the world are Soros-funded. Turkey's President Erdogan called Gezi Park protesters "çapulcu" (looters with nothing else to do), to undermine the meaning of the protests.

[6]The policy maker in our model can be interpreted as the rest of society.

[7]The presence of random participation costs is the only significant departure from Battaglini (2017), who assumes that participation is costless, $c \equiv 0$; we discuss the prior results in greater detail later.

of policy A.[8] In such equilibria, the policy maker chooses policy A if the turnout is large and policy B if the turnout is small. In particular, there is a tipping point for the turnout above which the policy maker chooses policy A. The participation of a citizen is critical exactly when the turnout is at the tipping point. Therefore, how likely the turnout is at the tipping point determines the effectiveness of participation and, hence, the participation incentives of the citizens.

Citizens decide whether to participate based on their costs, the information contained in their signal, and the anticipated effectiveness of their participation. The costs c capture participation motives that are unrelated to the effect of the protest on the policy maker's choice, with negative costs capturing direct benefits from the participation itself. For any given signal, a or b, a citizen participates if her cost is low enough, implying signal-dependent cost-cutoffs c_a and c_b, respectively. Since the protest is in favor of policy A, a citizen with signal a is more eager to "tip" the policy from B to A, compared to a citizen with signal b. Therefore, c_a is larger than c_b. So, citizens with high costs—larger than c_a—never participate, those with intermediate costs—between c_b and c_a—participate based on their signal (informative citizens), and those with low costs—smaller than c_b—always participate (activists). Because of the signal-dependent participation of the informative citizens, the distribution of the turnout differs across the two states, with larger participation in state α than in state β.

The policy maker faces an inference problem where he learns about the state from the realized turnout. The informativeness of the turnout as a signal is determined by the expected number of informative citizens relative to the expected number of activists.

At the one extreme, if there are many informative citizens relative to activists, the turnout is very informative, and the correct policy is likely to be chosen. In this case, it is very unlikely that the turnout is at the tipping point, which implies weak participation incentives. Moreover, we show that, if the number of informative citizens increases further, the probability that turnout is at the tipping point decreases, which decreases the incentives to participate even further. This is a *strategic substitution effect* that captures the natural free-rider problem among citizens when participation is costly. The free-rider problem limits how much information can be transmitted in equilibrium.

At the other extreme, if there are few informative citizens relative to activists, the turnout distributions are close to each other, and turnout contains little information. In this case, the policy maker is unlikely to react to turnout; hence, it is unlikely for the turnout to be at the tipping point. In this case, we show that if the number of informative citizens increases, then the probability that turnout is at the tipping point increases, which increases the incentives to participate, potentially inducing even more participation by informative citizens. This is a *strategic complementarity effect*. We show that this positive participation feedback can lead to a multiplicity of equilibria and also to the fragility of information transmission, with small changes in costs potentially leading to a full unraveling of participation by informative citizens.

To formalize these observations, we study a setting with a large population, which simplifies the analysis and allows us to give precise characterizations. In particular, we can utilize the central limit theorem to approximate the turnout distribution.

[8] All results will also hold for protests that are in favor of B.

When there are many citizens, the probability that a citizen can sway the policy maker's decision is small. Therefore, almost all citizens with negative costs are activists, while informative citizens have costs close to 0. The activists affect the inference problem by adding to the standard deviation of the turnout, while informative citizens affect the difference in the means of the turnout distributions. The power of this inference is then determined by the ratio of the difference in the expected turnout in the two states to the standard deviation of the turnout; we call this ratio the *informativeness* of the protest.

In our main result, we characterize the maximal equilibrium informativeness of protests in large populations. It depends on the cost distribution F only through its reverse hazard rate at 0, $f(0)/F(0)$. For a fixed citizen strategy, the expected number of informative citizens is proportional to $f(0)$. Hence, the density of the cost distribution at 0 scales the difference in the expected turnouts across the two states. The variance of the turnout (i.e., the noise) is approximately state independent and equal to the expected number of citizens with negative costs. Hence, the noise is proportional to $F(0)$. Our main result shows that, if the reverse hazard rate is smaller than a threshold, τ, then there is no information transmission in equilibrium.[9] If the reverse hazard rate is larger than τ, then some information is transmitted. At τ, there is a discontinuity in the maximal informativeness. Above τ, the maximal informativeness is increasing in the reverse hazard rate. Moreover, both the policy maker and the citizens are better off if the informativeness of the turnout increases. When the reverse hazard rate is large, turnout reveals the state almost perfectly. In particular, if there are no benefits from participation (no noise), that is, if $\underline{c} = 0 < \bar{c}$ and so $f(0)/F(0) = \infty$, then information aggregates, and the correct policy is chosen.

We conduct comparative statics for the maximal informativeness of the protest with respect to the informativeness of the citizens' signals and the policy maker's bias towards policy B. The maximal informativeness of the protest is increasing in the informativeness of the citizens' signals: If the citizens' signals are more informative in the Blackwell order, then the maximal informativeness of the protest increases. The effect of an increase in the policy maker's bias towards policy B on the maximal informativeness is ambiguous. If the reverse hazard rate is smaller than a cutoff, then increasing the bias reduces the maximal informativeness. If the reverse hazard rate is larger than the cutoff, then the maximal informativeness is single-peaked in the bias.

As previously argued, additional noise decreases the maximal informativeness of the protests. However, we identify a countervailing, positive effect of noise on citizens' participation incentives: If noise is small, an increase in noise (i.e., a higher $F(0)$ given a fixed $f(0)$) increases the participation rate of informative citizens. In other words, noise results in an *encouragement effect* that mobilizes citizens. When noise is not too large, the encouragement effect partly compensates for the adverse effect of noise on the informativeness of the turnout and leads to a nontrivial amount of information transmission in equilibrium.

Our results for informal elections with costs and noise contrast with those from previous work (Battaglini, 2017; Wolinsky, 2002; Levit and Malenko, 2011; Morgan and Stocken, 2008). This work demonstrated that informal political processes face a difficulty in information transmission: Information transmission is impossible—babbling is the unique equi-

[9] The threshold τ depends on the preferences of the citizens and the policy maker, the prior distribution of the states, and the signal distribution.

librium outcome—if the policy maker's preferences and the citizens' preferences differ too much or if each citizen's information is poorly informative. By way of contrast, we show that if there are only costs and no activists (i.e., $\underline{c} = 0 < \bar{c}$), information is fully aggregated with many citizens for all parameters. When we allow for activists, then equilibrium is still partially informative—provided the reverse hazard rate is above τ—and the informativeness changes continuously in the policy maker's bias. The babbling equilibria that arise in this case when the reverse hazard rate drops below τ are due to reasons different than those identified previously. In particular, babbling arises in our setting even if there is no bias and even if citizens are perfectly informed.

Whether participation in protests exhibits strategic substitutes or complements has been debated in political economy and empirically investigated. For example, in a field experiment on participation in a protest, Cantoni et al. (2017) find evidence of strategic substitution effects, with a citizen's participation probability decreasing in their belief about the participation level of others. Whereas in prior work on protests the form of the participation incentives is often assumed to be exogenous, in our model, the effectiveness of the turnout is endogenous, and each citizen can affect the outcome with a strictly positive probability. The complementarity and substitution effects are driven by how this probability changes with the citizens' and the policy maker's strategy.

Our model and results also speak to a variety of public debates. First, there was a recent discussion of an SEC ruling that was enacted in 2011 as part of the Dodd–Frank Act, which requires all publicly traded companies to have a nonbinding vote on executive compensation and golden parachute compensation.[10] An important policy discussion was with respect to whether institutional investors are allowed to abstain from participation (see Malenko and Malenko, Forthcoming for a more detailed discussion). In our model, mandatory participation would correspond effectively to zero participation costs. Thus, in view of our results, when there are (small) costs of participation, voluntary voting allows information transmission, whereas mandatory voting may lead to no information transmission when the board is biased. Second, many protests are often claimed to be astroturfing politics, such as "Soros-funded" protests, and the aim of such claims seems to be to undermine the meaning of the protests. Such claims draw attention to the underlying motives of the protesters and try to persuade the public that the protest has a lot of noise. Our results highlight that even when the protest movement contains much noise, it may still carry significant information. Furthermore, such claims may be aiming to utilize the fragility of the informativeness of protests in order to undermine them.

We discuss our assumptions and potential extensions in Section 9. In particular, in our main model, we formalize "noise" by allowing $\underline{c} < 0$, i.e., introducing citizens with negative costs (activists). This choice of modeling is made for expositional compactness. Alternative modeling choices—such as allowing for a random number of activists, randomly behaving poll subjects, or allowing counting errors—lead to similar results as we discuss in Subsection 9.2.

We discuss the related literature in more detail in Section 8. Unless otherwise noted, all proofs are in the Appendix. The Online Appendix contains proofs of Theorems for the case without activists in Section 7. Supplementary Material with the formal analysis of the

[10]See https://www.sec.gov/news/press/2011/2011-25.htm.

deterministic population size case is available on the authors' websites.

2 Model

A policy maker wishes to implement one of the two policies, A (reform) or B (status quo). There is also a population of citizens (experts, protesters, or shareholders), each of whom is privately informed about the state of the world. The population size is a Poisson distributed random variable with mean n. That is, the probability that there are $m \geq 0$ citizens is $e^{-n}\frac{n^m}{m!}$.[11]

The policy maker's and the citizens' preference over the policies depends on an unknown state of the world, $\omega \in \{\alpha, \beta\}$. Both the policy maker and the citizens prefer the policy that matches the state of the world, i.e., policy A in state α and policy B in state β. However, they have different preferences when the state is uncertain.

In particular, the policy maker's preferences are as follows: If the state is α, his payoff is $1-\mu \in (0, 1/2]$ if the outcome is A and 0 if it is B; if the state is β, his payoff is $\mu \in [1/2, 1)$ if the outcome is B and 0 if it is A. This payoff function implies that he prefers to implement A when his belief that the state is α is greater than μ, and he prefers to implement B when this belief is less than μ. The ex-ante probability that the state is α is equal to $q \in (0, \mu]$, i.e., the policy maker (weakly) prefers to implement policy B without further information and needs to see some evidence in favor of state α in order to implement policy A.

Citizens have common preferences: If the state is α, a citizen's payoff is 1 if the policy is A and 0 if it is B; if the state is β, her payoff is 1 if the outcome is B and 0 if it is A. Hence, citizens prefer policy A when they believe the state is α with a probability greater than 1/2, and they prefer policy B when they believe the state is α with a probability less than 1/2.[12]

The preferences of the policy maker and citizens are aligned when the state is known; however, when $\mu \neq 1/2$, the policy maker's and the citizens' preferences are misaligned when the state is uncertain. We assume that $\mu \geq 1/2$, which makes the policy maker biased towards policy B compared to the citizens (i.e., he is more conservative than the citizens in his willingness to implement a reform). The difference $\mu - 1/2$ measures the conflict of interest between the citizens and the policy maker.

Each citizen receives a binary signal, $\theta \in \{a, b\}$.[13] Conditional on the state ω, signals are identically and independently distributed across the population according to a probability distribution function $\mathbb{P}(\theta|\omega)$ that denotes the conditional probability that a citizen's signal is θ. We assume that $\mathbb{P}(\theta|\omega) > 0$ for $\theta = a, b$; $\omega = \alpha, \beta$. We also assume that the signal distribution satisfies the monotone likelihood ratio property (MLRP), i.e.,

$$\infty > \frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)} > 1 > \frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} > 0.$$

There is a protest movement in place, and each citizen decides whether to participate in

[11] Assuming that the population size is Poisson-distributed simplifies some of the analysis (see Myerson, 1998b), but our results do not rely on this specification. In the Supplementary Appendix, available on the authors' websites, we show that our results still hold when the number of citizens is deterministic.

[12] We assume that citizens care equally about the mistakes across the two states for simplicity. We discuss how our results change with more general citizen preferences in Subsection 9.5.

[13] Our results do not depend on the binary signal assumption, and we give a detailed discussion in Subsection 9.3.

the protest or not. Participation in the protest is costly; some citizens have positive costs while others have negative costs. Each citizen's cost c is a random variable drawn according to a cumulative distribution function (c.d.f.) F with support $[\underline{c}, \bar{c}]$, for some $\underline{c} < 0 < \bar{c}$, independently across the citizens. We assume that F is a strictly increasing and continuous function, and it admits a continuous density function denoted by f.[14]

Each citizen, after observing her signal and cost, chooses whether to participate in the protest or to abstain. The strategy states the probability that a citizen participates and we denote it by

$$\psi : \{a, b\} \times [\underline{c}, \bar{c}] \to [0, 1].$$

We denote by t the realized turnout, that is, the number of citizens who participate. Each citizen strategy ψ, together with signal distribution $\mathbb{P}$ and cost distribution F, induces a distribution of the turnout. In particular, the turnout is again Poisson-distributed (from the decomposition property of the Poisson distribution; see Myerson, 1998b), and we denote its mean in state $\omega = \alpha, \beta$ by $\lambda(\omega)$.[15]

The policy maker observes t and then chooses the policy. A strategy for the policy maker is a probability of choosing A for each t, and we denote it by

$$\rho : \mathbb{N} \to [0, 1].$$

The policy maker forms his beliefs about the state after observing the realization of the turnout, t. We assume that the policy maker does not observe the realization of the population size. This assumption does not change our results qualitatively but makes the analysis simpler.[16]

A symmetric Nash equilibrium is a strategy profile (ψ, ρ) such that ψ is a best reply to other citizens playing according to strategy ψ and the policy maker playing according to strategy ρ, and ρ is a best reply by the policy maker to the citizens' strategy ψ. Myerson (1998b) showed that in Poisson games, all Nash equilibria are symmetric; hence, focusing on symmetric Nash equilibria is without loss of generality.

Moreover, by the environmental equivalence property (Myerson, 1998b), for any citizen, the distribution of the number of other citizens is also Poisson distributed with mean n, as is the distribution of other citizens who participate in each state with mean $\lambda(\omega)$. Now, consider a citizen with a signal θ and cost c. For a given strategy profile (ψ, ρ), the payoff difference between participating in the protest and abstaining for such a citizen is given by

$$u(\theta, c) = \sum_{t \in \mathbb{N}} (\rho(t+1) - \rho(t)) \left(\mathbb{P}(\alpha|\theta) e^{-\lambda(\alpha)} \frac{\lambda(\alpha)^t}{t!} - \mathbb{P}(\beta|\theta) e^{-\lambda(\beta)} \frac{\lambda(\beta)^t}{t!} \right) - c. \quad (1)$$

A citizen joins the protest when $u(\theta, c) > 0$ and abstains when $u(\theta, c) < 0$. Notice that the net payoff from participation is decreasing with the cost c, so any best reply of a citizen to other citizens' strategy and to the policy maker's strategy has a cutoff structure: There exist

[14] The essential property we use is that F is atomless and f is continuous in a neighborhood of 0. We will also analyze the cases in which $\underline{c} = 0$, or F may have an atom later in Section 7.

[15] The mean is simply $\lambda(\omega) = n \left(\mathbb{P}(a|\omega) \int \psi(a, c) dF(c) + \mathbb{P}(b|\omega) \int \psi(b, c) dF(c) \right)$.

[16] Our results hold if n is publicly observed by the citizens and the policy maker; see the supplementary material on the authors' websites.

cutoffs (c_a, c_b) such that a citizen with a signal θ and cost c participates in the protest if $c < c_\theta$ and abstains if $c > c_\theta$. Henceforth, we identify a citizen strategy ψ that has a cutoff structure with the cutoff pair (c_a, c_b) that refers to the participation cutoffs of citizens with a and b signals. Note that the cost cutoffs satisfy $u(\theta, c_\theta) = 0$.

3 Responsive Equilibria

There is always an uninformative equilibrium (or babbling equilibrium) with the citizens protesting when $c < 0$, and abstaining when $c \geq 0$; i.e., $c_a = c_b = 0$. A citizen's behavior in such a strategy profile is independent of her signal, and hence the turnout is state-independent, i.e., $\lambda(\alpha) = \lambda(\beta)$. Therefore, turnout is uninformative about the state, and the policy maker implements policy B for each realization of the turnout, i.e., $\rho(t) = 0$ for every $t \geq 0$. Hence, $u(\theta, 0) = 0$ for $\theta = a, b$, and the citizens act only based on their cost.

We are interested in non-babbling equilibria in which $\lambda(\alpha) \neq \lambda(\beta)$. To avoid tedious case distinctions, in the main text, we will consider only non-babbling equilibria in which $\lambda(\alpha) > \lambda(\beta)$, and we will refer to those as "responsive equilibria". We will refer to citizens' strategies that induce turnouts $\lambda(\alpha) > \lambda(\beta)$ as responsive strategies. In responsive equilibria, citizens with a signals are more likely to join the protest than citizens with b signals.[17] Responsive equilibria may not exist for every n. We first present the properties of such equilibria when they exist, and then move to the comprehensive analysis of when such equilibria exist for large n in Section 5. Such equilibria also give a meaning to the protest being in favor of A, and we confirm this in the development below.[18]

In a responsive equilibrium, the turnout is Poisson-distributed with mean given by

$$\lambda(\omega) = n\left(\mathbb{P}(a|\omega)F\left(c_a\right) + \mathbb{P}(b|\omega)F\left(c_b\right)\right) \text{ for } \omega = \alpha, \beta. \tag{2}$$

For given $\lambda(\alpha)$ and $\lambda(\beta)$, the policy maker's posterior belief for a given realization of the turnout t, expressed as the likelihood ratio that the state is α, is given by

$$L(t) = \frac{q}{1-q}\frac{\mathbb{P}(t|\alpha)}{\mathbb{P}(t|\beta)} = \left(\frac{q}{1-q}\right)e^{-(\lambda(\alpha)-\lambda(\beta))}\left(\frac{\lambda(\alpha)}{\lambda(\beta)}\right)^t.$$

In a responsive equilibrium, the posterior likelihood ratio L is strictly increasing in t, $L(0) < \frac{q}{1-q}$, and $\lim_{t\to\infty} L(t) = \infty$. Consider the largest T such that $L(t) < \frac{\mu}{1-\mu}$ for all $t \leq T$. If $t \leq T$, the policy maker's posterior belief that the state is α is less than μ, and he implements policy B, and when $t > T+1$, his posterior belief that the state is α is greater than μ, and he implements A. If $t = T+1$, he implements A if $L(T+1) > \frac{\mu}{1-\mu}$, and is indifferent between the two policies if $L(T+1) = \frac{\mu}{1-\mu}$. Therefore, in any responsive equilibrium, $\rho(t)$ is an increasing function, with $\rho(t) = 0$ for $t \leq T$, $\rho(t) \in [0,1]$ for $t = T+1$, and $\rho(t) = 1$ for $t > T+1$. This confirms our intuition that in responsive equilibria, protests

[17] So, in a slight abuse of the term, equilibria with $\lambda(\alpha) < \lambda(\beta)$ are not responsive. We refer to these together with the responsive ones as non-babbling equilibria. Note also that $\lambda(\alpha) > \lambda(\beta)$ together with the MLRP implies that $F(c_a) > F(c_b)$. This requires $c_a > c_b$.

[18] We also analyze properties of equilibria in which citizens with b signals participate more often than a signals, i.e., $F(c_b) > F(c_a)$. The maximal informativeness of protests when n is large is identical in such equilibria and in responsive equilibria, as we show in Theorem 1.

carry a meaning being in favor of A, and that participation in the protest increases the chances of A being implemented.

In a responsive equilibrium, let

$$\mathbb{P}(\text{piv}|\omega) = \sum_{t\in\mathbb{N}} \left(\rho(t+1) - \rho(t)\right) e^{-\lambda(\omega)} \frac{\lambda(\omega)^t}{t!}, \tag{3}$$

be the probability that an additional protester changes the outcome in favor of A in state ω, that is, the probability of being *pivotal*. The payoff difference in equation (1) depends on the probability that the citizen is pivotal for the outcome in each state, and her cost. Equation (1) simplifies to

$$u(\theta, c) = \mathbb{P}(\alpha|\theta)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|\theta)\mathbb{P}(\text{piv}|\beta) - c.$$

Therefore, in a responsive equilibrium,

$$c_\theta = \mathbb{P}(\alpha|\theta)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|\theta)\mathbb{P}(\text{piv}|\beta) \;\text{ for } \theta = a, b. \tag{4}$$

In general, the cutoffs may be outside the support of F, i.e., we allow for $c_\theta < \underline{c}$ and for $c_\theta > \bar{c}$. Note also that if $c_a = c_b$, then $\lambda(\alpha) = \lambda(\beta)$. Therefore, in this case, the posterior likelihood ratio L is constant and is equal to $\frac{q}{1-q} \leq \frac{\mu}{1-\mu}$ for every t, delivering that $\rho(t) = 0$ for every t is a best reply for the policy maker, i.e., we obtain a babbling equilibrium.

Because the policy maker's strategy has a threshold structure, a citizen is pivotal in two events. The first pivotal event is when there are T other protesters, and the policy maker chooses A when there are $T+1$ protesters. The probability of this first event is $e^{-\lambda(\omega)} \frac{\lambda(\omega)^T}{T!} \rho(T+1)$. The second pivotal event is when there are $T+1$ other protesters, and the policy maker chooses B when there are $T+1$ protesters. The probability of this event is $e^{-\lambda(\omega)} \frac{\lambda(\omega)^{(T+1)}}{(T+1)!}(1-\rho(T+1))$. Therefore, equation (3) simplifies to

$$\mathbb{P}(\text{piv}|\omega) = e^{-\lambda(\omega)} \frac{\lambda(\omega)^T}{T!} \rho(T+1) + e^{-\lambda(\omega)} \frac{\lambda(\omega)^{(T+1)}}{(T+1)!}(1-\rho(T+1)).$$

In a responsive equilibrium, the incentives of the citizens to participate in the protest are determined by the probability of the pivotal events in states α and β, given by equation (4). Because $\mathbb{P}(\alpha|a) > \mathbb{P}(\alpha|b)$, and because in every responsive equilibrium $\mathbb{P}(\text{piv}|\alpha) > 0$, we have $c_a > c_b$. When c_a and/or c_b are in the interior of the support of F, this implies that $F(c_a) > F(c_b)$. This will be the case when $\mathbb{P}(\text{piv}|\alpha)$ and $\mathbb{P}(\text{piv}|\beta)$ are small. Hence, when the policy maker perceives the protest to be in favor of A, the citizen's best reply induces a higher participation rate in state α than in state β.

Thus, in responsive equilibria, citizens with costs between c_b and c_a participate in the protest based on their signal, those with costs less than c_b participate independent of their signals, and those with costs above c_a abstain regardless of their signals. In other words, citizens whose costs are between c_b and c_a provide information in the protest movement (*informative citizens*), whereas those with costs less than c_b add noise to the protest movement (*activists*); see Figure 1 for a depiction.

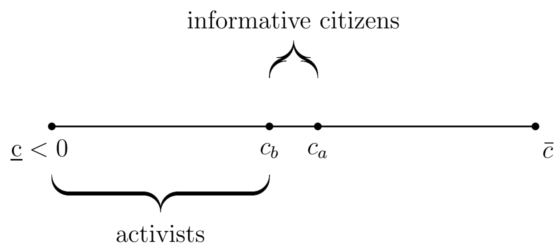


Figure 1: In responsive equilibria, citizens with costs between c_b and c_a participate if their signal is a, and do not participate if their signal is b.

4 Large Populations with Activists: Best Responses

We are interested in the properties of responsive equilibria as the expected number of citizens grows without bound. Fix all the parameters of the model except n (i.e., $\mathbb{P}$, F, q, μ). As a first step towards analyzing equilibrium, we derive here the best-responses of the policy maker and the citizens iteratively. Specifically, we consider some sequence of responsive cutoffs $\{c_{a,n}, c_{b,n}\}_n$ for which $c_{a,n}, c_{b,n} \to 0$ as $n \to \infty$. (As we will verify later, in equilibrium, cutoffs will indeed necessarily vanish to 0.) In the first subsection, we characterize the policy maker's best-response threshold $\hat{T}_n$ to the cutoff sequence. In the second subsection, we characterize the citizens' best response cutoffs $\{\hat{c}_{a,n}, \hat{c}_{b,n}\}_n$ to the cutoffs and the policy maker's best response, $\left\{c_{a,n}, c_{b,n}, \hat{T}_n\right\}_n$. An equilibrium must be a fixed point of the resulting "composite" best response mapping from cost cutoffs to cost cutoffs. We provide a closed-form characterization of this necessary condition for large n at the end of the section. An intermediate finding is the identification of regions with complementarities and free-riding, respectively, in citizens' participation decisions. In Section 5, we then use this condition to characterize the equilibrium informativeness.

4.1 Protest Informativeness and the Inference Problem

Given any responsive participation strategy, turnout is Poisson-distributed, and the mean of the distribution is state-dependent. Hence, the policy maker faces an inference problem where he observes the realization of the turnout, a single sample point. The inference problem of the policy maker is simplified by the fact that the Poisson distribution is well-approximated by the normal distribution.

In general, if some random variable $\tilde{N}$ follows a Poisson distribution with mean k, then the standard deviation of $\tilde{N}$ is $\sqrt{k}$, and for any $z \in \mathbb{R}$, $\lim_{k\to\infty} \Pr\left(\tilde{N} \leq k + z\sqrt{k}\right) = \Phi(z) \in (0,1)$, where Φ is the c.d.f. of the standard normal distribution.[19] Thus, the Poisson distribution becomes concentrated on an interval proportional to its standard deviation, $\sqrt{k}$, around its

[19] This can be seen by observing that $\tilde{N}$ is the sum of $\lfloor k \rfloor$ independent random variables, each of which is Poisson-distributed with mean $\frac{k}{\lfloor k \rfloor}$. Applying the central limit theorem and observing that $\lim_k \frac{k}{\lfloor k \rfloor} = 1$ delivers the approximation.

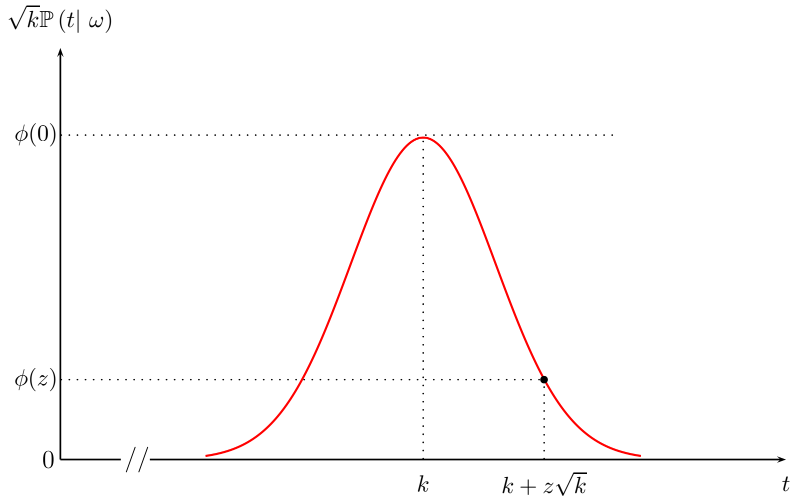


Figure 2: Visualization of the Normal approximation of Poisson distributions with mean k, given by equation 5.

mean k. Moreover, the approximation can be used locally as well, namely,[20]

$$\Pr\left\{\tilde{N} = k + z\sqrt{k}\right\} \approx \frac{\phi(z)}{\sqrt{k}}, \tag{5}$$

where ϕ is the density of the standard normal distribution.[21] Figure 2 illustrates the approximation.

Since the turnout in state ω is Poisson-distributed with mean $\lambda_n(\omega)$, the standard deviation in state ω is $\sqrt{\lambda_n(\omega)}$. Because $c_{a,n}, c_{b,n} \to 0$, we have

$$\sqrt{\lambda_n(\omega)} \approx \sqrt{nF(0)} := \sigma_n, \tag{6}$$

i.e., the expected turnout in each state grows approximately proportional to the expected number of citizens with negative costs. Therefore, there is more noise in the turnout distribution if $F(0)$ is larger. Moreover, the standard deviation of the turnout distributions are approximately equal in the two states. Yet, the difference in the expected protest sizes (normalized by the standard deviation) may be different from 0. Hence, the policy maker's inference problem is akin to an inference problem where the policy maker draws a single observation from a normal distribution with state-dependent mean and known standard deviation. In such an inference problem, the larger the difference in the means of the normal distributions, the more informative the sample draw is about the true distribution (see Figure 3). This motivates us to define the measure of informativeness of the protests as the difference in the expected turnout in the two states, normalized by the standard deviation

[20] For any two sequences of numbers $\{f_n\}_n, \{g_n\}_n$, $f_n \approx g_n$ means $\lim_{n\to\infty} \frac{f_n}{g_n} = 1$.

[21] Using again that $\tilde{N}$ is the sum of $\lfloor k \rfloor$ independent, Poisson-distributed random variables, the local central limit theorem delivers the approximation (see Theorem 1.1 of Davis and McDonald, 1995, or Gnedenko, 1948).

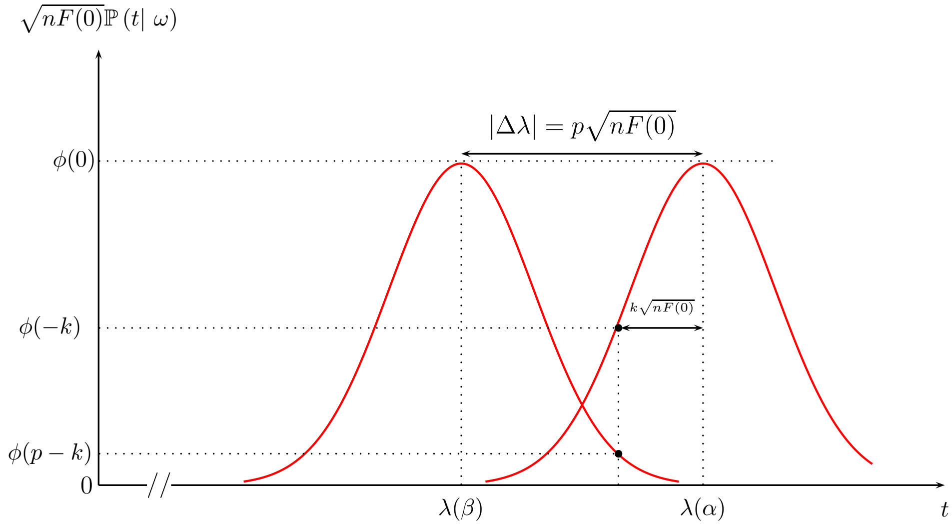


Figure 3: This figure illustrates the inference problem of the policy maker when he observes a turnout t that is k standard deviations away from the mean of the turnout in state α.

$\sigma_n = \sqrt{nF(0)}$, i.e., given a sequence of responsive cutoffs $c_{a,n}, c_{b,n} \to 0$, we define

$$p := \lim_{n\to\infty} \left| \frac{\lambda_n(\alpha) - \lambda_n(\beta)}{\sigma_n} \right| \tag{7}$$

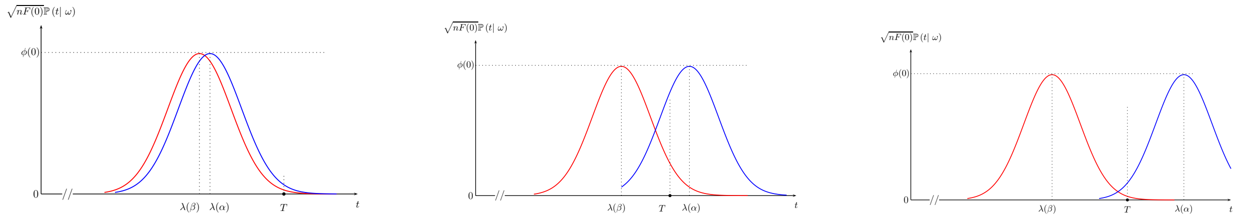
to be the informativeness of the protest when the limit in (7) exists. The information content of the protest is increasing in p in Blackwell order (Figure 4 illustrates how the threshold the policy maker uses for the policy choice and the probabilities of the correct outcomes change with p given the policy maker's best response).

The informativeness determines the normalized distance between the policy maker's threshold and the expected turnout. In particular, fix a responsive sequence of cutoffs with informativeness p, and take an arbitrary sequence of thresholds, $\{T'_n\}_n$, such that $\lim \frac{T'_n - \lambda_n(\alpha)}{\sigma_n} = k \in (-\infty, \infty)$. Then, using the local approximation from equation 5, the policy maker's posterior likelihood ratio when the turnout is T'_n is given by (Figure 3 provides an insight for this derivation)

$$\lim_{n\to\infty} L(T'_n) = \frac{q}{1-q}\frac{\phi(k)}{\phi(k+p)}. \tag{8}$$

When n is large, the policy maker is approximately indifferent between policies A and B at the optimal threshold. Therefore, if $\{\hat{T}_n\}_n$ is a sequence of best-response thresholds for the policy maker, then

$$\lim_{n\to\infty} L(\hat{T}_n) = \frac{\mu}{1-\mu}. \tag{9}$$



(a) Small p (b) Intermediate p (c) Large p

Figure 4: This panel illustrates the pivotal event, T, its probability in each state, and the probability of each policy choice as a function of p.

We can solve (8) and (9) for k using the analytic expression for ϕ, yielding:[22]

$$\kappa(p) := -\frac{p}{2} + \frac{1}{p}\ln\left(\frac{1-q}{q}\frac{\mu}{1-\mu}\right). \tag{10}$$

Thus, the normalized distance between the policy maker's best response threshold $\hat{T}_n$ and the expected turnout in state α is approximately $\kappa(p)$ for large n.

This also determines the probability of being pivotal in state α as (see Figure 3 again):

$$\mathbb{P}(\text{piv}|\alpha) \approx \mathbb{P}\left(t = \hat{T}_n|\alpha\right) \approx \frac{\phi(\kappa(p))}{\sigma_n}. \tag{11}$$

For state β, we obtain that $\lim \frac{\hat{T}_n - \lambda_n(\beta)}{\sigma_n} = \kappa(p) + p$, using equation (7). Therefore,

$$\mathbb{P}(\text{piv}|\beta) \approx \frac{\phi(\kappa(p)+p)}{\sigma_n}. \tag{12}$$

Two observations will be important. First, given some $p \in (0, \infty)$, the probability of being pivotal is on the order of $^1/_{\sigma_n}$. Thus, to a first approximation, the larger $F(0)$ and, hence σ_n, the smaller the probability that a citizen is pivotal. Second, the probability of being pivotal is non-monotone in p and, more specifically, hump-shaped. This is illustrated by Figure 4.

4.2 Citizens' Best-Response Cutoffs

We are now interested in the citizens' best-response cutoffs $\{\hat{c}_{a,n}, \hat{c}_{b,n}\}$ given some sequence of responsive cutoffs $\{c_{a,n}, c_{b,n}\}$, and the policy maker's best response $\hat{T}_n$ to it. As observed above, the policy maker's best response determines the probability of being pivotal in each state. The probabilities of being pivotal in each state, in turn, determine the citizens' cost cutoffs, as stated in equation (4). Finally, the cost cutoffs imply a new difference in the expected turnout.

In particular, from the policy maker's optimal choice of the threshold, the relative probability of being pivotal is:

$$\lim_{n\to\infty} \frac{\mathbb{P}(\text{piv}|\alpha)}{\mathbb{P}(\text{piv}|\beta)} = \frac{\mu}{1-\mu}.$$

[22] We use the convention that, when $\mu > q$, $\kappa(0) := \lim_{p\to 0}\kappa(p) = \infty$, and $\phi(\kappa(0)) = 0$. When $\mu = q$, then $\kappa(p) = -p/2$ for $p > 0$, so we use the convention that $\kappa(0) := \lim_{p\to 0}\kappa(p) = 0$; therefore, $\phi(\kappa(0)) > 0$.

Using this and the optimality condition (4) for the cost cutoffs, we get that the difference in the best-response cost cutoffs is:

$$\hat{c}_{a,n} - \hat{c}_{b,n} \approx (\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b)) \left(1 + \frac{q}{1-q}\frac{1-\mu}{\mu}\right) \mathbb{P}(\text{piv}|\alpha). \tag{13}$$

Thus, the expected number of informative citizens is proportional to the probability of being pivotal. This should not be a surprise, since the probability of being pivotal measures the effectiveness of the protest.

From here, the difference in the expected turnout in states α and β given the citizens' best response is:

$$\hat{\lambda}_n(\alpha) - \hat{\lambda}_n(\beta) = n \left(F\left(\hat{c}_{a,n}\right) - F\left(\hat{c}_{b,n}\right)\right) \left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right),$$

using equation (2). Using L'Hopital's rule and $\hat{c}_{a,n}, \hat{c}_{b,n} \to 0$, this implies

$$\hat{\lambda}_n(\alpha) - \hat{\lambda}_n(\beta) \approx nf(0) \left(\hat{c}_{a,n} - \hat{c}_{b,n}\right) \left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right). \tag{14}$$

Dividing both sides by σ_n and using equation (13), we get that the best response of the citizens implies the following new normalized difference in means,

$$\frac{\hat{\lambda}_n(\alpha) - \hat{\lambda}_n(\beta)}{\sigma_n} \approx \frac{nf(0)}{\sigma_n} Z\, \mathbb{P}(\text{piv}|\alpha), \tag{15}$$

with

$$Z := \left(1 + \frac{q}{1-q}\frac{1-\mu}{\mu}\right) \left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right) \left(\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b)\right).$$

Finally, substituting for $\mathbb{P}(\text{piv}|\alpha)$ from the optimality of the policy maker (11), we get the implied "best-response" informativeness $\hat{p}$,

$$\hat{p}(p) := \frac{f(0)}{F(0)} Z \phi\left(\kappa(p)\right). \tag{16}$$

In other words, for large n, if the policy maker and the citizens expect cutoffs $(c_{a,n}, c_{b,n})$ that imply an informativeness of approximately p and the policy maker chooses a best response threshold $\hat{T}_n$ to $(c_{a,n}, c_{b,n})$, then the citizen's best response to $(c_{a,n}, c_{b,n}, \hat{T}_n)$ implies that the expected informativeness is approximately $\hat{p}(p)$.

The informativeness of a responsive *equilibrium* sequence must be a fixed point of $\hat{p}$. Thus, the previous discussion is summarized by the following lemma.

Lemma 1. *For every responsive equilibrium sequence with informativeness p^*,*

$$\hat{p}(p^*) = p^*. \tag{17}$$

Therefore, the shape of $\hat{p}$ is critical for understanding the equilibrium properties and performing comparative statics, so we discuss its shape in the next subsection in more detail.

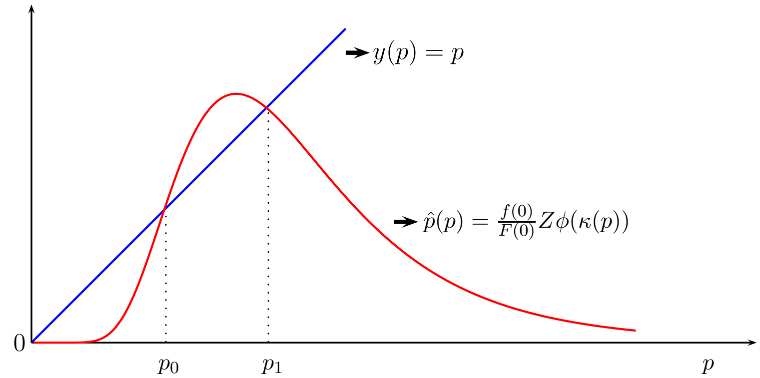


Figure 5: Equilibrium informativeness p is the largest intersection of $\frac{f(0)}{F(0)} Z\phi(\kappa(p))$ with the 45 degree line. The reverse hazard rate $\frac{f(0)}{F(0)}$ scales the function. (The figure is a sketch of the qualitative features of $\hat{p}$, not a computed example.)

4.3 The Participation Incentives: Free-Riding and Complementarities

Figure 5 illustrates the function $\hat{p}$ when $\mu > q$. The basic "hump shape" follows from the properties of the normal density ϕ and the monotonicity of κ.[23] From this, three observations about the citizens' participation incentives follow.

Stationarity. First, suppose that the difference of the original cutoffs is on the order of $1/\sigma_n$; that is, for some k we have $c_{a,n} - c_{b,n} \approx k/\sigma_n$. It follows from equation (14) that the expected number of informative citizens is on the order σ_n.[24] As a consequence, the informativeness is close to some interior number p. Then, the best-response threshold of the policy maker $\hat{T}_n$ is finitely many standard deviations σ_n away from the means. Hence, the probability of being pivotal is on the order $1/\sigma_n$; see (11). But for the citizens' best response, the difference of the cutoffs is proportional to the probability of being pivotal, by equation (13), and so the citizens' best response implies that the number of informative citizens is again on the order σ_n. So, in summary, we find that if the anticipated number of informative citizens is on the order of the standard deviation, σ_n, then the best-response number of informative citizens is also on the order of the standard deviation, σ_n.

Complementarity and Free-Riding. We now discuss the implications of an increase in the anticipated protest informativeness p. Figure 5 shows that $\hat{p}$ is first increasing and then decreasing. Put differently, for small anticipated p, an increase of p will increase the incentives for citizens to participate, and hence increase the number of informative citizens. We call this a *strategic complementarity effect*: If the policy maker anticipates a larger p, as implied by a larger number of informative citizens, and reacts optimally, then the citizens' best response will imply that more informative citizens participate. As p increases further, however, the participation incentives decrease, due to a free-riding problem.

To understand this further, recall that the number of informative citizens is proportional to the probability of being pivotal by equation (13). In addition, as illustrated by Figures 3 and 4, the probability of being pivotal is first increasing and then decreasing in p. Intuitively,

[23] In particular, $\hat{p}$ is continuous, $\hat{p}(0) = \hat{p}'(0) = 0$, $\lim_{p\to\infty} \hat{p}(p) = 0$, $\hat{p}$ is first increasing and then decreasing.

[24] It is on the order $n\frac{k}{\sigma_n} = n\frac{k}{\sqrt{nF(0)}}$, which has order σ_n.

for small p, the policy maker is initially unlikely to react to the protest because turnout is not very informative. However, as p increases, the turnout becomes more informative and the policy maker is more likely to be swayed by the turnout. In this region, the participation decision of the informative citizens are complementary. Finally, as p becomes very large, the participation of the "other" protesters is almost certain to ensure the correct choice and, hence, an individual protester has little incentive to participate (free riding).[25]

Reverse Hazard Rate. As suggested by the function $\hat{p}$, the cost distribution F enters the participation incentives only via its (reverse) hazard rate at 0, $\frac{f(0)}{F(0)}$. In particular, the reverse hazard rate "scales" the best response map $\hat{p}$: For any given p, the larger $\frac{f(0)}{F(0)}$, the larger the implied informativeness of the best response. To understand this, note the following: First, the density $f(0)$ enters only in equation (14) where it is scaling the best response cutoffs; for a given difference $\hat{c}_{a,n} - \hat{c}_{b,n}$, the larger the density, the larger the implied number of informative citizens. Second, $F(0)$ enters via σ_n in two ways. First, a larger σ_n mechanically reduces the informativeness of the protest for a given number of informative citizens because it appears in the denominator of $\hat{p}$; see equation (15). Second, a larger σ_n implies a small probability of being pivotal—see equation (11)—thereby reducing the participation incentives; see equation (13). In both cases, $\sqrt{F(0)}$ enters multiplicatively in the denominator and, so, $F(0)$ appears in $\hat{p}$.

5 Characterization of Equilibrium Informativeness

Here, we study the maximal information transmission that is sustainable in equilibrium. Specifically, we look for the largest informativeness of protests achievable in some equilibrium sequence. Our first result shows how the maximal informativeness depends on F only via its reverse hazard rate, $\frac{f(0)}{F(0)}$. Then, we provide further comparative statics with respect to the other parameters of the model (the distribution of the citizens' signals and the bias of the policy maker).

5.1 Main Result: Characterization

From our previous discussion, summarized in Lemma 1, we know that for any sequence of responsive cutoffs with informativeness $p > 0$, the composite best response implies a strictly positive informativeness, $\hat{p}(p) > 0$. However, this does not mean that there will always exist an equilibrium sequence with positive informativeness: it may be that for any p, we have $\hat{p}(p) < p$, which means that, starting from any level of informativeness, the iterated best response will lead to an ever-decreasing number of informative citizens, eventually taking us to the babbling equilibrium with $p = 0$. Indeed, as the next result shows, if $\frac{f(0)}{F(0)}$ is too small, then there is no responsive equilibrium sequence, and, in fact, for n large enough, babbling is the unique equilibrium outcome. However, if $\frac{f(0)}{F(0)}$ is sufficiently large, then there is an equilibrium with positive informativeness and, as $\frac{f(0)}{F(0)}$ grows further, the informativeness

[25] Note that the argument above is via the "composite" best response, $\hat{p}$. A similar observation is the following: Fix some strategy profile $(c_{a,n}, c_{b,n}, \hat{T}_n)$ such that the implied informativeness p_n is small and $\hat{T}_n$ is a best response threshold. Then, a marginal increase in $c_{a,n}$ increases the best response cutoff $\hat{c}_{a,n}$. This follows from the same idea: For initially small p_n, the best response cutoff $\hat{T}_n$ is in the right tail (see the left panel in Figure 4). In this case, an increase in $c_{a,n}$ implies an increase in $\mathbb{P}(\text{piv}|\alpha)$ because it moves the mean closer to threshold (keeping $c_{b,n}$ and $\hat{T}_n$ fixed). Conversely, for large p_n, an increase in $c_{a,n}$ implies a decrease in $\mathbb{P}(\text{piv}|\alpha)$. An analogous argument applies to the cutoff $c_{b,n}$.

increases without bound.

The previous section showed that the informativeness of any equilibrium sequence must be a fixed point of the (approximation of the) composite best responses stated in equation (17). Thus, the fixed points of this equation are candidates for equilibrium informativeness. Our main result shows that the largest fixed point is indeed an equilibrium outcome, and it spells out the implications of this for equilibrium informativeness. In particular, the maximal equilibrium informativeness depends on F only through its reverse hazard rate, i.e., $\frac{f(0)}{F(0)}$, and the result shows how the maximal equilibrium informativeness is changing as a function of $\frac{f(0)}{F(0)}$. To this end, we define $P\left(\frac{f(0)}{F(0)}\right)$ to be the largest $p-$solution of equation (17) when the other parameters $(q, \mu, \mathbb{P}(\theta|\omega))$ are fixed.

Theorem 1. *When the other parameters $(q, \mu, \mathbb{P}(\theta|\omega))$ are fixed, there is some cutoff $\tau \geq 0$ such that, for every cost distribution F with $\frac{f(0)}{F(0)} \neq \tau$, the sequence of equilibria with the maximal informativeness satisfies*

$$\lim_{n\to\infty}\left|\frac{\lambda_n(\alpha)-\lambda_n(\beta)}{\sigma_n}\right| = P\left(\frac{f(0)}{F(0)}\right).$$

The cutoff $\tau = 0$ if and only if $\mu = q$. Moreover, P is as follows:

1. *$P(x) = 0$ for $x < \tau$,*
2. *P is continuous and strictly increasing for $x > \tau$,*
3. *If $\mu > q$, that is, $\tau > 0$, then $\lim_{x\downarrow\tau^+} P(x) > 0$,*
4. *$\lim_{x\to\infty} P(x) = \infty$.*

Theorem 1 characterizes the informativeness of protests in relation to the cost distribution, F, when all other parameters of the model are fixed. In particular, for a fixed reverse hazard rate $\frac{f(0)}{F(0)} = x$ and n large, the difference in the expected turnout in the two states, normalized by the standard deviation of the turnout, is close to $P(x)$.

Figure 6 illustrates the properties of the function P: There is some τ such that, below it, P is constant and equal to zero, meaning there is no information transmission, (Item 1). In fact, when n is large, we show that babbling is the unique equilibrium in this case (Lemma 4, and the proof of Theorem 1 in the Appendix). Above τ, P is continuous and strictly increasing (Item 2). At τ, there is a discontinuity if $\mu > q$ (Item 3).[26] Finally, P grows without bound, i.e., protests become arbitrarily informative, as the reverse hazard rate grows without bound (Item 4).[27]

Figure 5 illustrates the function $\hat{p}$ and its intersection points with the function $y(p) := p$, when $\mu > q$. The properties of the P function follow from the hump shape of the function $\hat{p}$ and the fact that $\hat{p}$ is scaled by $\frac{f(0)}{F(0)}$, as previously discussed.[28]

[26]Because $\tau = 0$ if and only if $\mu = q$, the discontinuity appears only when $\mu > q$.

[27]We study the case $F(0) = 0$, corresponding to $\frac{f(0)}{F(0)} = \infty$, in Subsection 7.1.

[28]As mentioned before, $\hat{p}$ is continuous, $\hat{p}(0) = \hat{p}'(0) = 0$, $\lim_{p\to\infty} \hat{p}(p) = 0$, and $\hat{p}$ is first increasing and then decreasing.

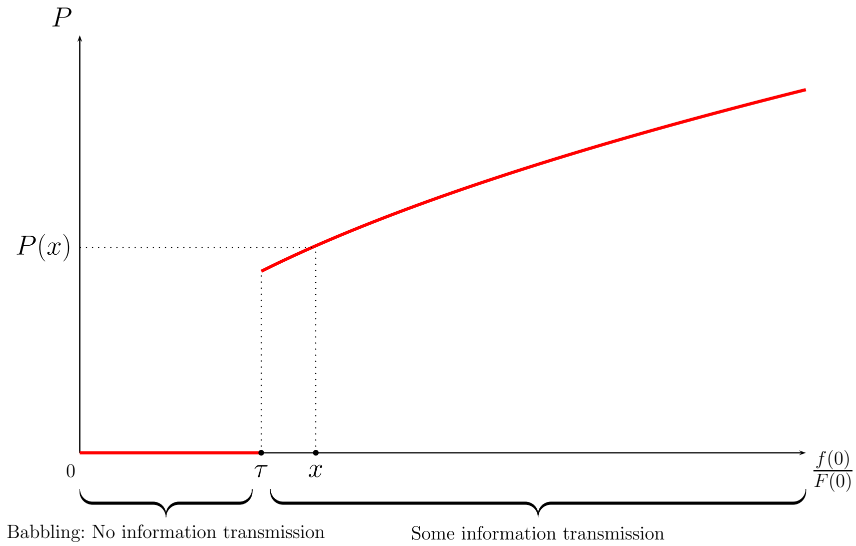


Figure 6: The maximal informativeness for n large as a function of the reverse hazard rate. If $\frac{f(0)}{F(0)} = x \in (\tau, \infty)$, then, as n increases, the equilibrium informativeness converges to $P(x) \in (0, \infty)$.

To see why, suppose $p_1 > 0$ is the largest intersection of $y(p) = \hat{p}(p)$. An increase in $\frac{f(0)}{F(0)}$ will scale up the function $\hat{p}$, and p_1 increases. On the other hand, if $\frac{f(0)}{F(0)}$ decreases sufficiently, eventually $\hat{p}$ and y will be tangent. The magnitude of $\frac{f(0)}{F(0)}$ that makes the two curves tangent is τ. For $\frac{f(0)}{F(0)} < \tau$, there is no longer any positive intersection point. Thus, there is a discontinuity at τ. This implies that informative protests may be fragile. Small changes in the cost distribution may unravel the informativeness of protests completely.

When $\mu = q$, the function $\hat{p}$ peaks at $p = 0$ at a positive number, and is decreasing. Therefore, $\tau = 0$ in this case.

Lemma 1 shows that the informativeness of an equilibrium sequence must be a solution of equation (17). When $\frac{f(0)}{F(0)} > \tau$, there are two positive solutions to equation (17). Both solutions are candidate equilibria. However, $\hat{p}$ is only an approximation of the actual best responses, hence it is not guaranteed that fixed points are equilibria. We show in the Appendix that the largest solution of equation (17) corresponds to the informativeness of some responsive equilibrium sequence. The existence proof uses a sequence of auxiliary games with restricted strategy spaces. We choose the restrictions in order to ensure that, along the equilibrium sequence of the auxiliary games, the restrictions do not bind for large n, and the informativeness of the equilibrium sequence converges to $P\left(\frac{f(0)}{F(0)}\right)$. What facilitates that the restrictions do not bind is the "pseudo-stability" of the approximate best response, $\hat{p}$, at the maximal solution of equation (17). The smaller positive fixed point of $\hat{p}$ lacks this pseudo-stability, hence our proof method does not apply for that candidate equilibrium. As a consequence, we do not know whether the smaller fixed point corresponds to an equilibrium sequence.

5.2 Policy Choice: Indeterminacy and Welfare

We now investigate the implications of our characterization for the equilibrium behavior and welfare.

Theorem 2. *For any cost distribution F with $\frac{f(0)}{F(0)} \neq \tau$, let $p = P\left(\frac{f(0)}{F(0)}\right)$. In the sequence of equilibria with maximally informative protests:*[29]

1. *(Policy choice)*

   (a) *The probability that A is chosen in state α converges to*

   $$1 - \Phi\left(\kappa\left(p\right)\right),$$

   (b) *The probability that B is chosen in state β converges to*

   $$\Phi\left(\kappa\left(p\right) + p\right).$$

2. *(Welfare) The expected payoffs of the policy maker and the citizens are strictly increasing in the informativeness of the protest p.*

The first part of the theorem follows from our previous discussions: As we stated earlier, the distance between the expected turnout and the policy maker's threshold, normalized by the standard deviation of the turnout, converges to $\kappa(p)$ in state α and $\kappa(p) + p$ in state β. Therefore, in state α, policy A is chosen with probability $1 - \Phi\left(\kappa\left(p\right)\right)$, and in state β, policy B is chosen with probability $\Phi\left(\kappa\left(p\right) + p\right)$. Therefore, when $\frac{f(0)}{F(0)} < \tau$, $p = P\left(\frac{f(0)}{F(0)}\right) = 0$, $\kappa(p) = \infty$, and policy B is chosen in both states. In this case, there is no information transmission in equilibrium. If $\frac{f(0)}{F(0)} > \tau$, then the policy is indeterminate conditional on the state even for large n since $1 - \Phi\left(\kappa\left(p\right)\right)$ and $\Phi\left(\kappa\left(p\right) + p\right)$ are both interior. The indeterminacy arises because the expected number of informative citizens is proportional to the the standard deviation of the turnout, i.e., σ_n.

In terms of welfare, the policy maker utilizes the information transmitted through the protests for a single-person decision problem. Because the information content of the protest is increasing in p in Blackwell order, the policy maker is better off when the informativeness of protests increases. Citizens' welfare consists of their costs from participation and the policy choice. Because the equilibrium cost cutoffs converge to 0, a change in the informativeness of the protests does not alter the expected cost from participation for large n. For the policy choice, an increase in p increases the probability that A is chosen in state α because κ is a decreasing function. An increase in p has an ambiguous effect on the probability that B is chosen in state β. However, the citizens are unambiguously better off when p increases. This is because $\mu > 0.5$, i.e., the policy maker is more concerned about a mistake in state β compared to the citizens and the policy maker's optimal choice implies that he is strictly better off when p increases.

[29] Recall that we use the convention that $\kappa(0) = \infty$ when $\mu > q$.

5.3 Encouragement Effect

The complementarity effect we discussed in Subsection 4.3 leads to the following comparative statics: For small noise, i.e., small $F(0)$, in responsive equilibrium sequences with maximal informativeness, the expected number of informative citizens increases with $F(0)$. We call this the encouragement effect of noise in the participation decisions of the informative citizens.

Theorem 3. *For any two cost distributions F^1, F^2 with $f^1(0) = f^2(0) > 0$, there exists $\bar{F} > 0$ such that $0 < F^1(0) < F^2(0) < \bar{F}$ implies that for any corresponding responsive equilibrium sequences with maximal informativeness $\left\{c^i_{a,n}, c^i_{b,n}, \rho^i_n\right\}$ for $i \in \{1, 2\}$,*

$$\lim_{n\to\infty} \frac{F^1\left(c^1_{a,n}\right) - F^1\left(c^1_{b,n}\right)}{F^2\left(c^2_{a,n}\right) - F^2\left(c^2_{b,n}\right)} < 1.$$

Note that the maximal informativeness, P, is decreasing in $F(0)$. Hence, additional noise unambiguously decreases the maximal informativeness of protests. However, when the initial noise is small, an increase in it leads to an increase in the probability of being pivotal, and thus an increase in the expected number of informative citizens. So, the citizens react to an increase in noise that partially offsets the impact of noise in the informativeness.[30] We thus conclude that when the noise in the turnout is sufficiently small, then an increase in the noise has an encouragement effect on the citizens to use their information: *Citizens join their voices to overcome the noise.*

5.4 Comparative Statics

We now investigate how the threshold τ and the function P change with μ and the informativeness of the signals. First, consider an increase in the informativeness of the citizens' signals in Blackwell order. Because we have 2 states and 2 signals, this amounts to an increase in $\frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)}$, and a decrease in $\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)}$.

Recall that the informativeness of the protests is given by the maximal solution of the equality (17):

$$p = \frac{f(0)}{F(0)}\left(1 + \frac{q}{1-q}\frac{1-\mu}{\mu}\right)\left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right)\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b)\right)\phi\left(\kappa(p)\right).$$

Inspection of this equation shows that the impact of such a change of the informativeness of the signals on the right-hand side of the equation depends on the term $\left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right)\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b)\right)$. However, this term is equal to the difference in the expected posterior beliefs of a citizen across states[31] and this difference is easily shown to be increasing when signals are more

[30] One may be reminded of the role of noise in the incentives to acquire information in noisy rational expectations equilibrium (Grossman and Stiglitz, 1980) and in social learning settings (Duffie et al., 2009), where the addition of noise to the signal of an aggregate statistic of agents' actions leads to more information acquisition.

[31] That is,

$$\left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right)\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b)\right) = \mathbb{P}(a|\alpha)\mathbb{P}(\alpha|a) + \mathbb{P}(b|\alpha)\mathbb{P}(\alpha|b) - \left(\mathbb{P}(a|\beta)\mathbb{P}(\alpha|a) + \mathbb{P}(b|\beta)\mathbb{P}(\alpha|b)\right).$$

informative in the Blackwell order. From this, it follows that the right-hand side of the equality ($p = \hat{p}(p)$) is indeed increasing in the informativeness of the citizens' signals, hence τ decreases while the informativeness of the protests increase with the informativeness of the citizens' signals. However, even when the signal distribution approaches the perfectly informative signal distribution, the informativeness of the protests remain bounded when $\frac{f(0)}{F(0)}$ is finite since $(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta))(\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b))$ is bounded by 1. (This observation implies, in particular, that our qualitative results do not depend sensitively on the citizens' update about the state from being pivotal, since our results also hold when citizens are certain of the state.)

Let $>_B$ denote the Blackwell order on distributions $\mathbb{P} : \{\alpha, \beta\} \to \Delta\{a, b\}$ that satisfy the MLRP condition, and let $P(x, \mathbb{P})$ denote the maximal informativeness of protests when the signal distribution is $\mathbb{P}$ and when the reverse hazard rate is x. Our previous discussion proves:

Theorem 4. *$\mathbb{P}_1 >_B \mathbb{P}_2$ implies $P(x; \mathbb{P}_1) > P(x; \mathbb{P}_2)$ for every $x \geq \tau(\mathbb{P}_1)$. Moreover, if $\mu > q$, then $\tau(\mathbb{P}_1) < \tau(\mathbb{P}_2)$.*

We now investigate how the policy makers bias, μ, affects the maximal informativeness of the protests.

Theorem 5. *Fixing $(q, \mathbb{P}, \frac{f(0)}{F(0)})$, the maximal informativeness of the protests P viewed as a function of μ satisfies the following comparative statics:*

1. *If $\frac{f(0)}{F(0)}(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta))(\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b)) < \frac{1}{\phi(1)}$, then $P(\mu)$ is decreasing in μ, and is maximized at $\mu = q$.*
2. *If $\frac{f(0)}{F(0)}(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta))(\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b)) > \frac{1}{\phi(1)}$, then $P(\mu)$ is single peaked, and is maximized at some $\mu \in (q, 1)$.*

Recall that the informativeness of the protests is given by the maximal solution of the equality

$$p = \frac{f(0)}{F(0)} \underbrace{\left(1 + \frac{q}{1-q}\frac{1-\mu}{\mu}\right)}_{A(\mu)} (\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta))(\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b))\,\phi(\kappa(p; \mu))$$

where $\kappa(p; \mu) = -\frac{p}{2} + \frac{1}{p}\ln\left(\frac{1-q}{q}\frac{\mu}{1-\mu}\right)$. Hence, the comparative statics depend on the impact of the term $\frac{1-\mu}{\mu}$ on the terms $A(\mu)$ and $\phi(\kappa(p;\mu))$. The term $A(\mu)$ is decreasing in μ while the impact of μ in the function $\phi(\kappa(p;\mu))$ is ambiguous because κ is increasing in μ while ϕ is a single-peaked function. Therefore, the comparative statics depend on the details of the other parameters of the model.

6 An Illustrative Numerical Example

Here, we consider a numerical example in which the citizens' participation costs are deterministic and equal to a constant $c > 0$. In addition, the citizens are perfectly informed about the state, i.e., $\mathbb{P}(a|\alpha) = \mathbb{P}(b|\beta) = 1$,[32] and $\mu > q = 1/2$.

[32] Recall that we observed in the discussion of Theorem 4 that our results continue to hold for perfectly informative signals.

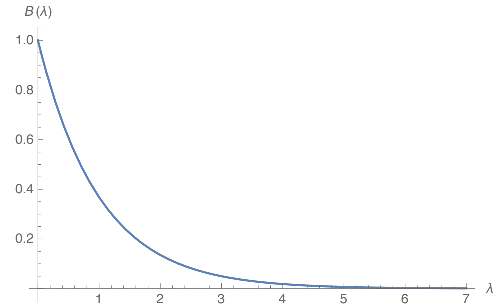


Figure 7: The benefit of participation in state α given expected turnout λ.

We look for a responsive equilibrium, i.e., an equilibrium in which $\lambda(\alpha) > \lambda(\beta)$. In any responsive equilibrium, citizens abstain in state β because the signal is perfectly informative and because the citizens prefer policy B in state β. Hence, $\lambda(\beta) = 0$. Therefore, in a responsive equilibrium, the policy maker's posterior belief that the state is α equals 1 if turnout is positive and is less than the prior (so less than μ) if turnout is 0. Hence, the policy choice is B if turnout is 0, and it is A if turnout is positive.

How do the citizens behave in state α? From a citizen's perspective, participation provides a public good to the rest of the population, but it is costly. Hence, the problem becomes a standard free-riding problem.

The benefit from participation (without the costs) for a given $\lambda(\alpha)$ is equal to the probability of being the only participant,

$$B(\lambda) := e^{-\lambda}.$$

Figure 7 shows the function B.

The function B is strictly decreasing. When $c < 1$, there is a unique $\lambda^* > 0$ such that $B(\lambda^*) = c$. If $n \geq \lambda^*$, then there is a unique responsive equilibrium in which a citizen with an a signal mixes between participating and abstaining, participating with a probability $\psi(a) = \frac{\lambda^*}{n}$. If $n < \lambda^*$, then all citizens join the protest in state α. We observe that the participation decision in state α exhibits a strategic substitutes effect: The higher $\lambda(\alpha)$ is, the lower the incentive to participate. Moreover, when $n \geq \lambda^*$, the expected turnout in state α is independent of n (it is stationary).

We now add noise to the turnout. A citizen is either an "activist" who always participates in the protest or an informative citizen who has the payoff function as described previously. The expected number of activists is m, and the expected number of informative citizens is n; so, the expected population size is $m + n$. In any responsive equilibrium, as before, informative citizens abstain in state β, hence we have $\lambda(\beta) = m$. Let λ be the expected number of informative citizens who participate in state α. Then, the posterior likelihood ratio that the state is α when the turnout is t is given by

$$L(t) = \frac{q}{1-q}\frac{e^{-(\lambda+m)}(\lambda+m)^t}{e^{-m}m^t} = e^{-\lambda}\left(\frac{\lambda+m}{m}\right)^t,$$

where we used $q = 1/2$ for the last equality. The threshold turnout T then becomes the

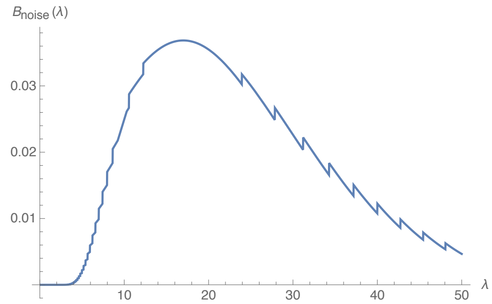


Figure 8: The benefit of participation in state α when the expected number of informative participants is λ. The parameters are $\mu = 0.8$ and $m = 100$. If $c = 0.01$, then the largest solution of $B_{noise}(\lambda) = 0.01$ is at $\lambda^* = 41.60415$.

largest t such that $L(t) \leq \frac{\mu}{1-\mu}$, which implies that

$$T(\lambda) = \lfloor \frac{\lambda + \ln\left(\frac{\mu}{1-\mu}\right)}{\ln\left(\frac{\lambda+m}{m}\right)} \rfloor.$$

Then, the benefit from participation in state α for a given λ is equal to[33] (see Figure 8)

$$B_{\text{noise}}(\lambda) := e^{-(\lambda+m)} \frac{(\lambda+m)^{T(\lambda)}}{T(\lambda)!}.$$

We observe that $\lim_{\lambda\to 0} B_{\text{noise}}(\lambda) = \lim_{\lambda\to\infty} B_{\text{noise}}(\lambda) = 0$, and $B_{\text{noise}}(\lambda)$ is single-peaked. The shape of B_{noise} highlights the complementarity and substitution effects: When λ is small, an increase leads to larger incentives to participate. When λ is large, an increase leads to smaller incentives to participate. When c is above the peak of B_{noise}, there is no responsive equilibrium. When c is smaller than the peak of B_{noise}, there are at least 2 solutions to $B_{\text{noise}}(\lambda) = c$.[34] Note that there is a discontinuity in the equilibrium informativeness as c increases above the peak (or, similarly, m increases slightly). The largest intersection point, λ^*, generates a higher turnout from the informative citizens compared to the case in which there is no noise, for a fixed but small c. Observe also that there is stationarity: When $n > \lambda^*$, in the equilibrium with maximal turnout, the expected turnout is independent of n.

For a numerical example, let $c = 0.01$. Then, the solution of $B(\lambda) = 0.01$ (no noise) is at $\lambda^* = 4.60517$. If $m = 100$, then the maximal solution of $B_{noise}(\lambda) = 0.01$ is obtained at $\lambda^* = 41.60415$. Observe that the equilibrium turnout by the informative citizens in state α is significantly higher when there is noise. In this case, the policy maker's threshold is $T = 123$. The equilibrium turnout distributions in states β and α are given in Figure 9.

[33]This is exact only when $T(\lambda) < \frac{\lambda+\ln\left(\frac{\mu}{1-\mu}\right)}{\ln(\lambda+m)}$. When $T(\lambda) = \frac{\lambda+\ln\left(\frac{\mu}{1-\mu}\right)}{\ln(\lambda+m)}$, the policy maker may be mixing between the two policies when turnout is $T(\lambda)$. Hence, B_{noise} is a correspondence, and not a function, and is depicted in Figure 8.

[34]The number of solutions to the equality $B_{\text{noise}}(\lambda) = c$ is typically 2, with the possibility of more solutions resulting from the policy maker's mixing.

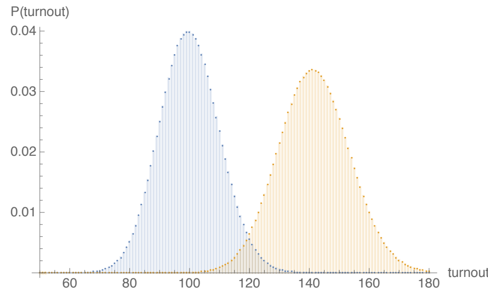


Figure 9: Turnout distributions in states β (the distribution with smaller mean) and α (the distribution with larger mean), when $m = 100$ and $\lambda = 41.60415$. The threshold turnout is $T = 123$.

7 Information Aggregation Without Activists

In the following two subsections, we study the case without activists (i.e., $\underline{c} = 0$). In the first subsection, we consider the case where costs are distributed without atoms on $[0, \bar{c}]$, and in the second and third subsections, the case with an atom at 0, including the case in which all citizens have 0 participation costs (costless participation). The analysis in this section provides some insights into the role of the costs and benefits of participation (noise) by considering only costly participation in isolation, and by comparing the results to the case with neither costs nor benefits.

7.1 Costly Protests

We analyze the case in which $\underline{c} = 0$ and $\bar{c} > 0$, i.e., there are still costs of participation but there are no longer non-informational benefits from participation in the protest movement. We maintain the assumptions that F is strictly increasing in the interval $[0, \bar{c}]$, has no atoms, and admits a density f with the following regularity property at 0:

Assumption 1. *For every $d > 1$*

$$\liminf_{c \to 0^+} \frac{F(dc)}{F(c)} > 1.$$

Note that any atomless and increasing cost distribution F with $f(0) \in (0, \infty)$ satisfies Assumption 1. Moreover, distributions of the form $F(c) = c^\gamma$ for any $\gamma \in (0, \infty)$ also satisfy the assumption.

In the analysis of the maximally informative equilibrium sequences when $\underline{c} < 0$, we showed that if $f(0) > 0$ and $F(0)$ is close to 0, then protests become arbitrarily informative; see the last item of Theorem 1. Therefore, one may conjecture that when $\underline{c} = 0$ (i.e., when $F(0) = 0$), information aggregation is possible. We show that this conjecture is indeed correct and that information also aggregates when $f(0) = 0$, provided that Assumption 1 holds. We devote a separate section to this part of the analysis because the proof technique is different and the strategic nature of the problem is different from the case in which $\underline{c} < 0$.

Take a sequence of strategy profiles $\{c_{a,n}, c_{b,n}, \rho_n\}$ where each strategy profile is a responsive equilibrium of a protest game in which the expected number of citizens is n, and along the sequence, all the parameters of the model except n (i.e., $\mathbb{P}$, F, q, μ) are fixed. We say that such a sequence is a large responsive equilibrium sequence if

$$\lim_{n\to\infty} \lambda_n(\alpha) = \infty.$$

Remark 1. Under some parameter values, one can show that there are responsive equilibrium sequences where, as $n \to \infty$, expected turnout goes to 0 in both states. Along such equilibrium sequences, protests become uninformative. When $\mu = q$, there are also responsive equilibrium sequences with $\lim \lambda_n(\alpha) \in (0, \infty)$. Hence, we focus on equilibria in which the expected turnout increases without bound.

Theorem 6. *Assume that $\underline{c} = 0 < \bar{c}$ and that F is atomless, strictly increasing and satisfies Assumption 1. Then:*

1. *There always exists a large responsive equilibrium sequence.*

2. *Every large responsive equilibrium sequence aggregates information: the probability that A is implemented in state α goes to 1, and the probability that B is implemented in state β goes to 1, i.e.,*
$$\lim_{n\to\infty} \mathbb{P}\left(\{t : \rho_n(t) = 1\}|\alpha\right) = 1,$$
$$\lim_{n\to\infty} \mathbb{P}\left(\{t : \rho_n(t) = 0\}|\beta\right) = 1.$$

3. *In every responsive equilibrium sequence,*
$$\limsup_{n\to\infty} \frac{\lambda_n(\omega)}{\ln n} < \infty.$$

Theorem 6 presents three results. First, a large responsive equilibrium sequence exists. Second, such equilibria aggregate information. These two results are as expected given our analysis for the case with $\underline{c} < 0$ and a large reverse hazard rate $\frac{f(0)}{F(0)}$.

The third result is that participation increases at a rate not faster than $\ln n$ in both states. This implies that the expected number of informative citizens also increases at a rate not more than $\ln n$. Comparing this rate with the corresponding rate when $\underline{c} < 0$, which is $\sqrt{n}$ (see the discussion in Section 4.3), we conclude that when there are only costs of protests, citizens in equilibrium have less incentive to participate. Broadly speaking, this is an instance of the "encouragement effect" of noise discussed in Section 5.3. Basically, protests are already very informative when participation is small (at the order of $\ln n$), so there is little incentive to increase their informativeness any further.

We now provide intuition for the information aggregation result. We start with two observations. First, cost cutoffs converge to 0 as n grows without bound. This is because the expected turnout grows without bound and so the pivot probability vanishes to 0. Our second observation is that the ratio of the expected protest sizes across the states stays bounded away from 1. This is because the optimality condition for the cost cutoffs (4) implies that $c_{a,n} > c_{b,n}\frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)}$, and so $\frac{c_{a,n}}{c_{b,n}}$ is bounded away from 1. When Assumption 1 holds, this implies that the ratio of the expected protest sizes $\frac{\lambda_n(\alpha)}{\lambda_n(\beta)}$ stays bounded away from 1. To see why information aggregates, recall that the standard deviation of the Poisson distribution is the square root of its mean, and the turnout in states α and β are apart from each other by a factor of the mean in state α, i.e., the distance in the means in standard deviations grows without bound. Hence, an application of Chebyshev's inequality delivers information aggregation because the policy maker is able to distinguish the two states with probability converging to 1 based on the realized turnout.

7.2 Costless Participation

We now consider a scenario in which $\underline{c} = \bar{c} = 0$, i.e., there are neither costs nor benefits of participating that are unrelated to the instrumental effect of the protest itself. This case was studied in Battaglini (2017). Because participation is costless for all citizens, a strategy for a citizen is simply the probability of participation based on her signal,

$$\psi : \{a, b\} \to [0, 1].$$

As before, an equilibrium is responsive if $\psi(a) > \psi(b)$, and an equilibrium is babbling if $\psi(a) = \psi(b)$. The following theorem provides a necessary and sufficient condition for the existence of a responsive equilibrium sequence that aggregates information; otherwise, the unique equilibrium is babbling. The main insights of the theorem are due to Battaglini (2017).

Theorem 7. *Assume that $\underline{c} = \bar{c} = 0$, i.e., participation in the protest is costless.*

1. *Battaglini (2017) If*
$$\frac{\mu}{1-\mu}\frac{\mathbb{P}(a|\beta)}{\mathbb{P}(a|\alpha)}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} > 1, \tag{18}$$
then babbling is the unique equilibrium outcome.

2. *If*
$$\frac{\mu}{1-\mu}\frac{\mathbb{P}(a|\beta)}{\mathbb{P}(a|\alpha)}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} < 1, \tag{19}$$
then there is a responsive equilibrium sequence that aggregates information.

The inequality (19) is satisfied if $\mu = \frac{1}{2}$ because the MLRP condition implies that $\frac{\mathbb{P}(a|\beta)}{\mathbb{P}(a|\alpha)} < 1$ and $\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} < 1$. Conversely, if $\mu > \frac{1}{2}$, inequality (18) is satisfied whenever the likelihood ratios are sufficiently close to 1 (i.e., signals are relatively uninformative).

The intuition for the first part of the theorem is that, when the conflict is large, or when the signals are imprecise, then citizens with b signals have strong incentives to participate because, conditional on being pivotal, their posterior belief that the state is α is larger than 1/2. This was shown in Battaglini (2017).[35] For the second claim, we construct a responsive equilibrium sequence. Hence, we show that the necessary condition that was previously identified is also tight.[36]

To see more precisely why the first claim is true, take any candidate for a responsive equilibrium, consisting of some participation probabilities for which the turnout is expected

[35] In fact, Battaglini (2017) shows that this result also holds if the cost distribution is discrete with finite support, provided there is one atom at 0. Intuitively, for large n, the cost cutoffs are close to 0 and so only citizens with 0 costs participate.

[36] Battaglini (2017) shows that, given any signal distribution, there exists some cutoff μ^* such that there is a responsive equilibrium for all $\mu < \mu^*$ and there does not exist a responsive equilibrium for all $\mu > \mu^*$, for large populations.

to be higher in state α than in state β, and the policy maker's best response threshold given those cutoffs. Then, a citizen is pivotal only when the policy maker is on the verge of choosing A, yet, without any further protester, his decision is B. More precisely, as noted before, a citizen is pivotal when there are T other protesters, and the policy maker's posterior belief when there are T protesters is close to μ (it is weakly smaller than μ when there are T protesters and it is weakly above μ when there are $T+1$ protesters). Now, if a b signal is not very informative relative to μ, then, when a citizen is pivotal because there are T other protesters, the probability that the state is α remains close to μ and, in particular, is strictly above $1/2$, even when the citizen has signal b. Therefore, conditional on being pivotal and her own signal, a citizen has strict incentives to participate not only with an a signal but also with a b signal. In other words, in any best response to any candidate for a responsive equilibrium, the citizens' participation decisions are **signal-independent**, and so, the turnout is state-independent as well. Thus, "babbling" is the unique equilibrium outcome.[37]

Battaglini's observation that, when inequality (18) holds, there is no information transmission in equilibrium is striking because, arguably, the model has little room for a conflict of interests between the policy maker and the citizens: The preferences are completely aligned under complete information about the state ω. Moreover, suppose that we interpret the realization of the citizens' signal profile as the *effective* state of the world. Then, the number of effective states in which the policy maker's and the citizens' preferred policies differ is bounded above, independently of n. Hence, both the policy maker and all citizens would be better off if the policy maker could delegate, before the protest, the policy decision to any one of the citizens. Likewise, if the policy maker could commit to a referendum with a fixed voting rule, then information would aggregate, and everybody would be better off (see Battaglini, 2017). Hence, the lack of commitment by the policy maker is detrimental to receiving informative advice.

In light of the difficulty of information transmission highlighted in Theorem 7, Theorem 6 shows that participation costs help information transmission and aggregation. The basic idea is the following: Whatever is the belief of a citizen conditional on being pivotal, a citizen with a signal a believes the state is α with a probability that is strictly higher than a citizen with a signal b. Hence, the cost that a citizen with a signal a is willing to incur to participate is strictly higher than the cost that a citizen with a signal b is willing to incur.[38] This induces participation decisions that are **signal-dependent**, hence turnout is informative about the state of the world. We discuss the relation of this observation to prior work on costs in voting—and in sender-receiver games more generally—in the literature review below in Section 8.

7.3 Cost Distributions with an Atom at 0

Finally, we consider the case in which participation is costless for some citizens while it is costly for others. In particular, $\underline{c} = 0$, $F(0) \in (0,1)$, and F is strictly increasing on $[0, \bar{c}]$ with a continuous density f.[39] We will see that the results for this case are, in a sense, "in

[37] The same argument also holds for strategy profiles where the turnout is expected to be larger in state β than in state α. Hence, there is no equilibrium in which the protest transmits information.

[38] Formally, as observed before, for any best response, $c_a > c_b \frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)}$.

[39] Here, let $f(0) := \lim_{\epsilon \downarrow 0} f(\epsilon)$.

between" the result for costly participation without noise from Theorem 6 and the result for costly participation with noise from Theorem 1.

Specifically, if there is an atom at 0, then the inequalities from Theorem 7 determine whether there is information aggregation (as in Theorem 6) and, if not, how much information can be transmitted (as in Theorem 1).

Theorem 8. *Assume that $\underline{c} = 0 < \bar{c}$, $F(0) \in (0, 1)$, and F is atomless with a continuous density on $(0, \bar{c}]$.*

1. *If inequality (19) holds, then there is a responsive equilibrium sequence that aggregates information.*

2. *If inequality (18) holds, then the maximally informative responsive equilibrium sequence generates protests with an informativeness given by the P function identified in Theorem 1.*

To gain some intuition, recall that, if F is atomless and if its support includes negative costs, then, in a responsive equilibrium, the expected number of informative citizens is equal to $n\,(F(c_a) - F(c_b))$. Moreover, when n is large, the expected number of informative citizens is approximately $nf(0)\,(c_a - c_b)$ when c_a and c_b are close to 0. This approximation is also valid when F has an atom at 0, provided that $c_b > 0$. As we show in the proof of Theorem 7, if inequality (18) holds, then $c_b > 0$ in every best response to any responsive strategy profile. (Citizens have a strict incentive to participate even with a b signal.) Thus, in this case, the analysis of the problem with an atom at 0 becomes identical to the analysis of the case $\underline{c} < 0 < \bar{c}$.

Conversely, if inequality (19) holds, then we showed that there is a responsive equilibrium in which citizens with b signals prefer to abstain, i.e., $c_b < 0$. In this case, citizens with no cost of participation use their signals in their participation decision, and the expected number of informative citizens is approximately $nF(0)$. This opens up the possibility that the ratio of the expected protest sizes in the two states stays bounded away from 1 as n grows, leading to information aggregation.

Finally, note that given the other parameters of the model (that is, given $(q, \mu, \mathbb{P}\,(\theta|\omega))$), if $F(0) > 0$ and $f(0)$ is sufficiently small, then the outcome is equivalent to the outcome with costless participation in Theorem 7.

8 Related Literature

Our paper is related to four strands of literature. The first one is on communication between multiple senders and a receiver, the second one is on information aggregation in elections, the third one is on costly voting, and the fourth one is on communication with money burning.

Communication models with multiple senders resemble voting models in that the senders send a message. Different from voting models, the receiver does not commit to a particular voting rule ex ante. The most closely related work is Battaglini (2017). He shows that when the citizens' information is poor, or when the policy maker and the citizens' preferences are not sufficiently aligned, then no information is transmitted in equilibrium (see Theorem 7). We build on his model and extend it by adding costs and benefits to participation. Wolinsky

(2002) studies a model of receiving advice from a group of experts who receive independent signals. He shows that information transmission is not possible if the preferences of the experts and the advisor are not aligned.[40] Morgan and Stocken (2008) study a model of polling in which the receiver makes a continuous policy choice after seeing the results of a poll obtained from a group of experts with heterogeneous preferences and with dispersed information. They show (in Proposition 13) that when there is a conflict of interest between the policy maker and the experts, the amount of information transmitted is limited regardless of the size of the population. In our model, citizens' preferences are homogeneous, the policy maker has a binary policy choice, and when participation is costless, a similar conclusion holds. Levit and Malenko (2011) model nonbinding shareholder voting where the board and the shareholders have a conflict of interest. They show that when the conflict is sufficiently large, the unique equilibrium is babbling.[41] Different from our model, in these other papers, there are no direct costs or benefits from either participation or from sending any of the messages.

The papers mentioned above propose certain policies or the availability of other means that facilitate information transmission and aggregation in such environments. Battaglini (2017) suggests that communication via social media may facilitate information aggregation by increasing the precision of signals among groups. Wolinsky (2002) studies the receiver's optimal elicitation mechanism. Morgan and Stocken (2008) show that the existence of citizens with sufficiently small conflicts of interest with the decision-maker facilitates information aggregation. Levit and Malenko (2011) suggest that the existence of an additional third party who can pressure the decision-maker for not following the advice of the shareholders may facilitate information aggregation. Our analysis complements these other proposed solutions and offers a simple and new channel by which costly participation may increase efficiency and information aggregation. Importantly, we show that participation costs help information transmission and aggregation regardless of the precision of citizens' signals and the size of the conflict of interest. Hence, we highlight that nonbinding voting may be an effective way to elicit dispersed information when participation or voting is costly and when voting is voluntary. Moreover, our main results are derived in the presence of "noise" in the communication via the presence of orthogonal participation benefits, and we characterize the effects of noise on the citizens' behavior and on information transmission.

Information aggregation in elections has been studied extensively. Notably, Austen-Smith and Banks (1996) show that sincere voting is typically not consistent with rational behavior, and Feddersen and Pesendorfer (1997) show that under any supermajority voting rule except unanimity, large elections aggregate information. In these models, voting is costless and the voting rule is fixed before the game, i.e., these models study costless formal elections. In relation to this literature, we study the information aggregation properties of informal elections, with or without participation costs and benefits.

Our model is closely related to costly voting models, especially our results regarding the case without noise in the later part of the paper in Section 7. Recall that, in private value elections, when voting is costless, the distribution of ordinal preferences determines the voting outcome. In the case of simple majority rule, this leads to the median voter theorem. When

[40] Gradwohl and Feddersen (2018) reach a similar conclusion in a related advisory committee setting.

[41] Yildirim (2012) also discusses the commitment problem of a policy maker in elections.

there are participation costs, the voting mechanism utilizes the costs in a way that is akin to monetary transfers and may elicit the intensity of the citizens' preferences. For example, in the case of the simple majority rule, Ledyard (1984) and Krishna and Morgan (2015) show that costly voting leads to outcomes that maximize the utilitarian welfare. Krishna and Morgan (2011) show that if the election has both private and common value components, then majority voting with costs results in the utilitarian outcome state by state.[42]

In our model, costs also help screen citizens' types. Different from the aforementioned papers, costs induce differential participation across citizens with different interim beliefs. Thus, turnout can identify the unknown state and aggregates information when participation is costly, while it would fail to do so in the benchmark model of costless participation. Therefore, in our model, costs lead to improvements that have a different nature, namely, by enabling information transmission to the policy maker. While our results from Section 7 for the case without noise are related to the costly voting literature, our main results account for the presence of noise. We study the interplay of costs and noise in shaping the participation incentives and information transmission, giving rise to complementarity and encouragement effects.

Our framework is an example of a sender–receiver game, first modeled by Crawford and Sobel (1982) with a single sender. They show that communication is limited when there are differences in the preferences of the sender and the receiver. Austen-Smith (1990, 1993) studies models in which there are multiple senders and the receiver does not commit to a voting rule ex ante. However, these papers focus on the case when the number of senders is small, and there is no cost of communication.[43] In sender–receiver games, the existence of purely dissipative signals, i.e., the possibility of "money burning," may increase the equilibrium amount of information transmission (see Austen-Smith et al., 2000; Kartik, 2007). Importantly, our finding from Section 7 that costs help information transmission is similar to the insight that is uncovered in this literature. The main differences from these papers are as follows: First, we consider a setting with a large number of senders. Second, the motives of the senders are unknown. Importantly, the citizens cannot choose the amount of costs they burn, and furthermore, they may have benefits from participation. Using the large number of senders, we show that information aggregates when costs are nonnegative. Finally, as in the case of costly voting, our main results regarding the presences of noise are new.[44]

[42] In another related paper, Krishna and Morgan (2012) study costly voting in a pure common-value model. They show that, despite the costs inducing a free-riding problem, information still aggregates as in the costless voting model. Borgers (2004) studies a symmetric private values setting and shows that costly voting is welfare-superior to a random decision despite the dissipation of its benefits through voting costs and excessive participation.

[43] See also Battaglini and Benabou (2003); Battaglini (2004) on the related topic. Likewise, there is literature on cheap talk with multiple senders, and in this literature, it is typically assumed that the sender's information is not dispersed (see Battaglini 2002).

[44] Other related papers on polling and protests (that either address different questions or work with different model assumptions) are Lohmann (1993), Lohmann (1994), Banerjee and Somanathan (2001), Cukierman (1991), and McKelvey and Ordeshook (1985).

9 Discussion and Conclusion

9.1 Deterministic Population Size

We used a model in which the number of citizens is Poisson-distributed because calculations related to probability of pivotal events become simpler in this setting. However, this may not be an ideal assumption in some of the applications we consider, such as shareholder voting, in which the number of shareholders is deterministic, or in surveys, in which the number of survey requests sent out is known to the policy maker. If we assume a deterministic population size with n citizens, then our results do not change qualitatively. There are, however, two differences that are noteworthy. The first one is that the policy maker, after observing the turnout t, would also know the number of absentees, which is $n-t$. The second one is that the standard deviation of the turnout is approximately $\sqrt{nF(0)(1-F(0))}$. In the Supplementary Appendix, we show that our main results continue to hold if the population size is deterministic, with the main difference being in the equation that gives the maximal informativeness of the protests (i.e., equation (17)). We show that the maximal informativeness of protests depends on $\frac{f(0)}{F(0)(1-F(0))}$ in this case.

9.2 Modeling the Noise

In our model, the main source of noise in the turnout is the assumption that $\underline{c} < 0$. This modeling choice leads to protest sizes that are of the same order of magnitude as n, having a standard deviation on the order of $\sqrt{n}$.

An alternative way of modeling noise is by assuming that there are activists who always join the protest. More precisely, suppose that a citizen is either an activist who always joins the protest or a non-activist who receives an informative signal and incurs a random positive cost of participation with distribution F. The number of non-activists, t_{na}, is Poisson-distributed with mean n, while the number of activists, t_a, is Poisson-distributed with mean nr for some $r > 0$. The policy maker observes the sum of the number of non-activists who choose to participate, and the number of activists t_a, and chooses the policy. The equilibria of this model produces results that are qualitatively similar to the results in our model when n is large. (In fact, we present a simplified version of this setup in our illustrative example in Section 6.)

Another way of modeling noise would be to consider a setting without activists (i.e., $\underline{c}$=0), but assume that the policy maker observes the turnout with noise. Suppose that when the turnout is t, the policy maker observes a size $\tilde{t} = t + \tilde{t}_1$, where $\tilde{t}_1 = t_{noise} - nr$ and t_{noise} is Poisson-distributed with mean nr for some $r > 0$. This model again produces equilibrium properties that are very similar to our model.

9.3 Multiple Signals

We assume that citizens have binary signals. Our results do not rely on this assumption and can easily be generalized to a signal space with an arbitrary finite number of elements. To see how we can accommodate multiple signals, suppose that there are K signals, $\{1, 2, ..., K\}$, and suppose that the probability distribution over the signals satisfies the monotone likelihood ratio property, i.e., $\frac{\mathbb{P}(i|\alpha)}{\mathbb{P}(i|\beta)}$ is strictly decreasing in i, and the bounded likelihood ratio property that $\mathbb{P}(i|\omega) \in (0, 1)$ for each $i = 1, 2, ..., K,$ and $\omega = \alpha, \beta$. In a responsive equilibrium of the model when $\underline{c} < 0 < \bar{c}$, citizens follow a cutoff strategy $\{c_i\}_{i\in\{1,2,...,K\}}$ where c_i

is decreasing in i and is given by

$$c_{i,n} = \mathbb{P}(\alpha|i)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|i)\mathbb{P}(\text{piv}|\beta).$$

Take a responsive equilibrium sequence with

$$\lim \frac{\lambda_n(\alpha) - \lambda_n(\beta)}{\sqrt{nF(0)}} = p \in (0, \infty).$$

Then, similar to what we show for the binary signal case in item 1 of Lemma 3 in the Appendix, for every $i \in \{1, 2, ..., K\}$, $\lim \frac{c_{i,n}}{c_{1,n}} = \frac{\mathbb{P}(\alpha|i)\frac{1-q}{q}\frac{\mu}{1-\mu} - \mathbb{P}(\beta|i)}{\mathbb{P}(\alpha|1)\frac{1-q}{q}\frac{\mu}{1-\mu} - \mathbb{P}(\beta|1)} =: \gamma_i$. Therefore,

$$p = \lim \frac{\lambda_n(\alpha) - \lambda_n(\beta)}{\sqrt{nF(0)}} = \lim n \frac{f(0)}{\sqrt{nF(0)}} c_{1,n} \sum_{i=1,..,K} \left((\mathbb{P}(i|\alpha) - \mathbb{P}(i|\beta)) \gamma_i \right). \tag{20}$$

Recall equations (11) and (12) that imply $\lim \sqrt{nF(0)}\mathbb{P}(\text{piv}|\alpha) = \phi(\kappa(p))$ and $\lim \sqrt{nF(0)}\mathbb{P}(\text{piv}|\beta) = \phi(\kappa(p) + p)$. Therefore,

$$\lim \sqrt{nF(0)} c_{1,n} = \phi(\kappa(p)) \left(\mathbb{P}(\alpha|1) - \mathbb{P}(\beta|1) \frac{q}{1-q} \frac{1-\mu}{\mu} \right). \tag{21}$$

Combining equations (20) and (21), we have

$$p = \frac{f(0)}{F(0)} \phi(\kappa(p)) \left(\mathbb{P}(\alpha|1) - \mathbb{P}(\beta|1) \frac{q}{1-q} \frac{1-\mu}{\mu} \right) \left(\sum_{i=1,..,K} \left((\mathbb{P}(i|\alpha) - \mathbb{P}(i|\beta)) \gamma_i \right) \right). \tag{22}$$

The right-hand side of equation (22) is the analogue of equation (17) when there are multiple signals. The difference between the two equations is a multiplicative constant, i.e., equation (22) is of the form $\frac{f(0)}{F(0)}\phi(\kappa(p))Z$ for some constant Z that does not depend on F, and hence, the qualitative features of the maximally informative equilibrium sequences identified in Theorem 1 hold when there are multiple signals.

9.4 Multiple Messages

Our analysis studied a protest model in which a citizen can choose either to participate or abstain. In applications such as polls, nonbinding shareholder voting, and surveys, a participant can choose one of many messages to communicate her opinion. In nonbinding shareholder voting, a shareholder may abstain or choose to cast a vote for approval or disapproval of a new policy. When there is a protest movement or a petition for a policy, occasionally there is another protest or petition for the alternative policy. Surveys conducted by the management of a company, or polls, typically contain multiple choices for each question.[45]

[45] In fact, Wolinsky (2002), Morgan and Stocken (2008), and Levit and Malenko (2011) observe similar results with costless participation in settings in which an expert can provide one of multiple messages, express different responses in a poll, or vote in favor or against a proposal, respectively.

In our model, if the number of absentees is also observed, then abstention can be interpreted as a second message. Note that the number of absentees is observed if the number of citizens is deterministic, and we show that our results go through without much alteration in this case in the Supplementary Appendix. When there are nonnegative costs of participation, then the interpretation is that sending one message (participation) is costly, while sending the second message (abstention) is costless. When costs of participation can be positive or negative, then the interpretation is that citizens may have non-informational motives to support one policy over the other.

In order to facilitate the applications mentioned above with the abstention option, suppose that each citizen, after observing a signal and participation cost (if participation is costly), chooses whether to participate together with a message $m \in \{m_A, m_B\}$ or to abstain. The policy maker observes the number of citizens that sent each message, t_A and t_B, and then chooses the policy. To avoid tedious case distinctions, we focus on "monotone" equilibria in which the probability that A is chosen is weakly increasing in t_A and weakly decreasing in t_B.

Our first observation is that, in this model, there is always an equilibrium in which one of the messages (for example, m_B) conveys no information, and the policy maker ignores t_B in his policy choice. Such equilibria (partially babbling) always exist with costly or costless participation and simply replicate the responsive equilibria of the model where the citizens have a single message. Hence, when information aggregates with a single message (for example, if participation is costless and inequality (19) holds, or if participation is costly and cost distribution satisfies the assumptions made in Section 7), then there is a partially babbling equilibrium sequence that aggregates information.

Next, suppose that participation is costless and inequality (18) holds. Then, there is a unique equilibrium outcome, and it is babbling outcome. To see this, note that, first, any partially babbling equilibrium is babbling; this follows from the analysis in Section 7.3. Second, any monotone equilibrium is partially babbling. This is because, conditional on being pivotal with a message m_B, the posterior belief that the state is α is at least μ. Therefore, a citizen with signal b has a posterior likelihood ratio that the state is α conditional on being pivotal for message m_B that is at least

$$\frac{\mu}{1-\mu}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} > \frac{\mu}{1-\mu}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)}\frac{\mathbb{P}(a|\beta)}{\mathbb{P}(a|\alpha)} > 1.$$

Therefore, no citizen sends the message m_B in equilibrium.

Finally, suppose that the support of the cost distribution for participation, F, is $[\underline{c}, \bar{c}]$, with $\underline{c} < 0 < \bar{c}$, and the costs are independent of the message the citizen sends. This model is similar to the model with a single message where the cost distribution has an atom at 0. If inequality (18) holds, then any monotone equilibrium is partially babbling, and the maximal informativeness is given by $P\left(\frac{f(0)}{F(0)}\right)$. In other words, the conclusions of Theorem 8 hold. If $\frac{\mu}{1-\mu}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} < 1$, then there is a monotone equilibrium sequence that aggregates information. When $\frac{\mu}{1-\mu}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)}\frac{\mathbb{P}(a|\beta)}{\mathbb{P}(a|\alpha)} < 1 < \frac{\mu}{1-\mu}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)}$, we conjecture that there is a monotone equilibrium sequence that aggregates information, but we have not been able to show this.

9.5 General Citizen Preferences

We assumed that citizens' preferences are symmetric across states. However, we can relax this assumption. More generally, suppose that a citizen's payoff is as follows: If the state is α, her payoff is r if the policy is A and 0 if it is B; if the state is β, her payoff is $1-r$ if the outcome is B and 0 if it is A. In this case, citizens' best reply cutoffs—given previously by equation (4)—change to

$$c_\theta = \mathbb{P}(\alpha|\theta)\mathbb{P}(\text{piv}|\alpha)r - \mathbb{P}(\beta|\theta)\mathbb{P}(\text{piv}|\beta)(1-r) \text{ for } \theta = a, b.$$

Hence, we obtain

$$c_{a,n} - c_{b.n} \approx (\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b)) \left(r + (1-r)\frac{q}{1-q}\frac{1-\mu}{\mu} \right) \mathbb{P}(\text{piv}|\alpha).$$

Moreover, $\kappa(p)$ is unchanged; hence,

$$\lim_{n\to\infty} \sqrt{n}\,(c_{a,n} - c_{b.n}) = \frac{1}{\sqrt{F(0)}} (\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b)) \left(r + (1-r)\frac{q}{1-q}\frac{1-\mu}{\mu} \right) \phi\left(\kappa\left(p\right)\right).$$

The definition of p is unchanged; hence,

$$\lim_{n\to\infty} \sqrt{n}\,(c_{a,n} - c_{b.n}) = \frac{p\sqrt{F(0)}}{f(0)\left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right)}.$$

Therefore, the limit equilibrium informativeness p must be a solution of the equation

$$p = \frac{f(0)}{F(0)} \left(r + (1-r)\frac{q}{1-q}\frac{1-\mu}{\mu} \right) (\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta))\, (\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b))\, \phi\left(\kappa(p)\right).$$

Therefore, changing the payoff function of the citizens results only in a change in the multiplicative constant term in the function $\hat{p}(p)$. Therefore, all of our results regarding the characterization of the maximally informative equilibria continue to hold qualitatively. However, not all the conclusions of Theorem 5 hold if $r \neq 0.5$. In particular, if $r \neq 0.5$, then for all but possibly one value of $\frac{f(0)}{F(0)}$, the μ that maximizes p is not equal to q. Similarly, for all but possibly one value of $\frac{f(0)}{F(0)}$, the μ that maximizes p is not equal to $\frac{1-r}{r}$ either. Thus, in general, maximizing the informativeness of the protests requires some conflict between the policy maker and the citizens as well as the policy maker not being indifferent between two policies ex ante. Similarly, the welfare conclusion of Theorem 2 for the citizens does not necessarily hold if the citizens' preferences are sufficiently different from the policy maker's.

9.6 Preference Heterogeneity

A key feature of models of nonbinding voting that prevent information transmission is the existence of a conflict of interest between the citizens and the policy maker. If citizen preferences are heterogeneous, and if the support of the preference distribution includes preferences that are aligned with that of the policy maker, then information transmission is possible, and information aggregates as the expected population size grows without bound. This insight was shown in Morgan and Stocken (2008) in a model of polls. However, if there is a conflict of interest between the policy maker and every citizen, then information transmission is not possible when there are no costs of participation and the citizens' information is poor. In contrast, in our model when there are nonnegative costs of participation, information transmission and aggregation is possible, even if there is preference heterogeneity. Conversely, our main results regarding the effects of noise on the informativeness of protests also hold if there is no policy maker bias, $\mu = \frac{1}{2}$.

Regarding our results for the case without noise in Section 7.3, note the following. With preference heterogeneity, information still aggregates by the following intuition: In a putative responsive equilibrium, conditional on being pivotal, holding constant the preference of a citizen, the citizen has strictly more incentives to participate when receiving signal a compared to when signal b is received. Therefore, in such an equilibrium, the turnout is expected to be larger in state α than in state β. To show that such an equilibrium exists and aggregates information when n is large, we can use the same method we use to prove Theorem 6.

9.7 Privately Informed Policy Maker

We have assumed throughout the paper that the policy maker is uninformed. In fact, one reason why a policy maker may wish to hold a nonbinding vote instead of committing to a fixed voting rule is to utilize the flexibility offered by nonbinding voting, which allows him to incorporate private information in the policy choice. Suppose that the policy maker receives a signal from a finite set of signals, and suppose that the signal is imperfectly informative about the state. If participation is costless, then the reasoning that delivers uniqueness of the babbling equilibrium when the conflict of interest is large (see Theorem 7) continues to hold. Likewise, when participation is costly (but there are no benefits) and the cost distribution F satisfies the assumptions made in Subsection 7.1, then conditional on being pivotal, a citizen with signal a has strictly more incentive to participate than a citizen with signal b. Hence, the same reasoning for the existence of responsive equilibrium sequences and information aggregation also applies here without alteration, even if the policy maker has private information. Finally, we expect the basic qualitative features of our main characterization result in Theorem 1 to extend to the case in which there are both costs and benefits to participation (noise).

9.8 Conclusion

Citizens affect policies through formal political processes, such as elections, and what we call informal political processes, such as protests, polls, petitions, and referenda. Condorcet's Jury Theorem has shown that formal elections effectively aggregate information. The robustness of these results has been studied extensively, and it is considered as a rationale for why elections are commonly used to select candidates or policies. Despite their frequent occurrence, there are relatively few similar results for informal elections.

Recent results have shown that informal elections do not aggregate information robustly and have identified an important difficulty in information transmission. Our first substantive contribution is to show that participation costs may help overcome this difficulty. Hence, with costs, participation is a more effective way of transmitting information and influencing policy. Our second substantive contribution is to introduce noise to informal elections via activist citizens and to study the interplay of costs and noise in shaping the participation incentives and informativeness of the process. One may expect noise to be bad for participation incentives. However, we show that, surprisingly, noise can increase participation via an encouragement effect that compensates at least partly for the adverse effect of noise on the informativeness of turnout.

In a formal election, the policy maker is bound by the voting rule, while in an informal election, such as nonbinding voting or polling, the policy maker retains some flexibility. One reason why informal elections may be preferred to formal elections is that this flexibility allows the policy maker to adjust the decision based on private information or the anticipated amount of noise in the turnout. For example, if the voting rule is fixed, and if the expected number of activists is large, then the activists determine the policy. If the voting rule is not fixed, then the policy maker can "deduct" the expected number of activists from the turnout and make a decision based on his inference.

References

Acharya, A. (2016): "Information aggregation failure in a model of social mobility," *Games and Economic Behavior*, 100, 257–272.

Ali, S. N., M. Mihm, and L. Siga (2017): "The perverse politics of polarization," Tech. rep.

Austen-Smith, D. (1990): "Information transmission in debate," *American Journal of political science*, 124–152.

——— (1993): "Interested experts and policy advice: Multiple referrals under open rule," *Games and Economic Behavior*, 5, 3–43.

Austen-Smith, D. and J. S. Banks (1996): "Information aggregation, rationality, and the Condorcet jury theorem," *American Political Science Review*, 34–45.

Austen-Smith, D., J. S. Banks, et al. (2000): "Cheap talk and burned money," *Journal of Economic Theory*, 91, 1–16.

Banerjee, A. and R. Somanathan (2001): "A simple model of voice," *The Quarterly Journal of Economics*, 116, 189–227.

Battaglini, M. (2002): "Multiple referrals and multidimensional cheap talk," *Econometrica*, 70, 1379–1401.

——— (2004): "Policy advice with imperfectly informed experts," *Advances in theoretical Economics*, 4, 1–32.

——— (2017): "Public protests and policy making," *The Quarterly Journal of Economics*, 132, 485–549.

Battaglini, M. and R. Benabou (2003): "Trust, coordination, and the industrial organization of political activism," *Journal of the European Economic Association*, 1, 851–889.

Bhattacharya, S. (2013): "Preference monotonicity and information aggregation in elections," *Econometrica*, 81, 1229–1247.

Borgers, T. (2004): "Costly voting," *The American Economic Review*, 94, 57–66.

Cannone, C. (2017): "A short note on Poisson tail bounds," Tech. rep., Stanford University.

Cantoni, D., D. Y. Yang, N. Yuchtman, and Y. J. Zhang (2017): "Protests as Strategic Games: Experimental Evidence from Hong Kong's Democracy Movement," .

Crawford, V. P. and J. Sobel (1982): "Strategic information transmission," *Econometrica*, 1431–1451.

Cukierman, A. (1991): "Asymmetric information and the electoral momentum of public opinion polls," *Public Choice*, 70, 181–213.

Davis, B. and D. McDonald (1995): "An elementary proof of the local central limit theorem," *Journal of Theoretical Probability*, 8, 693–701.

Duffie, D., S. Malamud, and G. Manso (2009): "Information percolation with equilibrium search dynamics," *Econometrica*, 77, 1513–1574.

Ekmekci, M. and S. Lauermann (Forthcoming): "Manipulated electorates and information aggregation," *The Review of Economic Studies*.

Feddersen, T. and W. Pesendorfer (1997): "Voting behavior and information aggregation in elections with private information," *Econometrica*, 1029–1058.

Gnedenko, B. V. (1948): "On a local limit theorem of the theory of probability," *Uspekhi Matematicheskikh Nauk*, 3, 187–194.

Gradwohl, R. and T. Feddersen (2018): "Persuasion and transparency," *The Journal of Politics*, 80, forthcoming.

Grossman, S. J. and J. E. Stiglitz (1980): "On the impossibility of informationally efficient markets," *The American economic review*, 70, 393–408.

Kartik, N. (2007): "A note on cheap talk and burned money," *Journal of Economic Theory*, 136, 749–758.

Krishna, V. and J. Morgan (2011): "Overcoming ideological bias in elections," *Journal of Political Economy*, 119, 183–211.

——— (2012): "Voluntary voting: Costs and benefits," *Journal of Economic Theory*, 147, 2083–2123.

——— (2015): "Majority rule and utilitarian welfare," *American Economic Journal: Microeconomics*, 7, 339–75.

Ledyard, J. O. (1984): "The pure theory of large two-candidate elections," *Public choice*, 44, 7–41.

Levit, D. and N. Malenko (2011): "Nonbinding voting for shareholder proposals," *The Journal of Finance*, 66, 1579–1614.

Lohmann, S. (1993): "A Signaling Model of Informative and Manipulative Political Action." *American Political Science Review*, 87, 319–333.

——— (1994): "Information aggregation through costly political action," *The American Economic Review*, 518–530.

Malenko, A. and N. Malenko (Forthcoming): "Proxy advisory firms: The economics of selling information to voters," *Journal of Finance*.

Mandler, M. (2012): "The fragility of information aggregation in large elections," *Games and Economic Behavior*, 74, 257–268.

McKelvey, R. D. and P. C. Ordeshook (1985): "Elections with limited information: A fulfilled expectations model using contemporaneous poll and endorsement data as information sources," *Journal of Economic Theory*, 36, 55–85.

Morgan, J. and P. C. Stocken (2008): "Information aggregation in polls," *The American Economic Review*, 98, 864–896.

Myerson, R. B. (1998a): "Extended Poisson games and the Condorcet jury theorem," *Games and Economic Behavior*, 25, 111–131.

——— (1998b): "Population uncertainty and Poisson games," *International Journal of Game Theory*, 27, 375–392.

Razin, R. (2003): "Signaling and election motivations in a voting model with common values and responsive candidates," *Econometrica*, 71, 1083–1119.

Wolinsky, A. (2002): "Eliciting information from multiple experts," *Games and Economic Behavior*, 41, 141–160.

Yildirim, H. (2012): "Time-Consistent Majority Rules with Interdependent Valuations," Tech. rep.

10 Regular Appendix: Proofs for the Case with Activists

In this Appendix, we first analyze the properties of responsive equilibrium sequences. In particular, in Lemma 4, we show that the informativeness of a responsive equilibrium sequence must be a $p-$solution of equation (17). This brings us to examine the properties of the $p-$solutions to equation (17), which we do in Lemma 5. We then show in Lemma 7 that when equation (17) has a positive $p-$solution, then there exists a responsive equilibrium sequence with informativeness equal to the largest $p-$solution of equation (17). We then summarize the proofs of Theorem 1 and 2 by giving references to the set of lemmas that conclude the proof of each result. We then provide the proofs of Theorems 3, 4 and 5 about the comparative statics.

10.1 Proofs for the General Characterization

Recall $\kappa(p) = -\frac{p}{2} + \frac{1}{p}\ln\left(\frac{1-q}{q}\frac{\mu}{1-\mu}\right)$, and equation (17), i.e.,

$$p = \frac{f(0)}{F(0)}\left(1 + \frac{q}{1-q}\frac{1-\mu}{\mu}\right)(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta))(\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b))\,\phi\left(\kappa(p)\right) =: g(p).$$

Lemma 2. *Take a sequence of strategy profiles $\{c_{a,n}, c_{b,n}, \rho_n\}_{n\in\mathbb{N}}$ where each ρ_n is a best response of the policy maker to the citizens' strategy $(c_{a,n}, c_{b,n})$ when the mean of the population size is n. If $\lim c_{a,n} = \lim c_{b,n} = 0$, and if*

$$\lim_{n\to\infty} \frac{\lambda_n(\alpha) - \lambda_n(\beta)}{\sqrt{nF(0)}} = p \in (0, \infty),$$

then:

1. $\lim_{n\to\infty} \frac{T_n - \lambda(\alpha)}{\sqrt{nF(0)}} = \kappa(p)$, *where T_n is the threshold of ρ_n,*
2. $\lim_{n\to\infty} \sqrt{nF(0)}\mathbb{P}(piv|\alpha) = \phi(\kappa(p))$,
3. $\lim_{n\to\infty} \sqrt{nF(0)}\mathbb{P}(piv|\beta) = \phi(\kappa(p) + p)$.

Proof. Recall that

$$\lambda_n(\alpha) = n\left(F(c_{a,n})\mathbb{P}(a|\alpha) + F(c_{b,n})\mathbb{P}(b|\alpha)\right)$$
$$\lambda_n(\beta) = n\left(F(c_{a,n})\mathbb{P}(a|\beta) + F(c_{b,n})\mathbb{P}(b|\beta)\right)$$

Let

$$\Delta\lambda_n := \lambda_n(\alpha) - \lambda_n(\beta) = n(F(c_{a,n}) - F(c_{b,n}))(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)). \tag{23}$$

Note that because $c_{a,n}, c_{b,n} \to 0$, and because F is continuously differentiable at 0, $\lim \frac{1}{n}\frac{\Delta\lambda_n}{(c_{a,n} - c_{b,n})} = f(0)(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta))$, i.e., $\Delta\lambda_n$ is approximately proportional to $(c_{a,n} - c_{b,n})$. More importantly, $\lim \frac{\Delta\lambda_n}{\lambda_n(\alpha)} = \lim \frac{\Delta\lambda_n}{\lambda_n(\beta)} = \lim \frac{\Delta\lambda_n}{nF(0)} = 0$. For $t \in \mathbb{R}_+$, let

$$L(t) := \frac{q}{1-q}\left(1 - \frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)^{-t} e^{-\Delta\lambda_n}.$$

Let $t_n \in \mathbb{R}_+$ be the solution of the following equality:

$$L(t_n) = \frac{\mu}{1-\mu}.$$

A positive solution exists because $\Delta\lambda_n > 0$, and thus $L(0) < \frac{q}{1-q} \leq \frac{\mu}{1-\mu}$, and L is an increasing function without bound. The policy maker's best reply, ρ_n, will choose policy A for every $t > t_n$, policy B for every $t < t_n$ and if t_n is an integer, then the policy maker is indifferent between two policies when $t = t_n$ and $\rho_n(t_n) \in [0,1]$. Taking the natural logs of both sides, we get

$$-t_n \ln\left(1 - \frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right) - \Delta\lambda_n = \ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right). \tag{24}$$

Then,

$$\frac{t_n - \lambda_n(\alpha)}{\sqrt{\lambda_n(\alpha)}} = \frac{\dfrac{-\Delta\lambda_n}{\ln\left(1-\frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)} + \dfrac{-\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)}{\ln\left(1-\frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)} - \lambda_n(\alpha)}{\sqrt{\lambda_n(\alpha)}}.$$

In order to show the first claim that $\lim_n \frac{T_n - \lambda(\alpha)}{\sqrt{nF(0)}} = \kappa(p)$ for the threshold T_n of ρ_n, first note that $|t_n - T_n| \leq 1$, and second note that $\lim \frac{\lambda_n(\alpha)}{\sqrt{nF(0)}} = \lim \frac{\lambda_n(\beta)}{\sqrt{nF(0)}} = 1$. Hence,

$$\lim_{n\to\infty} \frac{T_n - \lambda(\alpha)}{\sqrt{nF(0)}} = \lim_{n\to\infty} \frac{t_n - \lambda_n(\alpha)}{\sqrt{\lambda_n(\alpha)}}.$$

We now proceed to show that

$$\lim \frac{t_n - \lambda_n(\alpha)}{\sqrt{\lambda_n(\alpha)}} = -\frac{p}{2} + \frac{1}{p}\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right) = \kappa(p).$$

To do so, we will first show (in Step 1 below) that

$$\lim \frac{\dfrac{-\Delta\lambda_n}{\ln\left(1-\frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)} - \lambda_n(\alpha)}{\sqrt{\lambda_n(\alpha)}} = -\frac{p}{2},$$

and second show (in Step 2 below) that

$$\lim \frac{\dfrac{-\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)}{\ln\left(1-\frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)}}{\sqrt{\lambda_n(\alpha)}} = \frac{1}{p}\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right).$$

Step 1:

$$\lim \frac{\frac{-\Delta\lambda_n}{\ln\left(1-\frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)} - \lambda_n(\alpha)}{\sqrt{\lambda_n(\alpha)}} = \lim \frac{\lambda_n(\alpha)\left(\frac{-\frac{\Delta\lambda_n}{\lambda_n(\alpha)}}{\ln\left(1-\frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)} - 1\right)}{\sqrt{\lambda_n(\alpha)}}.$$

Let $y_n := -\frac{\Delta\lambda_n}{\lambda_n(\alpha)}$. Observe that $\lim y_n = 0$. Note also that ln is analytic on $1+x$ for $|x|<1$, i.e.,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...+ \tag{25}$$

we get

$$\frac{y_n}{\ln(1+y_n)} - 1 = \frac{\frac{(y_n)^2}{2} - \frac{(y_n)^3}{3} + ...}{y_n - \frac{(y_n)^2}{2} + \frac{(y_n)^3}{3} - ...} = \frac{y_n}{2}\left(\frac{1-\frac{2(y_n)}{3} + ...}{1 - \frac{(y_n)}{2} + \frac{(y_n)^2}{3} - ...}\right).$$

Hence, we get

$$\lim \sqrt{\lambda_n(\alpha)}\left(\frac{y_n}{\ln(1+y_n)} - 1\right) = \lim \sqrt{\lambda_n(\alpha)}\frac{y_n}{2}.$$

Putting $y_n = -\frac{\Delta\lambda_n}{\lambda_n(\alpha)}$, we get

$$\lim \sqrt{\lambda_n(\alpha)}\frac{y_n}{2} = \lim \frac{-\Delta\lambda_n}{2\sqrt{\lambda_n(\alpha)}} = \lim \frac{-\Delta\lambda_n}{2\sqrt{nF(0)}} = -\frac{p}{2}.$$

Step 2:
First note $\lim_{x\to 0}\frac{1}{x}ln(1+x) = 1$. Therefore, we get

$$\lim \frac{\frac{-\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)}{\ln\left(1-\frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)}}{\sqrt{\lambda_n(\alpha)}} = -\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)\lim \frac{1}{\sqrt{\lambda_n(\alpha)}}\frac{1}{-\frac{\Delta\lambda_n}{\lambda_n(\alpha)}} = \frac{1}{p}\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right).$$

Hence, we conclude from the calculations in Steps 1 and 2 above that

$$\lim \frac{t_n - \lambda_n(\alpha)}{\sqrt{\lambda_n(\alpha)}} = -\frac{p}{2} + \ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)\frac{1}{p} = \kappa(p). \tag{26}$$

Because the pivotal events $T_n \in \mathbb{N}$ satisfy $|t_n - T_n| \le 1$, we have for every pivotal event T_n,

$$\lim \frac{T_n - \lambda_n(\alpha)}{\sqrt{\lambda_n(\alpha)}} = \kappa(p).$$

Note that if t_n is not an integer, then $T_n = \lfloor t_n \rfloor$, and[46]

$$\mathbb{P}(\text{piv}|\alpha) = e^{-\lambda_n(\alpha)} \frac{\lambda_n(\alpha)^{T_n}}{T_n!}.$$

If t_n is an integer, then

$$\mathbb{P}(\text{piv}|\alpha) = e^{-\lambda_n(\alpha)} \frac{\lambda_n(\alpha)^{t_n - 1}}{(t_n - 1)!} \rho_n(t_n) + e^{-\lambda_n(\alpha)} \frac{\lambda_n(\alpha)^{t_n}}{t_n!} \left(1 - \rho_n(t_n)\right).$$

Using the normal approximation of Poisson distribution, i.e., equation (5), if

$$\lim \frac{t_n - \lambda_n(\alpha)}{\sqrt{\lambda_n(\alpha)}} = -\frac{p}{2} + \ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)\frac{1}{p} = \kappa(p) \tag{27}$$

then,

$$\sqrt{\lambda_n(\alpha)}\mathbb{P}(\text{piv}|\alpha) \to \phi(\kappa(p)).$$

Because $\lim \frac{\sqrt{\lambda_n(\alpha)}}{\sqrt{nF(0)}} = 1$, we obtain

$$\sqrt{nF(0)}\mathbb{P}(\text{piv}|\alpha) \to \phi(\kappa(p)).$$

Because $\lim \frac{\lambda_n(\alpha) - \lambda_n(\beta)}{\sqrt{nF(0)}} = \lim \frac{\lambda_n(\alpha) - \lambda_n(\beta)}{\sqrt{\lambda_n(\alpha)}} = \lim \frac{\lambda_n(\alpha) - \lambda_n(\beta)}{\sqrt{\lambda_n(\beta)}} = p$, we obtain that

$$\lim \frac{t_n - \lambda_n(\beta)}{\sqrt{nF(0)}} = \kappa(p) + p.$$

Using the same argument, we obtain that

$$\sqrt{nF(0)}\mathbb{P}(\text{piv}|\beta) \to \phi(\kappa(p) + p).$$

□

Lemma 3. *Take a sequence of strategy profiles $\{c_{a,n}, c_{b,n}, \rho_n\}_{n\in\mathbb{R}_+}$ where each ρ_n is a best response of the policy maker to the symmetric citizen strategy $(c_{a,n}, c_{b,n})$ when the mean of the population size is n. Take another sequence of citizen strategy profiles $\{\hat{c}_{a,n}, \hat{c}_{b,n}\}_{n\in\mathbb{N}}$ where each $(\hat{c}_{a,n}, \hat{c}_{b,n})$ is a best reply of the citizens to $(c_{a,n}, c_{b,n}, \rho_n)$. If $\lim c_{a,n} = \lim c_{b,n} = 0$, and if*

$$\lim \frac{\lambda_n(\alpha) - \lambda_n(\beta)}{\sqrt{nF(0)}} = p \in (0, \infty),$$

then:

1. $\lim \frac{\hat{c}_{b,n}}{\hat{c}_{a,n}} = \frac{\mathbb{P}(\alpha|b)\frac{\mu}{1-\mu}\frac{1-q}{q} - \mathbb{P}(\beta|b)}{\mathbb{P}(\alpha|a)\frac{\mu}{1-\mu}\frac{1-q}{q} - \mathbb{P}(\beta|a)} =: \gamma$, *and* $\gamma \in (-\infty, 1)$.

[46] The term $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

2. $\lim \sqrt{nF(0)}\left(\hat{c}_{a,n} - \hat{c}_{b,n}\right) = \phi(\kappa(p))\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)\frac{1-\mu}{\mu}\frac{q}{1-q}\right)(1-\gamma)$.

Proof. Item 1: If ρ_n is a best reply to $(c_{a,n}, c_{b,n})$, then for every pivotal event T_n, $L(T_n) \leq \frac{\mu}{1-\mu} \leq L(T_{n+1})$. Moreover,

$$\ln L\left(T_{n+1}\right) - \ln L\left(T_n\right) = \ln \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}.$$

Because $\lim \frac{\lambda_n(\alpha)}{nF(0)} = \lim \frac{\lambda_n(\beta)}{nF(0)} = 1$, $\lim \ln \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} = 0$. Therefore, for each pivotal event T_n, $\lim \frac{q}{1-q}\frac{\mathbb{P}(t=T_n|\alpha)}{\mathbb{P}(t=T_n|\beta)} = \frac{\mu}{1-\mu}$. Since this is true for each pivotal event, and since there are at most two pivotal events for each n, we have

$$\lim \frac{q}{1-q}\frac{\mathbb{P}(\text{piv}|\alpha)}{\mathbb{P}(\text{piv}|\beta)} = \frac{\mu}{1-\mu}. \tag{28}$$

Recall that the best reply cost cutoffs of the citizens are given by:

$$\hat{c}_{a,n} = \mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta),$$

$$\hat{c}_{b,n} = \mathbb{P}(\alpha|b)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|b)\mathbb{P}(\text{piv}|\beta).$$

Dividing the cost cutoffs, and using equation (28) we obtain:

$$\begin{aligned}\lim \frac{\hat{c}_{b,n}}{\hat{c}_{a,n}} &= \lim \frac{\mathbb{P}(\alpha|b)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|b)\mathbb{P}(\text{piv}|\beta)}{\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta)} \\ &= \frac{\mathbb{P}(\alpha|b)\frac{\mu}{1-\mu}\frac{1-q}{q} - \mathbb{P}(\beta|b)}{\mathbb{P}(\alpha|a)\frac{\mu}{1-\mu}\frac{1-q}{q} - \mathbb{P}(\beta|a)}.\end{aligned}$$

Observe that the denominator of γ, $\mathbb{P}(\alpha|a)\frac{\mu}{1-\mu}\frac{1-q}{q} - \mathbb{P}(\beta|a) > 0$ because $\mu \geq 0.5$, and because of the MLRP condition. If the numerator is nonpositive, then $\gamma \in (-\infty, 0]$. If the numerator is positive, then because $\mathbb{P}(\alpha|a) < \mathbb{P}(\alpha|b)$, and because $\mathbb{P}(\beta|a) < \mathbb{P}(\beta|b)$, $\gamma \in (0,1)$.

Item 2:

From Lemma 2, $\lim \sqrt{nF(0)}\mathbb{P}(\text{piv}|\alpha) = \phi(\kappa(p))$, and $\lim \sqrt{nF(0)}\mathbb{P}(\text{piv}|\beta) = \phi(\kappa(p)+p)$. Observe also that

$$\frac{q}{1-q}\frac{\phi(\kappa(p))}{\phi(\kappa(p)+p)} = \frac{\mu}{1-\mu}.$$

Therefore,

$$\begin{aligned}\lim \sqrt{nF(0)}\hat{c}_{a,n} &= \lim \sqrt{nF(0)}\mathbb{P}(\text{piv}|\alpha)\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)\frac{1-\mu}{\mu}\frac{q}{1-q}\right) \\ &= \phi(\kappa(p))\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)\frac{1-\mu}{\mu}\frac{q}{1-q}\right).\end{aligned}$$

Because $\lim \frac{\hat{c}_{b,n}}{\hat{c}_{a,n}} = \gamma$, we obtain

$$\lim \sqrt{nF(0)}\left(\hat{c}_{a,n} - \hat{c}_{b,n}\right) = \phi(\kappa(p))\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)\frac{1-\mu}{\mu}\frac{q}{1-q}\right)(1-\gamma).$$

$\square$

Lemma 4. *Take a responsive equilibrium sequence with* $\lim \frac{\lambda_n(\alpha)-\lambda_n(\beta)}{\sqrt{nF(0)}} = p \in [0,\infty) \cup \{\infty\}$.

1. *If $p \in (0,\infty)$, then p is a solution of equation (17).*
2. $p < \infty$.
3. *If $\mu > q$, then $p > 0$. Hence, if equation (17) has no positive solution, then there exists no responsive equilibrium sequence.*

Proof. Pick a sequence of responsive equilibria $\{c_{a,n}, c_{b,n}, \rho_n\}_{n\in\mathbb{R}_+}$ with $\lim \frac{\lambda_n(\alpha)-\lambda_n(\beta)}{\sqrt{nF(0)}} = p$.

We start by showing that if $p \in (0,\infty)$, then p solves equation (17).

Recall equation (23)

$$\Delta\lambda_n = n\left(\left(F(c_{a,n}) - F(c_{b,n})\right)\left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right)\right).$$

In any responsive equilibrium sequence, $c_{a,n}, c_{b,n} \to 0$, therefore

$$\lim \frac{\Delta\lambda_n}{n\left(c_{a,n} - c_{b,n}\right)} = f(0)\left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right). \tag{29}$$

Combining the finding from Lemma 3 that

$$\lim \sqrt{nF(0)}\left(c_{a,n} - c_{b,n}\right) = \phi(\kappa(p))\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)\frac{1-\mu}{\mu}\frac{q}{1-q}\right)(1-\gamma),$$

with equation (29) we obtain

$$\lim \frac{\Delta\lambda_n}{\sqrt{nF(0)}} = \frac{f(0)}{F(0)}\phi(\kappa(p))\left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right)\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)\frac{1-\mu}{\mu}\frac{q}{1-q}\right)(1-\gamma).$$

Because $\lim \frac{\Delta\lambda_n}{\sqrt{nF(0)}} = p$, we get that

$$p = \frac{f(0)}{F(0)}\phi(\kappa(p))\left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right)\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)\frac{1-\mu}{\mu}\frac{q}{1-q}\right)(1-\gamma),$$

i.e., p solves equation (17).

We now show that in any responsive equilibrium sequence, $p < \infty$. Suppose on the way to a contradiction that $p = \infty$. Then, repeating the calculations in steps 1 and 2 of the proof of Lemma 2, we obtain that $\lim \frac{t_n - \lambda_n(\alpha)}{\sqrt{\lambda_n(\alpha)}} = -\infty$. Therefore, $\sqrt{nF(0)}\mathbb{P}(\text{piv}|\alpha) \to 0$. A similar calculation shows that $\sqrt{nF(0)}\mathbb{P}(\text{piv}|\beta) \to 0$. Therefore, $\lim \sqrt{nF(0)}\,(c_{a,n} - c_{b,n}) = 0$. Because $f(0) < \infty$, we have $\lim \frac{\lambda_n(\alpha)-\lambda_n(\beta)}{\sqrt{nF(0)}} = 0$, which contradicts that $p = \infty$.

We now show that if $\mu > q$, then in any responsive equilibrium sequence, $p > 0$. This implies that if there is no positive $p-$solution of (17), then there is no responsive equilibrium sequence.

Suppose to the contrary that the sequence of responsive equilibria $\{c_{a,n}, c_{b,n}, \rho_n\}_{n\in\mathbb{R}_+}$ satisfies the equality $\lim \frac{\Delta\lambda_n}{\sqrt{nF(0)}} = 0$. Observe that if $\lim \frac{\Delta\lambda_n}{\sqrt{nF(0)}} = 0$, then $\lim \sqrt{n}\,(c_{a,n} - c_{b,n}) = 0$. Because $\lim \frac{c_{b,n}}{c_{a,n}} = \gamma \in (-\infty, 1)$, we have that $\lim \sqrt{n}c_{a,n} = 0$.

We will first show that $(T_n - \lambda_n(\alpha))\,c_{a,n} \to K \in (0,\infty)$.

From equation (24) we get

$$\begin{aligned} t_n - \lambda_n(\alpha) &= \frac{-\Delta\lambda_n}{\ln\left(1 - \frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)} + \frac{-\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)}{\ln\left(1 - \frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)} - \lambda_n(\alpha) \\ &= \lambda_n(\alpha)\left(\frac{-\frac{\Delta\lambda_n}{\lambda_n(\alpha)} - \ln\left(1 - \frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)}{\ln\left(1 - \frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)}\right) + \frac{-\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)}{\ln\left(1 - \frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)}. \end{aligned}$$

Letting again $y_n := -\frac{\Delta\lambda_n}{\lambda_n(\alpha)}$, and using equation (25), we obtain that

$$\lim c_{a,n}\lambda_n(\alpha)\left(\frac{-\frac{\Delta\lambda_n}{\lambda_n(\alpha)} - \ln\left(1 - \frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)}{\ln\left(1 - \frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)}\right) = \lim c_{a,n}\lambda_n(\alpha)\frac{\Delta\lambda_n}{2\lambda_n(\alpha)} = \lim\,(c_{a,n})^2\, n \to 0.$$

Observe that using equation (25), we obtain that

$$\lim c_{a,n}\frac{-\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)}{\ln\left(1 - \frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right)} = \lim c_{a,n}\frac{\lambda_n(\alpha)}{\Delta\lambda_n}\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right) \to K \tag{30}$$

for some $K \in (0,\infty)$. $K \in (0,\infty)$ because $\mu > q$, $\Delta\lambda_n$ is approximately $nf(0)(c_{a,n} - c_{b,n})$, and because $\lim \frac{c_{b,n}}{c_{a,n}} = \gamma \in (-\infty, 1)$. Because $|t_n - T_n| \leq 1$, and because $c_{a,n} \to 0$, we have,

$$(T_n - \lambda_n(\alpha))\,c_{a,n} \to K \in (0,\infty). \tag{31}$$

We will now show that $(T_n - \lambda_n(\alpha))\,\mathbb{P}(\text{piv}|\alpha) \to 0$.

Observe that because $\sqrt{n}c_{a,n} \to 0$, equation (31) implies that

$$\frac{(T_n - \lambda_n(\alpha))}{\sqrt{\lambda_n(\alpha)}} \to \infty. \tag{32}$$

Because the density of the Poisson distribution is decreasing when above the mean,

$$\sum_{t > \frac{\lambda_n(\alpha)+T_n}{2}} e^{-\lambda_n(\alpha)}\frac{(\lambda_n(\alpha))^t}{t!} > \frac{1}{2}\,(T_n - \lambda_n(\alpha))\,e^{-\lambda_n(\alpha)}\frac{(\lambda_n(\alpha))^{T_n}}{T_n!}.$$

Moreover, Chebyshev's inequality together with the equation (32) delivers that

$$\sum_{t>\frac{\lambda_n(\alpha)+T_n}{2}} e^{-\lambda_n(\alpha)} \frac{(\lambda_n(\alpha))^t}{t!} \to 0.$$

Hence, we obtain that

$$(T_n - \lambda_n(\alpha))\, e^{-\lambda_n(\alpha)} \frac{(\lambda_n(\alpha))^{T_n}}{T_n!} \to 0.$$

Therefore,

$$(T_n - \lambda_n(\alpha))\, \mathbb{P}(\text{piv}|\alpha) \to 0. \tag{33}$$

Equalities (31) and (33) imply that $\lim \frac{c_{a,n}}{\mathbb{P}(\text{piv}|\alpha)} = \infty$, which contradicts to $\lim \frac{c_{a,n}}{\mathbb{P}(\text{piv}|\alpha)} = \mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a) \frac{1-\mu}{\mu} \frac{q}{1-q} < \infty$. Therefore, in any responsive equilibrium sequence, $\lim \frac{\Delta\lambda_n}{\sqrt{nF(0)}} = p > 0$, which concludes the proof of the last claim of the lemma. □

We now present an auxiliary result that characterizes the properties of the $p-$solutions of equation (17).

Lemma 5.

1. *There exists a function $P : \mathbb{R}_+ \to \mathbb{R}_+$, with $P(x)$ the largest $p-$solution of equation (17) when $x = \frac{f(0)}{F(0)}$.*

2. *If $\mu = q$, then equation (17) has a unique solution if $\frac{f(0)}{F(0)} > 0$. For $x > 0$, $P(x) > 0$.*

3. *If $\mu > q$, then there is a $\tau > 0$ such that:*

   (a) $P(x) = 0$ for $x < \tau$, i.e., equation (17) has a unique solution with $p = 0$ if $\frac{f(0)}{F(0)} < \tau$.

   (b) $P(x) > 0$ for $x > \tau$, and equation (17) has two positive solutions if $\frac{f(0)}{F(0)} > \tau$.

4. *$P(x)$ is continuous and strictly increasing in x for $x > \tau$.*

5. *$\lim_{x\downarrow\tau} P(x) > 0$ if $\mu > q$.*

6. *$\lim_{x\to\infty} P(x) = \infty$.*

Proof. We first argue that when $q < \mu$, depending on $\frac{f(0)}{F(0)}$, the curves $y(p) = p$ and

$$g(p) = \frac{f(0)}{F(0)} \left(1 + \frac{q}{1-q}\frac{1-\mu}{\mu}\right) (\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta))\, (\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b))\, \phi\,(\kappa(p))$$

may have two positive intersection points, the curves may be tangent to each other at a positive p, or they may never intersect at a positive p. We will investigate the properties of the two curves when $q = \mu$, after analyzing the case $q < \mu$.

Taking the natural logs of the two sides of equation (17), we get

$$\ln p = K + \ln\left(\frac{f(0)}{F(0)}\right) - \frac{1}{2}\kappa(p)^2,$$

where K is a constant (that depends on q, μ, and $\mathbb{P}$).

Let $f_1(p) := \ln p$, $f_2(p) := K + \ln\left(\frac{f(0)}{F(0)}\right) - \frac{1}{2}\kappa(p)^2$. Because $\kappa(p)$ is a decreasing function, $f_2(p)$ is a single-peaked function. The derivative of $f_1(p)$ with respect to p is

$$f_1'(p) = \frac{1}{p}.$$

Note that $-\kappa(p)^2 = -\frac{p^2}{4} - \frac{m^2}{p^2} + m$, where $m := \ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)$. Note that if $\mu > q$, then $m > 0$. Hence,

$$f_2'(p) = -\frac{p}{4} + \frac{m^2}{p^3}.$$

We now investigate the sign of

$$f_1'(p) - f_2'(p) := \frac{1}{p} + \frac{p}{4} - \frac{m^2}{p^3} = \frac{1}{p^3}\left(p^2 + \frac{p^4}{4} - m^2\right).$$

The term $\left(p^2 + \frac{p^4}{4} - m^2\right)$ is continuous and strictly increasing in p, and is negative at $p = 0$. Hence, there is some $p^{**} > 0$ such that $f_1'(p^{**}) = f_2'(p^{**})$, and $f_1'(p) - f_2'(p)$ is strictly negative for $p \in (0, p^{**})$ and is strictly positive for $p > p^{**}$. Moreover, there is some $\epsilon > 0$ such that $f_1(p) > f_2(p)$ for all $p \in (0, \epsilon)$. Therefore, if the two curves have a positive intersection point at some $p > 0$, it has to be that there is some $p_0 \in (0, p^{**}]$ where they intersect. If $p_0 < p^{**}$, then $f_1(p)$ crosses $f_2(p)$ from above.

Because $f_1'(p) - f_2'(p)$ is negative in p for $p \in (0, p^{**})$, there is no other intersection point in the interval $(0, p^{**}]$. If $p_0 < p^{**}$, then $f_1(p^{**}) < f_2(p^{**})$, and $f_1'(p) - f_2'(p)$ is strictly positive for $p > p^{**}$, and converges to ∞ with p, so they intersect exactly once in the interval (p^{**}, ∞), at some p_1. Moreover, at p_1, $f_1(p)$ crosses $f_2(p)$ from below. If $p_0 = p^{**}$, then the two curves never intersect again, and because $f_1'(p^{**}) = f_2'(p^{**})$ in this case, p^{**} is a tangency point of the two curves. We conclude that the two curves intersect at most twice, and hence there is a function $P : \mathbb{R}_+ \to \mathbb{R}_+$, with $P(x)$ the largest $p-$solution of equation (17) when $x = \frac{f(0)}{F(0)}$.

Suppose $0 < p_0 < p_1$ are two intersection points of the two curves for some $\frac{f(0)}{F(0)}$. Take some $\frac{\tilde{f}(0)}{\tilde{F}(0)} > \frac{f(0)}{F(0)}$, and let $h(p; y) := K + \ln(y) - \frac{1}{2}\kappa(p)^2$. Note that $h\left(p_1; \frac{\tilde{f}(0)}{\tilde{F}(0)}\right) > f_1(p_1)$, and $h\left(p_0; \frac{\tilde{f}(0)}{\tilde{F}(0)}\right) > f_1(p_0)$. Because $f_1(p)$ diverges to $+\infty$ while $h\left(p; \frac{\tilde{f}(0)}{\tilde{F}(0)}\right)$ diverges to $-\infty$ with p, and because $\lim_{p\to 0} f_1(p) - h\left(p; \frac{\tilde{f}(0)}{\tilde{F}(0)}\right) = \infty$, the two curves f_1 and h also have two intersection points $\tilde{p}_0, \tilde{p}_1$ with $0 < \tilde{p}_0 < p_0 < p_1 < \tilde{p}_1$. Notice also that for $p > 0$, $h(p; y)$ is increasing without bound in y, hence $P(x) > 0$ for some $x > 0$. This proves that there exists a $\tau \geq 0$ such that there are two positive p-solutions to equation (17) if and only if $\frac{f(0)}{F(0)} > \tau$, and that $P(x)$ is strictly increasing for $x > \tau$. A similar argument shows that if the two curves intersect only once, i.e., if $p_0 = p^{**}$ for some $\frac{f(0)}{F(0)}$, then $\frac{\tilde{f}(0)}{\tilde{F}(0)} > \frac{f(0)}{F(0)}$ implies

$h\left(p_0; \frac{\tilde{f}(0)}{\tilde{F}(0)}\right) > f_1(p_0)$. Therefore, the two curves have two positive intersection points when $\frac{\tilde{f}(0)}{\tilde{F}(0)} > \frac{f(0)}{F(0)}$. Likewise, if $\frac{\tilde{f}(0)}{\tilde{F}(0)} < \frac{f(0)}{F(0)}$, then $h\left(p; \frac{\tilde{f}(0)}{\tilde{F}(0)}\right) < f_1(p)$ for every $p > 0$, hence the two curves have no positive intersection point. We thus conclude that $P(x) > 0$ if and only if $x \geq \tau$.

To see that $\tau > 0$, note $f_1(p) - h(p; y)$ achieves its minimum at $p^{**} > 0$ independently of y, and $h(p^{**}; y)$ is an increasing function of y and approaches $-\infty$ as y goes to 0. Therefore, there exists $\epsilon > 0$ such that $f_1(p) - h(p; y) > 0$ for all $p > 0$, $y \in [0, \epsilon)$.

Because $\kappa(p)$ diverges to $-\infty$ with p, $\lim_{p\to\infty} \phi(\kappa(p)) = 0$. Moreover, for every $p > 0$, $\lim_{y\to\infty} h(p; y) = \infty$ and $\lim_{y\to\infty} f_1(p) - h(p; y) = -\infty$. Therefore,

$$\lim_{x\to\infty} P(x) = \infty.$$

We now show that $P(x)$ is continuous when $x > \tau$. Observe that the function $z(p, y) := f_1(p) - h(p; y)$ is continuously differentiable for $p > 0, y > 0$, and $\frac{\partial z(p,y)}{\partial p} \neq 0$ for all $p > p^{**}$. Because the largest solution of $z(p(y), y) = 0$ is greater than p^{**} when $y > \tau$, by the implicit function theorem, $P(x)$ is differentiable (hence continuous) for $x > \tau$.

We now show that $\lim_{x\downarrow\tau} P(x) > 0$. Note that $z(p^{**}, \tau) = 0$, and that $P(x) > p^{**}$ for $x > \tau$. We have already shown that $p^{**} > 0$, hence this completes the proof of this claim.

We now turn to the case when $\mu = q$, i.e., when $m = 0$. In this case,

$$f_1'(p) - f_2'(p) := \frac{1}{p} + \frac{p}{4} = \frac{1}{p^3}\left(p^2 + \frac{p^4}{4}\right).$$

Therefore, $f_1'(p) > f_2'(p)$ for every $p > 0$. Moreover, $\lim_{p\to 0} f_1(p) - f_2(p) = -\infty$ and $\lim_{p\to\infty} f_1(p) - f_2(p) = \infty$. Therefore, f_1 crosses f_2 from below only once at a positive p for every $\frac{f(0)}{F(0)} > 0$, and therefore, $\tau = 0$. The other claims of the lemma (items 4 and 6) for the case $\mu = q$ follow from a similar analysis of the case $\mu > q$. □

Lemma 6. *If $\frac{f(0)}{F(0)} < \tau$, then there is no responsive equilibrium sequence.*

Proof. Note that if $\mu = q$, then $\tau = 0$, and the statement is vacuously true. If $\mu > q$, then it follows from Lemma 5 that if $\frac{f(0)}{F(0)} < \tau$, there is no positive $p-$solution of equation (17). By Lemma 4, if $\mu > q$, and if there is a responsive equilibrium sequence, then $p = \lim \frac{\Delta\lambda_n}{\sqrt{nF(0)}} > 0$ and solves equation (17), which is not possible when $\frac{f(0)}{F(0)} < \tau$. □

Lemma 7. *If equation (17) has two positive $p-$solutions (i.e., when $\frac{f(0)}{F(0)} > \tau$) or if $\mu = q$, then there exists a responsive equilibrium sequence with informativeness $P\left(\frac{f(0)}{F(0)}\right)$, i.e., the maximal informativeness of the protests is given by the largest solution of equation (17).*

Proof. Recall that for $p \in (0, \infty)$, $\kappa(p) := -\frac{p}{2} + \frac{1}{p}\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)$, $\phi(\cdot)$ and $\Phi(\cdot)$ denote the density and cumulative distribution for the standard normal distribution. Recall that

$$\gamma := \frac{\dfrac{\mu}{1-\mu}\dfrac{1-q}{q}\mathbb{P}(\alpha|b) - \mathbb{P}(\beta|b)}{\dfrac{\mu}{1-\mu}\dfrac{1-q}{q}\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)}.$$

Note that $\gamma < 1$ by Lemma 3.

The existence of a solution $p > 0$ of equation (17) determines whether there exists a responsive equilibrium sequence or not. Recall that if $\frac{f(0)}{F(0)} > \tau$, then equation (17) has two positive solutions, $0 < p_0 < p_1$ as shown in Lemma 5, when $\mu > q$. When $\mu = q$, let $p_0 = 0$, and set p_1 to be the unique positive solution of equation (17).

Auxiliary Game with Restricted Strategies

We now proceed by defining an auxiliary protest game in which the strategies of the citizens (c_a, c_b) are restricted to be in a strict subset of $[\underline{c}, \bar{c}]^2$, and the policy maker's choices are restricted such that he implements A when $t \geq 2n$. We will show that the auxiliary game has an equilibrium. We will also show that when n is large, all equilibria of the auxiliary game will have the property that the equilibrium strategies of the citizens are in the strict interior of their restricted strategy set, and that policy maker's choice $T_n < 2n$. This allows us to conclude that the equilibria of the auxiliary game are also equilibria of the protest game when n is large.

We start with describing the boundaries of the restricted strategy sets of the citizens. Let

$$c_0 := \frac{p_0 + p_1}{2} \sqrt{F(0)} \frac{1}{f(0)} \frac{1}{\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)} \frac{1}{1-\gamma}.$$

The term c_0 is defined so that if along a sequence $c_{a,n} = \frac{c_0}{\sqrt{n}}$, and if $\lim \frac{c_{b,n}}{c_{a,n}} = \gamma$, then $\lim \frac{\lambda_n(\alpha) - \lambda_n(\alpha)}{\sqrt{nF(0)}} = \frac{p_0+p_1}{2}$, which will imply that in the auxiliary game, the sequence of best responses of citizens $\{\hat{c}_{a,n}, \hat{c}_{b,n}\}$ to such a sequence of strategy profiles have $\hat{c}_{a,n} > \frac{c_0}{\sqrt{n}}$ when n is large. Hence there is no such an equilibrium sequence of a sequence of auxiliary games, as we show in Lemma 4.

Consider the following restricted set of citizen strategies:

$$C^n := \left\{ (c_a, c_b) \in [\underline{c}, \bar{c}]^2 : c_a \in \left[\frac{c_0}{\sqrt{n}}, \frac{1}{n^{1/4}} \right], c_b \in [c_a(\gamma - \varepsilon), c_a(\gamma + \varepsilon)] \right\},$$

where the term $\varepsilon > 0$ satisfies the following restrictions:

- $\gamma + \varepsilon < 1$,
- $\sqrt{F(0)} c_0 < \left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a) \frac{1-\mu}{\mu} \frac{q}{1-q} \right) \phi(\kappa(p))$ for all $p \in \left[\frac{p_0+p_1}{2} \frac{1-\gamma}{1-(\gamma-\varepsilon)}, \frac{p_0+p_1}{2} \frac{1-\gamma}{1-(\gamma+\varepsilon)} \right]$.

Existence of such a $\varepsilon > 0$ is guaranteed, because $\gamma < 1$, and because when $\mu > q$,

$$\sqrt{F(0)} c_0 < \left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a) \frac{1-\mu}{\mu} \frac{q}{1-q} \right) \phi\left(\kappa\left(\frac{p_0 + p_1}{2} \right) \right)$$

(which follows because for all $p \in (p_0, p_1)$, the right-hand side of equation (17) is larger than the side of that equation, see also Figure 5 and Lemma 5).

Consider the set of restricted strategies for the policy maker:

$$\Psi^n := \{ l : \{0, 1, ..., 2n-1\} \to [0, 1] \mid l(t) \text{ is nondecreasing.} \}$$

For every $l \in \Psi^n$, let the extension of l,$\bar{l} : \mathbb{N} \to [0,1]$ such that $\bar{l}(t) = l(t)$ for $t < 2n$, and $\bar{l}(t) = 1$ for all $t \geq 2n$. Observe that $\bar{l}$ is a nondecreasing function.

Each $c_a, c_b, \bar{l}$ induces pivot probabilities in state $\omega = \alpha, \beta$ as given in the main text through equality (3), where $\lambda(\omega) > 0$ is found by equation (2).

Note that $(C^n \times \Psi^n)$ is a closed and convex subset of $\mathbb{R}^{2+2n}$. Let

$$\Gamma^n : (C^n \times \Psi^n) \mapsto (C^n \times \Psi^n)$$

be a correspondence that

$$(c'_a, c'_b, l') \in \Gamma(c_a, c_b, l) \text{ if and only if}$$

$$c'_a = \max\left\{\min\left\{\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta), \frac{1}{n^{1/4}}\right\}, \frac{c_0}{\sqrt{n}}\right\},$$
$$c'_b = \max\left\{\min\left\{\mathbb{P}(\alpha|b)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|b)\mathbb{P}(\text{piv}|\beta), (\gamma+\epsilon)c'_a\right\}, (\gamma-\epsilon)c'_a\right\},$$
$$\exists \bar{l} \in BR_{DM}(c_a, c_b) \text{ such that } l'(t) = \bar{l}(t) \text{ for all } t < 2n.$$

Where, $\mathbb{P}(\text{piv}|\omega)$ is calculated using $c_a, c_b, \bar{l}$, and $BR_{DM}(c_a, c_b)$ denotes the set of best replies of the policy maker to citizen strategies with cutoffs c_a and c_b.

Claim 1. Γ^n has a fixed point for every n.

Proof. Note that, by construction, $(c'_a, c'_b) \in C^n$ and are uniquely determined from $c_a, c_b, \bar{l}$, and any $\bar{l} \in BR_{DM}(c_a, c_b)$ is nondecreasing and $\bar{l}(0) = 0$, so any l' in the image of the correspondence is in Ψ^n. $\Gamma^n(c_a, c_b, l)$ is a closed and convex set because c'_a, c'_b are uniquely chosen, and the projection of $BR_{DM}(c_a, c_b)$ on $t \in \{0, 1, .., 2n-1\}$ is a closed and convex set. Γ^n is upper hemicontinuous because $\mathbb{P}(\text{piv}|\alpha)$ is a continuous function of c_a, c_b, l, and BR_{DM} is upper hemicontinuous. C^n and Ψ^n are closed convex sets in $\mathbb{R}^2$ and $\mathbb{R}^{2\text{n}}$. Hence, Kakutani's fixed-point theorem delivers that there exists a fixed point of Γ^n for every n. $\square$

Fixed points of Γ^n correspond to equilibria of the protest game when n is large.

For each fixed point (c_a, c_b, l) of Γ^n, associate a strategy profile $(c_a, c_b, \bar{l})$ of the protest model. We now claim and proceed to show that there exists $\bar{n} \in \mathbb{N}$ such that $n > \bar{n}$ implies any strategy profile $\left(c_{a,n}, c_{b,n}, \bar{l}_n\right)$ associated with a fixed point of Γ^n is a Bayesian Nash equilibrium of the protest model. In particular, we will show that in any sequence of fixed points of $\{\Gamma^n\}_n$, when n is large, the restrictions on the citizens' strategies do not bind, and that $T_n < 2n$ with the following three lemmas.

Claim 2. There is $\bar{n} \in \mathbb{N}$ such that $n > \bar{n}$ implies $L_n(2n) > \frac{\mu}{1-\mu}$ at any fixed point (c_a, c_b, l) of Γ^n.

Proof. Note that

$$\ln\left(L_n(2n)\right) = \ln\left(\frac{q}{1-q}\right) - \Delta\lambda_n - 2n\ln\left(1 - \frac{\Delta\lambda_n}{\lambda_n(\alpha)}\right).$$

Because when $(c_a, c_b) \in C^n$, $\frac{\Delta\lambda_n(\alpha)}{\lambda_n(\alpha)} \to 0$, by equation (25) we have that

$$\ln\left(L_n(2n)\right) \approx \ln\left(\frac{q}{1-q}\right) - \Delta\lambda_n + 2n\frac{\Delta\lambda_n}{\lambda_n(\alpha)}.$$

Because $\frac{2n}{\lambda_n(\alpha)} > 2$, and because $\Delta\lambda_n \to \infty$, we have that $L_n(2n) \to \infty$, proving the claim. $\square$

Claim 3. If $\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta) \in \left(\frac{c_0}{\sqrt{n}}, \frac{1}{n^{1/4}}\right)$ for every n larger than a cutoff $\bar{n}_0$, then there exists $\bar{n}_1$ such that for all $n > \bar{n}_1$, $c_{b,n} \in (c_{a,n}(\gamma-\varepsilon), c_{a,n}(\gamma+\varepsilon))$.

Proof. Suppose $\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta) \in \left(\frac{c_0}{\sqrt{n}}, \frac{1}{n^{1/4}}\right)$.

Because

$$\frac{q}{1-q}\frac{\mathbb{P}(\text{piv}|\alpha)}{\mathbb{P}(\text{piv}|\beta)} \to \frac{\mu}{1-\mu},$$

we have

$$\lim \frac{\mathbb{P}(\alpha|b)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|b)\mathbb{P}(\text{piv}|\beta)}{\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta)} = \gamma \in (\gamma-\varepsilon, \gamma+\varepsilon). \tag{34}$$

Therefore, $c_{b,n} = \mathbb{P}(\alpha|b)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|b)\mathbb{P}(\text{piv}|\beta)$ and $c_{b,n} \in (c_{a,n}(\gamma-\varepsilon), c_{a,n}(\gamma+\varepsilon))$ for every n larger than some cutoff $\bar{n}_1$. $\square$

Claim 4. There is $\bar{n} > 0$ such that $n > \bar{n}$ implies $\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta) \in \left(\frac{c_0}{\sqrt{n}}, \frac{1}{n^{1/4}}\right)$.

Proof. Suppose on the way to a contradiction that the claim is false and first assume that there is a subsequence with $\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta) \le \frac{c_0}{\sqrt{n}}$ for every n along the subsequence. Because $c_{b,n} \in [(\gamma-\epsilon)\, c_{a,n}, (\gamma+\epsilon)\, c_{a,n}]$, we have along a convergent subsequence

$$\lim \frac{\Delta\lambda_n}{\sqrt{\lambda_n(\alpha)}} = \lim \frac{\Delta\lambda_n}{\sqrt{nF(0)}} = p$$

where $p \in \left[\frac{p_0+p_1}{2}\frac{1-\gamma}{1-(\gamma-\varepsilon)}, \frac{p_0+p_1}{2}\frac{1-\gamma}{1-(\gamma+\varepsilon)}\right]$ due to our choice of ε. Therefore, by Lemma 3,

$$\sqrt{nF(0)}\mathbb{P}(\text{piv}|\alpha) \to \phi(\kappa(p)).$$

Hence,

$$\sqrt{n}\left(\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta)\right) \to \frac{1}{\sqrt{F(0)}}\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)\frac{1-\mu}{\mu}\frac{q}{1-q}\right)\phi(\kappa(p)).$$

Because $p \in \left[\frac{p_0+p_1}{2}\frac{1-\gamma}{1-(\gamma-\varepsilon)}, \frac{p_0+p_1}{2}\frac{1-\gamma}{1-(\gamma+\varepsilon)}\right]$, and because

$$\sqrt{F(0)}c_0 < \left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)\frac{1-\mu}{\mu}\frac{q}{1-q}\right)\phi(\kappa(p))$$

for all

$$p \in \left[\frac{p_0+p_1}{2}\frac{1-\gamma}{1-(\gamma-\varepsilon)}, \frac{p_0+p_1}{2}\frac{1-\gamma}{1-(\gamma+\varepsilon)}\right],$$

we have

$$\begin{aligned}\lim \sqrt{n}\left(\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta)\right) &= \frac{1}{\sqrt{F(0)}}\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)\frac{1-\mu}{\mu}\frac{q}{1-q}\right)\phi(\kappa(p)) \\ &> c_0 = \sqrt{n}c_{a,n}.\end{aligned}$$

This implies when n is large, $(\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta)) > \frac{c_0}{\sqrt{n}}$, leading to a contradiction.

To show that the upper bound is not binding, notice that if along a subsequence we have $\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta) = \frac{1}{n^{1/4}}$, then

$$\lim \frac{\Delta\lambda_n}{\sqrt{\lambda_n(\alpha)}} = \infty,$$

and

$$\sqrt{nF(0)}\mathbb{P}(\text{piv}|\alpha) \to 0.$$

However, then

$$\lim \sqrt{n}\left(\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta)\right) = 0,$$

leading to a contradiction. □

Claims 2, 3 and 4 together imply that there exists a $\bar{n}$ such that the restrictions on the strategy sets C^n and Ψ^n do not bind in any fixed point of Γ^n for every $n > \bar{n}$. This establishes that the fixed points of Γ^n are responsive equilibria of the protest game. □

Lemma 8. *All responsive equilibrium sequences associated with fixed points of* $\{\Gamma^n\}_n$ *have the feature that* $\lim \sqrt{n}c_{a,n} = \frac{c_0}{p_0+p_1}p_1$, $\lim \frac{c_{b,n}}{c_{a,n}} = \gamma$, *and* $\lim \frac{\Delta\lambda_n}{\sqrt{nF(0)}} = p_1$.

Proof. We have already shown that $\lim \frac{c_{b,n}}{c_{a,n}} = \gamma$ in equation (34).

Because

$$\lim \sqrt{n}c_{a,n} \geq c_0 = \frac{p_0+p_1}{2}\sqrt{F(0)}\frac{1}{f(0)}\frac{1}{\mathbb{P}(a|\alpha)-\mathbb{P}(a|\beta)}\frac{1}{1-\gamma},$$

and because $\lim \frac{c_{b,n}}{c_{a,n}} = \gamma$,

$$\lim \frac{\Delta\lambda_n}{\sqrt{\lambda_n(\alpha)}} = \lim \frac{\Delta\lambda_n}{\sqrt{nF(0)}} = p \geq \frac{p_0+p_1}{2} > 0.$$

Because in all responsive equilibrium sequences the term p is a solution of equation (17), and because $p \geq \frac{p_0+p_1}{2}$, we conclude that $p = p_1$. Because $\lim \frac{c_{b,n}}{c_{a,n}} = \gamma$, we conclude that

$$\lim \sqrt{n}c_{a,n} = \frac{c_0}{p_0+p_1}p_1.$$

□

Lemma 9. *All responsive equilibrium sequences associated with fixed points of* $\{\Gamma^n\}_n$ *have the feature that the probability that* A *is selected in state* α *converges to* $1-\Phi(\kappa(p_1))$*, and the probability that* B *is selected in state* β *converges to* $\Phi(\kappa(p_1)+p_1)$.

Proof. Because

$$\frac{T_n-\lambda_n(\alpha)}{\sqrt{\lambda_n(\alpha)}} \to \kappa(p_1),$$

applying the normal approximation to Poisson distribution we get

$$\mathbb{P}(t > T_n|\alpha) \to 1-\Phi(\kappa(p_1)).$$

Note that

$$\frac{T_n-\lambda_n(\beta)}{\sqrt{\lambda_n(\beta)}} = \frac{T_n-\lambda_n(\alpha)+\Delta\lambda_n}{\sqrt{\lambda_n(\alpha)}} \to \kappa(p_1)+p_1.$$

Hence, applying the normal approximation to Poisson distribution we get

$$\mathbb{P}(t \le T_n|\beta) \to \Phi(\kappa(p_1)+p_1).$$

□

Lemma 10. $\lim_{p\to\infty} 1-\Phi(\kappa(p)) = \lim_{p\to\infty}\Phi(\kappa(p)+p) = 1.$

Proof. Recall

$$\kappa(p) = -\frac{p}{2}+\frac{1}{p}\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)$$

$\lim_{p\to\infty}\kappa(p) = -\infty$, and $\lim_{p\to\infty}\kappa(p)+p = \infty$, therefore $\lim_{p\to\infty} 1-\Phi(\kappa(p)) = \lim_{p\to\infty}\Phi(\kappa(p)+p) = 1$. □

Proof of Theorem 1:

Proof. We will first prove the Theorem along responsive equilibrium sequences. Then, we will argue that the informativeness of all equilibrium sequences are bounded above by the P function.

If $\mu = q$, equation (17) has a unique positive solution by Lemma 5, $P\left(\frac{f(0)}{F(0)}\right)$. Moreover, there exists a responsive equilibrium sequence with informativeness $P\left(\frac{f(0)}{F(0)}\right)$ by Lemma 7. By Lemma 5, it follows that $\tau = 0$, items 2 and 4.

If $\mu > q$, then equation (17) has two positive solutions if $\frac{f(0)}{F(0)} > \tau > 0$, and has no positive solution if $\frac{f(0)}{F(0)} < \tau$, by Lemma 5. By Lemma 6, if $\frac{f(0)}{F(0)} < \tau$, then there is no responsive equilibrium sequence, hence $P\left(\frac{f(0)}{F(0)}\right) = 0$, proving the first item. By Lemma 7, if $\frac{f(0)}{F(0)} > \tau$, then there exists a responsive equilibrium sequence with informativeness $P\left(\frac{f(0)}{F(0)}\right)$. By Lemma 5, items 2, 3 and 4 follow.

We will now show that the informativeness of all equilibrium sequences are bounded above by the P function. This is true along responsive equilibrium sequences, since any P is the largest solution of equation (17), and the informativeness of every responsive equilibrium sequence is a solution of equation (17). Suppose there is an equilibrium sequence with $\lim \frac{\lambda_n(\alpha)-\lambda_n(\beta)}{\sqrt{nF(0)}} = p < 0$. Then, there is a $\bar{n}$ such that for all $n > \bar{n}$, $\lambda_n(\beta) > \lambda_n(\alpha)$. In this case, the policy maker's best reply $\rho_n(t)$ is a nonincreasing function, and there is a T_n such that $\rho_n(t) = 1$ for all $t < T_n$, and $\rho_n(t) = 0$ for all $t > T_n$. In this case, the net payoff from participation becomes

$$u(\theta, c) = \mathbb{P}(\beta|\theta)\mathbb{P}(\text{piv}|\beta) - \mathbb{P}(\alpha|\theta)\mathbb{P}(\text{piv}|\alpha) - c,$$

where $\mathbb{P}(\text{piv}|\omega) = \mathbb{P}(T_n - 1|\omega)(1 - \rho_n(T_n)) + \mathbb{P}(T_n|\omega)\rho_n(T_n)$.

With these changes, Lemma 2 goes through under the hypothesis that $\lim \frac{\lambda_n(\alpha)-\lambda_n(\beta)}{\sqrt{nF(0)}} = p < 0$. Likewise, the second claim in Lemma 3 changes to $\lim \sqrt{nF(0)}\,(\hat{c}_{a,n} - \hat{c}_{b,n}) = -\phi(\kappa(p))\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\beta|a)\frac{1-\mu}{\mu}\frac{q}{1-q}\right)(1-\gamma)$, and the proof goes through without any major alteration. Therefore, Lemma 4 changes without any major alteration in the proof to the following claim:

Claim 5. Take a sequence of equilibria $\{c_{a,n}, c_{b,n}, \rho_n\}$ with $\lim \frac{\lambda_n(\alpha)-\lambda_n(\beta)}{\sqrt{nF(0)}} = p \leq 0$.

1. If $p \in (-\infty, 0)$, then p is a solution of the following equation:

$$|p| = \frac{f(0)}{F(0)}\left(1 + \frac{q}{1-q}\frac{1-\mu}{\mu}\right)(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta))(\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b))\,\phi(\kappa(p)) = g(p). \tag{35}$$

2. $p > -\infty$.

3. If $\mu > q$, then $p < 0$. Hence, if equation (35) has no solution, then there exists n^* such that for all $n > n^*$, babbling is the unique equilibrium.

Because $\phi(\kappa(p)) = \phi(\kappa(-p))$, if $p < 0$ is a solution of equation (35), then $-p > 0$ is a solution of equation (17). Hence, for the smallest p solution of equation (35), $-p$ is the largest solution of equation (17). Because we showed the existence of a responsive equilibrium sequence with informativeness equal to the maximal p solution of equation (17), there does not exist any informative equilibrium sequence that has informativeness greater than the maximal p-solution of equation (17). ☐

Proof of Theorem 2:

Proof. The first claim: If $\frac{f(0)}{F(0)} < \tau$, then $P = 0$, and babbling is the unique equilibrium for all n sufficiently large, and the claim follows. If $\frac{f(0)}{F(0)} > \tau$, then the claim follows from Lemma 9.

The second claim: The claim that the expected payoff of the policy maker is increasing in p follows from the information content of the turnout increasing in Blackwell order in p, and from the policy maker's choice being a single person decision problem. For the citizens, in any equilibrium sequences, the cost cutoffs converge to 0, hence the costs and benefits of participation do not change with p. To compare the citizens' expected payoff from policy choice with p, note that the policy maker's expected payoff along an equilibrium sequence with informativeness p converges to:

$$U_{PM}(p) := q(1-\mu)\left(1-\Phi(\kappa(p))\right) + (1-q)\mu\Phi\left(\kappa(p)+p\right),$$

while the citizens' expected payoff along the same sequence converges to (not counting the cost and benefit of participation):

$$U_c(p) := q\left(1-\Phi(\kappa(p))\right) + (1-q)\Phi\left(\kappa(p)+p\right).$$

The function $U_{PM}(p)$ is strictly increasing in p as we argued previously. Moreover, because κ is decreasing in p, $1-\Phi(\kappa(p))$ is increasing in p. Finally, because $\mu \geq 1/2$, $U'_{PM}(p) > 0$ implies $U'_c(p) > 0$, completing the proof. □

10.2 Proofs for the Comparative Statics

Proof of Theorem 3:

Proof. Recall that $\lim \sqrt{n}\left(F\left(c_{a,n}\right) - F\left(c_{b,n}\right)\right) = \lim \sqrt{n} f(0)\left(c_{a,n} - c_{b,n}\right)$. Using the definition of informativeness, equation (7) and the difference in the expected turnouts given by equation (14), we obtain that

$$\lim \frac{n\left(F\left(c_{a,n}\right) - F\left(c_{b,n}\right)\right)}{\sqrt{n}} = \frac{P\left(\frac{f(0)}{F(0)}\right)\sqrt{F(0)}}{\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)}.$$

Thus, $\lim \sqrt{n}\left(F\left(c_{a,n}\right) - F\left(c_{b,n}\right)\right)$ is proportional to $P\left(\frac{f(0)}{F(0)}\right)\sqrt{F(0)}$. Hence, to prove the theorem, it suffices to show that $\frac{\partial\left(P\left(\frac{f(0)}{F}\right)\sqrt{F}\right)}{\partial F} > 0$ when F is smaller than a cutoff, $\bar{F} > 0$.

Recall that $P\left(\frac{f(0)}{F}\right)$ is the largest p solution of the equation $p = \frac{f(0)}{F} Z\phi(\kappa(p))$, for some constant Z. Rewriting the equation in logs, we have

$$\ln p = \ln \frac{f(0)}{F} + \ln Z - \frac{1}{2}\kappa(p)^2 - \ln \sqrt{2\pi}. \tag{36}$$

At the largest solution (i.e., $P\left(\frac{f(0)}{F}\right)$, when this is strictly positive), we have that the curve $\ln p$ crosses the curve $\ln \frac{f(0)}{F} + \ln Z - \frac{1}{2}\kappa(p)^2 - \ln \sqrt{2\pi}$ from below, i.e., the derivative of the left-hand side of equation (36) with respect to p is greater than the derivative of the right-hand side of equation (36) with respect to p, at $P\left(\frac{f(0)}{F}\right)$. Hence, at $p = P\left(\frac{f(0)}{F}\right)$,

$$\frac{1}{p} > -\kappa(p)\kappa'(p).$$

Taking the total derivative of equation (36) with respect to p and F, we obtain

$$\frac{1}{p}dp = -\frac{1}{F}dF - \kappa(p)\kappa'(p)dp.$$

Rearranging, we have

$$\frac{dp}{dF} = \frac{-\frac{1}{F}}{\frac{1}{p} + \kappa(p)\kappa'(p)} < 0.$$

The sign of the expression $\frac{\partial\left(p\sqrt{F}\right)}{\partial F}$ is equal to the sign of the expression $\frac{\partial\left(\ln p + \ln\sqrt{F}\right)}{\partial F}$. The latter is equal to

$$\frac{1}{p}\frac{dp}{dF} + \frac{1}{2}\frac{1}{F} = \frac{1}{F}\left(\frac{1}{2} - \frac{1}{p + \kappa(p)\kappa'(p)}\right).$$

Note that $p + \kappa(p)\kappa'(p) = 1 + \frac{p^2}{4} - \frac{1}{p^2}\left(\ln\left(\frac{\mu}{1-\mu}\frac{1-q}{q}\right)\right)^2$, and is an increasing function of p, without bound. Hence, there exists some $\bar{p} > 0$ such that when $p > \bar{p}$, $\frac{\partial\left(p\sqrt{F}\right)}{\partial F} > 0$. Observing that $P\left(\frac{f(0)}{F}\right)$ is monotone increasing in $\frac{1}{F}$, and without bound, we obtain that there exists some $\bar{F} > 0$ such that if $F(0) < \bar{F}$, then $\frac{\partial\left(P\left(\frac{f(0)}{F}\right)\sqrt{F}\right)}{\partial F} > 0$. □

Proof of Theorem 4:

Proof. We will show that if $\mathbb{P}_1 >_B \mathbb{P}_2$, then

$$\left(\mathbb{P}_1(a|\alpha) - \mathbb{P}_1(a|\beta)\right)\left(\mathbb{P}_1(\alpha|a) - \mathbb{P}_1(\alpha|b)\right) > \left(\mathbb{P}_2(a|\alpha) - \mathbb{P}_2(a|\beta)\right)\left(\mathbb{P}_2(\alpha|a) - \mathbb{P}_2(\alpha|b)\right).$$

This will suffice to prove the two claims, because the expression $\left(\mathbb{P}_1(a|\alpha) - \mathbb{P}_1(a|\beta)\right)\left(\mathbb{P}_1(\alpha|a) - \mathbb{P}_1(\alpha|b)\right)$ is a constant that multiplies the function g, and if the multiplier increases, then the largest intersection point of the function with the 45 degree line increases. Likewise, when the function g increases, then the cutoff τ on the reverse hazard rate that makes this function intersect with the 45 degree line at a tangency point decreases. We now proceed to show the inequality above. If $\mathbb{P}_1 >_B \mathbb{P}_2$, then

$$\frac{\mathbb{P}_1(a|\alpha)}{\mathbb{P}_1(a|\beta)} \geq \frac{\mathbb{P}_2(a|\alpha)}{\mathbb{P}_2(a|\beta)},$$

$$\frac{\mathbb{P}_1(b|\alpha)}{\mathbb{P}_1(b|\beta)} \leq \frac{\mathbb{P}_2(b|\alpha)}{\mathbb{P}_2(b|\beta)},$$

with at least one of the inequalities being a strict inequality. Note first that, $\left(\mathbb{P}_1(\alpha|a) - \mathbb{P}_1(\alpha|b)\right) > \left(\mathbb{P}_2(\alpha|a) - \mathbb{P}_2(\alpha|b)\right)$, due to the inequalities above. We now proceed to show that $\left(\mathbb{P}_1(a|\alpha) - \mathbb{P}_1(a|\beta)\right) \geq \left(\mathbb{P}_2(a|\alpha) - \mathbb{P}_2(a|\beta)\right)$. Note that

$$\mathbb{P}_1(a|\alpha) - \mathbb{P}_1(a|\beta) = \mathbb{P}_1(b|\beta) - \mathbb{P}_1(b|\alpha),$$

and

$$\mathbb{P}_1(a|\alpha) - \mathbb{P}_1(a|\beta) = \mathbb{P}_1(a|\beta)\left(\frac{\mathbb{P}_1(a|\alpha)}{\mathbb{P}_1(a|\beta)} - 1\right),$$

$$\mathbb{P}_1(b|\beta) - \mathbb{P}_1(b|\alpha) = \mathbb{P}_1(b|\beta)\left(1 - \frac{\mathbb{P}_1(b|\alpha)}{\mathbb{P}_1(b|\beta)}\right).$$

Because $\mathbb{P}_1(a|\beta)+\mathbb{P}_1(b|\beta) = 1 = \mathbb{P}_2(a|\beta)+\mathbb{P}_2(b|\beta)$, either $\mathbb{P}_1(a|\beta) \geq \mathbb{P}_2(a|\beta)$, or $\mathbb{P}_1(b|\beta) \geq \mathbb{P}_2(b|\beta)$. Suppose $\mathbb{P}_1(a|\beta) \geq \mathbb{P}_2(a|\beta)$. Then, because $\left(\frac{\mathbb{P}_1(a|\alpha)}{\mathbb{P}_1(a|\beta)} - 1\right) \geq \left(\frac{\mathbb{P}_2(a|\alpha)}{\mathbb{P}_2(a|\beta)} - 1\right)$, we have $(\mathbb{P}_1(a|\alpha) - \mathbb{P}_1(a|\beta)) \geq (\mathbb{P}_2(a|\alpha) - \mathbb{P}_2(a|\beta))$. If $\mathbb{P}_1(b|\beta) \geq \mathbb{P}_2(b|\beta)$, then $\mathbb{P}_1(a|\alpha) - \mathbb{P}_1(a|\beta) = \mathbb{P}_1(b|\beta)\left(1 - \frac{\mathbb{P}_1(b|\alpha)}{\mathbb{P}_1(b|\beta)}\right) \geq \mathbb{P}_2(b|\beta)\left(1 - \frac{\mathbb{P}_2(b|\alpha)}{\mathbb{P}_2(b|\beta)}\right) = (\mathbb{P}_2(a|\alpha) - \mathbb{P}_2(a|\beta))$, completing the proof. □

Proof of Theorem 5:

Proof. Theorem 5 states comparative statics of the function P with respect to μ. We start by investigating the relationship between maximal information transmission in equilibrium and the policy maker's indifference belief μ. A simplified version of the equation (17) gives us the following condition for the maximal informativeness of protests, p.

Recall

$$m := \frac{1-\mu}{\mu}\frac{q}{1-q},$$

$$p = K(1+m)\phi\left(\frac{p}{2} + \frac{1}{p}\ln m\right) \tag{37}$$

where[47]

$$K := \frac{f(0)}{F(0)}\left(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)\right)\left(\mathbb{P}(\alpha|a) - \mathbb{P}(\alpha|b)\right).$$

Our analysis will utilize the following properties of the function

$$R(p,m) := K(1+m)\phi\left(\frac{p}{2} + \frac{1}{p}\ln m\right).$$

1. $R(p,m) = R(p,1/m)$.
2. $R_2(p,m) = 0$ for $m = 1$, where R_2 denotes the derivative of R with respect to its second argument.
3. $R(p,m)$ viewed as a function of m is single-peaked for $p < 2$, hence by the previous property, is maximized at $m = 1$.
4. $R_2(p,m) = 0$ has 3 solutions for $p > 2$: at $m = 1$, at $m = y$ for some $y < 1$, and at $m = 1/y > 1$. $R(p,m)$ is maximized at $m = y$ or $m = 1/y$.
5. $\max_m R(p,m) - p$ is a decreasing function of p.

These 5 properties give us the following Proposition, which proves Theorem 5.

Proposition.

1. *If $K < \phi(1)^{-1}$, then the largest solution of (37) viewed as a function of $m = \frac{1-\mu}{\mu}\frac{q}{1-q}$ is maximized at $m^* = 1$, i.e., $\mu = q$, and is increasing in m for $m \leq 1$ (or equivalently, decreasing in μ). At the maximized value, $P < 2$.*

[47] Note the expression in equation (17) has the inverse of m, and we use the identity $\ln\frac{1}{x} = -\ln x$ to derive equation (37).

2. *If $K > \phi(1)^{-1}$, then largest solution of (37) viewed as a function of m is maximized at some $y < 1$ (or at $1/y$) that solves $R_2(p^*, m^*) = 0$ and $R(p^*, m^*) = p^*$ simultaneously for $p^* > 2$.*

Proof. If $K < \phi(1)^{-1}$, then for $p < 2$, R is maximized at $m = 1$ by property 3, and when $K < \phi(1)^{-1}$, at $m = 1$, $R(p, 1) = p$ has a unique solution at $p^* < 2$. From property 5, we have that $\max_m R(m, p) - p$ is decreasing in p, hence, there is no $p > p^*$ that can solve (37) for any $m \geq 0$.

If $K > \phi(1)^{-1}$, at $m = 1$, the solution of (37) has $P(\mu) > 2$. Hence, $R(P(\mu), m)$ is maximized at some $m = y < 1$, and $R(P(q), y) > R(P(q), 1) = P(q)$. From property 5, we have that the maximum p is achieved when $m^*(p^*) \in \arg\max R(m, p^*)$ and $p^* = R(p^*, m^*(p^*))$. □

Proof of Property 1

Follows from the following equality:

$$(1+m)\phi\left(\frac{p}{2} + \frac{1}{p}\ln m\right) = \frac{(1+m)}{\sqrt{m}}\frac{1}{\sqrt{2\pi}}e^{-\left(p^2/8+\frac{1}{2p^2}(\ln m)^2\right)} = \frac{\left(1+\frac{1}{m}\right)}{\sqrt{\frac{1}{m}}}\frac{1}{\sqrt{2\pi}}e^{-\left(p^2/8+\frac{1}{2p^2}\left(\ln\frac{1}{m}\right)^2\right)},$$

which uses the identity $\frac{(1+m)}{\sqrt{m}} = \frac{\left(1+\frac{1}{m}\right)}{\sqrt{\frac{1}{m}}}$.

Proof of Property 2

For some constant $B := \frac{1}{\sqrt{2\pi}}K$,

$$\ln R(p, m) = \ln B + \ln(1+m) - \frac{1}{2}\left(p/2 + 1/p \ln m\right)^2.$$

Taking the derivative with respect to m, we obtain:

$$\frac{d \ln R(p, m)}{dm} = \frac{1}{1+m} - \left(p/2 + 1/p \ln m\right)\left(\frac{1}{pm}\right).$$

Clearly, for $m = 1$

$$\frac{d \ln R(p, m)}{dm} = 0.$$

Proof of Property 3

We will argue that for every $p < 2$, $\frac{d \ln R(p,m)}{dm} > 0$ for all $m < 1$. This will also imply (due to property 1) that $\frac{d \ln R(p,m)}{dm} < 0$ for $m > 1$, hence $R(p, \cdot)$ is single-peaked, and is maximized at $m = 1$.

Recall that

$$\begin{aligned}\frac{d \ln R(p, m)}{dm} &= \frac{1}{1+m} - \left(p/2 + 1/p \ln m\right)\left(\frac{1}{pm}\right) \\ &= \frac{m-1}{m+1}\frac{1}{2m} - \frac{1}{p^2}\frac{\ln m}{m}.\end{aligned}$$

For $m < 1$, we want to show that

$$\frac{m-1}{m+1}\frac{1}{2m} - \frac{1}{p^2}\frac{\ln m}{m} > 0,$$

which is equivalent to showing that

$$-\frac{1}{p^2}\frac{\ln m}{m} > \frac{1-m}{m+1}\frac{1}{2m},$$

which in turn is equivalent to showing that

$$-\frac{1}{p^2}\ln m > \frac{1-m}{m+1}\frac{1}{2}.$$

Because $\ln m < 0$, showing the inequality at $p = 2$ suffices. So we need to show

$$-\frac{1}{4}\ln m > \frac{1-m}{m+1}\frac{1}{2},$$

or equivalently

$$-\frac{1}{2}\ln m > \frac{1-m}{m+1}$$

for $m < 1$. Consider the function

$$t(m) := \frac{1-m}{m+1} + \frac{1}{2}\ln m.$$

Note that $t(1) = 0$, so if we show that $t'(m) > 0$ for $m < 1$, we will be done. Indeed,

$$t'(m) = \frac{-2m}{(m+1)^2} + \frac{1}{2m} = \frac{1}{2m}\left(1 - \left(\frac{2m}{m+1}\right)^2\right) > 0$$

because $2m < m + 1$.

Proof of Property 4

We have shown that $R_2(p, m) = 0$ for $m = 1$. Likewise, because

$$\frac{d\ln R(p,m)}{dm} = \frac{m-1}{m+1}\frac{1}{2m} - \frac{1}{p^2}\frac{\ln m}{m},$$

when $\frac{d\ln R(p,m)}{dm}|_{\bar{m}} = 0$, we have that $\frac{d\ln R(p,m)}{dm}|_{\frac{1}{\bar{m}}} = 0$. So we want to show that for $p > 2$, there is a unique $y < 1$ with $\frac{d\ln R(p,m)}{dm} = 0$ at $m = y$. Note that rewriting the condition $\frac{d\ln R(p,m)}{dm} = 0$, we get

$$p^2 = \frac{2(m+1)}{m-1}\ln m.$$

The function $z(m) := \frac{2(m+1)}{m-1}\ln m$ has the property that $\lim_{m\to 0} z(m) = \infty$, and $\lim_{m\to 1} z(m) = 4$ (obtained using L'Hopital's rule). What we will show is that $z'(m) < 0$ for $m < 1$, hence $z(m) = p^2$ has a unique solution for $m < 1$, and only when $p > 2$. To show that $z'(m) < 0$, we use the following equalities on the sign of $z'(m)$, which follows from usual calculations:

$$sign(z'(m)) = sign\left(\left(\left(1+\frac{2}{m-1}\right)\ln m\right)'\right) = sign\left(-\frac{m+1}{m}+\frac{2\ln m}{m-1}\right) = sign\left(\frac{m^2-1}{m}-2\ln m\right).$$

Let

$$\eta(m) := \frac{m^2-1}{m} - 2\ln m.$$

Note that $\eta(1) = 0$. We will show that $\eta'(m) > 0$ which will allow us to infer that $\eta(m) < 0$, hence $z'(m) < 0$:

$$\eta'(m) = 1 + \frac{1}{m^2} - \frac{2}{m} = \frac{1}{m^2}(m-1)^2 > 0.$$

To see that $R(p,\cdot)$ is maximized at $m < 1$, note that $\ln R(p,m)$ is decreasing in m in a small neighborhood of 1 when $m < 1$, and $\ln R(p,m)$ is increasing in m for sufficiently small m. Hence, R is maximized at some $\bar{m}$, and the value at $\bar{m}$ is higher than the value at $m = 1$.

Proof of Property 5

Let $m(p)$ be the maximizer of $R(p,m)$ at some p. For $p < 2$, the claim is clearly true because then $m(p) = 1$, and $\phi(\frac{p}{2})$ is decreasing in p. We want to show that

$$\frac{dR(p,m(p))}{dp} - 1 < 0,$$

or equivalently by the envelope theorem,

$$-K(1+m)\left(\frac{p}{2}+\frac{1}{p}\ln m\right)\left(\frac{1}{2}-\frac{1}{p^2}\ln m\right)\phi\left(\frac{p}{2}+\frac{1}{p}\ln m\right) - 1 < 0.$$

Note that at $m(p)$, we have

$$\frac{1}{1+m} - \left(\frac{p}{2}+\frac{1}{p}\ln m\right)\left(\frac{1}{pm}\right) = 0,$$

i.e., $\frac{p}{2}+\frac{1}{p}\ln m = \frac{pm}{m+1} > 0$. Also notice that $\frac{1}{p}\ln m = \frac{pm}{m+1} - \frac{p}{2}$, hence

$$\frac{1}{2} - \frac{1}{p^2}\ln m = \frac{1}{p}\left(\frac{p}{2} - \left(\frac{pm}{m+1} - \frac{p}{2}\right)\right) = \frac{1}{m+1} > 0.$$

Thus, we have

$$-K(1+m)\left(\frac{p}{2}+\frac{1}{p}\ln m\right)\left(\frac{1}{2}-\frac{1}{p^2}\ln m\right)\phi\left(\frac{p}{2}+\frac{1}{p}\ln m\right) - 1 < 0.$$

□

11 Online Appendix: Proofs for the Case without Activists

11.1 Proof of Theorem 6

We proceed first by showing properties of large responsive equilibrium sequences, i.e., the second claim of the theorem, and then proceed to showing the existence of such an equilibrium sequence, i..e, the first claim of the theorem. We finally show the third claim of the theorem. For the following, fix a large responsive equilibrium sequence, $\{c_{a,n}, c_{b,n}, \rho_n\}_n$.

Lemma 11. $\liminf \frac{F(c_{a,n})}{F(c_{b,n})} > 1$.

Proof. Recall that

$$c_\theta = \mathbb{P}(\alpha|\theta)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|\theta)\mathbb{P}(\text{piv}|\beta) \text{ for } \theta = a, b.$$

Therefore, when $\underline{c} = 0$, $c_b \leq 0$ implies $F(c_b) = 0$. Because in a responsive equilibrium $c_{a,n} \geq c_{b,n}$, and because large responsive equilibrium sequences must have $F(c_{\theta,n}) > 0$ for some θ when n is large, we have $c_{a,n} > 0$.

Notice that in any responsive equilibrium, if $c_a = c_b$, then $\mathbb{P}(\text{piv}|\alpha) = \mathbb{P}(\text{piv}|\beta) = 0$, due to the MLRP condition. Hence, when $c_a = c_b$, we have $c_a = c_b = 0$. This together with $c_{a,n} > 0$ for n large imply that $\mathbb{P}(\text{piv}|\alpha) > 0$. The MLRP condition implies that $c_{a,n} > c_{b,n}$ for large n. If $c_{b,n} \leq 0$ along the (sub)sequence we are looking, then the claim is true along such a sequence, because $c_{a,n} > 0$, hence $F(c_{a,n}) > 0$ while $F(c_{b,n}) = 0$, and the claim is true. So assume that $c_{b,n} > 0$ for every n along a subsequence.

Because $c_{b,n} \in (0, \bar{c})$ when n is large, we have

$$c_{b,n} = \mathbb{P}(\alpha|b)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|b)\mathbb{P}(\text{piv}|\beta).$$

Because $c_{b,n} > 0$, we have

$$\mathbb{P}(\text{piv}|\alpha) > \mathbb{P}(\text{piv}|\beta)\frac{\mathbb{P}(\beta|b)}{\mathbb{P}(\alpha|b)}.$$

Rewriting the indifference condition for an a signal with cost c_a,

$$\begin{aligned} c_{a,n} &= \mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta) \\ &= \frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)}c_{b,n} + \frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)}\mathbb{P}(\beta|b)\mathbb{P}(\text{piv}|\beta) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta). \end{aligned}$$

Because $\frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)}\mathbb{P}(\beta|b) > \mathbb{P}(\beta|a)$ (which follows because $\frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)} > 1$ and $\frac{\mathbb{P}(\beta|a)}{\mathbb{P}(\beta|b)} < 1$), we have that

$$c_{a,n} > \frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)}c_{b,n}.$$

Because $c_{a,n}, c_{b,n} \to 0$, because $c_{a,n} > \frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)}c_{b,n}$, and because $\frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)} > 1$, by assumption (1) we have

$$\liminf \frac{F(c_{a,n})}{F(c_{b,n})} > 1.$$

□

Lemma 12.

1. $\liminf \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} > \kappa$ *for some* $\kappa > 1$.

2. $\limsup \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} \leq \frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)}$.

Proof. If for some n, $c_{b.n} \leq 0$, then $F(c_{b,n}) = 0$, and we have

$$\frac{\lambda(\alpha)}{\lambda(\beta)} = \frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)} > 1.$$

If $c_{b,n} \leq 0$ for every n larger than a cutoff $\bar{n}$, then both claims are true.

Now assume $c_{b,n} > 0$ for every n along a (sub)sequence of equilibria.

We then have

$$\begin{aligned}
\frac{\lambda(\alpha)}{\lambda(\beta)} &= \frac{F(c_b) + \mathbb{P}(a|\alpha)(F(c_a) - F(c_b))}{F(c_b) + \mathbb{P}(a|\beta)(F(c_a) - F(c_b))} \\
&= 1 + \frac{(F(c_a) - F(c_b))(\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta))}{F(c_b) + \mathbb{P}(a|\beta)(F(c_a) - F(c_b))} \\
&= 1 + \frac{\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)}{\frac{F(c_b)}{F(c_a) - F(c_b)} + \mathbb{P}(a|\beta)}
\end{aligned}$$

<u>Proof of $\liminf \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} > \kappa > 1$</u>:

In the proof of Lemma 11 we showed that if $c_{b,n} > 0$ for every n larger then a cutoff, then $\liminf \frac{c_{a,n}}{c_{b,n}} > \frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)} > 1$. Therefore, assumption (1) implies

$$\limsup \frac{F(c_{b,n})}{F(c_{a,n}) - F(c_{b,n})} = \limsup \frac{1}{\frac{F(c_{a,n})}{F(c_{b,n})} - 1} < \infty,$$

i.e., this term is bounded. Because $\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta) > 0$, we have

$$\frac{\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)}{\frac{F(c_{b,n})}{F(c_{a,n}) - F(c_{b,n})} + \mathbb{P}(a|\beta)}$$

is bounded away from 0, proving the first part of the claim $\liminf \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} > \kappa$ for some $\kappa > 1$. Going forward, this is very important for some information revelation, since in equilibrium, expected turnout depends on the state, and hence the policy maker learns something from the turnout.

<u>Proof of</u> $\limsup \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} \leq \frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)}$:

Because $c_{a,n}, c_{b,n} \to 0$, and because the sequence is a large responsive equilibrium sequence, we have $F(c_{a,n}) > F(c_{b,n})$ for n larger than some $\bar{n}$. Therefore,

$$\limsup \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} = 1 + \limsup \frac{\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)}{\frac{F(c_{b,n})}{F(c_{a,n}) - F(c_{b,n})} + \mathbb{P}(a|\beta)} \leq 1 + \frac{\mathbb{P}(a|\alpha) - \mathbb{P}(a|\beta)}{\mathbb{P}(a|\beta)} = \frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)}.$$

$\square$

Lemma 13. *$\lim \lambda_n(\beta) = \infty$.*

Proof. The signals have bounded information, i.e., $\frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)} < \infty$, and $\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} > 0$, therefore when $\lambda_n(\alpha) \to \infty$, $\lambda_n(\beta) \to \infty$. $\square$

Lemma 14. *There is $y \in (0,1)$ such that when n is sufficiently large, for every pivotal event T_n, $\lambda_n(\alpha)(1-y) > T_n > \lambda_n(\beta)(1+y)$.*

Proof. We start noting that for $x > 0$,

$$\ln x \leq x - 1,$$

and $\ln x = x - 1$ if and only if $x = 1$. Note also that $\frac{x-1}{\ln x}$ is a strictly increasing function of x for $x > 1$, and is a strictly decreasing function of x for $x < 1$. Recall that we have $\liminf \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} = \kappa > 1$.

Using the policy maker's optimality, we have $L(T_n) \leq \frac{\mu}{1-\mu} < L(T_n+1)$, and taking the log's of the term $L(\cdot)$ gives us

$$\lambda_n(\beta)\left(1 - \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}\right) + T_n \ln \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} \leq \ln \frac{\mu}{1-\mu} - \ln \frac{q}{1-q} < \lambda_n(\beta)\left(1 - \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}\right) + (T_n+1) \ln \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}.$$

or equivalently,

$$\left(1 - \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}\right) + \frac{T_n}{\lambda_n(\beta)} \ln \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} \leq \frac{\ln \frac{\mu}{1-\mu} - \ln \frac{q}{1-q}}{\lambda_n(\beta)} < \left(1 - \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}\right) + \frac{T_n+1}{\lambda_n(\beta)} \ln \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}.$$

Because $\lambda_n(\beta) \to \infty$ as shown in Lemma 13, and because $\ln \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} \leq \ln \frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)}$, and hence is bounded from above, we have

$$\lim \left(\left(1 - \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}\right) + \frac{T_n}{\lambda_n(\beta)} \ln \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}\right) = 0.$$

Hence,

$$\liminf \frac{T_n}{\lambda_n(\beta)} = \liminf \frac{\frac{\lambda_n(\alpha)}{\lambda_n(\beta)} - 1}{\ln \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}}.$$

Because $\liminf \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} > 1$, because $\frac{x-1}{\ln x} > 1$ and because for $x > 1$, $\frac{x-1}{\ln x}$ is increasing in x, relabeling $x = \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}$ gives that

$$\liminf \left(\frac{\frac{\lambda_n(\alpha)}{\lambda_n(\beta)} - 1}{\ln \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}}\right) > 1.$$

Therefore, $\liminf \frac{T_n}{\lambda_n(\beta)} > 1$.

A similar argument delivers that $T_n < \lambda_n(\alpha)(1-y)$ for some $y > 0$. $\square$

Lemma 15. *Every large responsive equilibrium sequence aggregates information: the probability that A is implemented in state α goes to 1, and the probability that B is implemented in state β goes to 1, i.e.,*

$$\lim_{n\to\infty} \mathbb{P}\left(\{t : \rho_n(t) = 1\}|\alpha\right) = 1,$$
$$\lim_{n\to\infty} \mathbb{P}\left(\{t : \rho_n(t) = 0\}|\beta\right) = 1.$$

Proof. We will show that in every large equilibrium sequence, $\lim_n \mathbb{P}(t > T_n + 1|\alpha) = 1$ and $\lim_n \mathbb{P}(t \leq T_n|\beta) = 1$, which will complete the proof due to the monotonicity of $\rho_n(\cdot)$.

We will use Chebyshev's inequality. Note that $E(n|\omega) = \lambda_n(\omega)$, and $Var(n|\omega) = \lambda_n(\omega)$. Therefore,

$$\mathbb{P}\left(|\frac{t}{\lambda(\omega)} - 1| \geq y\right) \leq \frac{1}{y^2\lambda(\omega)}.$$

Because for some $y > 0$, $(1-y)\lambda_n(\beta) < T_n < (1-y)\lambda_n(\alpha)$ by Lemma 14, we have that

$$\lim \mathbb{P}(t > T_n + 1|\alpha) \geq \lim \mathbb{P}(t \geq (1-y)\lambda_n(\alpha)|\alpha) \geq 1 - \frac{1}{y^2\lambda_n(\alpha)}.$$

Because $\lambda_n(\alpha) \to \infty$, the right-hand side term goes to 1. A similar argument proves the second claim. □

We now proceed to show the existence of a large responsive equilibrium sequence.

Proposition 1. *There exists a large responsive equilibrium sequence if $\mu > \frac{1}{2}$.*

Proof. The proof structure is similar to the proof of Lemma 7. We first define an auxiliary game with restricted strategy sets, and show that when n is large, the restrictions do not bind for the equilibria of the auxiliary game.

Auxiliary Game with Restricted Strategy Sets

Consider the sets

$$C^n := \left\{(c_a, c_b) \in [0,\bar{c}]^2 : F(c_a) \geq 1/n, c_b \leq \frac{c_a}{1+\epsilon}\right\},$$
$$\Psi^n := \{l{:}\{0,1,...,2n-1\}{\to}[0,1] : l(t) \text{ is nondecreasing.}\},$$

where $\epsilon \in \left(0, \frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)} - 1\right)$. For every $l \in \Psi^n$, let the extension of l be $\bar{l} : \mathbb{N} \to [0,1]$ such that $\bar{l}(t) = l(t)$ for $t < 2n$, and $\bar{l}(t) = 1$ for all $t \geq 2n$. Observe that $\bar{l}$ is a nondecreasing function.

Each $c_a, c_b, \bar{l}$ induces pivot probabilities in state $\omega = \alpha, \beta$ as given in the main text through equality (3), where $\lambda(\omega)$ is found by equation 2. Note that $(C^n \times \Psi^n)$ is a closed and convex subset of $\mathbb{R}^{2n+2}$. Let

$$\Gamma^n : (C^n \times \Psi^n) \mapsto (C^n \times \Psi^n)$$

be a correspondence that

$$(c'_a, c'_b, l') \in \Gamma^n(c_a, c_b, l) \text{ iff}$$

$$c_a' = \min\left\{\max\left\{\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta), F^{-1}\left(\frac{1}{n}\right)\right\}, \bar{c}\right\}$$

$$c_b' = \min\left\{\max\left\{\mathbb{P}(\alpha|b)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|b)\mathbb{P}(\text{piv}|\beta), 0\right\}, \frac{1}{1+\epsilon}c_a'\right\}$$

$$\exists \bar{l} \in BR_{DM}(c_a, c_b) \text{ such that } l'(t) = \bar{l}(t) \text{ for all } t < 2n.$$

Where, $\mathbb{P}(\text{piv}|\omega)$ is calculated using $c_a, c_b, \bar{l}$, and $BR_{DM}(c_a, c_b)$ denotes the set of best replies of the policy maker to citizen strategies with cutoffs c_a and c_b.

Lemma 16. Γ^n *has a fixed point for every* n.

Proof. Note that, by construction, $(c_a', c_b') \in C^n$ and is uniquely determined by $c_a, c_b, \bar{l}$. Moreover, any $\bar{l} \in BR_{DM}(c_a, c_b)$ is nondecreasing and $\bar{l}(0) = 0$, so any l' in the image of the correspondence is in Ψ^n. $\Gamma^n(c_a, c_b, l)$ is closed and convex set because c_a', c_b' are uniquely chosen, and the projection of $BR_{DM}(c_a, c_b)$ on $t \in \{0, 1, .., 2n-1\}$ is a closed and convex set. Γ^n is upper-hemicontinuous because, F is continuous, $\mathbb{P}(\text{piv}|\alpha)$ is a continuous function of $F(c_a), F(c_b), l$, and BR_{DM} is upper-hemicontinuous. C^n and Ψ^n are closed convex sets in $\mathbb{R}^2$ and $\mathbb{R}^{2\text{n}}$. Hence, Kakutani's fixed-point theorem delivers us a fixed point of Γ^n, for every n. ☐

We now proceed to show that the fixed points of Γ^ncorrespond to equilibria of the protest game when n is large.

Fixed points of Γ^n correspond to equilibria of the protest game when n is large

For each fixed point (c_a, c_b, l) of Γ^n, associate a strategy profile $(c_a, c_b, \bar{l})$ of the protest model. We will show that there exists $\bar{n} \in \mathbb{N}$ such that $n > \bar{n}$ implies any strategy profile $\left(c_{a,n}, c_{b,n}, \bar{l}_n\right)$ associated with a fixed point of Γ^n is a Nash equilibrium of the protest model.

Suppose strategy profile $\left(c_{a,n}, c_{b,n}, \bar{l}_n\right)$ is associated with a fixed point $(c_{a,n}, c_{b,n}, l_n)$ of Γ^n. Because $c_{a,n} \geq c_{b,n}(1+\epsilon)$, and because $q \leq \mu$, $\bar{l}_n(0) = 0$. Because $\bar{l}_n$ is nondecreasing, and because $\bar{l}_n(2n) = 1$, $\bar{l}$ is not constant everywhere. Moreover, because $F(c_{a,n}) \geq 1/n$, $\liminf \lambda_n(\omega) > 0$ for $\omega = \alpha, \beta$. We will show in Lemmata 17, 18 and 19 when n is sufficiently large, any strategy profile $\left(c_{a,n}, c_{b,n}, \bar{l}_n\right)$ associated with a fixed point of Γ^n satisfies the following inequalities:

$$F(c_{a,n}) > 1/n,$$

$$c_{a,n} > c_{b,n}(1+\epsilon),$$

$$L_n(t) > \frac{\mu}{1-\mu} \text{ for all } t \geq 2n,$$

i.e., the strategy profile $\left(c_{a,n}, c_{b,n}, \bar{l}_n\right)$ is a responsive equilibrium of the protest game when n is large.

Lemma 17. $\lim nF(c_{a,n}) = \infty$, *and* $\lim(c_{a,n} + c_{b,n}) = 0$. *Therefore, there exists* $\bar{n}$ *such that* $n > \bar{n}$ *implies* $F(c_{a,n}) > 1/n$ *for every* $c_{a.n}$ *that is part of a fixed point of* Γ^n.

Proof. First part: Suppose the contrary, and suppose that $\lim nF(c_{a,n}) = k < \infty$. Because $c_{a,n} \geq c_{b,n}(1+\epsilon)$, $\lim nF(c_{b,n}) \leq k$. Then, $\lambda_n(\omega)$ stays bounded from above for each state ω. Because $\frac{\lambda_n(\alpha)}{\lambda_n(\beta)}$ is bounded from below and away from 1, any sequence of pivotal events T_n stays bounded. Moreover, because $F(c_{a,n}) \geq 1/n$, $\liminf \lambda_n(\omega) > 0$. Moreover, because $\lambda_n(\alpha) > \lambda_n(\beta)$ and because $q \leq \mu$, $L(0) < \frac{\mu}{1-\mu}$, and hence $\mathbb{P}(\text{piv}|\omega)$ stays bounded from below by a strictly positive number.

Observe that

$$\begin{aligned}\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta) &= \mathbb{P}(\text{piv}|\beta)\mathbb{P}(\beta|a)\left(\frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\beta|a)}\frac{\mathbb{P}(\text{piv}|\alpha)}{\mathbb{P}(\text{piv}|\beta)} - 1\right)\\ &\geq \mathbb{P}(\text{piv}|\beta)\mathbb{P}(\beta|a)\left(\frac{\mu}{1-\mu} - 1\right),\end{aligned}$$

where the last inequality follows because $\frac{q}{1-q}\frac{\mathbb{P}(\text{piv}|\alpha)}{\mathbb{P}(\text{piv}|\beta)}\frac{\lambda_n(\alpha)}{\lambda_n(\beta)} \geq \frac{\mu}{1-\mu}$, and because $\frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)} \geq \frac{\lambda_n(\alpha)}{\lambda_n(\beta)}$. Because $\mu > \frac{1}{2}$, $\left(\frac{\mu}{1-\mu} - 1\right) > 0$, and if $\mathbb{P}(\text{piv}|\beta)$ is bounded from below by a strictly positive number, then

$$\liminf\left(\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta)\right) > 0.$$

Therefore, $\liminf c_{a,n} > 0$, which contradicts to $\lim nF(c_{a,n}) = k < \infty$.

Second part: Suppose the contrary. Then, $\lambda_n(\omega)$ diverges. Observe that $\mathbb{P}(\text{piv}|\omega) \leq \max_{t\in\mathbb{N}} e^{-\lambda(\omega)}\frac{\lambda(\omega)^t}{t!}$, and $\lim_{\lambda\to\infty}\max_{t\in\mathbb{N}} e^{-\lambda}\frac{\lambda^t}{t!} = 0$. Therefore, $\mathbb{P}(\text{piv}|\omega) \to 0$. Then,

$$\limsup\left(\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - (\beta|a)\mathbb{P}(\text{piv}|\beta)\right) = 0,$$

hence, $\lim c_{a,n} = 0$. Because $0 \leq c_{b,n} < c_{a,n}$, we get $\lim(c_{a,n} + c_{b,n}) = 0$. $\square$

Lemma 18. *$c_{a,n} > (1+\epsilon)\,c_{b,n}$ for every $n > 0$.*

Proof. If $c_{b,n} = 0$, then $c_{a,n} \geq 1/n > (1+\epsilon)c_{b,n}$. Suppose $c_{b,n} > 0$. Note that,

$$\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta) > \mathbb{P}(\alpha|b)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|b)\mathbb{P}(\text{piv}|\beta),$$

therefore, when $c_{b,n} > 0$, $\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta) > 0$. Observe that by MLRP, we have,

$$\frac{\mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta)}{\mathbb{P}(\alpha|b)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|b)\mathbb{P}(\text{piv}|\beta)} > \frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)}.$$

Therefore, when $\epsilon \in \left(0, \frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\alpha|b)} - 1\right)$, $c_{a,n} > (1+\epsilon)c_{b,n}$. $\square$

Lemma 19. *$L_n(t) > \frac{\mu}{1-\mu}$ for all $t \geq 2n$ for all n sufficiently large.*

Proof. Because $c_{a,n} \geq c_{b,n}(1+\epsilon)$, and because $\lim(c_{a,n} + c_{b,n}) = 0$, $\liminf\frac{\lambda_n(\alpha)}{\lambda_n(\beta)} > 1$. An argument that is analogous to the one used for Lemma 17 delivers that $\lambda_n(\omega) \to \infty$. From Lemma 14, it follows that there is $y > 0$ such that $L(t) > \frac{\mu}{1-\mu}$ for all $t \geq \lambda_n(\alpha)(1-y)$, for all n when n is sufficiently large. Because $2n > \lambda_n(\alpha)$, the result follows. $\square$

Lemmata 17, 18 and 19 imply that there exists a $\bar{n}$ such that for every $n > \bar{n}$, any strategy profile $\left(c_{a,n}, c_{b,n}, \bar{l}_n\right)$ associated with a fixed point of Γ^n is an equilibrium of the protest game. This completes the proof of Proposition 1, i.e., existence of a large responsive equilibrium sequence when $\mu > 1/2$. □

Proposition 2. *There exists a large responsive equilibrium sequence if $\mu = \frac{1}{2}$.*

Proof. The proof is by construction. For the following, let $l := \frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)}$. By the MLRP, $l > 1$.

Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers where for each $k \geq 1$,

$$z_k := \frac{k \ln l + \ln\left(\frac{q}{1-q}\right)}{1 - l^{-1}}.$$

Note that $\{z_k\}_{k\in\mathbb{N}}$ is a strictly increasing sequence without any upper bound, hence there exists a $\bar{k} \geq 1$ such that $z_k > 0$ for all $k > \bar{k}$. We will show that there exists a sequence $\{n_k\}_{k\in\mathbb{N},k>\bar{k}}$ of positive real numbers such that for every $n > n_k$, there exists an equilibrium in which $c_{b,n} \leq 0$, $c_{a,n} = F^{-1}\left(\frac{z_k}{n\mathbb{P}(a|\alpha)}\right)$, $\rho_n(t) = 0$ if $t < k$, $\rho_n(t) = 1$ if $t > k$ and $\rho_n(k) \in [0,1]$. This suffices to prove the Proposition, because then for each $\lambda > 0$, there exists a $\bar{n}$ such that for every $n > \bar{n}$, there exists an equilibrium in which $\lambda_n(\alpha) > \lambda$.

To this end, fix $k \geq \bar{k}$, and let $p_k(\alpha) = e^{-z_k}\frac{(z_k)^k}{k!}$, $p_k(\beta) = e^{-\left(\frac{z_k}{l}\right)}\frac{\left(\frac{z_k}{l}\right)^k}{k!}$,

$$\tilde{n}_k := \frac{z_k}{\mathbb{P}(a|\alpha)F\left(\mathbb{P}(\alpha|a)p_k(\alpha) - \mathbb{P}(\beta|a)p_k(\beta)\right)},$$

and

$$n_k := \max\left\{\tilde{n}_k, \frac{\mathbb{P}(a|\alpha)}{z_k F\left(\bar{c}\right)}\right\}.$$

Let $\rho_n(k) \in [0,1]$ be the unique solution of

$$F\left((1-\rho_n(k))\left(\mathbb{P}(\alpha|a)p_k(\alpha) - \mathbb{P}(\beta|a)p_k(\beta)\right)\right) = \frac{z_k}{\mathbb{P}(a|\alpha)n}.$$

The solution exists and is unique because $n_k \leq n$, F is strictly increasing and is continuous with $F(0) = 0$.

Step 1: Policy maker's optimality:

If $c_{b,n} \leq 0$, and if $c_{a,n} = F^{-1}\left(\frac{z_k}{n\mathbb{P}(a|\alpha)}\right) \leq \bar{c}$, then $\lambda_n(\alpha) = nF(c_{a,n})\mathbb{P}(a|\alpha) = z_k$, and $\lambda_n(\beta) = nF(c_{a,n})\mathbb{P}(a|\beta) = \frac{\lambda_n(\alpha)}{l} = \frac{z_k}{l}$. Therefore,

$$L(k) = \frac{q}{1-q}e^{-(\lambda_n(\alpha)-\lambda_n(\beta))}\left(\frac{\lambda_n(\alpha)}{\lambda_n(\beta)}\right)^k = \frac{q}{1-q}e^{-\left(z_k(1-l^{-1})\right)}l^k = 1.$$

Because $\mu = \frac{1}{2}$, the policy maker is indifferent between the two policies if there are k protesters, hence ρ_n is a best reply to $(c_{a,n}, c_{b.n})$.

Step 2: Citizens' optimality.

The best reply of the citizens with a b signal is to abstain, because $\mathbb{P}(\alpha|\text{piv}, b) < \frac{1}{2}$. This is because the pivotal events are when $T_n = k-1$ and when $T_n = k$, and $L(k) = 1$, while $L(k-1) < 1$.

For citizens with an a signal, notice that $\mathbb{P}(\text{piv}|\alpha) = e^{-z_k}\frac{(z_k)^{k-1}}{(k-1)!}\rho_n(k) + e^{-z_k}\frac{(z_k)^k}{k!}(1-\rho_n(k))$ while $\mathbb{P}(\text{piv}|\beta) = e^{-\left(\frac{z_k}{l}\right)}\frac{\left(\frac{z_k}{l}\right)^{k-1}}{(k-1)!}\rho_n(k) + e^{-\left(\frac{z_k}{l}\right)}\frac{\left(\frac{z_k}{l}\right)^k}{k!}(1-\rho_n(k))$. The incentives of a signals to participate is given by

$$\hat{c}_{a,n} := \mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta)$$
$$= \rho_n(k)\left(\mathbb{P}(\alpha|a)e^{-z_k}\frac{(z_k)^{k-1}}{(k-1)!}\rho_n(k) - \mathbb{P}(\beta|a)e^{-\left(\frac{z_k}{l}\right)}\frac{\left(\frac{z_k}{l}\right)^{k-1}}{(k-1)!}\rho_n(k)\right)$$
$$+ (1-\rho_n(k))\left(\mathbb{P}(\alpha|a)p_k(\alpha) - \mathbb{P}(\beta|a)p_k(\beta)\right).$$

Because $\frac{\mathbb{P}(\alpha|a)}{\mathbb{P}(\beta|a)} = \frac{q}{1-q}\frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)}$ and because of the indifference of the policy maker at $t = k-1$, we have

$$\left(\mathbb{P}(\alpha|a)e^{-z_k}\frac{(z_k)^{k-1}}{(k-1)!}\rho_n(k) - \mathbb{P}(\beta|a)e^{-\left(\frac{z_k}{l}\right)}\frac{\left(\frac{z_k}{l}\right)^{k-1}}{(k-1)!}\rho_n(k)\right) = 0.$$

Therefore,

$$\hat{c}_{a,n} = \mathbb{P}(\alpha|a)\mathbb{P}(\text{piv}|\alpha) - \mathbb{P}(\beta|a)\mathbb{P}(\text{piv}|\beta) = (1-\rho_n(k))\left(\mathbb{P}(\alpha|a)p_k(\alpha) - \mathbb{P}(\beta|a)p_k(\beta)\right).$$

Note that $c_{a,n} = F^{-1}\left(\frac{z_k}{n\mathbb{P}(a|\alpha)}\right) = (1-\rho_n(k))\left(\mathbb{P}(\alpha|a)p_k(\alpha) - \mathbb{P}(\beta|a)p_k(\beta)\right) = \hat{c}_{b,n}$, completing the proof that $c_{a,n} = F^{-1}\left(\frac{z_k}{n\mathbb{P}(a|\alpha)}\right)$ is a best reply to the strategy profile $(c_{a,n}, c_{b,n}, \rho_n)$. ☐

Lemma 20. *In any responsive equilibrium sequence, $\{c_{a,n}, c_{b,n}, \rho_n\}_n$, $\limsup \frac{\lambda_n(\omega)}{\ln n} < \infty$.*

Proof. Suppose to the contrary that there is a subsequence of responsive equilibrium sequence with $\lim \frac{\lambda_n(\omega)}{\ln n} > z$ for every $z > 0$.

By Lemma 14, in such a sequence, there is a $y > 0$ such that when n is large, the pivotal event $T_n < (1-y)\lambda_n(\alpha)$. The following result on the tail properties of the Poisson distribution is shown in Cannone (2017), as Fact 6 of Theorem 1:

$$\mathbb{P}\left(t \le (1-y)\lambda_n(\alpha)|\alpha\right) \le e^{-\left(\frac{(y\lambda_n(\alpha))^2}{\lambda_n(\alpha)(1+y)}\right)}.$$

(The inequality is analogous to Hoeffding's inequality for independent Bernoulli random variables.) Therefore,

$$c_{a,n} \le \mathbb{P}\left(\alpha|a\right)\mathbb{P}(\text{piv}|\alpha) \le \mathbb{P}\left(\alpha|a\right)e^{-\left(\frac{y^2\lambda_n(\alpha)}{1+y}\right)}$$

when n is sufficiently large. Therefore, $\limsup nc_{a,n} \le \limsup ne^{-\left(\frac{y^2\lambda_n(\alpha)}{1+y}\right)}$. Now, if the initial hypothesis holds for $z = 2\frac{1+y}{y^2}$, then $\lambda_n(\omega) > 2\frac{1+y}{y^2}\ln n$, implies that $e^{-\left(\frac{y^2\lambda_n(\alpha)}{1+y}\right)} <$

$e^{-2\ln n} = \frac{1}{n^2}$. Hence, from the displayed equation, $\limsup nc_{a,n} \leq \limsup \frac{n}{n^2} = 0$, which contradicts $\lim \lambda_n(\omega) > \lim 2\frac{1+y}{y^2}\ln n = \infty$. $\square$

11.2 Proof of Theorem 8

Proof. The second part of the Theorem follows because if condition 18 holds, then, $\gamma > 0$, and hence in any responsive equilibrium $c_{b,n} > 0$. Therefore, the analysis is identical to the case in which $\underline{c} < 0$ and $\gamma > 0$.

We now proceed to show the first item of the Theorem, i.e., when condition 19 holds.

Observe that if

$$\frac{\mu}{1-\mu}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} \leq 1,$$

then there is a responsive equilibrium sequence in which for large n, b signals abstain, and all a signals with 0 costs (and costs below a positive number) participate. This is not straightforward, but the argument is analogous to the existence of a large responsive equilibrium when $F(0) = 0$, so we skip the formal proof. Such equilibrium sequences aggregate information, because $\liminf \frac{\lambda_n(\alpha)}{\lambda_n(\beta)} > 1$, and because $\lambda_n(\omega) \to \infty$ for $\omega = \alpha, \beta$. So, we now consider the case in which

$$\frac{\mu}{1-\mu}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} > 1.$$

For $x, y \in [0,1]$, satisfying $1 - y > x$, let

$$\frac{1}{n}\lambda(\omega; y, x) := (1-y)F(0)P(a|\omega) + xF(0)P(b|\omega)$$

Lemma 21. *The following inequalities hold:*

$$\frac{d}{dx}\frac{\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right)}{\frac{\lambda(\alpha;y,x)-\lambda(\beta;y,x)}{n}} < 0,$$

and

$$\frac{d}{dy}\frac{\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right)}{\frac{\lambda(\alpha;y,x)-\lambda(\beta;y,x)}{n}} > 0.$$

Proof. We start by calculating the derivatives of the following functions:

$$\frac{d}{dx}\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right) = nF(0)\left(\frac{P(b|\alpha)}{\lambda(\alpha;y,x)} - \frac{P(b|\beta)}{\lambda(\beta;y,x)}\right)$$

$$\frac{d}{dy}\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right) = -nF(0)\left(\frac{P(a|\alpha)}{\lambda(\alpha;y,x)} - \frac{P(a|\beta)}{\lambda(\beta;y,x)}\right)$$

$$\frac{d}{dx}\left(\lambda(\alpha;y,x) - \lambda(\beta;y,x)\right) = nF(0)\left(P(b|\alpha) - P(b|\beta)\right)$$

$$\frac{d}{dy}\left(\lambda(\alpha;y,x) - \lambda(\beta;y,x)\right) = -nF(0)\left(P(a|\alpha) - P(a|\beta)\right)$$

Now taking the derivative $\frac{d}{dx}\frac{\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right)}{\frac{\lambda(\alpha;y,x)-\lambda(\beta;y,x)}{n}}$ we get

$$\begin{aligned}
\frac{d}{dx}\frac{\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right)}{\frac{\lambda(\alpha;y,x)-\lambda(\beta;y,x)}{n}} &= n^2\frac{F(0)\left(\frac{P(b|\alpha)}{\lambda(\alpha;y,x)}-\frac{P(b|\beta)}{\lambda(\beta;y,x)}\right)(\lambda(\alpha;y,x)-\lambda(\beta;y,x))-F(0)\left(P(b|\alpha)-P(b|\beta)\right)\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right)}{(\lambda(\alpha;y,x)-\lambda(\beta;y,x))^2} \\
&= \frac{n^2F(0)}{(\lambda(\alpha;y,x)-\lambda(\beta;y,x))^2}\left(P(b|\alpha)\left(1-\frac{\lambda(\beta;y,x)}{\lambda(\alpha;y,x)}+\ln\left(\frac{\lambda(\beta;y,x)}{\lambda(\alpha;y,x)}\right)\right)+P(b|\beta)\left(1-\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}+\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right)\right)\right).
\end{aligned}$$

Note that for any $z>0$, $z-1\geq \ln z$, with equality satisfied only at $z=1$.Because $x+y<1$, we have $0<\frac{\lambda(\beta;y,x)}{\lambda(\alpha;y,x)}<1$. Therefore,

$$1-\frac{\lambda(\beta;y,x)}{\lambda(\alpha;y,x)}+\ln\left(\frac{\lambda(\beta;y,x)}{\lambda(\alpha;y,x)}\right)<0,$$

and

$$1-\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}+\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right)<0.$$

Therefore, $\frac{d}{dx}\frac{\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right)}{\frac{\lambda(\alpha;y,x)-\lambda(\beta;y,x)}{n}}<0$. Moreover, note that $\frac{n^2F(0)}{(\lambda(\alpha;y,x)-\lambda(\beta;y,x))^2}$ is independent of n, and is bounded away from 0 and ∞. Doing the similar exercise for the second claim, we have

$$\begin{aligned}
\frac{d}{dx}\frac{\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right)}{\frac{\lambda(\alpha;y,x)-\lambda(\beta;y,x)}{n}} &= -n^2\frac{F(0)\left(\frac{P(a|\alpha)}{\lambda(\alpha;y,x)}-\frac{P(a|\beta)}{\lambda(\beta;y,x)}\right)(\lambda(\alpha;y,x)-\lambda(\beta;y,x))-F(0)\left(P(a|\alpha)-P(a|\beta)\right)\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right)}{(\lambda(\alpha;y,x)-\lambda(\beta;y,x))^2} \\
&= \frac{-n^2F(0)}{(\lambda(\alpha;y,x)-\lambda(\beta;y,x))^2}\left(P(a|\alpha)\left(1-\frac{\lambda(\beta;y,x)}{\lambda(\alpha;y,x)}+\ln\left(\frac{\lambda(\beta;y,x)}{\lambda(\alpha;y,x)}\right)\right)+P(a|\beta)\left(1-\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}+\ln\left(\frac{\lambda(\alpha;y,x)}{\lambda(\beta;y,x)}\right)\right)\right) \\
&> 0.
\end{aligned}$$

□

We now proceed by setting the boundaries of the citizens' restricted strategies. Note that, by Lemma 21, we have that for some $\epsilon_1,\epsilon_2>0$:

$$\frac{\ln\left(\frac{\lambda(\alpha;\epsilon_2,0)}{\lambda(\beta;\epsilon_2,0)}\right)}{\frac{\lambda(\alpha;\epsilon_2,0)-\lambda(\beta;\epsilon_2,0)}{n}}>\frac{\ln\left(\frac{\lambda(\alpha;0,0)}{\lambda(\beta;0,0)}\right)}{\frac{\lambda(\alpha;0,0)-\lambda(\beta;0,0)}{n}}>\frac{\ln\left(\frac{\lambda(\alpha;0,\epsilon_1)}{\lambda(\beta;0,\epsilon_1)}\right)}{\frac{\lambda(\alpha;0,\epsilon_1)-\lambda(\beta;0,\epsilon_1)}{n}},$$

and

$$\frac{\ln\left(\frac{\lambda(\alpha;0,0)}{\lambda(\beta;0,0)}\right)}{\frac{\lambda(\alpha;0,0)-\lambda(\beta;0,0)}{n}}=\frac{\ln\left(\frac{\lambda(\alpha;\epsilon_2,\epsilon_1)}{\lambda(\beta;\epsilon_2,\epsilon_1)}\right)}{\frac{\lambda(\alpha;\epsilon_2,\epsilon_1)-\lambda(\beta;\epsilon_2,\epsilon_1)}{n}}$$

There exists a $\xi>0$ and a $\psi\in(0,1)$ such that

$$\psi > (1+\xi)\frac{\frac{\lambda(\alpha;0,0)-\lambda(\beta;0,0)}{n}}{\ln\left(\frac{\lambda(\alpha;0,0)}{\lambda(\beta;0,0)}\right)} = (1+\xi)\frac{\ln\left(\frac{\lambda(\alpha;\epsilon_2,\epsilon_1)}{\lambda(\beta;\epsilon_2,\epsilon_1)}\right)}{\frac{\lambda(\alpha;\epsilon_2,\epsilon_1)-\lambda(\beta;\epsilon_2,\epsilon_1)}{n}},$$
$$\psi < (1-\xi)\frac{\frac{\lambda(\alpha;0,\epsilon_1)-\lambda(\beta;0,\epsilon_1)}{n}}{\ln\left(\frac{\lambda(\alpha;0,\epsilon_1)}{\lambda(\beta;0,\epsilon_1)}\right)}.$$

Let T_n be the closest integer to ψn (in case of a tie, choose one of the closest integers arbitrarily).

Consider an auxiliary game in which the auxiliary policy maker implements A iff the turnout is at least T_n, b signals with 0 costs participate with a probability that is bounded above by ϵ_1, and b signals with positive costs abstain, and a signals with 0 costs participate with a probability at least $1-\epsilon_2$.

This auxiliary game has an equilibrium, in an analogous way to our existence results for responsive equilibria.

We will show that when n is large, the restrictions on citizen strategies are not binding, and that when there are T_n protesters, the policy maker's belief that the state is α is at least μ, and when there are T_n-1 protesters, his belief that the state is α is not more than μ, which justifies this strategy being an equilibrium strategy (i.e., a best reply) in the protest game.

Let $(c_{a,n}, c_{b,n}, y_n, x_n)$ denote that equilibrium strategy of the citizens in the auxiliary game, with the understanding that the first two arguments represent the cost cutoffs, and the last two are randomizations (in line with the definition of x, y above) at cost 0 for a and b signals respectively.

The first observation is that because $\psi \in (0,1)$, $T_n \to \infty$, hence pivot probabilities converge to 0, and so $c_{a,n}, c_{b,n} \to 0$.

Step 1: $x_n = 0$ and $y_n \to y \in [0,\epsilon_2]$ is not an equilibrium configuration:

Suppose on the way to a contradiction that there is a sequence of equilibria of the auxiliary game with $x_n = 0$ and $y_n \to y \in [0,\epsilon_2]$. Then,

$$\begin{aligned}
\lim \frac{1}{n}\ln L(T_n) &= \lim\left(\frac{1}{n}\ln\frac{q}{1-q} - \frac{1}{n}(\lambda(\alpha)-\lambda(\beta)) + \frac{1}{n}T_n\ln(\frac{\lambda(\alpha)}{\lambda(\beta)})\right) \\
&= \lim\left(-\frac{1}{n}(\lambda(\alpha)-\lambda(\beta)) + \psi\ln(\frac{\lambda(\alpha)}{\lambda(\beta)})\right) \\
&= -\frac{1}{n}(\lambda(\alpha;y,0)-\lambda(\beta;y,0)) + \psi\ln(\frac{\lambda(\alpha;y,0)}{\lambda(\beta;y,0)}) > 0
\end{aligned}$$

Where we have used $c_{a,n}, c_{b,n} \to 0$,

$$\psi > (1+\xi)\frac{\frac{\lambda(\alpha;0,0)-\lambda(\beta;0,0)}{n}}{\ln\left(\frac{\lambda(\alpha;0,0)}{\lambda(\beta;0,0)}\right)}$$

and that

$$\frac{\frac{\lambda(\alpha;0,0)-\lambda(\beta;0,0)}{n}}{\ln\left(\frac{\lambda(\alpha;0,0)}{\lambda(\beta;0,0)}\right)} \geq \frac{\frac{\lambda(\alpha;y,0)-\lambda(\beta;y,0)}{n}}{\ln\left(\frac{\lambda(\alpha;y,0)}{\lambda(\beta;y,0)}\right)},$$

for all $y \in [0, \epsilon_2]$.

Therefore, $\ln L(T_n) \to \infty$, which contradicts that $x_n = 0$ is a best reply for the citizens with b signals in the auxiliary game.

Step 2: $x_n = \epsilon_1$ is not an equilibrium configuration:

Note that it cannot be that $x_n = \epsilon_1$ and $y_n > 0$, since if $x_n > 0$, then $y_n = 0$ since if b signal is willing to participate, best reply of an a signal with no cost cannot be to stay out. Therefore, if $x_n = \epsilon_1$, then $y_n = 0$.

Suppose on the way to a contradiction that there is a sequence of equilibria of the auxiliary game with $x_n = \epsilon_1$ and $y_n = 0$. Then,

$$\begin{aligned}
\lim \frac{1}{n} \ln L(T_n) &= \lim\left(\frac{1}{n}\ln\frac{q}{1-q} - \frac{1}{n}(\lambda(\alpha)-\lambda(\beta)) + \frac{1}{n}T_n \ln(\frac{\lambda(\alpha)}{\lambda(\beta)})\right)\\
&= \lim\left(-\frac{1}{n}(\lambda(\alpha)-\lambda(\beta)) + \psi\ln(\frac{\lambda(\alpha)}{\lambda(\beta)})\right)\\
&= -\frac{1}{n}(\lambda(\alpha;0,\epsilon_1)-\lambda(\beta;0,\epsilon_1)) + \psi\ln(\frac{\lambda(\alpha;0,\epsilon_1)}{\lambda(\beta;0,\epsilon_1)}) < 0
\end{aligned}$$

Where again we have used $c_{a,n}, c_{b,n} \to 0$, and

$$\psi < (1-\xi)\frac{\frac{\lambda(\alpha;0,\epsilon_1)-\lambda(\beta;0,\epsilon_1)}{n}}{\ln\left(\frac{\lambda(\alpha;0,\epsilon_1)}{\lambda(\beta;0,\epsilon_1)}\right)}.$$

Therefore, $\ln L(T_n) \to -\infty$, which contradicts that $x_n = \epsilon_1$ is a best reply for the citizens with b signals in the auxiliary game.

From Steps 1 and 2 above, we conclude that the constraints on the randomizations of citizens with 0 costs are not binding, and $x_n \in (0, \epsilon_1)$ for all n larger than a cutoff. Therefore, b signals with 0 costs are indifferent between participating and abstaining, hence, $\mathbb{P}(\alpha|t = T_n - 1, b) = \frac{q}{1-q}\frac{\mathbb{P}(t=T_n-1|\alpha)}{\mathbb{P}(t=T_n-1|\beta)}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} = 1$.

Step 3: Policy maker's optimality:

Because $\frac{\mu}{1-\mu}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} > 1$, $\frac{q}{1-q}\frac{\mathbb{P}(t=T_n-1|\alpha)}{\mathbb{P}(t=T_n-1|\beta)} < \frac{\mu}{1-\mu}$, i.e., $\mathbb{P}(\alpha|t = T_n - 1) < \mu$. Hence, it is optimal for the policy maker to implement policy B when there are less than T_n protesters. To see that $\mathbb{P}(\alpha|t = T_n) \geq \mu$, notice that, if ϵ_1 is sufficiently small, then, $\frac{\lambda_n(\alpha)}{\lambda_n(\beta)}$ is close to $\frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)}$, hence

$$\frac{\mathbb{P}(t = T_n|\alpha)}{\mathbb{P}(t = T_n|\beta)} = \frac{\mathbb{P}(t = T_{n-1}|\alpha)}{\mathbb{P}(t = T_{n-1}|\beta)}\frac{\lambda_n(\alpha)}{\lambda_n(\beta)} = \left(\frac{q}{1-q}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)}\right)^{-1}\frac{\lambda_n(\alpha)}{\lambda_n(\beta)}$$

which can be made arbitrarily close to $\left(\frac{q}{1-q}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)}\right)^{-1}\frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)}$ by choosing ϵ_1 arbitrarily small. Because $L(T_n) = \frac{q}{1-q}\frac{\mathbb{P}(t=T_n|\alpha)}{\mathbb{P}(t=T_n|\beta)}$, when ϵ_1 is small, $L(T_n)$ is arbitrarily close to $\left(\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)}\frac{\mathbb{P}(a|\beta)}{\mathbb{P}(a|\alpha)}\right)^{-1}$.

Because the assumed condition is

$$\frac{\mu}{1-\mu}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)}\frac{\mathbb{P}(a|\beta)}{\mathbb{P}(a|\alpha)} < 1,$$

By choosing ϵ_1 small, we can ensure that $L(T_n) > \frac{\mu}{1-\mu}$. □

11.3 Proof of Theorem 7

The first part is proved in Battaglini (2017), so we skip it. We will show the second part of the theorem.

If

$$\frac{\mu}{1-\mu}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} \leq 1,$$

clearly there is an equilibrium that aggregates information with a signals participating, and b signals abstaining (Battaglini, 2017 also notes this, so we skip the proof). So we now consider the case where

$$\frac{\mu}{1-\mu}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} > 1.$$

We will construct a responsive equilibrium sequence in which $\psi(a) = 1$, i.e., citizens with a signals participate, and $\psi(b) = x \in [0, \epsilon]$ for some $\epsilon > 0$, and ϵ arbitrarily small. Let $\lambda_n(\omega; x) := n(\mathbb{P}(a|\omega) + x\mathbb{P}(b|\omega))$. Such a sequence of equilibria aggregates information, since the ratio of expected protest sizes in the two states stays bounded away from 1, and since expected protest sizes increase without bound.

For every n, consider the integer T_n found by the following equality:

$$T_n := \min\left\{t \in \mathbb{N} : \frac{q}{1-q}e^{-(\lambda_n(\alpha;0)-\lambda_n(\beta;0))}\left(\frac{\lambda_n(\alpha;0)}{\lambda_n(\beta;0)}\right)^t > \frac{\mu}{1-\mu}\right\}$$

We will show the following inequality shortly:

$$\frac{d}{dx}\frac{\ln\left(\frac{\lambda_n(\alpha;x)}{\lambda_n(\beta;x)}\right)}{\frac{\lambda_n(\alpha;x)-\lambda_n(\beta;x)}{n}}|_{x=0} < 0. \tag{38}$$

Then, for some $\epsilon > 0$,

$$\lim \frac{q}{1-q}e^{-(\lambda_n(\alpha;\epsilon)-\lambda_n(\beta;\epsilon))}\left(\frac{\lambda_n(\alpha;\epsilon)}{\lambda_n(\beta;\epsilon)}\right)^{T_n} = 0$$

Because $\frac{\lambda_n(\omega;x)}{n}$ is continuous in x, by the intermediate value theorem, there is some $\{\epsilon_n\}$ with $\epsilon_n < \epsilon$ such that for every large n we have

$$\frac{q}{1-q}e^{-(\lambda_n(\alpha;\epsilon_n)-\lambda_n(\beta;\epsilon_n))}\left(\frac{\lambda(\alpha;\epsilon_n)}{\lambda(\beta;\epsilon_n)}\right)^{T_n-1}\frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} = 1.$$

It is easy to verify that the strategy profile in which $\psi(a) = 1$ and $\psi(b) = \epsilon_n$, and the policy maker chooses policy A iff when the turnout is at least T_n is a Nash equilibrium, if $\frac{\lambda_n(\alpha;\epsilon)}{\lambda_n(\beta;\epsilon)} \frac{\mathbb{P}(b|\beta)}{\mathbb{P}(b|\alpha)} > \frac{\mu}{1-\mu}$. This last inequality is true when ϵ is chosen small enough because $\frac{\lambda_n(\alpha;0)}{\lambda_n(\beta;0)} = \frac{\mathbb{P}(a|\alpha)}{\mathbb{P}(a|\beta)}$, and because of the assumption that $\frac{\mu}{1-\mu} \frac{\mathbb{P}(b|\alpha)}{\mathbb{P}(b|\beta)} \frac{\mathbb{P}(a|\beta)}{\mathbb{P}(a|\alpha)} < 1$.

We now show that inequality (38) holds.

$$\frac{d}{dx} \ln\left(\frac{\lambda_n(\alpha;x)}{\lambda_n(\beta;x)}\right) = n\left(\frac{\mathbb{P}(b|\alpha)}{\lambda_n(\alpha;x)} - \frac{\mathbb{P}(b|\beta)}{\lambda_n(\beta;x)}\right)$$

$$\frac{d}{dx}\left(\lambda_n(\alpha;x) - \lambda_n(\beta;x)\right) = n\left(\mathbb{P}(b|\alpha) - \mathbb{P}(b|\beta)\right)$$

$$\begin{aligned}\frac{d}{dx} \frac{\ln\left(\frac{\lambda_n(\alpha;x)}{\lambda_n(\beta;x)}\right)}{\frac{\lambda(\alpha;x,y)-\lambda(\beta;x,y)}{n}} &= \frac{n^2\left(\frac{\mathbb{P}(b|\alpha)}{\lambda_n(\alpha;x)} - \frac{\mathbb{P}(b|\beta)}{\lambda_n(\beta;x)}\right)\left(\lambda_n(\alpha;x) - \lambda_n(\beta;x)\right) - n\left(\mathbb{P}(b|\alpha) - \mathbb{P}(b|\beta)\right)\ln\left(\frac{\lambda_n(\alpha;x)}{\lambda_n(\beta;x)}\right)}{\left(\lambda_n(\alpha;x) - \lambda_n(\beta;x)\right)^2} \\ &= \frac{n^2}{\left(\lambda_n(\alpha;x) - \lambda_n(\beta;x)\right)^2}\left(\mathbb{P}(b|\alpha)\left(1 - \frac{\lambda_n(\beta;x)}{\lambda_n(\alpha;x)} + \ln\left(\frac{\lambda_n(\beta;x)}{\lambda_n(\alpha;x)}\right)\right) + \mathbb{P}(b|\beta)\left(1 - \frac{\lambda_n(\alpha;x)}{\lambda_n(\beta;x)} + \ln\left(\frac{\lambda_n(\alpha;x)}{\lambda_n(\beta;x)}\right)\right)\right)\end{aligned}$$

Note that for any $z > 0$, $z - 1 \geq \ln z$, with equality satisfied only at $z = 1$. When $x < 1$, we have $0 < \frac{\lambda_n(\beta;x)}{\lambda_n(\alpha;x)} < 1$, therefore,

$$1 - \frac{\lambda_n(\beta;x)}{\lambda_n(\alpha;x)} + \ln\left(\frac{\lambda_n(\beta;x)}{\lambda_n(\alpha;x)}\right) < 0$$

and

$$1 - \frac{\lambda_n(\alpha;x)}{\lambda_n(\beta;x)} + \ln\left(\frac{\lambda_n(\alpha;x)}{\lambda_n(\beta;x)}\right) < 0$$

Therefore, $\frac{d}{dx} \frac{\ln\left(\frac{\lambda_n(\alpha;x)}{\lambda_n(\beta;x)}\right)}{\frac{\lambda_n(\alpha;x)-\lambda_n(\beta;x)}{n}} < 0$.