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Dynamic Expert Incentives: Complementarity and Substitutability in Information Acquisition

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Dynamic Expert Incentives: Complementarity and Substitutability in Information Acquisition*

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Abstract

We consider a model of dynamic expertise, in which two experts with the same bias exert efforts over time to discover the state of the world and are able to send verifiable messages about the discovery to a decision maker. We propose a definition of strategic complementarity and substitutability in this setting and find that the experts' information acquisition decisions are always substitutes when the experts are homogeneous, but sometimes complements when the experts are heterogeneous.

1 Introduction

Decision makers, from corporate CEOs to judges and political leaders, often find themselves in need of experts' recommendations or testimonies. Experts, who are rarely disinterested

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and often have their own agendas, in turn need to put in effort to conduct researches on the relevant issues. Resource will only be spent if it helps to advance the expert's own agenda.

In this paper, we try to understand an expert's incentive to conduct costly research and acquire policy-relevant information and how the presence of another expert with similar interest may affect his incentive. More specifically, we are interested in a situation in which researches are conducted over time and the decision maker constantly updates her policy based on the (verifiable) evidences provided by the experts. Such an environment is especially relevant to public policy decision making, as the government often needs to act without full knowledge of the exact situation. Since the experts are biased, they only reveal favorable evidences to the decision maker. The absence of evidence, therefore, is interpreted as bad evidence. This leads the decision maker's belief and the experts' incentives to acquire information to evolve over time.

We focus on experts with identical linear preference over the decision maker's belief but (possibly) different costs of effort and find that dynamic incentives have important implications on the strategic interactions of the experts. When the experts are homogeneous, their information acquisition decisions are substitutes as they free ride on each other's effort and reduce their own efforts relative to the single-expert benchmark. However, when the experts are heterogeneous, information acquisition decisions can be complementary. More specifically, the introduction of a more efficient expert can have a "motivating" effect on a less efficient one. This is because, with some probability, the more efficient expert may fail to find any evidence, this makes the decision maker more skeptical about the state and creates a gap between the experts' and the decision maker's beliefs. This gap in beliefs in turn motivates the less efficient expert, who would not exert effort if he were by himself, to acquire information to "correct" it.

Our results have some implications for the problems of optimal choice of expert panel composition and optimal task assignment to experts. For example, if hiring an efficient expert is costly, a decision maker could introduce heterogeneity into the expert panel and explore the "motivating" effect to economize costs. As another example, when a decision maker faces multiple decision problems and has a group of heterogeneous experts to assign, it may be beneficial for the decision maker to pair a “novice” with a “veteran” to an individual task.

1.1 Related Literature

Our paper belongs to the literature of “persuasion games” pioneered by Grossman and Hart (1980), Grossman (1981), and Milgrom (1981). However, most of the models in this literature assume either single expert (e.g., Che and Kartik 2009; Henry 2009) or exogenous information (e.g., Bhattacharya and Mukherjee 2013). The closest paper to ours in this literature is Kartik, Lee, and Suen (2017) who consider the static counterpart of our model and show that information acquisition decisions are strategic substitutes. We show that dynamic revelations of information can turn information acquisition decisions to strategic complements. Moreover, casting our model in continuous-time also makes the expert’s interactions very different from those in Kartik, Lee, and Suen (2017). In our model, information arrives in a Poisson process so that the experts never make a discovery at the same time. In contrast, the strategic substitutability in Kartik, Lee, and Suen (2017) in the identical bias case stems from the reduction in the value of information when the experts reveal favorable information simultaneously. Yang (2018) also demonstrates a substitutability result in a static persuasion game, but with perfectly informative signals and experts with monotone and oppositely biased preferences. Che and Kartik (2009) consider a single-expert persuasion game and show that difference between the DM’s and the expert’s beliefs can incentivize the expert to acquire information. In our model, the experts’ incentives to acquire information evolve over time as the DM’s posterior belief changes for a similar reason.

Outside the persuasion game literature, Gul and Pesendorfer (2012) study a dynamic model of political campaigns in which two parties provide costly information to a decision maker and find that the parties’ actions are strategic substitutes. They focus on parties with opposite biases. As a result, the parties compete over influence on the decision maker as opposed to free-riding on each other as in our case. Moreover, in their model, the parties cannot withhold information from the decision maker and the players are, as a result, always symmetrically informed. In contrast, the focus of the current paper is to demonstrate how selective disclosure affects experts' incentives to provide efforts over time.

Our paper is also related to the literature of strategic experimentation. In particular, the experts' incentives to exert effort increase over time as in the exponential bandit models with "bad news" studied by Keller and Rady (2015) and Bonatti and Hörner (2017). In these models, an agent's (public or private) experimentation leads to an increase in the agent's belief in the absence of bad news. In contrast, in our model, the experts do not learn from their actions in the absence of news, rather, the decision maker's belief evolves endogenously from equilibrium expectations and the experts' incentive to hide unfavorable information. Since the experts' efforts are not observed by the decision maker, unless there is a disclosed discovery, the evolution of decision maker's belief is independent of the experts' actions. This leads to the different dynamics and results in our model.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 analyzes the single-expert benchmark. Section 4 presents the main results. Section 5 considers a variant of the model in which the experts do not share information about bad discoveries. Section 6 concludes. Most of the proofs are relegated to the Appendix.

2 The Model

Time is continuous and the horizon is infinite. There is an unknown and persistent state of the world, $\omega \in \{0, 1\}$. With prior probability $p_0 \in (0, 1)$, $\omega = 1$. We also say that the state is good (bad) if $\omega = 1$ ($\omega = 0$). The game has three players: two experts and a decision maker (DM, hereafter). All players are Bayesian.

At each time t, expert i exerts effort $a_{i,t} \in [0, 1]$ at cost $c_i a_{i,t}$ to search for an evidence about state of the world, where $c_i > 0$. Effort is unobservable to the other players and the instantaneous probability of obtaining the evidence is $\lambda a_{i,t}$. For simplicity, we assume that the evidence, if obtained, is conclusive. Moreover, an expert can reveal the evidence to the DM only if he is in possession of it. That is, the experts may feign ignorance but cannot fabricate evidence. We further assume that the experts' instantaneous payoffs are linear and increasing in the DM's belief. Moreover, the experts discount future payoffs at rate $r > 0$. Let $\tilde{p}$ and $\tilde{a}_i$ denote the stochastic processes of the DM's belief and expert i's effort, respectively. The payoff of expert i is given by

$$E\left[\int_0^\infty e^{-rt}\left(\tilde{p}_t - c_i \tilde{a}_{i,t}\right) dt\right].$$

Suppose an expert discovers the state at time t. We assume that the discovery automatically becomes common knowledge among the experts. This is either because the experts work together as a close team so that any discovery will inevitably becomes known to each other or because the expert who made the discovery sends a private (cheap talk) message to inform the other of the discovery. In Section 5, we consider a variant of the model in which bad discoveries are private to the experts.

Suppose the expert discovers that the state is good at time t, it is dominant for him to cease any effort and disclose the finding immediately to the DM. On the other hand, if he found that the state is bad, it is dominant for him to cease any effort and withhold the information for good. As a result, a strategy of expert i in this game can be reduced to a time-dependent path of effort $\{a_{i,t}\}_{t\geq 0}$ such that *if no discovery was made prior to time t, expert i exerts instantaneous effort $a_{i,t}$ at time t.* We assume that $\{a_{i,t}\}_{t\geq 0}$ is càdlàg, or, right continuous with left limits.[1] We use $\left\{a^*_{i,t}\right\}_{t\geq 0}$ to denote expert i's equilibrium effort path in the single-expert benchmark and $\left\{a^{**}_{i,t}\right\}_{t\geq 0}$ in the two-expert case.

Upon disclosure of a favorable evidence, the DM's belief jumps to 1. Thus, to describe the dynamics of the DM's belief, we only need to calculate the law of motion when there is no disclosure. Given the equilibrium strategies $\left\{\left(a^{**}_{1,t}, a^{**}_{2,t}\right)\right\}_{t\geq 0}$, the DM's belief evolves according to

$$\dot{p}_t = -\lambda\left(a^{**}_{1,t} + a^{**}_{2,t}\right) p_t\left(1 - p_t\right). \tag{1}$$

Given the equilibrium expectation $\left\{\left(a^{**}_{1,t}, a^{**}_{2,t}\right)\right\}_{t\geq 0}$, suppose a discovery has not been made by time t, expert i receives the instantaneous benefit p_t, exerts instantaneous effort $a_{i,t}$, and incurs the instantaneous cost $c_i a_{i,t}$. With instantaneous probability $\lambda\left(a_{i,t} + a^{**}_{j,t}\right)$, a discovery is made. In this event, expert i receives $\frac{1}{r}$ if the state is good, and the contin-

[1] This assumption allows us to pin down $a^{**}_{i,t}$ when it is not determined by (4).

uation payoff

$$W_t \equiv \int_t^{\infty} e^{-r(s-t)} p_s ds \tag{2}$$

if the state is bad. Since the probability that a discovery has not been made by time t is given by $\exp\{-\int_0^t \lambda \left(a_{i,s} + a_{j,s}^{**}\right) ds\}$, expert i's problem can be formulated as

$$\max_{\{a_{i,t}\}_{t\geq 0}} \int_0^{\infty} e^{-\int_0^t \{r+\lambda(a_{i,s}+a_{j,s}^{**})\}ds} \left\{p_t - c_i a_{i,t} + \lambda \left(a_{i,t} + a_{j,t}^{**}\right) \left(\frac{p_0}{r} + (1-p_0) W_t\right)\right\} dt, \tag{3}$$

Our equilibrium notion is as follows.

Definition 1 *A strategy profile* $\left\{\left(a_{1,t}^{**}, a_{2,t}^{**}\right)\right\}_{t\geq 0}$ *form an equilibrium if and only if it solves (3) for* $i = 1, 2$.

Notice that we have chosen to not refer to the DM's belief in the definition of an equilibrium. However, we have assumed that the DM's belief is *consistent* by assuming that it satisfies (1) and that the DM's belief jumps to 1 after a good discovery. We assume that the latter holds even if the experts are not expected to possess information in equilibrium. This reflects our modelling assumption that evidences are *hard*.

2.1 Preliminary Analysis

Denote the probability that no discovery has been made by time t by

$$x_{i,t} \equiv e^{-\int_0^t \lambda(a_{i,s}+a_{j,s}^{**})ds}.$$

We have

$$\dot{x}_{i,t} = -\lambda \left(a_{i,t} + a_{j,t}^{**}\right) x_{i,t}.$$

The current-value Hamiltonian for expert i's problem is

$$H_{i,t} = x_{i,t} \left\{p_t - c_i a_{i,t} + \lambda \left(a_{i,t} + a_{j,t}^{**}\right) \left(\frac{p_0}{r} + (1-p_0) W_t\right)\right\} - \gamma_{i,t} \lambda \left(a_{i,t} + a_{j,t}^{**}\right) x_{i,t}.$$

Define expert i's *instantaneous incentive to acquire information* at time t by

$$B_{i,t} \equiv \frac{p_0}{r} + (1 - p_0) W_t - \gamma_{i,t}.$$

We also call $\lambda B_{i,t}$ expert i's *instantaneous marginal benefit* at time t. Pontryagin's maximum principle requires that

$$a_{i,t}^{**} = \begin{cases} 0 & \text{if } \lambda B_{i,t} < c_i, \\ 1 & \text{if } \lambda B_{i,t} > c_i. \end{cases} \tag{4}$$

Moreover, there exists a continuous function $\gamma_{i,t}$ whose right derivative satisfies

$$\begin{aligned} \dot{\gamma}_{i,t} &= r\gamma_{i,t} - \frac{\partial H_{i,t}}{\partial x_{i,t}} \\ &= r\gamma_{i,t} - \left(p_t - c_i a_{i,t}^{**} + \lambda \left(a_{i,t}^{**} + a_{j,t}^{**}\right) B_{i,t}\right). \end{aligned} \tag{5}$$

Finally, if $\left\{x_{i,t}^{**}\right\}_{t \geq 0}$ is the optimal trajectory, then, the transversality condition,

$$\lim_{t \to \infty} e^{-rt} \gamma_{i,t} \left(x_{i,t}^{**} - x_{i,t}\right) \leq 0, \tag{6}$$

must be satisfied for all feasible trajectories $\{x_{i,t}\}_{t \geq 0}$. In this model, conditions (4)–(6) are both necessary and sufficient for optimality. This is because the Hamiltonian is linear in the state variable $x_{i,t}$ and the optimal control $a_{i,t}^{**}$ does not depend directly on $x_{i,t}$. As a result, the maximized Hamiltonian is linear and, thus, concave in $x_{i,t}$. Sufficiency then follows from the Arrow sufficiency theorem.

The right derivative of $B_{i,t}$ satisfies

$$\dot{B}_{i,t} = -p_0 (1 - p_t) + (\lambda B_{i,t} - c_i) a_{i,t}^{**} + \left(r + \lambda a_{j,t}^{**}\right) B_{i,t}. \tag{7}$$

Notice that $\{B_{i,t}\}_{t \geq 0}$ is continuous and, by (4), $\left\{a_{i,t}^{**}\right\}_{t \geq 0}$ has jumps only when $\lambda B_{i,t} = c_i$. Thus, $\left\{\dot{B}_{i,t}\right\}_{t \geq 0}$ is continuous whenever $\left\{a_{j,t}^{**}\right\}_{t \geq 0}$ is continuous. Moreover, since $\left\{a_{j,t}^{**}\right\}_{t \geq 0}$

is càdlàg, $\left\{\dot{B}_{i,t}\right\}_{t\geq 0}$ is also càdlàg. Solving (7) for $B_{i,t}$, we have

$$B_{i,t} = \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{i,z}^{**}+a_{j,z}^{**}\right)\right)dz} \left(\frac{p_0\left(1-p_0\right)}{p_0 e^{-\lambda \int_0^s \left(a_{i,z}^{**}+a_{j,z}^{**}\right)dz} + 1 - p_0} + c_i a_{i,s}^{**} \right) ds.^2 \qquad (8)$$

Moreover, suppose that $\dot{a}_{i,t}^{**}$ and $\dot{a}_{j,t}^{**}$ exist. Denote the right derivative of $\dot{B}_{i,t}$ by $\ddot{B}_{i,t}$, we have,

$$\ddot{B}_{i,t} = p_0 \dot{p}_t + \left(\lambda B_{i,t} - c_i\right) \dot{a}_{i,t}^{**} + \lambda \dot{a}_{j,t}^{**} B_{i,t} + \left(r + \lambda \left(a_{i,t}^{**} + a_{j,t}^{**}\right)\right) \dot{B}_{i,t}.$$

By continuity of $B_{i,t}$ and (4), $\lambda B_{i,t} - c_i \neq 0$ implies that $\dot{a}_{i,t}^{*} = 0$. Thus,

$$\ddot{B}_{i,t} = p_0 \dot{p}_t + \lambda \dot{a}_{j,t}^{**} B_{i,t} + \left(r + \lambda \left(a_{i,t}^{**} + a_{j,t}^{**}\right)\right) \dot{B}_{i,t}, \qquad (9)$$

whenever $\dot{a}_{j,t}^{**}$ exists.

3 The Single-expert Case

We first solve for the single-expert case to establish a benchmark for comparison. Suppose expert i is the only expert, the formulation in the previous section applies if we simply set the other expert's effort to zero, i.e., $a_{j,t}^{**} = 0$. We use $\left\{a_{i,t}^{*}\right\}_{t\geq 0}$ to denote expert i's equilibrium effort in the single-expert case.

By (4), an expert's optimal effort at time t depends on the *instantaneous incentive to acquire information*, $B_{i,t}$. The following lemma shows that when there is only one expert, the path $\{B_{i,t}\}_{t\geq 0}$ has a simple pattern.

Lemma 1 *Suppose expert i is the only expert. If expert i ever exerts any effort in equilibrium, then the equilibrium instantaneous incentive to acquire information $B_{i,t}$ is strictly increasing over time. Otherwise, it remains constant.*

Intuitively, by (1), the DM's belief weakly decreases over time in the absence of a discovery. As a result, the incentive to acquire information must be weakly increasing.

[2] (8) is derived by assuming $\lim_{t\to\infty} e^{-\int_0^t \left(r+\lambda\left(a_{i,z}^{**}+a_{j,z}^{**}\right)\right)dz} B_{i,t} = 0$, which in turn can be derived from (4) and (6).

On one hand, if the expert is not expected to acquire information in equilibrium, the DM's belief remains constant and so does the expert's incentive to acquire information. On the other hand, if the expert is expected to acquire information in the equilibrium, the DM's belief strictly decreases and thus the expert's incentive to acquire information strictly increases. Therefore, once the expert starts to acquire information, he must have the incentive to continue to do so until a discovery is made.

Lemma 1 helps us to narrow down the possible equilibria to just three simple cases. In Proposition 1, we summarize all the possible equilibria in the single-expert case. To proceed, we define the two cutoffs that are crucial to the equilibrium characterization.

Definition 2 $\overline{c}$ *is defined by*

$$\overline{c} \equiv \lambda \left(\frac{r+\lambda}{r} \right) \int_0^\infty e^{-(r+\lambda)s} \left(\frac{p_0 (1-p_0)}{p_0 e^{-\lambda s} + 1 - p_0} \right) ds. \tag{10}$$

Suppose that the DM expects full effort at all time, we can use (8) to calculate the incentive to exert effort at time 0. By Lemma 1, $B_{i,0} \geq \frac{c_i}{\lambda}$ is necessary and sufficient for exerting full effort at all time t to be optimal. $\overline{c}$ is simply the cost level with which the expert is indifferent between exerting and not exerting effort at time 0 given such an expectation. For any cost level $c_i \leq \overline{c}$, a *full-effort equilibrium* exists.

Definition 3 $\underline{c}$ *is defined by*

$$\underline{c} \equiv \lambda \frac{p_0 (1-p_0)}{r}. \tag{11}$$

Suppose that the DM expects no effort at all time, (1) implies that for all $t \geq 0$, $p_t = p_0$. If the expert deviates from the expectation by acquiring information for a small interval of time dt, the probability that the expert finds out that the state is good is approximately $p_0 \lambda dt$, in that case, the expert's payoff increases from $\frac{p_0}{r}$ to $\frac{1}{r}$. For all cost level $c_i \geq \underline{c}$, such a deviation is not profitable and a *no-effort equilibrium* exists.

In the proof of Proposition 1, we show that $0 < \underline{c} < \overline{c}$. Intuitively, the DM's expectation of full effort provides the expert extra incentive to acquire information. Given the two cutoffs, we have

Proposition 1 *Suppose expert i is the only expert.*

1. *A no-effort equilibrium in which expert i never exerts any effort exists if and only if $c_i \geq \underline{c}$.*
2. *A full-effort equilibrium in which expert i exerts full effort until a discovery is made exists if and only if $c_i \leq \overline{c}$.*
3. *For all $\tau > 0$, there exists an delayed-effort equilibrium in which expert i exerts no effort until τ and then exerts full effort afterward if and only if $c_i = \overline{c}$.*

Moreover, these are all the equilibria.

Figure 1 summarizes the results in Proposition 1.

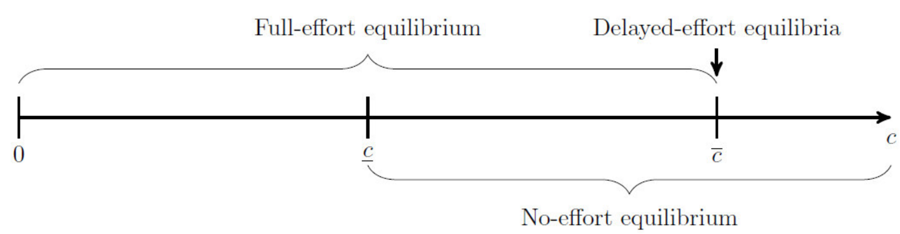

Figure 1: Equilibria in the single-expert case

Proposition 1 illustrates the self-fulfilling property of equilibrium expectations. When the cost level is intermediate, i.e., $\underline{c} \leq c_i < \overline{c}$, two equilibria coexist. In equilibrium, the expert finds exerting full effort optimal if the DM expects full effort and exerting no effort optimal if the DM expects no effort. When the cost is high enough, i.e., $c_i > \overline{c}$, the expert could not benefit from exerting any effort to obtain information and the unique equilibrium involves no effort and an expectation of no effort. The expert gets "paid" according to the prior belief p_0. When the cost is low enough, i.e., $c_i < \underline{c}$, the expert would always deviate to acquire information even if the DM expects no effort. The unique equilibrium thus involves full effort and an expectation of full effort. The expert exerts full effort from time 0, and the DM's belief decreases accordingly over time. Finally, when c_i is exactly equal to $\overline{c}$, another type of equilibrium, which we call *delayed-effort equilibrium*, emerges. In this type of equilibrium, the DM expects the expert to start exerting full effort at some $\tau > 0$ but none before. For such an expectation to be correct, the expert has to be indifferent between exerting or not exerting effort at τ. Otherwise, the continuity of $B_{i,t}$ implies that the expert would either exert effort before τ or delay effort after τ. Moreover, delaying must be beneficial to the expert. Otherwise, the expert would act immediately at time 0. The first condition requires $c_i = \overline{c}$, as the DM's posterior belief at τ is simply the prior. The second condition requires $c_i \geq \underline{c}$, but it is automatically satisfied if $c_i = \overline{c}$. For any $\tau > 0$, there exists a delayed-effort equilibrium with "starting time" τ if $c_i = \overline{c}$. This is again due to the self-fulfilling nature of equilibrium expectations.

4 The Two-expert Case

We now move to the two-expert case. Without loss of generality, we assume $c_1 \leq c_2$. We will focus on a class of equilibria that we call *regular* equilibrium. It requires the more efficient expert to exert weakly more effort at each point in time in equilibrium.

Definition 4 (Regular strategy profile) *Suppose $c_i \leq c_j$, a strategy profile $\{(a_{1,t}, a_{2,t})\}_{t\geq 0}$ is regular if and only if $a_{i,t} \geq a_{j,t}$ for all $t \geq 0$. An equilibrium consisting of regular strategy profile is a regular equilibrium.*

Notice that when the experts' costs are symmetric, i.e., $c_1 = c_2$, regularity implies symmetry, i.e., $a_{1,t} = a_{2,t}$. Given a regular equilibrium candidate $\left\{\left(a^{**}_{1,t}, a^{**}_{2,t}\right)\right\}_{t\geq 0}$, it can be show that

$$\lambda B_{1,t} - c_1 \geq \lambda B_{2,t} - c_2,$$

for all $t \geq 0$. This suggests that a regular equilibrium exists. Intuitively, the expert with a lower cost of effort should exert more effort in equilibrium. The following lemma formalizes this intuition.

Lemma 2 *There exists a regular equilibrium.*[3]

[3] In general, nonregular equilibrium may also exist. This includes, for example, the limit equilibrium of a sequence of regular equilibria, as $c_1 \to c_2$, which is in general not symmetric.

In Propositions 2 and 4, we prove Lemma 2 by explicit constructions.

The aim of this paper is to compare the incentives to acquire information in the single-expert and two-expert cases. However, an expert's strategies in these two situations are two different stochastic processes, which do not have a natural ordering. We formulate a definition of complementarity and substitutability that we believe to be appropriate in our setting. The idea is to compare the effort paths *conditional on the absence of a discovery.*

Definition 5 *The experts' information acquisition decisions are substitutes (complements) if and only if there exists an equilibrium* $\left\{\left(a_{1,t}^{**}, a_{2,t}^{**}\right)\right\}_{t\geq 0}$ *in the two-expert game such that for all equilibria* $\left\{a_{1,t}^{*}\right\}_{t\geq 0}$ *and* $\left\{a_{2,t}^{*}\right\}_{t\geq 0}$ *in the single-expert games, we have,*

$$a_{i,t}^{**} \leq a_{i,t}^{*} \ (a_{i,t}^{**} \geq a_{i,t}^{*}) \tag{12}$$

for all $t \geq 0$ *and* $i \in \{1, 2\}$*. The decisions are* strict *substitutes (complements) if and only if for some* $i \in \{1, 2\}$*, all equilibria* $\left\{a_{i,t}^{*}\right\}_{t\geq 0}$*, (12) also holds strictly for some* $t \geq 0$*.*

To illustrate why this definition of substitutes and complements is useful in our setting, notice that a discovery perfectly reveals the state in this model, therefore, only efforts exerted by the experts when a discovery has not been made is relevant to the DM. Let $\tilde{\theta}_i^*$ and $\tilde{\theta}^{**}$ be random times of discovery *conditional on the fact that the state is good* in the single-expert and two-expert cases, respectively. It is easy to see that,

Lemma 3 *Suppose experts' information acquisition decisions are substitutes (complements), then, there exists an equilibrium* $\left\{\left(a_{1,t}^{**}, a_{2,t}^{**}\right)\right\}_{t\geq 0}$ *in the two-expert game such that for all equilibria* $\left\{a_{1,t}^{*}\right\}_{t\geq 0}$ *and* $\left\{a_{2,t}^{*}\right\}_{t\geq 0}$ *in the single-expert games, the random variable* $\tilde{\theta}^{**}$ *(*$\min\left\{\tilde{\theta}_1^*, \tilde{\theta}_2^*\right\}$*) first-order stochastically dominates the random variable* $\min\left\{\tilde{\theta}_1^*, \tilde{\theta}_2^*\right\}$ *(*$\tilde{\theta}^{**}$*).*

Intuitively, if information acquisition decisions are substitutes, discovery is delayed if the experts "work together," as they reduce their efforts. On the other hand, if information acquisition decisions are complements, increased efforts speed up the discovery in the two-expert game.

4.1 Symmetric Experts

In this subsection, we characterize the set of regular (symmetric) equilibria when the experts are homogeneous, i.e., $c_1 = c_2 = c$, in Proposition 2, and then conclude that the experts' information acquisition decisions are substitutes in Proposition 3. To simplify the notations, we drop all the identity subscripts in this subsection and denote, for example, a symmetric equilibrium simply by $\{a_t^{**}\}_{t\geq 0}$.

Lemma 4 *Suppose $c_1 = c_2 = c$. The effort level in any regular equilibrium is increasing in time, i.e., $a_t^{**} \geq a_s^{**}$, for all $t \geq s \geq 0$.*

To understand Lemma 4, notice that (4) implies that whenever $\{a_t^{**}\}_{t\geq 0}$ varies, it must be at a point when $B_t = \frac{c}{\lambda}$. Consider some open interval I such that for all $t \in I$, $B_t = \frac{c}{\lambda}$. Then, (7) implies that, all $t \in I$,

$$a_t^{**} = \frac{r}{\lambda c}\left(\lambda \frac{p_0\left(1-p_t\right)}{r} - c\right), \tag{13}$$

which is increasing. Thus, $\{a_t^{**}\}_{t\geq 0}$ cannot be strictly decreasing on an interval. In the proof of Lemma 4, we further show that it is impossible to have $\{a_t^{**}\}_{t\geq 0}$ strictly decreasing *at a point*.

To proceed, we define two cost cutoffs as in the single-expert case. The first cut-off $\check{c}$ is the counterpart of $\overline{c}$ in the two-expert case, below which both experts are willing to exert full effort if the DM expects full effort from both experts.

Definition 6 *$\check{c}$ is defined by*

$$\check{c} \equiv \frac{\lambda\left(r+2\lambda\right)}{r+\lambda}\int_0^\infty e^{-(r+2\lambda)s}\left(\frac{p_0\left(1-p_0\right)}{p_0 e^{-2\lambda s}+1-p_0}\right)ds. \tag{14}$$

Similar to the single-expert case, a full-effort equilibrium exists if and only if $c \leq \check{c}$. We show in the proof of Proposition 2 that $\check{c} < \overline{c}$. This is due to the expert's incentive to free ride on the other expert's effort. Notice that when $c \in (\check{c}, \overline{c}]$, full effort equilibrium ceases to exist after a symmetric expert is introduced.

The other cut-off $\hat{c}$ determines the asymptotic effort levels in an equilibrium with strict positive level of effort.

Definition 7 *$\hat{c}$ is defined by*

$$\hat{c} \equiv \frac{\lambda p_0}{r+\lambda}. \tag{15}$$

To understand the meaning of $\hat{c}$, suppose that the DM believes that the state is bad for sure, i.e., $p_t = 0$, and expert j always exerts full effort. If expert i exerts no effort, his payoff is given by

$$\int_0^{\infty} e^{-(r+\lambda)s} \lambda p_0 \left(\frac{1}{r}\right) ds = \frac{\lambda p_0}{r\,(r+\lambda)}. \tag{16}$$

On the other hand, if expert i exerts full effort until a discovery is made, his payoff is given by

$$\int_0^{\infty} e^{-(r+2\lambda)s} \left(2\lambda p_0 \left(\frac{1}{r}\right) - c_i\right) ds = \frac{2\lambda p_0 - r c_i}{r\,(r+2\lambda)}. \tag{17}$$

Comparing (16) and (17), we find that exerting full effort strictly dominates exerting no effort if and only if $c_i < \hat{c}$. Thus, if $c_i < \hat{c}$, given that expert j always exerts full effort, expert i must also exert full effort for t large enough. Thus, the cutoff $\hat{c}$ determines whether full effort can be achieved asymptotically in the two-expert case. We have,

Proposition 2 *Suppose $c_1 = c_2 = c$.*

1. *A no-effort equilibrium in which the experts never exert any effort exists if and only if $c \geq \underline{c}$.*

2. *A full-effort equilibrium in which the experts exert full effort until a discovery is made exists if and only if $c \leq \check{c}$.*

3. *For all $\tau > 0$, there exists a delayed-effort equilibrium in which the experts exert no effort until τ and then exert full effort afterward if and only if $\check{c} \geq \underline{c}$ and $c = \check{c}$.*

4. *There exists a partial-effort equilibrium such that*

$$a_t^{**} = \begin{cases} \frac{r}{\lambda c}\left(\lambda \frac{p_0(1-p_t)}{r} - c\right) > 0 & \text{if } 0 \leq t < t^*, \\ 1 & \text{if } t \geq t^*, \end{cases} \tag{18}$$

for some $t^* \in (0, \infty]$, *if and only if* $\check{c} < c < \underline{c}$. *Moreover, if* $c \geq \hat{c}$, $t^* = \infty$. *If* $c < \hat{c}$, $t^* < \infty$ *satisfies*

$$\frac{\lambda (r + 2\lambda)}{r + \lambda} \int_0^\infty e^{-(r+2\lambda)s} \left(\frac{p_0 (1 - p_{t^*})}{p_{t^*} e^{-2\lambda s} + 1 - p_{t^*}} \right) ds = c. \tag{19}$$

Moreover, these are all the regular equilibria.

Figure 2 summarizes Proposition 2 and illustrates the types of equilibrium depending on whether the incentive to free ride is severe, i.e., $\check{c} < \underline{c}$.

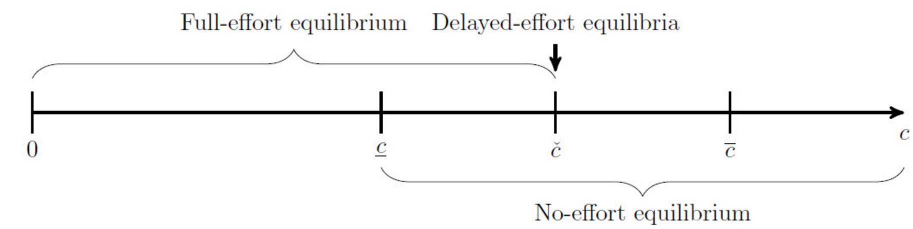

Figure 2(a): Regular equilibria with symmetric experts ($\underline{c} < \check{c}$)

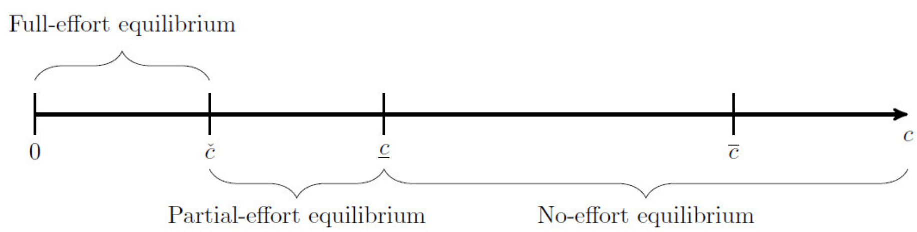

Figure 2(b): Regular equilibria with symmetric experts ($\check{c} < \underline{c}$)

In the situation illustrated by Figure 2(a), the incentive to free ride is not severe, as a result, the intuitions we get from the single-expert case apply and there is still a range of cost under which full-effort and no-effort equilibria coexist. However, when the incentive to free ride is severe, this region may disappear, as illustrated in Figure 2(b). This is because, although the decision maker's full effort expectation provides an additional incentive for the experts to work, in equilibrium, it is dominated by the incentive to free ride on the other expert's effort, as a result, the incentive to acquire information *at time 0* is less in a full-effort equilibrium than a no-effort equilibrium, i.e., $\check{c} < \underline{c}$. In this case, delayed-effort equilibria disappear. This is because, when $c < \underline{c}$, the experts would choose to exert effort even if they are expected to delay. However, another type of equilibrium, which we call *partial-effort equilibrium*, appears. In a partial-effort equilibrium, the experts exert partial effort, i.e., $a_t^{**} \in (0,1)$, in the beginning of the game, but may start to exert full effort at some $t^* < \infty$ if $c < \hat{c}$. Figure 3 provides a numerical illustration of a partial-effort equilibrium.

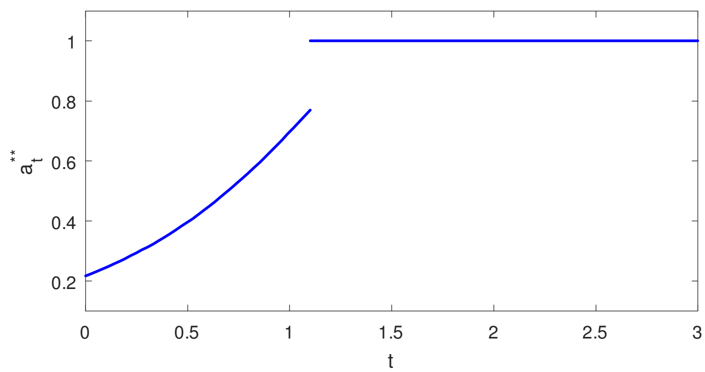

Figure 3: A partial-effort equilibrium.[4]

Parameter values: $p_0 = 0.5$, $\lambda = 1$, $r = 1$, $c_1 = c_2 = 0.2055$.

Given the equilibrium characterization, we have the following result on the substitutability of experts' information acquisition decisions.

Proposition 3 *Suppose $c_1 = c_2 = c$. The experts' information acquisition decisions are substitutes. Moreover, if $c \in (\check{c}, \underline{c})$, the experts' information acquisition decisions are strict substitutes.*

To see the results, we consider two cases. When $c \geq \underline{c}$, by Proposition 2, there exists a no-effort equilibrium in the two-expert case, so efforts are substitutes by definition of substitutes in Definition 5. When $c < \underline{c}$, by Proposition 1, the *unique* equilibrium in the single-expert case involves full effort, so efforts are substitutes by definition as well. Moreover, if $c \in (\check{c}, \underline{c})$, the regular equilibrium in the two-expert case involves partial effort given by (18). Since $a_{i,t}^* > a_t^{**}$ for all $t \in [0, t^*)$, efforts are strict substitutes.

[4] Notice that the equilibrium effort a_t^{**} necessarily jumps upward at t^*. To see that, observe that $\dot{B}_t = 0$ for all $t < t^*$. Moreover, by the logic of Lemma 1, we must have $\dot{B}_{t^*} > 0$. (7) then implies that a_t^{**} must jump at t^*.

With homogeneous experts, free-riding occurs at all cost levels. We obtain a result analogous to the static setting of Kartik, Lee, and Suen (2017). In the next subsection, we look at heterogeneous experts. With heterogeneous agents, the dynamic feature of our model introduces the possibility of complementarity between experts' information acquisition decisions.

4.2 Asymmetric Experts

In this subsection, we characterize the set of regular equilibria when the experts are heterogeneous, i.e., $c_1 < c_2$. We show that the substitutability result in Proposition 3 no longer holds when the costs are asymmetric.

We begin the analysis by observing that, in contrast to the case with symmetric experts, the regularity assumption allows us to focus only on strategies with different "starting times" at which the expert starts to exert full effort.

Lemma 5 *Suppose $c_1 < c_2$. In any regular equilibrium $\left\{\left(a_{1,t}^{**}, a_{2,t}^{**}\right)\right\}_{t\geq 0}$, expert i's effort satisfies*

$$a_{i,t}^{**} = \begin{cases} 0 & \text{if } t < t_i^*, \\ 1 & \text{if } t \geq t_i^*, \end{cases} \tag{20}$$

for some $t_i^* \in [0, \infty]$. Moreover, $t_1^* \leq t_2^*$.

Intuitively, when the costs are heterogeneous, in a regular equilibrium, whenever expert 1 is indifferent between exerting and not exerting effort, expert 2 must strictly prefer to not exert effort. Thus, it is impossible for expert 2 to use an interior effort level to keep expert 1 indifferent for a positive amount of time as in a partial-effort equilibrium. As a result, at each point of time, expert 1 either exerts full effort or no effort. Moreover, in equilibrium, once expert 1 starts to exert effort, he must continue to do so. To see that, suppose expert 1 stops to exert effort at some t after he started, regularity implies that expert 2 must have stopped exerting effort already and expert 1 is, in effect, facing a single-expert problem. Lemma 1 suggests that expert 1 cannot cease to exert effort. Similar reasoning suggests that expert 2's equilibrium strategy must also take the form in (20).

When $c_1 < c_2$, no-effort equilibrium remains the unique equilibrium when both experts' costs are high enough. Likewise, when the costs are low, an equilibrium must involve full effort. However, when the costs are intermediate, heterogeneity in costs leads to heterogeneity in starting times in a regular equilibrium. We have,

Proposition 4 *Suppose $c_1 < c_2$.*

1. *A no-effort equilibrium in which the experts never exert any effort exists if and only if $c_1 \geq \underline{c}$.*

2. *There exists an increasing function $c_f(c_2) : [c_1, \infty) \to [\underline{c}, \overline{c}]$ such that*

   (a) *A regular equilibrium in which expert 1 exerts full effort until a discovery is made exists if and only if $c_1 \leq c_f(c_2)$.*

   (b) *For any $t_1^* > 0$, there exists an equilibrium in which expert 1 exerts no effort until t_1^* and then exerts full effort afterward if and only if $c_1 = c_f(c_2)$.*

   (c) *If $c_2 \geq \hat{c}$, then, $c_f(c_2) = \overline{c}$ and expert 2 never exerts any effort in a regular equilibrium. If $c_2 < \hat{c}$, then, $c_f(c_2) < \overline{c}$, moreover, if expert 1 ever exerts any effort in a regular equilibrium, expert 2 exerts no effort until some $t_2^* < \infty$ and full effort afterward; if expert 1 never exerts any effort, expert 2 also exerts no effort.*

Moreover, these are all the regular equilibria.

Figure 4(a) illustrates expert 1's behavior in regular equilibria. Notice that unlike the symmetric-expert case, expert 1 must exert full effort in a regular equilibrium if $c_1 < \underline{c}$. Figure 4(b) illustrates expert 2's asymptotic effort in regular equilibria in which expert 1 exerts nonzero effort. Like the symmetric-expert case, full effort is achieved if the experts' costs are lower than the cutoff $\hat{c}$.

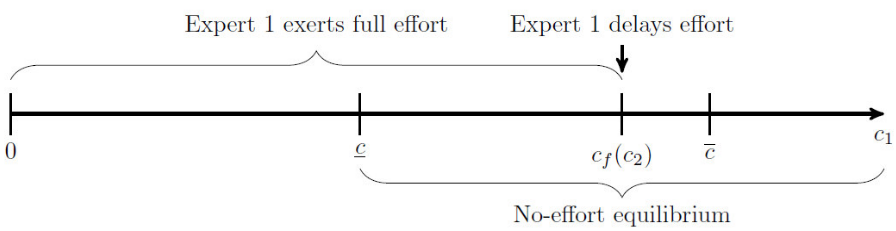

Figure 4(a): Expert 1's effort in regular equilibria ($c_1 < c_2$)

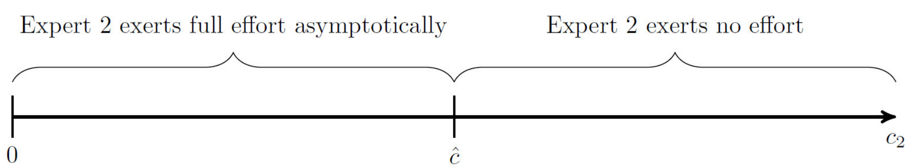

Figure 4(b): Expert 2's effort in regular equilibria in which expert 1 exerts nonzero effort ($c_1 < c_2$)

One of the consequences of Proposition 4 is that a decrease in the less efficient expert's cost may not be beneficial to the DM. This is because, as c_2 decreases, $c_f(c_2)$ decreases. This means that the range of cost, within which an equilibrium in which expert 1 exerts full effort exists, shrinks. For a concrete example, suppose $\overline{c} < \hat{c}$, a case that is satisfied when p_0 is close to 1. Suppose further that $c_1 = \overline{c}$. By Proposition 4, if $c_2 \geq \hat{c}$, then, $c_f(c_2) = \overline{c}$, there exists an equilibrium in which expert 1 exerts full effort and expert 2 exerts no effort. However, if $\overline{c} < c_2 < \hat{c}$, then, $c_f(c_2) < \overline{c}$, the only regular equilibrium is the one in which both experts exert no effort.

Corollary 1 follows directly from Propositions 1 and 4.

Corollary 1 *Suppose c_1 and c_2 satisfy*

$$c_1 < \underline{c} < \overline{c} < c_2 < \hat{c}. \tag{21}$$

Then, the experts' information acquisition decisions are strict complements.

Notice that $\underline{c} < \overline{c}$. Moreover, $\overline{c} < \hat{c}$ if p_0 is close enough to 1. In the situation described by Corollary 1, expert 2 is too inefficient in information acquisition, i.e., $\overline{c} < c_2$, for an equilibrium with positive effort to exist if he is the only expert that the DM consults. In the single-expert equilibrium with only expert 2, the DM's belief stays constant at p_0.

However, if the DM also consults expert 1, who cannot commit to not exerting effort in any equilibrium, i.e., $c_1 < \underline{c}$, then, in the absence of a discovery, the DM's belief decreases and this provides extra incentive for expert 2 to acquire information. Since $c_2 < \hat{c}$, the experts' asymptotic efforts in the two-expert game equal to 1. Thus, in expectation, expert 2 exerts strictly more effort in the two-expert game than in the single-expert game.

It is now clear that the introduction of two heterogeneous experts allows the more efficient expert to provide the less efficient expert with extra incentive to acquire information by driving down the DM's belief over time. Because of the dynamic nature of this extra incentive, it is not surprising that complementarity cannot occur in the static game.

5 Information Withholding Experts

In the previous sections, we assume that information is fully shared among the experts. This greatly simplifies the equilibrium analysis as the private beliefs of the experts stay constant over time. When bad information is not shared, conditional on the absence of a discovery, an expert's belief decreases over time as he attaches an increasing probability to the event that the other expert has made a *bad* discovery. This reduces the experts' incentives to acquire information in the two-expert case. In particular, it is now impossible to have both experts exerting effort indefinitely as this would imply that the private beliefs of the experts approach zero, which in turn destroys the incentives to exerting effort. A complete characterization of the equilibria in this game is more difficult as the equilibrium strategies are in general not monotone. Nevertheless, we show in this section that information acquisition decisions can still be complementary, in spite of the aforementioned negative effect.

Under the new assumption, the DM's (public) belief of the state at time t conditional on no good discovery remains governed by (1). The experts' beliefs, however, become private and are no longer constant over time. Let $q_{i,t}$ denote expert i's private belief of the state at time t conditional on no discovery by expert i and no good discovery by expert j, $q_{i,t}$ evolves according to

$$\dot{q}_{i,t} = -\lambda a^{**}_{j,t} q_{i,t} \left(1 - q_{i,t}\right),$$

where $q_{i,0} = p_0$. Thus, expert i's problem becomes

$$\max_{\{a_{i,t}\}_{t\geq 0}} \int_0^\infty e^{-\int_0^t \left\{r+\lambda\left(a_{i,s}+q_{i,s}a_{j,s}^{**}\right)\right\}ds} \left\{p_t - c_i a_{i,t} + \lambda a_{i,t}\left(\frac{q_{i,t}}{r} + (1-q_{i,t})\,W_t\right) + \lambda a_{j,t}^{**}\frac{q_{i,t}}{r}\right\} dt, \tag{22}$$ [5]

where p_t and W_t remain given by (1) and (2), respectively.

The next lemma shows that in the case with heterogeneous experts, the more efficient expert's equilibrium effort is still specified by a "starting time" in a regular equilibrium. Moreover, even though it is impossible to have both experts exerting positive efforts asymptotically in equilibrium, the more efficient expert would still exert full effort in a regular equilibrium if his cost of effort is smaller than $\underline{c}$, as in the case with information sharing experts.

Lemma 6 *Suppose $c_1 < c_2$. In any regular equilibrium $\left\{\left(a_{1,t}^{**}, a_{2,t}^{**}\right)\right\}_{t\geq 0}$ with information withholding experts, expert 1's effort satisfies*

$$a_{1,t}^{**} = \begin{cases} 0 & \text{if } t < t_1^*, \\ 1 & \text{if } t \geq t_1^*, \end{cases} \tag{23}$$

for some $t_1^* \in [0, \infty]$. Moreover, if $c_1 < \underline{c}$, $t_1^* = 0$.

In Proposition 5, we provide a result analogous to Corollary 1 and demonstrate the key observation of this paper survives alternative information sharing assumption.

Proposition 5 *Suppose c_1 and c_2 satisfy $c_1 < \underline{c} < \bar{c} < c_2$. Then, for any regular equilibrium $\left\{\left(a_{1,t}^{**}, a_{2,t}^{**}\right)\right\}_{t\geq 0}$ with information withholding experts and for all $t \geq 0$,*

$$a_{1,t}^{**} = a_{1,t}^* = 1, \tag{24}$$

$$a_{2,t}^{**} \geq a_{2,t}^* = 0. \tag{25}$$

[5] Notice that conditional on a bad state, expert j's effort does not matter. As a result, expert i's problem is identical to one in which expert j exerts effort $a_{j,t}^{**}$ *until a good discovery is made.*

Moreover, if there exists $t' \geq 0$ that satisfies

$$\int_0^\infty \frac{e^{-(r+\lambda)s-\lambda t'} p_0 (1-p_0)}{\left(p_0 e^{-\lambda t'} + 1 - p_0\right)\left(p_0 e^{-\lambda(s+t')} + 1 - p_0\right)} ds > \frac{c_2}{\lambda}, \tag{26}$$

then, the inequality (25) also holds strictly for some $t \geq 0$.

To see the result, notice that, by Proposition 1, $\{a^*_{1,t}\}_{t\geq 0} = \{1\}_{t\geq 0}$ and $\{a^*_{2,t}\}_{t\geq 0} = \{0\}_{t\geq 0}$. Moreover, by Lemma 6, $c_1 < \underline{c}$ implies that expert 1 must exert full effort in any regular equilibrium. Suppose $\{(1,0)\}_{t\geq 0}$ is an equilibrium, expert 2's instantaneous incentive at time t' can be shown to be

$$B_{2,t'} = \int_0^\infty e^{-(r+\lambda)s} p_{t'} \left(1 - p_{t'+s}\right) ds,$$

which is equivalent to the left hand side of (26). Thus, if (26) is satisfied at some $t' \geq 0$, expert 2 must exert some effort in any regular equilibrium.

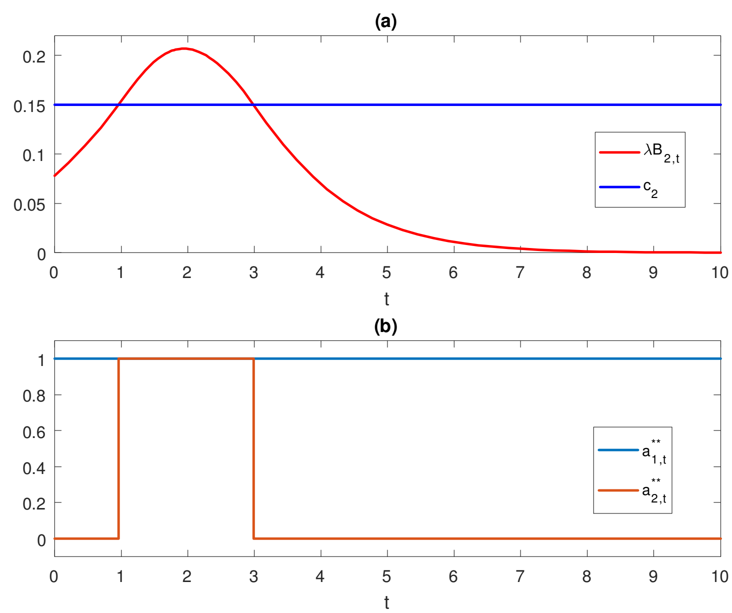

Figure 5. A numerical example illustrating the complementarity in information acquisition with information withholding experts. Parameter values: $p_0 = 0.9$, $\lambda = 1$, $r = 1$, $c_1 < \underline{c} = 0.09$, $c_2 = 0.15$.[6]

[6]It can be numerically verified that $c_2 > \bar{c} = 0.1488$ and that (26) holds for $t' = 2$. Thus, the result in Proposition 5 applies in this case.

Figure 5 illustrates the result in Proposition 5 by means of a numerical example. In this example, $\{a^*_{1,t}\}_{t\geq 0} = \{1\}_{t\geq 0}$ and $\{a^*_{2,t}\}_{t\geq 0} = \{0\}_{t\geq 0}$. However, in the regular equilibrium illustrated in Figure 5(b), expert 1's effort remains at 1 and expert 2 also exerts some effort. Information acquisition decisions are thus complements. Notice that in this example, only expert 1 exerts effort asymptotically and expert 2's instantaneous marginal benefit $\lambda B_{2,t}$ is not increasing with time.

6 Conclusion

In this paper, we study complementarity and substitutability in experts' information acquisition decisions in a dynamic model of persuasion. The experts' ability to hide unfavorable information leads to accumulation of private information over time. This has important implications on the strategic interactions of the experts. We find that, in our model, information acquisition decisions are always substitutes when the experts are homogeneous, but sometimes complements when the experts are heterogeneous.

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7 Appendix

Proof of Lemma 1. We first show that for all $t \geq 0$, $\dot{B}_{i,t} \geq 0$. Suppose not, there exists $t' \geq 0$ such that $\dot{B}_{i,t'} < 0$. Since $a_{j,t}^{**} = \dot{a}_{j,t}^{**} = 0$, (9) implies that $\dot{B}_{i,t}$ is continuous and $\ddot{B}_{i,t}$ always exists. Then, for all $t \geq t'$,

$$\begin{aligned}
\ddot{B}_{i,t} &= -p_0 \lambda a_{i,t}^{*} p_t (1 - p_t) + \left(r + \lambda a_{i,t}^{*}\right) \dot{B}_{i,t} \\
&\leq \left(r + \lambda a_{i,t}^{*}\right) \dot{B}_{i,t} \\
&< 0.
\end{aligned}$$

This implies that as $t \to \infty$, $B_{i,t} \to -\infty$. However, by (8), $B_{i,t}$ is positive. Thus, we must have for all $t \geq 0$, $\dot{B}_{i,t} \geq 0$. Next, suppose $B_{i,t}$ is not strictly increasing over time and let $t' \geq 0$ to be the earliest time such that $\dot{B}_{i,t'} = 0$, we would like to show that for all $t \geq 0$, $a_{i,t}^{*} = 0$. Notice that if $\dot{B}_{i,t} = 0$, then,

$$\ddot{B}_{i,t} = -p_0 \lambda a_{i,t}^{*} p_t (1 - p_t) + \left(r + \lambda a_{i,t}^{*}\right) \dot{B}_{i,t} \leq 0.$$

Since $\dot{B}_{i,t} \geq 0$ for all $t \geq 0$, this implies that $\dot{B}_{i,t} = \ddot{B}_{i,t} = a_{i,t}^{*} = 0$ for all $t \geq t'$. Next, we want to show that $t' = 0$. Suppose not, there exists $t'' \in (0, t')$ such that for all $t \in (t'', t')$, $\ddot{B}_{i,t} \leq 0$. Since $\dot{B}_{i,t} > 0$, (9) implies that $a_{i,t}^{*} > 0$ for all $t \in (t'', t')$. $\dot{B}_{i,t} > 0$ for all $t \in (t'', t')$ also implies that we must have $a_{i,t'}^{*} = 1$. But this is a contradiction as we have shown that $a_{i,t'}^{*} = 0$. As a result, $t' = 0$ and $a_{i,t}^{*} = 0$ for all $t \geq 0$. Finally, if $a_{i,t}^{*} = 0$ for all $t \geq 0$, then by (8), for all $t \geq 0$,

$$B_{i,t} = \frac{p_0 (1 - p_0)}{r},$$

which is a constant. ■

Proof of Proposition 1. We first show that $\underline{c} < \overline{c}$.

$$\begin{aligned} \overline{c} &\equiv \lambda\left(\frac{r+\lambda}{r}\right)\int_0^\infty e^{-(r+\lambda)s}\left(\frac{p_0\,(1-p_0)}{p_0e^{-\lambda s}+1-p_0}\right)ds \\ &> p_0\,(1-p_0)\,\lambda\left(\frac{r+\lambda}{r}\right)\int_0^\infty e^{-(r+\lambda)s}ds \\ &= \lambda\frac{p_0\,(1-p_0)}{r} \\ &\equiv \underline{c}. \end{aligned}$$

Next, by Lemma 1, either we have for all $t \geq 0$, $a^*_{i,t} = 0$ or $B_{i,t}$ is strictly increasing. In the first case, we have, for all $t \geq 0$,

$$B_{i,t} = \frac{p_0\,(1-p_0)}{r} \leq \frac{c_i}{\lambda},$$

as $a^*_{i,t} = 0$. Clearly, $c_i \geq \underline{c}$ is also sufficient for the existence of no-effort equilibrium.

In the second case, effort is nonzero and $\lambda B_{i,t}$ either crosses c at one point or never does. Thus, (4) implies there exists a $\tau < \infty$ such that

$$a^*_{i,t} = \begin{cases} 0 & \text{if } t < \tau, \\ 1 & \text{if } t \geq \tau. \end{cases}$$

We must have

$$B_{i,\tau} = \int_\tau^\infty e^{-(r+\lambda)(s-\tau)}\left(\frac{p_0\,(1-p_0)}{p_0e^{-\lambda(s-\tau)}+1-p_0}+c_i\right)ds \geq \frac{c_i}{\lambda}, \tag{27}$$

which is equivalent to $c_i \leq \overline{c}$. Moreover, if $\tau > 0$, (27) must also hold with equality. Moreover, since

$$B_{i,t} = \begin{cases} \frac{p_0(1-p_0)}{r}\left(1-e^{-r(\tau-t)}\right)+e^{-r(\tau-t)}\frac{c_i}{\lambda} & \text{if } t < \tau, \\ \int_t^\infty e^{-(r+\lambda)(s-t)}\left(\frac{p_0(1-p_0)}{p_0e^{-\lambda(s-\tau)}+1-p_0}+c_i\right)ds & \text{if } t \geq \tau, \end{cases}$$

is strictly increasing, these conditions are also sufficient for the existence of the corresponding types of equilibrium. ■

Proof of Lemma 3. The result follows directly from the fact that for all $t \geq 0$, $\Pr\left(\tilde{\theta}^{**} > t\right) = e^{-\lambda \int_0^t \left(a_{1,s}^{**} + a_{2,s}^{**}\right) ds}$ and $\Pr\left(\min\left\{\tilde{\theta}_1^{*}, \tilde{\theta}_2^{*}\right\} > t\right) = e^{-\lambda \int_0^t \left(a_{1,s}^{*} + a_{2,s}^{*}\right) ds}$. ■

Proof of Lemma 4. We have shown in the main text that $\{a_t^{**}\}_{t\geq 0}$ cannot be strictly decreasing on an interval. We now show that $\{a_t^{**}\}_{t\geq 0}$ cannot be strictly decreasing at a point. Suppose $\{a_t^{**}\}_{t\geq 0}$ is strictly decreasing at some $t' > 0$. We must have $B_{t'} = \frac{c}{\lambda}$, $\lim_{t\uparrow t'} \dot{B}_t \leq 0$ and $\dot{B}_{t'} \leq 0$. Suppose $\dot{B}_{t'} = 0$, then (7) implies

$$\lim_{t\uparrow t'} \dot{B}_t = \dot{B}_{t'} + \left(\lim_{t\uparrow t'} a_t^{**} - a_{t'}^{**}\right) c > 0,$$

which is a contradiction. Next, suppose $\dot{B}_{t'} < 0$, then, $a_{t'}^{**} = 0$. Thus, for all $t > t'$,

$$\begin{aligned} \dot{B}_t &< 0, \\ \ddot{B}_t &= r\dot{B}_t < 0, \end{aligned}$$

as $a_t^{**} = 0$. As a result, $B_t \to -\infty$ as $t \to \infty$, which is impossible. Thus, $\{a_t^{**}\}_{t\geq 0}$ cannot be strictly decreasing at t'. ■

Proof of Proposition 2. The proof consists of three parts. First, we show that $\check{c} < \overline{c}$, $\check{c} < \hat{c}$ and that both $\underline{c} < \check{c}$ and $\underline{c} > \check{c}$ are possible under different parametric assumptions. Second, we check that the strategies listed in Proposition 2 are equilibria given the stated conditions and show that they can only be supported as equilibria under these conditions. Third, we show that the equilibria listed in Proposition 2 exhaust all the regular equilibria.

1. To show that $\check{c} < \overline{c}$,

$$\begin{aligned} \check{c} &\equiv \lambda \left(\frac{r+2\lambda}{r+\lambda}\right) \int_0^\infty e^{-(r+2\lambda)s} \left(\frac{p_0(1-p_0)}{p_0 e^{-2\lambda s} + 1 - p_0}\right) ds \\ &< \lambda \left(\frac{r+\lambda}{r}\right) \int_0^\infty e^{-(r+2\lambda)s} \left(\frac{p_0(1-p_0)}{p_0 e^{-2\lambda s} + 1 - p_0}\right) ds \\ &< \lambda \left(\frac{r+\lambda}{r}\right) \int_0^\infty e^{-(r+\lambda)s} \left(\frac{p_0(1-p_0)}{p_0 e^{-\lambda s} + 1 - p_0}\right) ds \\ &\equiv \overline{c}. \end{aligned}$$

To show that $\check{c} < \hat{c}$,

$$\begin{aligned}
\check{c} &\equiv \frac{\lambda(r+2\lambda)}{r+\lambda}\int_0^\infty e^{-(r+2\lambda)s}\left(\frac{p_0(1-p_0)}{p_0e^{-2\lambda s}+1-p_0}\right)ds \\
&< p_0\frac{\lambda(r+2\lambda)}{r+\lambda}\int_0^\infty e^{-(r+2\lambda)s}ds \\
&= \frac{\lambda p_0}{r+\lambda} \\
&\equiv \hat{c}.
\end{aligned}$$

To show that it is possible to have $\underline{c} < \check{c}$, consider p_0 large but not equal to 1.

$$\begin{aligned}
\lim_{p_0\to 1}\frac{\check{c}}{\underline{c}} &= \lim_{p_0\to 1}\frac{\frac{\lambda(r+2\lambda)}{r+\lambda}\int_0^\infty e^{-(r+2\lambda)s}\left(\frac{p_0(1-p_0)}{p_0e^{-2\lambda s}+1-p_0}\right)ds}{\lambda\frac{p_0(1-p_0)}{r}} \\
&= \frac{r(r+2\lambda)}{r+\lambda}\int_0^\infty e^{-rs}ds \\
&= \frac{r+2\lambda}{r+\lambda} \\
&> 1.
\end{aligned}$$

To show that it is possible to have $\underline{c} > \check{c}$, consider p_0 small but not equal to 0.

$$\begin{aligned}
\lim_{p_0\to 0}\frac{\check{c}}{\underline{c}} &= \lim_{p_0\to 0}\frac{r(r+2\lambda)}{r+\lambda}\int_0^\infty e^{-(r+2\lambda)s}\left(\frac{1}{p_0e^{-2\lambda s}+1-p_0}\right)ds \\
&= \frac{r(r+2\lambda)}{r+\lambda}\int_0^\infty e^{-(r+2\lambda)s}ds \\
&= \frac{r}{r+\lambda} \\
&< 1.
\end{aligned}$$

2. Case 1. Given the strategy $\{a_t^{**}\}_{t\geq 0} = \{0\}_{t\geq 0}$, we have, for all $t \geq 0$,

$$B_t = \frac{p_0(1-p_0)}{r},$$

as in the single-expert case. Thus, a no-effort equilibrium exists if and only if $c \geq \underline{c}$.

Case 2. Given the strategy $\{a_t^{**}\}_{t\geq 0} = \{1\}_{t\geq 0}$, we must have $\dot{B}_t > 0$ for all $t \geq 0$.

Otherwise, suppose $\dot{B}_\tau \le 0$ for some $\tau \ge 0$. Then, for all $t > \tau$,

$$\begin{aligned} \dot{B}_t &< 0, \\ \ddot{B}_t &= -2p_0\lambda p_t (1 - p_t) + (r + 2\lambda) \dot{B}_t < 0. \end{aligned}$$

As a result, $B_t \to -\infty$ as $t \to \infty$, which is impossible. Thus, a full-effort equilibrium exists if and only if $B_0 \ge \frac{c}{\lambda}$, which is equivalent to $c \le \check{c}$.

Case 3. For all $\tau > 0$, consider the strategy

$$a_t^{**} = \begin{cases} 0 & \text{if } t < \tau, \\ 1 & \text{if } t \ge \tau. \end{cases}$$

Continuity of B_t implies that we must have $\lambda B_\tau = c$ for $\{a_t^{**}\}_{t \ge 0}$ to be an equilibrium. Since $a_t^{**} = 0$ for all $t < \tau$, $p_\tau = p_0$. We must have $c = \check{c}$. Moreover,

$$\lim_{t \uparrow \tau} \dot{B}_t = -p_0 (1 - p_0) + \frac{rc}{\lambda},$$

which is nonnegative if and only if $c \ge \underline{c}$. Thus, we also must have $c \ge \underline{c}$ for $\{a_t^{**}\}_{t \ge 0}$ to be an equilibrium. For sufficiency, consider $c = \check{c} \ge \underline{c}$. By Case 2, $c = \check{c}$ implies that $a_t^{**} = 1$ is optimal for all $t \ge \tau$. For $t < \tau$, we have, $\lim_{t \uparrow \tau} \dot{B}_t \ge 0$ and

$$\ddot{B}_t = r\dot{B}_t$$

for all $t < \tau$. This means that $\dot{B}_t$ is nonnegative for all $t < \tau$. Thus, for all $t < \tau$,

$$B_t \le B_\tau = \frac{c}{\lambda}.$$

$a_t^{**} = 0$ is optimal for all $t < \tau$.

Case 4. Given $t^* > 0$, consider the strategy

$$a_t^{**} = \begin{cases} \frac{r}{\lambda c}\left(\lambda \frac{p_0(1-p_t)}{r} - c\right) > 0 & \text{if } t < t^*, \\ 1 & \text{if } t \geq t^*. \end{cases}$$

By the same argument in Case 2, we must have $\dot{B}_t > 0$ for all $t \geq t^*$, if $B_{t^*} = \frac{c}{\lambda}$. Moreover, by construction, $\dot{B}_t = 0$ for all $t < t^*$. Thus, $\{a_t^{**}\}_{t\geq 0}$ is an equilibrium if and only if

$$a_0^{**} > 0, \tag{28}$$

$$a_{t^*}^{**} < 1, \tag{29}$$

and

$$B_{t^*} = \int_0^\infty e^{-(r+2\lambda)s}\left(\frac{p_0(1-p_{t^*})}{p_{t^*}e^{-2\lambda s} + 1 - p_{t^*}} + c\right) ds = \frac{c}{\lambda}, \text{ if } t^* < \infty. \tag{30}$$

We would like to check that the conditions in Proposition 2 are necessary and sufficient for (28)–(30). Notice first that (28) is equivalent to $c < \underline{c}$.

Next, suppose $t^* = \infty$, then,

$$\lim_{t\to\infty} a_t^{**} = \frac{p_0}{c} - \frac{r}{\lambda} \leq 1$$

if and only if $c \geq \hat{c}$.

Suppose $t^* < \infty$, (30) is equivalent to (19), which has a unique solution $p_{t^*} \in (0, p_0)$ if and only if $\check{c} < c < \hat{c}$. Moreover, (29) is implied by (19), as

$$\begin{aligned} a_{t^*}^{**} &= \frac{r}{c}\left(\frac{p_0(1-p_{t^*})}{r} - \frac{c}{\lambda}\right) \\ &< \frac{r}{c}\left(\frac{r+2\lambda}{r}\int_0^\infty e^{-(r+2\lambda)s}\left(\frac{p_0(1-p_{t^*})}{p_{t^*}e^{-2\lambda s}+1-p_{t^*}}\right)ds - \frac{c}{\lambda}\right) \\ &= 1, \end{aligned}$$

where the last equality follows from (19).

3. By Lemma 4, a symmetric equilibrium strategy must be increasing. Suppose the experts use an increasing strategy $\{a_t^{**}\}_{t\geq 0}$. Let $\tau = \inf\{t \in \mathbb{R}_+ : a_t^{**} > 0\}$ and $t^* = \inf\{t \in \mathbb{R}_+ : a_t^{**} = 1\}$, where $\inf \phi \equiv \infty$. Since a_t^{**} satisfies (13) whenever $a_t^{**} \notin \{0,1\}$. In equilibrium, we must have

$$a_t^{**} = \begin{cases} 0 & \text{if } t < \tau, \\ \frac{r}{\lambda c}\left(\lambda \frac{p_0(1-p_t)}{r} - c\right) > 0 & \text{if } \tau < t < t^*, \\ 1 & \text{if } t \geq t^*. \end{cases}$$

for some $\tau \leq t^*$. We claim that either $\tau = 0$ or $\tau = t^*$. Suppose $\tau < t^*$, then,

$$a_\tau^{**} = \frac{r}{\lambda c}\left(\lambda \frac{p_0\left(1-p_0\right)}{r} - c\right) > 0,$$

$\tau > 0$ implies that

$$\lim_{t\uparrow\tau} \dot{B}_t = -p_0\left(1-p_0\right) + r\frac{c}{\lambda} < 0,$$

as $B_\tau = \frac{c}{\lambda}$. This means that the experts have incentives to exert efforts before τ, which is a contradiction.

■

Proof of Proposition 3. In text. ■

Proof of Lemma 5. Suppose $c_1 < c_2$ and the equilibrium $\left\{\left(a_{1,t}^{**}, a_{2,t}^{**}\right)\right\}_{t\geq 0}$ is regular. First, we would like to show that $\lambda B_{1,t} - c_1 > \lambda B_{2,t} - c_2$.

$$\begin{aligned}
& \lambda B_{1,t} - c_1 \\
= \; & \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+a_{2,z}^{**}\right)\right)dz} \left(p_0\left(1-p_s\right) + c_1 a_{1,s}^{**}\right) ds - c_1 \\
\geq \; & \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+a_{2,z}^{**}\right)\right)dz} \left(p_0\left(1-p_s\right) + c_1 a_{2,s}^{**}\right) ds - c_1 \\
= \; & \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+a_{2,z}^{**}\right)\right)dz} p_0\left(1-p_s\right) ds - c_1\left(1 - \lambda\int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+a_{2,z}^{**}\right)\right)dz} a_{2,s}^{**} ds\right) \\
> \; & \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+a_{2,z}^{**}\right)\right)dz} p_0\left(1-p_s\right) ds - c_2\left(1 - \lambda\int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+a_{2,z}^{**}\right)\right)dz} a_{2,s}^{**} ds\right) \\
= \; & \lambda B_{2,t} - c_2.
\end{aligned}$$

The last inequality follows immediately if $a_{2,s}^{**} = 0$ for all $s \in (t, \infty)$. Otherwise,

$$\begin{aligned} & \lambda \int_t^\infty e^{-\int_t^s \left(r + \lambda \left(a_{1,z}^{**} + a_{2,z}^{**}\right)\right) dz} a_{2,s}^{**} ds \\ < & \lambda \int_t^\infty e^{-\int_t^s \lambda a_{2,z}^{**} dz} a_{2,s}^{**} ds \\ = & -\int_t^\infty d\left(e^{-\int_t^s \lambda a_{2,z}^{**} dz}\right) \\ = & 1 - e^{-\int_t^\infty \lambda a_{2,z}^{**} dz} \\ \leq & 1. \end{aligned}$$

Next, suppose there exists some open interval I such that $\lambda B_{1,t} - c_1 = 0$ for all $t \in I$, then, $\lambda B_{2,t} - c_2 < 0$ and $a_{2,t}^{**} = 0$ for all $t \in I$. $\dot{B}_{1,t} = 0$ implies that for all $t \in I$,

$$\frac{\lambda p_0 (1 - p_t)}{r} = c_1.$$

Thus, $a_{1,t}^{**} = 0$ for all $t \in I$. Next, suppose $\lambda B_{1,\tau} - c_1 = 0$ for some $\tau \in (0, \infty)$, we would like to show that it is impossible to have $a_{1,\tau}^{**} = 0$ but $\lim_{t \uparrow \tau} a_{1,\tau}^{**} > 0$. Since $\lambda B_{2,\tau} - c_2 < 0$, $\dot{a}_{2,\tau}^{**} = 0$. We must have $\lim_{t \uparrow \tau} \dot{B}_{1,t} = \dot{B}_{1,\tau} < 0$. This implies that for all $t \geq \tau$,

$$\begin{aligned} \dot{B}_{1,t} &< 0, \\ \ddot{B}_{1,t} &= r\dot{B}_{1,t} < 0, \end{aligned}$$

as $a_{1,t}^{**} = a_{2,t}^{**} = 0$. As a result, $\lim_{t \to \infty} B_{1,t} = -\infty$, which is a contradiction. Thus, expert 1 must use a strategy given by (20).

A similar argument establishes that expert 2 must also use a strategy given by (20). Since $\lambda B_{1,t} - c_1 > \lambda B_{2,t} - c_2$, expert 2 must not exert any effort for $t < t_1^*$. ■

Proof of Proposition 4. Suppose $c_1 < c_2$. Lemma 5 implies that any regular equilibrium must take the form of one of the listed equilibria. We would like to show that the stated conditions are necessary and sufficient for the corresponding types of equilibria.

1. Since $c_1 < c_2$, Proposition 1 implies that a no-effort equilibrium exists if and only if $c_1 \geq \underline{c}$.

2. Given c_2, let

$$\breve{t}_2^*(c_2) \in \inf\left\{t \in \mathbb{R}_+ : \int_0^\infty e^{-(r+2\lambda)s}\left(\frac{p_0(1-p_0)}{p_0 e^{-\lambda t - 2\lambda s} + 1 - p_0} + c_2\right) ds \geq \frac{c_2}{\lambda}\right\},$$

where $\inf \phi = \infty$. That is, $\breve{t}_2^*(c_2)$ is the earliest time when expert 2's marginal benefit exceeds marginal cost provided that expert 1 always exerts effort. Let $\breve{c}_f(c_2)$ be the unique root of the function H defined by

$$H(c) \equiv \int_0^\infty e^{-(r+\lambda)s - \lambda\left(s - \breve{t}_2^*(c_2)\right)^+}\left(\frac{p_0(1-p_0)}{p_0 e^{-\lambda\left(s + \left(s-\breve{t}_2^*(c_2)\right)^+\right)} + 1 - p_0} + c\right) ds - \frac{c}{\lambda}.$$

Given c_1, define the function $c_f : [c_1, \infty) \to \mathbb{R}$ by

$$c_f(c_2) \equiv \max\left\{\underline{c}, \breve{c}_f(c_2)\right\}.$$

Since $H(c)$ is decreasing in c and increasing in c_2, $c_f(c_2)$ is increasing. To show that $c_f(c_2) \leq \overline{c}$, consider

$$\begin{aligned} H(\overline{c}) &= \int_0^\infty e^{-(r+\lambda)s - \lambda\left(s - \breve{t}_2^*(c_2)\right)^+}\left(\frac{p_0(1-p_0)}{p_0 e^{-\lambda\left(s + \left(s-\breve{t}_2^*(c_2)\right)^+\right)} + 1 - p_0} + \overline{c}\right) ds - \frac{\overline{c}}{\lambda} \\ &\leq \int_0^\infty e^{-(r+\lambda)s}\left(\frac{p_0(1-p_0)}{p_0 e^{-\lambda s} + 1 - p_0} + \overline{c}\right) ds - \frac{\overline{c}}{\lambda} \\ &= 0. \end{aligned}$$

Thus, $\breve{c}_f(c_2) \leq \overline{c}$. Since $\underline{c} < \overline{c}$, $c_f(c_2) \leq \overline{c}$.

(a) Since expert 2 would start exerting effort at $\breve{t}_2^*(c_2)$ if expert 1 always exerts effort (see 2.(c) of the current proof), a regular equilibrium with $t_1^* = 0$ exists *only if* $c_1 \leq \breve{c}_f(c_2)$. This is because $\lambda B_{1,t} - c_1$ is decreasing with c_1 and expert 1 with cost $\breve{c}_f(c_2)$ is exactly indifferent between exerting and not exerting effort at time 0. Next, we would like to show that $c_1 \leq \breve{c}_f(c_2)$ if and only if $c_1 \leq c_f(c_2)$, provided that $c_1 < c_2$. Suppose $\breve{c}_f(c_2) \neq c_f(c_2)$, then $\breve{c}_f(c_2) < \underline{c}$, we would like to show that we must have, in this case, $c_2 < \breve{c}_f(c_2)$, so that $c_1 < c_2 < \breve{c}_f(c_2) <$

$c_f(c_2)$. First, notice that, in this case, we must have $\breve{t}_2^*(c_2) < \infty$. Otherwise, $\breve{c}_f(c_2) = \bar{c} > \underline{c}$.

Next, by definition,

$$\lambda \int_0^\infty e^{-(r+\lambda)s-\lambda\left(s-\breve{t}_2^*(c_2)\right)^+}\left(p_0 \frac{1-p_0}{p_0 e^{-\lambda\left(s+\left(s-\breve{t}_2^*(c_2)\right)^+\right)}+1-p_0}+\breve{c}_f(c_2)\right) ds - \breve{c}_f(c_2) = 0.$$

Divide the integral into two, we have

$$\begin{aligned} & \lambda \int_0^{\breve{t}_2^*(c_2)} e^{-(r+\lambda)s}\left(\frac{p_0(1-p_0)}{p_0 e^{-\lambda s}+1-p_0}+\breve{c}_f(c_2)\right) ds + e^{-(r+\lambda)\breve{t}_2^*(c_2)} \times \\ & \lambda \int_{\breve{t}_2^*(c_2)}^\infty e^{-(r+2\lambda)\left(s-\breve{t}_2^*(c_2)\right)}\left(\frac{p_0(1-p_0)}{p_0 e^{-\lambda\left(2s-\breve{t}_2^*(c_2)\right)}+1-p_0}+\breve{c}_f(c_2)\right) ds - \breve{c}_f(c_2) \\ = & \ 0. \end{aligned}$$

Replace the integrand inside the first integral by the low bound $p_0(1-p_0) + \breve{c}_f(c_2)$, we have

$$\begin{aligned} & \frac{\lambda}{r+\lambda}\left(p_0(1-p_0)+\breve{c}_f(c_2)\right)\left(1-e^{-(r+\lambda)\breve{t}_2^*(c_2)}\right) + e^{-(r+\lambda)\breve{t}_2^*(c_2)} \times \\ & \lambda \int_{\breve{t}_2^*(c_2)}^\infty e^{-(r+2\lambda)\left(s-\breve{t}_2^*(c_2)\right)}\left(\frac{p_0(1-p_0)}{p_0 e^{-\lambda\left(2s-\breve{t}_2^*(c_2)\right)}+1-p_0}+\breve{c}_f(c_2)\right) ds - \breve{c}_f(c_2) \\ < & \ 0. \end{aligned}$$

Rearranging terms, we have

$$\begin{aligned} & \frac{r}{r+\lambda}\left(1-e^{-(r+\lambda)\breve{t}_2^*(c_2)}\right)\left(\frac{\lambda p_0(1-p_0)}{r}-\breve{c}_f(c_2)\right) + e^{-(r+\lambda)\breve{t}_2^*(c_2)} \times \\ & \left(\lambda \int_{\breve{t}_2^*(c_2)}^\infty e^{-(r+2\lambda)\left(s-\breve{t}_2^*(c_2)\right)}\left(\frac{p_0(1-p_0)}{p_0 e^{-\lambda\left(2s-\breve{t}_2^*(c_2)\right)}+1-p_0}+\breve{c}_f(c_2)\right) ds - \breve{c}_f(c_2)\right) \\ < & \ 0. \end{aligned}$$

$\breve{c}_f(c_2) < \underline{c}$ means that the first term is strictly positive, thus,

$$\lambda \int_{\breve{t}_2^*(c_2)}^\infty e^{-(r+2\lambda)\left(s-\breve{t}_2^*(c_2)\right)}\left(\frac{p_0(1-p_0)}{p_0 e^{-\lambda\left(2s-\breve{t}_2^*(c_2)\right)}+1-p_0}+\breve{c}_f(c_2)\right) ds - \breve{c}_f(c_2) < 0,$$

which means that $c_2 < \breve{c}_f(c_2)$.

Next, we would like to show that $c_1 \le c_f(c_2)$ is also sufficient. As shown, $c_1 \le c_f(c_2)$ if and only if $c_1 \le \breve{c}_f(c_2)$. Suppose expert 1 and 2 start exerting full effort at $t_1^* = 0$ and $t_2^* = \breve{t}_2^*(c_2)$, respectively, then $c_1 \le \breve{c}_f(c_2)$ implies that $\lambda B_{1,0} - c_1 \ge 0$. Moreover, for all $t \ge t_2^*$, $\lambda B_{1,t} - c_1 > \lambda B_{2,t} - c_2 \ge 0$. For all $t < t_2^*$, (9) implies that

$$\ddot{B}_{1,t} = -p_0 \lambda p_t (1 - p_t) + (r + \lambda) \dot{B}_{1,t}.$$

Thus, if $\lambda B_{1,t} - c_1$ meets 0 at any $t' \in [0, t_2^*)$, we must have $\dot{B}_{1,t'} > 0$. Otherwise, $\dot{B}_{1,t} < 0$ and $\ddot{B}_{1,t} < 0$ for all $t \in (t', t_2^*)$, contradicting $\lambda B_{1,t_2^*} - c_1 > 0$. As a result, for all $t \ge 0$, $\lambda B_{1,t} - c_1 \ge 0$. It is optimal for expert 1 to exert full effort.

(b) In order to have an equilibrium with $t_1^* \in (0, \infty)$ as expert 1's starting time, we must have $\lambda B_{1,t_1^*} = c_1$. Since $a_{1,t}^{**} = a_{2,t}^{**} = 0$ for all $t < t_1^*$, $p_{t_1^*} = p_0$. We must have $c_1 = \breve{c}_f(c_2)$. Moreover,

$$\dot{B}_{1,t_1^*} = -p_0 (1 - p_0) + \frac{r c_1}{\lambda},$$

which is nonnegative if and only if $c_1 \ge \underline{c}$. Thus, we must have $c_1 = c_f(c_2)$.

Conversely, if $c_1 = c_f(c_2)$, for any $t_1^* \in (0, \infty)$, we will show that there is an equilibrium with t_1^* as expert 1's starting time. Suppose expert 2 uses the starting time $t_1^* + \breve{t}_2^*(c_2)$, which will be shown to be optimal in 2.(c) of the current proof. By the same analysis as in 2.(a) in the current proof, exerting full effort for $t \ge t_1^*$ is optimal for expert 1. Finally, for all $t < t_1^*$,

$$\dot{B}_{1,t} = e^{-r\left(t_1^* - t\right)} \dot{B}_{1,t_1^*} \ge 0.$$

Since $B_{1,t_1^*} = \frac{c_1}{\lambda}$, exerting no effort for $t < t_1^*$ is optimal for expert 1.

(c) Notice that $\breve{t}_2^*(c_2)$ is increasing in c_2 and is finite if and only if $c_2 < \hat{c}$.

If $c_2 \ge \hat{c}$, $\breve{t}_2^*(c_2) = \infty$, $H(\overline{c}) = 0$. As $\underline{c} < \overline{c}$, $c_f(c_2) = \overline{c}$. We first show that expert 2 never exerts any effort in a regular equilibrium. By Lemma 5, we can focus on strategies with different starting times. If $t_1^* = \infty$, Lemma 5 implies that we must also have $t_2^* = \infty$. Suppose $t_1^* < \infty$, then, $c_2 \geq \hat{c}$ implies that for all $t > t_1^*$,

$$\int_0^\infty e^{-(r+2\lambda)s}\left(\frac{p_0\left(1-p_0\right)}{p_0 e^{-\lambda\left(t-t_1^*\right)-2\lambda s}+1-p_0}+c_2\right)ds < \frac{c_2}{\lambda}.$$

Thus, the only plausible equilibrium starting time is $t_2^* = \infty$. Next, we show that never exerting any effort is expert 2's best response. Suppose expert 1 starts exerting full effort at some $t_1^* \geq 0$ in a regular equilibrium, then, for all $t \leq t_1^*$, it is optimal for expert 2 to exert no effort. Moreover, $c_2 \geq \hat{c}$ implies that for $t > t_1^*$,

$$\begin{aligned} B_{2,t} &= \int_t^\infty e^{-(r+\lambda)(s-t)}\left(\frac{p_0\left(1-p_0\right)}{p_0 e^{-\lambda\left(s-t_1^*\right)}+1-p_0}\right)ds \\ &< p_0\int_0^\infty e^{-(r+\lambda)s}ds \\ &\leq \frac{p_0}{r+\lambda} \\ &\leq \frac{c_2}{\lambda}. \end{aligned}$$

Thus, it is optimal for expert 2 to exert no effort for all $t \geq 0$.

Next, consider $c_2 < \hat{c}$. In this case, $H\left(\bar{c}\right) < 0$ and $c_f\left(c_2\right) < \bar{c}$. Suppose expert 1 exerts full effort after $t_1^* < \infty$ and expert 2 exerts full effort after $t_2^* = t_1^* + \check{t}_2^*\left(c_2\right) < \infty$. We would like to verify that expert 2 is playing a best response. By the choice of t_2^*, $B_{2,t_2^*} = \frac{c_2}{\lambda}$. Moreover, for all $t > t_2^*$,

$$\begin{aligned} & B_{2,t} \\ =& \int_0^\infty e^{-(r+2\lambda)s}\left(\frac{p_0\left(1-p_0\right)}{p_0 e^{-\lambda\left(t-t_1^*\right)-\lambda\left(t-t_2^*\right)-2\lambda s}+1-p_0}+c_2\right)ds \\ >& \int_0^\infty e^{-(r+2\lambda)s}\left(\frac{p_0\left(1-p_0\right)}{p_0 e^{-\lambda\check{t}_2^*\left(c_2\right)-2\lambda s}+1-p_0}+c_2\right)ds \\ =& \frac{c_2}{\lambda}. \end{aligned}$$

Thus, it is optimal for expert 2 to exert full effort for $t \geq t_2^*$. Since $\lambda B_{1,t} - c_1 > \lambda B_{2,t} - c_2$ in a regular equilibrium, it is also optimal for expert 2 to exert no effort for $t \leq t_1^*$. Next, suppose that it is not optimal for expert 2 to exert no effort for some $t_1^* < t < t_2^*$. We must have $\dot{B}_{2,t'} < 0$ for some $t' < t_2^*$, otherwise, for all $B_{2,t} \leq B_{2,t_2^*} = \frac{c_2}{\lambda}$. But, since

$$\ddot{B}_{2,t} = \begin{cases} -p_0 \lambda p_t \left(1 - p_t\right) + \left(r + \lambda\right) \dot{B}_{2,t} & \text{if } t_1^* < t < t_2^*, \\ -2p_0 \lambda p_t \left(1 - p_t\right) + \left(r + 2\lambda\right) \dot{B}_{2,t} & \text{if } t_2^* \leq t. \end{cases}$$

$\dot{B}_{2,t'} < 0$ implies that for all $t \geq t'$, $\dot{B}_{2,t} < 0$ and $\lim_{t \to \infty} B_{2,t} = -\infty$, which is a contradiction. Thus, the starting time t_2^* is optimal for expert 2.

■

Proof of Corollary 1. By (10), (11) and (15), as $p_0 \to 1$, $\overline{c}, \underline{c} \to 0$, but $\hat{c} \to \frac{\lambda}{r+\lambda}$. Thus, there exist parameters p_0, c_1, and c_2 such that (21) is satisfied. By Proposition 1, the single-expert cases have unique equilibrium, $\left\{a_{1,t}^*\right\}_{t \geq 0} = \{1\}_{t \geq 0}$ and $\left\{a_{2,t}^*\right\}_{t \geq 0} = \{0\}_{t \geq 0}$. By Proposition 4, there exists an equilibrium in the two-expert game such that $\left\{a_{1,t}^{**}\right\}_{t \geq 0} = \{1\}_{t \geq 0}$ and

$$a_{2,t}^{**} = \begin{cases} 0 & \text{if } t < t_2^*, \\ 1 & \text{if } t \geq t_2^*. \end{cases}$$

for some $t_2^* \in [0, \infty)$. Since $a_{2,t}^{**} > a_{2,t}^*$ for all $t \geq t_2^*$, the experts' information acquisition decisions are strictly complementary. ■

Preliminaries for information withholding experts. The current-value Hamiltonian for expert i's problem is

$$H_{i,t} = x_{i,t} \left\{ p_t - c_i a_{i,t} + \lambda a_{i,t} \left(\frac{q_{i,t}}{r} + (1 - q_{i,t}) W_t \right) + \lambda a^{**}_{j,t} \left(\frac{q_{i,t}}{r} \right) \right\} - \gamma_{i,t} \lambda \left(a_{i,t} + q_{i,t} a^{**}_{j,t} \right) x_{i,t}.$$

As before, define $B_{i,t} \equiv \frac{q_{i,t}}{r} + (1 - q_{i,t}) W_t - \gamma_{i,t}$, by Pontryagin's maximum principle, we have,

$$a^{**}_{i,t} = \begin{cases} 0 & \text{if } \lambda B_{i,t} < c_i, \\ 1 & \text{if } \lambda B_{i,t} > c_i. \end{cases} \tag{31}$$

Moreover, the continuous function $\gamma_{i,t}$ satisfies

$$\dot{\gamma}_{i,t} = r \gamma_{i,t} - \frac{\partial H_{i,t}}{\partial x_{i,t}}. \tag{32}$$

Finally, if $\left\{ x^{**}_{i,t} \right\}_{t \geq 0}$ is the optimal trajectory, then, the transversality condition,

$$\lim_{t \to \infty} e^{-rt} \gamma_{i,t} \left(x^{**}_{i,t} - x_{i,t} \right) \leq 0, \tag{33}$$

must be satisfied for all feasible trajectories $\{x_{i,t}\}_{t \geq 0}$. By the Arrow sufficiency theorem, conditions (31)–(33) are both necessary and sufficient for optimality. We have

$$\begin{aligned} \dot{B}_{i,t} &= -q_{i,t} (1 - p_t) + (\lambda B_{i,t} - c_i) a^{**}_{i,t} + \left(r + \lambda q_{i,t} a^{**}_{j,t} \right) B_{i,t} \\ &= -q_{i,t} (1 - p_t) - c_i a^{**}_{i,t} + \left(r + \lambda \left(a^{**}_{i,t} + q_{i,t} a^{**}_{j,t} \right) \right) B_{i,t}, \end{aligned} \tag{34}$$

which has the solution

$$B_{i,t} = \int_t^\infty e^{-\int_t^s \left(r + \lambda \left(a^{**}_{i,z} + q_{i,z} a^{**}_{j,z} \right) \right) dz} \left(q_{i,s} (1 - p_s) - c_i a^{**}_{i,s} \right) ds. \tag{35}$$

Suppose $\dot{a}^{**}_{j,t}$ exists, differentiate (34), we have

$$\ddot{B}_{i,t} = q_{i,t} \dot{p}_t - \dot{q}_{i,t} (1 - p_t) + \lambda \left(\dot{q}_{i,t} a^{**}_{j,t} + q_{i,t} \dot{a}^{**}_{j,t} \right) B_{i,t} + \left(r + \lambda \left(a^{**}_{i,t} + q_{i,t} a^{**}_{j,t} \right) \right) \dot{B}_{i,t}. \tag{36}$$

Proof of Lemma 6. The proof is similar to that of Lemma 5. Suppose $c_1 < c_2$ and the equilibrium $\left\{\left(a_{1,t}^{**}, a_{2,t}^{**}\right)\right\}_{t\geq 0}$ is regular. We first show that $\lambda B_{1,t} - c_1 > \lambda B_{2,t} - c_2$. By (35),

$$\begin{aligned}
& \lambda B_{1,t} - c_1 \\
= \;& \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+q_{1,z}a_{2,z}^{**}\right)\right)dz} \left(q_{1,s}\left(1-p_s\right) + c_1 a_{1,s}^{**}\right) ds - c_1 \\
= \;& \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+a_{2,z}^{**}\right)\right)dz} q_{1,t}\left(1-p_s\right) ds - c_1 \left(1 - \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+q_{1,z}a_{2,z}^{**}\right)\right)dz} a_{1,s}^{**} ds\right) \\
\geq \;& \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+a_{2,z}^{**}\right)\right)dz} q_{2,t}\left(1-p_s\right) ds - c_1 \left(1 - \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+q_{1,z}a_{2,z}^{**}\right)\right)dz} a_{1,s}^{**} ds\right) \\
> \;& \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+a_{2,z}^{**}\right)\right)dz} q_{2,t}\left(1-p_s\right) ds - c_2 \left(1 - \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+q_{1,z}a_{2,z}^{**}\right)\right)dz} a_{1,s}^{**} ds\right) \\
\geq \;& \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+a_{2,z}^{**}\right)\right)dz} q_{2,t}\left(1-p_s\right) ds - c_2 \left(1 - \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{2,z}^{**}+q_{2,z}a_{1,z}^{**}\right)\right)dz} a_{2,s}^{**} ds\right) \\
= \;& \lambda B_{2,t} - c_2,
\end{aligned}$$

where the last inequality follows from the fact that

$$\begin{aligned}
& \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+q_{1,z}a_{2,z}^{**}\right)\right)dz} a_{1,s}^{**} ds \\
\geq \;& \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{1,z}^{**}+q_{2,z}a_{1,z}^{**}\right)\right)dz} a_{1,s}^{**} ds \\
= \;& -\int_t^\infty e^{-\int_t^s \left(r+\lambda q_{2,z}a_{1,z}^{**}\right)dz} d\left(e^{-\lambda \int_t^s a_{1,z}^{**} dz}\right) \\
= \;& 1 + \int_t^\infty e^{-\lambda \int_t^s a_{1,z}^{**} dz} d\left(e^{-\int_t^s \left(r+\lambda q_{2,z}a_{1,z}^{**}\right)dz}\right) \\
\geq \;& 1 + \int_t^\infty e^{-\lambda \int_t^s a_{2,z}^{**} dz} d\left(e^{-\int_t^s \left(r+\lambda q_{2,z}a_{1,z}^{**}\right)dz}\right) \\
= \;& \lambda \int_t^\infty e^{-\int_t^s \left(r+\lambda\left(a_{2,z}^{**}+q_{2,z}a_{1,z}^{**}\right)\right)dz} a_{2,s}^{**} ds.
\end{aligned}$$

Suppose there exists some open interval I such that $\lambda B_{1,t} - c_1 = 0$ for all $t \in I$, then, $\lambda B_{2,t} - c_2 < 0$ and $a_{2,t}^{**} = 0$ for all $t \in I$. $\dot{B}_{1,t} = 0$ implies that for all $t \in I$,

$$\frac{\lambda q_{1,t}\left(1-p_t\right)}{r} = c_1.$$

Thus, $a_{1,t}^{**} = 0$ for all $t \in I$. It is impossible to have $a_{1,t}^{**} \in (0,1)$. Next, suppose $\lambda B_{1,\tau} - c_1 = 0$ for some $\tau \in (0,\infty)$, we would like to show that it is impossible to have $a_{1,\tau}^{**} = 0$ but

$\lim_{t\uparrow\tau} a^{**}_{1,\tau} = 1$. Since $\lambda B_{2,\tau} - c_2 < 0$, we have $\dot{a}^{**}_{2,\tau} = 0$. (34) implies that $\lim_{t\uparrow\tau} \dot{B}_{1,t} = \dot{B}_{1,\tau} < 0$. By (36), $\dot{B}_{1,\tau} < 0$ implies that for all $t \geq \tau$,

$$\begin{aligned} \dot{B}_{1,t} &< 0, \\ \ddot{B}_{1,t} &= r\dot{B}_{1,t} < 0, \end{aligned}$$

as $a^{**}_{1,t} = a^{**}_{2,t} = 0$. As a result, $\lim_{t\to\infty} B_{1,t} = -\infty$, which is a contradiction. Thus, expert 1 must use a strategy given by (23).

Finally, suppose $t^*_1 = \infty$, we must have

$$B_{1,0} = \frac{p_0\left(1-p_0\right)}{r} \leq \frac{c_1}{\lambda},$$

which is equivalent to $c_1 \geq \underline{c}$. Similarly, if $t^*_1 \in (0,\infty)$, $a^{**}_{1,t} = a^{**}_{2,t} = 0$ for all $t < t^*_1$. Moreover, $B_{1,t^*_1} = \frac{c_1}{\lambda}$. We must have,

$$\dot{B}_{1,t^*_1} = -p_0\left(1-p_0\right) + \frac{rc_1}{\lambda} \geq 0,$$

which implies that $c_1 \geq \underline{c}$. Thus, if $c_1 < \underline{c}$, $t^*_1 = 0$ is the only plausible starting time. ■

Proof of Proposition 5. We only need to show that if $\{(1,0)\}_{t\geq 0}$ is an equilibrium, $B_{2,t'}$ is equivalent to the left hand side of (26). The rest is explained in the main text. Since $\left\{a^*_{2,t}\right\}_{t\geq 0} = \{0\}_{t\geq 0}$, $q_{2,t} = p_t$ for all $t \geq 0$. By (35),

$$\begin{aligned} B_{2,t'} &= \int_{t'}^{\infty} e^{-\int_{t'}^{s}(r+\lambda p_z)dz} p_s\left(1-p_s\right)ds \\ &= \int_{t'}^{\infty} e^{-r(s-t')}\left(\frac{p_0 e^{-\lambda s}+1-p_0}{p_0 e^{-\lambda t'}+1-p_0}\right) p_s\left(1-p_s\right)ds \\ &= \int_{t'}^{\infty} e^{-(r+\lambda)(s-t')} p_{t'}\left(1-p_s\right)ds \\ &= \int_{0}^{\infty} e^{-(r+\lambda)s} p_{t'}\left(1-p_{t'+s}\right)ds \\ &= \int_{0}^{\infty} \frac{e^{-(r+\lambda)s-\lambda t'} p_0\left(1-p_0\right)}{\left(p_0 e^{-\lambda t'}+1-p_0\right)\left(p_0 e^{-\lambda(s+t')}+1-p_0\right)} ds. \end{aligned}$$

■