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A Common-Value Auction with State-Dependent Participation

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A Common-Value Auction with State-Dependent Participation*

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This paper analyzes a common-value, first-price auction with state-dependent participation. The number of bidders, which is unobservable to them, depends on the true value. For exogenously given participation patterns that involve many bidders in each state, the bidding equilibrium may be of a "pooling" type—with high probability, the winning bid is the same across states and is below the ex-ante expected value—or of a "partially separating" type—with no significant atoms in the winning bid distribution and an expected winning bid increasing in the true value. Which of these forms will arise is determined by the likelihood ratio at the top of the signal distribution and the participation across states. When the state-dependent participation is endogenized as the strategic solicitation by an informed seller who bears a small cost for each solicited bidder, an equilibrium of the separating type always exists and is unique of this type; for certain signal distributions there also exist equilibria of the pooling type.

In various auctions and similar trading scenarios, the extent of participation may be correlated with information that is relevant for the bidding. This might be the case when the decisions on the costly recruitment of participants are made by an informed seller, or when the participants are induced to participate by the value of correlated outside options.

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Strategic participants take this into account and it may affect behavior and the resulting prices. The main objective of this paper is to shed light on these considerations that are obviously present in many different scenarios, be it a sale of an asset of uncertain value or the shopping around of a venture by an entrepreneur to potential lenders.

Price formation with state-dependent participation can take different forms. This paper explores it by studying auctions with state-dependent participation, where the number of bidders varies across states, and bidders can learn about the state from their participation. We view the auction model as a convenient abstraction of a free-form price-formation process that takes place in a decentralized market environment, rather than in a formal mechanism. The specific auction format and some of the other features are selected to facilitate the clear exposition of the insights concerning the strategic effects of state-dependent participation rather than tailored to fit a specific application.

Specifically, we analyze a first-price auction for a single good with two value-states, ℓ and h, such that the common value of the good, v_ω, $\omega = \ell, h$, satisfies $v_h > v_\ell$. In state ω, there are n_ω bidders. They do not observe ω or n_ω but get private, conditionally independent signals that are drawn from a distribution G_ω with support $[\underline{x}, \bar{x}]$ and density g_ω. This is the same basic model as in our companion paper, Lauermann and Wolinsky (2017).[1] In this world, bidders obtain information about the total participation (and hence the state) through their own presence at the auction. This augments their private signal information, and the compound posterior likelihood ratio of the states depends both on the signal likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ (as it would in a standard auction environment) and on the participation ratio $\frac{n_h}{n_\ell}$. The objective of this paper is to explore the implications of this feature.

We first take the state-dependent participation, (n_ℓ, n_h), as exogenously given and explore the bidding equilibria of this game. Our main characterization result (Theorem 1) concerns the general form of the bidding equilibrium when there are many bidders in each state. Specifically, assuming that the likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ is increasing, the general form of bidding equilibria when n_ℓ and n_h are sufficiently large is determined by the compound posterior likelihood ratio $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$. If this ratio is below 1, then any bidding equilibrium is necessarily of a pooling type: there is some $\bar{b}$ such that, with probability close to 1, the winning bid is equal to $\bar{b}$ in both states, and $\bar{b}$ is below the ex-ante expected value. If this ratio is above 1, then any bidding equilibrium is of a separating type: there are no significant atoms in the winning bid distribution, and the expected winning bid is higher in state h than in ℓ. Thus, in this case, it resembles the equilibrium of an ordinary common-value auction.

These results are explained by the form of the "winner's inference," $\frac{\Pr(\text{all other bids} \leq b|h)}{\Pr(\text{all other bids} \leq b|\ell)}$, given a common bidding strategy β. When there are many bidders, for a strictly increas-

[1] We discuss this relationship later. For now, we note that there is essentially no overlap in results.

ing bidding strategy β to be an equilibrium, the expected value conditional on winning must be increasing in the bid. But this is the case only if this ratio is increasing in b. The analysis will show that, given a common, strictly increasing bidding strategy β, the winner's inference is increasing if and only if $\frac{G_h(x)^{n_h-1}}{G_\ell(x)^{n_\ell-1}}$ is increasing in x. And, for x near $\bar{x}$ and for large n_ℓ and n_h, this is determined by whether $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$ is above or below 1.

In the second part of the paper, the state-dependent participation is endogenized. An informed seller knows ω and invites n_ω bidders at a constant cost per bidder.[2] Costly strategic solicitation is interesting in its own right, in that stimulating participation is an important element of the seller's activities in bidding scenarios. We focus on the case of a small solicitation cost, which naturally results in large participation. Theorem 2 characterizes the limit outcomes of the equilibria of the full game as the solicitation costs vanish. In particular, it establishes that when the limit equilibrium is of the separating type—which requires $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\lim\frac{n_h}{n_\ell} > 1$ by Theorem 1—then the optimal solicitation pins down the ratio $\frac{n_h}{n_\ell}$ uniquely, and hence determines a unique limit distribution of the winning bid.

Our final two main results concern the existence[3] of equilibrium: Theorem 3 establishes that there always exists an equilibrium of the separating type with $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell} > 1$. Theorem 4 establishes the existence of a nontrivial pooling equilibrium for a class of discrete signals.

The extent of information aggregation by the price—as reflected by the closeness of the price to the true value—depends on the form of the equilibrium. The price aggregates no information in the pooling equilibrium and aggregates some information in the separating equilibria (in the sense that the distribution of the winning bid in state h stochastically dominates that of state ℓ). The extent of information aggregation in the separating equilibria increases in the ratio $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$. In an ordinary large common-value auction without state-dependent participation, the price aggregates only the bidders' information, and the extent of information aggregation depends on the informativeness of the private signals as captured by $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$ (Wilson (1977) and Milgrom (1979)). With state-dependent participation, the seller's information is also aggregated into the price via $\frac{n_h}{n_\ell}$, either dampening or enhancing the effect of the bidders' information. For a given value of $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$, the larger $\frac{n_h}{n_\ell}$ is, the more information is incorporated into the price. In particular, the price aggregates information better than it does in a large ordinary auction with the same signal structure whenever $\frac{n_h}{n_\ell}$ is larger than 1, and it aggregates information worse when $\frac{n_h}{n_\ell}$ is smaller than 1.[4]

[2] State-dependent participation may arise for a number of different reasons. In Murto and Välimäki (2015a), it is the result of partially informed bidders' costly entry decisions; in Atakan and Ekmekci (2016), bidders' entry decisions differ across states due to differences in the value of outside options. A range of behavioral considerations might have a similar effect as well.

[3] The statements on existence pertain to the limit with respect to the step size of a finite grid of the set of feasible bids.

[4] More precisely, the limit price distribution is shown to be Blackwell more informative about the state as the limit ratio of $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$ increases.

We analyze large, first-price auctions in a binary-state world and, strictly speaking, the results apply to that environment. However, this model is just a means to illustrate the main insights concerning the effects of state-dependent participation that are likely to be relevant for a broader set of environments. In Sections 7.3 and 7.5, after having presented our model and analysis, we will discuss in some more detail this speculative claim concerning the wider scope.

1.1 Literature Connections

Lauermann and Wolinsky (2017) uses the same model as this paper and shares with it some auxiliary observations. Our unapologetic use of the same model reflects our view of it as a fundamental model of an important situation that is not exhausted after one use. Otherwise, the overlap of the two papers is minimal. The main result from Lauermann and Wolinsky (2017) is that there exists an equilibrium with an atom at the top of the bid distribution when the bidders' signals are binary. This corresponds to a special case of Theorem 4 in the current paper. The other results of the current paper have no counterpart in Lauermann and Wolinsky (2017). In particular, the inevitability of atoms for certain participation patterns is introduced here for the first time, as is everything about the partially separating equilibria, including their general form, their existence and uniqueness under endogenous solicitation, and the corresponding insights regarding information aggregation.

From the perspective of auction theory, the closest papers are Murto and Valimaki (2015a) and Atakan and Ekmekci (2016). They also have a common-value auction with state-dependent participation,[5] but they explore other mechanisms that generate it.

Our discussion of information aggregation continues the discussion of this question by Milgrom (1979) and Wilson (1977) in the context of an ordinary common-value auction. Translated to the two-state model considered here, Milgrom's (1979) result is that the winning bid in an ordinary common-value auction approaches the true value as the number of bidders grows if and only if the likelihood ratio of the two states is unbounded over the support of the signal distribution. Our analysis recognizes the additional information due to the state-dependent strategic solicitation and points out that this solicitation may dampen or enhance the extent to which bidders' information is incorporated into prices.

Broecker (1990) and Riordan (1993) model competition among incompletely informed banks over the business of potential borrowers as an ordinary auction—the borrowers contact all the banks for quotes. This and our companion paper recognize that such competition may be significantly affected when borrowers choose how many banks to contact based on their private information. The inevitability of atoms established in the present paper implies that contact patterns of a broad class will *necessarily* result in banks

[5] Remark to the referees: These papers became available after earlier versions of our paper.

pooling on a unique quote, which resembles a collusive outcome.

For the special case of $n_\ell = n_h = n$, the result in Theorem 1 concerning the form of the winning bid distribution is essentially implied by the analysis of Murto and Valimaki (2015b).

In markets of the sort we are interested in, the contacts made by agents do not always follow a rigid protocol—sometimes they are indeed simultaneous, as in the present model, sometimes sequential, and sometimes a combination of the two. We explored the sequential scenario in Lauermann and Wolinsky (2016). A central qualitative difference is due to the absence of direct price competition in the sequential-search-with-bargaining model. In it, uninformed agents with promising signals cannot actively overbid, and, therefore, the extent of information aggregation is determined by the interaction of search and the signal technology. In contrast, the auction setting assigns a prominent role to price competition. The uninformed may try to evade adverse selection by bidding more aggressively, and in the process inject their information into the price. This explains why, with sufficiently informative signals, the partially separating equilibrium with bidder solicitation is nearly competitive and also aggregates information well, unlike the corresponding unique equilibrium of the search model.

2 The Bidding Game and Preliminary Characterization

This section and the following one discuss the bidding behavior for an exogenously given pattern of state-dependent participation. The solicitation by an informed seller is one possible such scenario. But, as mentioned above, state-dependent participation may arise for other reasons as well. Therefore, the understanding of this situation is both of interest in its own right and used as a building block for the subsequent analysis of endogenous solicitation.

2.1 The Bidding Game and its Equilibrium

Basics.—This is a single-good, common-value, first-price auction environment with two underlying states, h and ℓ. There are N potential bidders (buyers). The common values of the good for all potential bidders in the two states are v_ℓ and v_h, respectively, with $0 \leq v_\ell < v_h$.

Nature draws a state $\omega \in \{\ell, h\}$ with prior probabilities $\rho_\ell > 0$ and $\rho_h > 0$, $\rho_\ell + \rho_h = 1$, and, in state ω, randomly draws n_ω bidders from the pool, $1 \leq n_\omega \leq N$. A bold $\mathbf{n}$ denotes the vector (n_ℓ, n_h).[6]

Each of the n_ω bidders observes a private signal $x \in [\underline{x}, \bar{x}]$. Conditional on the state

[6] The solicitation is modeled here as a move of nature to focus on the bidding, but, as mentioned above, it can be endogenized in several ways, and we will do this later.

$\omega \in \{\ell, h\}$, signals are independently and identically distributed according to a cumulative distribution function (c.d.f.) G_ω. A bidder does not observe ω or n_ω, but she believes that her probability of being invited to the auction in state ω is $\frac{n_\omega}{N}$.

The n_ω bidders simultaneously submit bids $b \in [0, v_h]$. The highest bid wins, and ties are broken randomly with equal probabilities. If the winning bid is p in state $\omega \in \{h, \ell\}$, then the payoffs are $v_\omega - p$ for the winning bidder and zero for all others. We call this the "bidding game" and denote it by $\Gamma_0(\mathbf{n}, N)$. The ordinary common-value auction is a special case with $n_\ell = n_h$.

The Signal.—The signal distributions G_ω, $\omega \in \{\ell, h\}$, have no atoms and strictly positive densities g_ω on an identical support, $[\underline{x}, \bar{x}] \subset \mathbb{R}$. The likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ is nondecreasing and right-continuous, with $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = \lim_{x \to \bar{x}} \frac{g_h(x)}{g_\ell(x)}$. This is the (weak) monotone likelihood ratio property (MLRP): larger values of x indicate a (weakly) higher likelihood of the higher state.[7] The signals are nontrivial and boundedly informative, i.e.,

$$0 < \frac{g_h(\underline{x})}{g_\ell(\underline{x})} < 1 < \frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \infty.$$

Expected Payoffs.—A bidder's posterior probability of ω, conditional on being solicited and receiving signal x, is

$$\Pr[\omega|x, \text{sol}; \mathbf{n}] \triangleq \frac{\rho_\omega g_\omega(x) \frac{n_\omega}{N}}{\rho_\ell g_\ell(x) \frac{n_\ell}{N} + \rho_h g_h(x) \frac{n_h}{N}},$$

where ρ_ω, $g_\omega(x)$, and $\frac{n_\omega}{N}$, respectively, reflect the information contained in the prior belief, in the signal x, and in the bidder being invited. We use "sol" to denote the event that the bidder was solicited. Notice that N cancels out, and, hence, it does not play any role in the analysis.

A bidding strategy β prescribes a bid as a function of the signal realization,

$$\beta : [\underline{x}, \bar{x}] \to [0, v_h].$$

We study strategies that are symmetric and pure.

Given a bidding strategy β employed by n other bidders, the probability of winning with a bid b in state ω is $\pi_\omega(b; \beta, n)$. From here on, $(\beta, \mathbf{n})$ and (β, n_ω) will typically be suppressed from the arguments, and we write expressions such as $\Pr[\omega|x,\text{sol}]$ and $\pi_\omega(b)$ with the understanding that they depend on a specific profile $(\beta, \mathbf{n})$.

Given the bidding strategy β and the participation $\mathbf{n} = (n_\ell, n_h)$, the expected payoff to a bidder who bids b, conditional on participating and observing the signal x, is

$$U(b|x, \text{sol}) = \Pr[\text{win at } b|x, \text{sol}] \left(\mathbb{E}[v|x, \text{sol,win at } b] - b\right), \tag{1}$$

[7] With a weak MLRP, discrete signals are a special case of our model.

where

$$\Pr\left[\text{win at } b|x\text{,sol}\right] = \frac{\rho_\ell g_\ell\left(x\right) n_\ell \pi_\ell\left(b\right) + \rho_h g_h\left(x\right) n_h \pi_h\left(b\right)}{\rho_\ell g_\ell\left(x\right) n_\ell + \rho_h g_h\left(x\right) n_h}, \tag{2}$$

and

$$\mathbb{E}[v|x\text{,sol,win at } b] = \frac{\rho_\ell g_\ell\left(x\right) n_\ell \pi_\ell\left(b\right) v_\ell + \rho_h g_h\left(x\right) n_h \pi_h\left(b\right) v_h}{\rho_\ell g_\ell\left(x\right) n_\ell \pi_\ell\left(b\right) + \rho_h g_h\left(x\right) n_h \pi_h\left(b\right)}, \tag{3}$$

where $(\beta, \mathbf{n})$ is suppressed from the arguments of $\mathbb{E}\left[v|...\right]$ and $\Pr\left[\text{win at } b|...\right]$, according to the convention adopted above.

Let $\mathbb{E}\left[v\right]$, without any conditioning, denote the expected ex-ante value of the good:

$$\mathbb{E}\left[v\right] = \rho_\ell v_\ell + \rho_h v_h.$$

Bidding Equilibrium.—A *bidding equilibrium* of $\Gamma_0\left(\mathbf{n}, N\right)$ is a bidding strategy β such that $b = \beta\left(x\right)$ maximizes $U(\cdot|x\text{,sol})$ for all x.

2.2 Monotonicity of the Bidding Equilibrium

When at least two bidders participate in the auction in both states, a bidding equilibrium strategy β is monotonic in the following sense. If the likelihood ratio $\frac{g_h}{g_\ell}$ is strictly increasing everywhere, then a bidding equilibrium β is necessarily nondecreasing. If $\frac{g_h}{g_\ell}$ is constant over some interval, then β need not be monotonic over it, since all those signals contain the same information. However, in this case there is an equivalent monotone bidding equilibrium that is obtained by reordering the bids over such intervals.

A bidding equilibrium $\widetilde{\beta}$ is said to be *equivalent* to a bidding equilibrium β if the implied joint distributions over bids and states are identical.

Proposition 1 **(Monotonicity of Bidding Equilibrium)** *Suppose that $n_\omega \geq 2$, $\omega = \ell, h$, and β is a bidding equilibrium.*

1. *If $x' > x$, then $U(\beta\left(x'\right)|x'\text{,sol}) \geq U(\beta\left(x\right)|x\text{,sol})$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.*

2. *There exists an equivalent bidding equilibrium $\widetilde{\beta}$, such that $\widetilde{\beta}$ is nondecreasing on $[\underline{x}, \bar{x}]$ and coincides with β over intervals over which $\frac{g_h}{g_\ell}$ is strictly increasing.*

This proposition is not proved separately since it is a special case of a more general version called Proposition 1', which will be stated and proved later in Section B.2. The main observation that is used in the proof is that given any β—monotone or not—the payoffs $U(b|x\text{,sol})$ satisfy a single crossing condition in b and x when $n_\omega \geq 2$. The single-crossing property also implies that the restriction to pure strategies is without loss of generality. The two-state assumption plays an important role in establishing this property.

Monotonicity is not immediate, since the signals inform bidders also about the number of competitors rather than just about the value. If fewer buyers are solicited when $\omega = h$, a higher signal implies both a higher value and less competition. The following example illustrates this consideration and also clarifies why the assumption that at least two bidders are solicited in both states is needed for establishing monotonicity.

Example of a Non-monotone Bidding Equilibrium: Let $[\underline{x}, \bar{x}] = [0, 1]$, with $g_h(x) = 2x$ and $g_\ell(x) = 2 - 2x$. Thus, the signals $x = 1$ and $x = 0$ reveal the state.[8] Suppose that $v_\ell > 0$, $n_h = 1$, and $n_\ell = 100$. It follows that $\pi_h(b; \beta, 1) = 1$ for all $b \geq 0$. Hence, $\beta(1) = 0$ in every bidding equilibrium. So, if β were weakly increasing, then $\beta(x) = 0$ for all x. However, this strategy cannot be an equilibrium. At $x = 0$, the expected payoff from bidding $b = 0$ is $\frac{1}{100} v_\ell$, whereas the expected payoff from bidding $b' = \varepsilon$ is $v_\ell - \varepsilon$. Because $v_\ell > 0$, a deviation to b' is profitable for small ε. Thus, in this example, there is no weakly increasing bidding equilibrium.

In light of Proposition 1, from now on, whenever $n_\omega \geq 2$, $\omega = \ell, h$, attention is confined to nondecreasing bidding equilibria.

3 Characterization of Bidding Equilibria with Many Bidders

This section characterizes bidding equilibria when there are many bidders in each state. From a substantive point of view, the many bidders case is the relevant case for the questions of competitiveness and information aggregation in markets. From an analytical point of view, this case makes it easier to get clean characterization results and identify the underlying economic mechanism.

3.1 Preliminaries

We look at a sequence of bidding games

$$\Gamma_0(N^k, \mathbf{n}^k) \text{ s.t. } \lim_{k\to\infty} n_\omega^k = \infty \text{ for } \omega = \ell, h, \text{ and } \lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} \text{ exists,}$$

and a corresponding sequence of bidding equilibria β^k, and consider the limits of equilibrium magnitudes as $k \to \infty$.[9]

With many bidders, only bids associated with signals that are sufficiently close to $\bar{x}$ have a significant probability of winning. Therefore, the object of interest is the equilib-

[8] The example violates the bounded likelihood-ratio assumption. This simplifies the argument but is not essential.

[9] By assumption, $N^k \geq n_\omega^k$ for $\omega = \ell, h$, and so $N^k \to \infty$.

rium distribution of the *winning* bid in state ω, namely,

$$F_\omega(p|\beta,n) \triangleq (G_\omega(\{x : \beta(x) \le p\}))^n,$$

and its pointwise limit,[10] rather than the distribution of all the bids.

The notation's density is reduced as follows. First, when we discuss a fixed sequence $\{(\beta^k, \mathbf{n}^k)\}_{k=1}^{\infty}$, then magnitudes induced by $(\beta^k, \mathbf{n}^k)$ are typically written as $U^k(b|x,\text{sol})$, $\pi^k_\omega(b)$, $\mathbb{E}^k[v|x,\text{sol,win at } b]$, $F^k_\omega(p)$, etc. (rather than as $U(b|x,\text{sol};\beta^k,\mathbf{n}^k)$, $F_\omega(p|\beta^k,n^k_\omega)$, etc.). Second, the term *limit* (and the operator lim) always refer to a limit over a subsequence such that all the magnitudes of interest are converging; we will not repeat this qualification each time. Third, since nearly all limits are with respect to k, we generally omit the delimiter $k \to \infty$. Finally, we sometimes use the abbreviations

$$\bar{g} \triangleq \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \text{ and } \rho \triangleq \frac{\rho_h}{\rho_\ell}.$$

The following lemma links the first-order statistics of the signals in the two states (in the limit). It relies on the fact that, when n^k_ω is large and x close to $\bar{x}$, the number of bidders with signals $\ge x$ is approximately Poisson-distributed with parameter $n^k_\omega(1 - G_\omega(x))$. Therefore, $\Pr(\text{no signal} \ge x) = \left(G_\omega(x^k)\right)^{n^k_\omega} \approx e^{-n^k_\omega(1-G_\omega(x))}$.

Lemma 1 (Poisson-Approximation.) *Consider some sequence* $(x^k, \mathbf{n}^k)$ *with* $\min\{n^k_\ell, n^k_h\} \to \infty$ *and* $\lim \frac{n^k_h}{n^k_\ell} = r < \infty$. *If*

$$\lim \left(G_\ell(x^k)\right)^{n^k_\ell} = q$$

for some $q \in [0,1]$, *then*

$$\lim \left(G_h(x^k)\right)^{n^k_h} = q^{\bar{g}r}.$$

Proof of Lemma 1. Let $Q_\omega \triangleq \lim\left(1 - G_\omega(x^k)\right) n^k_\omega \in [0,\infty) \cup \infty$. Observe that

$$\lim \left(G_\omega(x^k)\right)^{n^k_\omega} = \lim(1 - \frac{1 - G_\omega(x^k)}{n^k_\omega} n^k_\omega)^{n^k_\omega} = e^{-Q_\omega}.$$

The lemma clearly holds with $q = 0$ if $\lim x^k < \bar{x}$. So, suppose $\lim x^k = \bar{x}$. Then, $\lim \frac{1-G_h(x^k)}{1-G_\ell(x^k)} = \bar{g}$, and so we have $Q_h = Q_\ell \bar{g} \lim(n^k_h/n^k_\ell)$. Therefore, $q = e^{-Q_\ell}$ implies

$$\lim \left(G_h(x^k)\right)^{n^k_h} = e^{-Q_h} = e^{Q_\ell \bar{g} \lim(n^k_h/n^k_\ell)} = q^{\bar{g}r}. \quad \blacksquare$$

[10] By Helly's selection theorem, every sequence of cumulative distribution functions has a pointwise everywhere convergent subsequence, which can be adjusted to be a c.d.f..

3.2 Winning Bid Distribution: Pooling vs. Separating

Our main characterization result shows that the form of the winning bid distributions F_ω^k is determined by $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} \lim \frac{n_h^k}{n_\ell^k}$. For large k, F_ω^k exhibits a large atom at the top if

$$\frac{g_h(\bar{x})}{g_\ell(\bar{x})} \lim \frac{n_h^k}{n_\ell^k} < 1, \tag{4}$$

and is essentially free of atoms if the reverse inequality holds. In state ω, the expected number of bidders with signals in a small ε-neighborhood of $\bar{x}$ is $n_\omega^k (1 - G_\omega(\bar{x} - \varepsilon)) \approx n_\omega^k g_\omega(\bar{x}) \varepsilon$, which is also the expected number of effective bidders when the n_ω^k's are large. Thus, the direction of the inequality in (4) is determined by whether this expected number is higher in state h or in state ℓ. This hints at why the inequality between $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} \lim \frac{n_h^k}{n_\ell^k}$ and 1 matters: when everybody uses the same strictly increasing bidding strategy and there are more (effective) bidders in state h, a bidder who wins at a higher bid is more optimistic about the value than a bidder who wins at a lower bid, while the opposite is true if there are more (effective) bidders in state ℓ. This, in turn, determines whether such a strictly increasing strategy can be sustained as an equilibrium or that pooling is unavoidable. Recall the abbreviations $\bar{g} \triangleq \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$ and $\rho \triangleq \frac{\rho_h}{\rho_\ell}$.

Theorem 1 (Large Bidding Equilibrium) *Consider a sequence of bidding games $\Gamma_0(\mathbf{n}^k, N^k)$ for which $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\lim \frac{n_h^k}{n_\ell^k} = r$, and a corresponding sequence of bidding equilibria β^k.*

1. *If $\bar{g}r < 1$, then, for large enough k, there is a bid $\bar{b}^k$ at which F_ω^k has an atom, i.e.,*

$$\lim \Pr(\text{winning bid } = \bar{b}^k | \omega) \triangleq \lim dF_\omega^k(\bar{b}^k) = 1 \qquad \text{for } \omega = \ell, h,$$

and

$$\lim \mathbb{E}[v|\bar{x}, sol] \equiv \frac{v_\ell + \rho\bar{g}rv_h}{1 + \rho\bar{g}r} \leq \lim \bar{b}^k \leq \mathbb{E}[v].$$

2. *If $\bar{g}r > 1$, then F_ω^k converges pointwise to $\Phi_\omega(\cdot|r)$, where*

$$\Phi_\ell(p|r) \triangleq \begin{cases} 1 & \text{if } \quad p \geq \frac{v_\ell + \rho\bar{g}rv_h}{1+\rho\bar{g}r}, \\ \left(\frac{1}{\rho\bar{g}r}\frac{p - v_\ell}{v_h - p}\right)^{\frac{1}{\bar{g}r-1}} & \text{if } \quad v_\ell < p \leq \frac{v_\ell + \rho\bar{g}rv_h}{1+\rho\bar{g}r}, \\ 0 & \text{if } \quad p \leq v_\ell, \end{cases} \tag{5}$$

and

$$\Phi_h(\cdot|r) \triangleq (\Phi_\ell(\cdot|r))^{\bar{g}r}. \tag{6}$$

This proof and all subsequent ones that do not appear immediately after the statements of the results are relegated to the Appendix. Part 1 does not mean that most bidders are bidding $\bar{b}^k$, but rather that the winning bid is very likely to be $\bar{b}^k$.

3.3 Intuition for Theorem 1

Recall from (1) that given a common bidding strategy β and a signal x, the payoff of bidding b is

$$\Pr\left[\text{win at } b|x,\text{sol}\right]\left(\mathbb{E}[v|x,\text{sol,win at } b]-b\right),$$

Thus, the trade-off between bidding b_1 and bidding $b_2 > b_1$ in the range of β consists of weighing the benefit of a lower payment at b_1 against the respective winning probabilities and the value estimate conditional on winning at each of the two bids. Given this β, the "winner's inference" is

$$\frac{\pi_h(b)}{\pi_\ell(b)}=\frac{G_h(\beta^{-1}(b))^{n_h-1}}{G_\ell(\beta^{-1}(b))^{n_\ell-1}}.$$

If $\frac{\pi_h}{\pi_\ell}$ is increasing, then the value estimate increases from b_1 to b_2, and, since the winning probability also increases, it is fairly immediate that one can construct a strictly increasing bidding equilibrium. It is just a matter of finding the right slope for β so that the higher payment conditional on winning is made worthwhile by the higher winning probability and the higher expected value conditional on winning.

If $\frac{\pi_h}{\pi_\ell}$ is decreasing, however, it is no longer obvious that a strictly increasing β may be sustained in equilibrium, since the "winner's inference" works in the opposite direction. The value estimate conditional on winning decreases from b_1 to b_2 and counters the benefit of a higher probability of winning at b_2. Consequently, it is not obvious which effect dominates. The argument presented below shows that, when the number of bidders is large, the adverse "winner's inference" effect necessarily offsets the favorable increase in probability of winning, and precludes strictly increasing bidding equilibria.

When everybody bids using the same strictly increasing bidding strategy, $\frac{\pi_h(b)}{\pi_\ell(b)}$ is increasing if and only if $\frac{G_h(x)^{n_h-1}}{G_\ell(x)^{n_\ell-1}}$ is increasing in x. By Lemma 1, for x near $\bar{x}$ and large n_ℓ and n_h, this is determined by whether $\bar{g}r$ is above or below 1, and, therefore, this inequality determines the form of the equilibrium. Next, we provide further details on how the general intuition just outlined is completed to establish the theorem.

Consider first Case 1: $\bar{g}r < 1$. The reason for the inevitability of the atom is that, if $\beta^k(x)$ were strictly increasing for large k, then the effect of the decreasing "winner's inference" $\frac{\pi_h^k(b)}{\pi_\ell^k(b)}$ explained above would swamp the effect of the higher winning probability. Specifically, we show that a bidder with the highest possible signal, $\bar{x}$, has an incentive to deviate to a lower bid in order to benefit from the higher value conditional on winning (i.e., the *winner's blessing* is stronger at a lower bid). First note that, for strictly increasing β^k, x close to $\bar{x}$, and large k, it follows from $\pi_\omega^k\left(\beta^k(x)\right)=\left(G_\omega(x)\right)^{n_\omega^k}$, Lemma 1, and $\bar{g}r<1$ that

$$\frac{\pi_h^k\left(\beta^k(x)\right)}{\pi_\ell^k\left(\beta^k(x)\right)}>1=\frac{\pi_h^k\left(\beta^k(\bar{x})\right)}{\pi_\ell^k\left(\beta^k(\bar{x})\right)}. \tag{7}$$

This inequality, together with $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} > \frac{g_h(x)}{g_\ell(x)}$, implies via (3) that

$$\mathbb{E}^k[v|\bar{x}, \text{sol, win at } \beta^k(x)] > \mathbb{E}^k[v|\bar{x}, \text{sol, win at } \beta^k(\bar{x})]. \tag{8}$$

Therefore, by deviating to $\beta^k(x)$, a bidder with signal $\bar{x}$ increases the expected payoff conditional on winning (i.e., the expected value conditional on winning is higher while $\beta^k(x) < \beta^k(\bar{x})$). The following argument shows that this effect swamps the effect of the lower probability of winning at $\beta^k(x)$. First, as $n^k_\omega \to \infty$, the payoffs of all bidders converge to 0. Therefore, for any x with a significant probability of winning,

$$\beta^k(x) \approx \mathbb{E}^k[v|x, \text{sol,win at } \beta^k(x)], \tag{9}$$

which applies to $x = \bar{x}$ as well. Next, let the sequence of signals $(x^k(q))$ converging to $\bar{x}$, be defined by $\pi^k_\ell\left(\beta^k(x^k(q))\right) = q \in (0,1)$. Observe that it follows from Lemma 1 that (7) and hence (8) hold strictly in the limit, taken over this sequence[11], while the probability of winning is $q > 0$. Therefore, conditional on winning with bid $\beta^k(x^k(q))$, bidder $\bar{x}$'s payoff satisfies

$$\lim\left[\mathbb{E}^k[v|\bar{x}, \text{sol, win at } \beta^k(x^k(q))] - \beta^k(x^k(q))\right] > 0.$$

Since the limit of the probability of winning is strictly positive as well (it is $q > 0$ in state ℓ and $q^{r\bar{g}} > 0$ in state h), it follows that the expected payoff from bidding $\beta^k(x^k(q))$ is strictly positive in the limit. However, by (9), the expected payoff from bidding $\beta^k(\bar{x})$ vanishes to zero, and, hence, bidding $\beta^k(x^k(q))$ must be a profitable deviation for k large enough. This rules out the possibility of a strictly increasing bidding equilibrium β^k, for large k, and implies that every bidding equilibrium strategy must have some atom. The Appendix shows that this atom must in fact dominate the winning bid distribution.

Finally, since for large k the winning bid is almost certainly $\bar{b}^k$ in both states, bidders' individual rationality dictates that $\bar{b}^k \leq \mathbb{E}[v]$. This is compatible with equilibrium, since, in this case, just being included in the auction already involves a "participation curse" that depresses the expected value estimate held by any bidder: for large k, $\rho\bar{g}\frac{n^k_h}{n^k_\ell} < \rho$, and hence

$$\mathbb{E}^k[v|\bar{x}, sol] \equiv \frac{v_\ell + \rho\bar{g}\frac{n^k_h}{n^k_\ell}v_h}{1 + \rho\bar{g}\frac{n^k_h}{n^k_\ell}v_h} < \frac{v_\ell + \rho v_h}{1 + \rho v_h} \equiv \mathbb{E}[v].$$

Therefore, bidders do not have an incentive to overbid the atom at $\bar{b}^k$, even if it is strictly below $\mathbb{E}[v]$, provided that it is not below $\mathbb{E}^k[v|\bar{x}\text{,sol}]$, which is indeed the lower bound on $\bar{b}^k$ stated in the theorem.

Now, consider Case 2: $\bar{g}r > 1$. The insights of Part 2 of Theorem 1 are, first, the

[11] Since by the definition of x^k, $\lim \pi^k_\ell\left(\beta^k(x^k(q))\right) = q$, Lemma 1 implies $\lim \pi^k_h\left(\beta^k(x^k(q))\right) = q^{\bar{g}\lim(n^k_h/n^k_\ell)}$. This and $\bar{g}\lim(n^k_h/n^k_\ell) < 1$ imply that $\lim \pi^k_h\left(\beta^k(x^k(q))\right) > \lim \pi^k_\ell\left(\beta^k(x^k(q))\right)$.

absence of an atom in the limit of the winning bid distribution, $\lim F_\omega^k$; and second, its specific form Φ_ω. The former is somewhat easier to explain for the special case of $n_h \geq n_\ell$. In this case, the absence of an atom can be established without the large-numbers assumption.[12]

Proposition 2 (**No Atoms if** $n_h \geq n_\ell$) *Suppose that* β *is a bidding equilibrium of* $\Gamma_0(N, \mathbf{n})$*, with* $n_h \geq n_\ell \geq 2$.

- *If* $n_h > n_\ell$*, then* β *is strictly increasing.*
- *If* $n_h = n_\ell$*, then, for* $\hat{x} \triangleq \sup\left\{x | \frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}\right\}$*, the following hold:*
  - β *is constant on* $(\underline{x}, \hat{x})$*;*
  - β *is strictly increasing on* $[\hat{x}, \bar{x}]$*.*

The intuition for the case of $n_h > n_\ell$ is as follows. Suppose a bidding equilibrium β has an atom at some $\bar{b}$. Let

$$\begin{aligned} x_-\left(\bar{b}\right) &\triangleq \inf\left\{x \in [\underline{x}, \bar{x}] \,|\beta\left(x\right) \geq \bar{b}\right\}, \text{ and} \\ x_+\left(\bar{b}\right) &\triangleq \sup\left\{x \in [\underline{x}, \bar{x}] \,|\beta\left(x\right) \leq \bar{b}\right\}. \end{aligned} \tag{10}$$

Thus,

$$\Pr(\text{winning bid } = \bar{b}|\omega) = G_\omega(x_+(\bar{b}))^{n_\omega} - G_\omega(x_-(\bar{b}))^{n_\omega} > 0.$$

The argument is the reverse of the one presented for the previous case. Here, the winner's inference is shown to be increasing between $\bar{b}$ and $\bar{b} + \varepsilon$, i.e.,

$$\frac{\pi_h\left(\bar{b}+\varepsilon\right)}{\pi_\ell\left(\bar{b}+\varepsilon\right)} > \frac{\pi_h\left(\bar{b}\right)}{\pi_\ell\left(\bar{b}\right)},$$

for ε small enough. Showing this requires some work, but intuitively this is because, conditional on the event that the highest other bid is $\bar{b}$ (which is when the difference between $\bar{b}$ and $\bar{b}+\varepsilon$ matters) the bid $\bar{b}+\varepsilon$ wins for sure in both states, whereas the bid $\bar{b}$ is less likely to win in state h than in state ℓ. This can be shown from $n_h > n_\ell$ and from the following expression,

$$\Pr(\beta(x) = \bar{b}|\beta(x) \leq \bar{b}, \omega) = \frac{G_\omega\left(x_+(\bar{b})\right) - G_\omega\left(x_-(\bar{b})\right)}{G_\omega\left(x_+(\bar{b})\right)}, \tag{11}$$

[12] The proposition is an extension to $n_h \geq n_\ell$ of similar results for $n_h = n_\ell$ in Lauermann and Wolinsky (2017) for binary signals and elsewhere in the literature (e.g., Rodriguez, 2000 and McAdams, 2007).

being weakly higher for $\omega = h$ than for $\omega = \ell$.[13] Now $\frac{\pi_h(\bar{b}+\varepsilon)}{\pi_\ell(\bar{b}+\varepsilon)} > \frac{\pi_h(\bar{b})}{\pi_\ell(\bar{b})}$ implies via (3) that

$$\lim_{\varepsilon \to 0} \mathbb{E}[v|x,\text{sol,win at } \bar{b}+\varepsilon] > \mathbb{E}[v|x,\text{sol,win at } \bar{b}]. \tag{12}$$

Therefore, slightly overbidding $\bar{b}$ is profitable, since it strictly increases both the expected value conditional on winning by (12) and the probability of winning.

Now, when there are many participants, the ratio of the numbers of bidders with signals near $\bar{x}$ is roughly $\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$. Since these bidders are the effective participants, $\lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} \equiv \bar{g}r > 1$ is in some sense the counterpart of $n_h > n_\ell$, and a similar intuition applies.

The functional form of $\lim F_\omega^k$ is pinned down by the absence of atoms together with bidders' zero payoff in the limit. Assume that, for large k, there are no atoms, that is, $F_\omega^k(\beta^k(x)) = (G_\omega(x))^{n_\omega^k}$; and assume for now that $\lim F_\omega^k$ is continuous and strictly increasing. Using the absence of atoms and recalling (3), the bidders' "zero payoff" observation (9) implies that, for sufficiently high k and x,

$$\beta^k(x) \approx \frac{v_\ell + \rho \frac{g_h(x)}{g_\ell(x)} \frac{n_h^k}{n_\ell^k} \frac{F_h^k(\beta^k(x))}{F_\ell^k(\beta^k(x))} v_h}{1 + \rho \frac{g_h(x)}{g_\ell(x)} \frac{n_h^k}{n_\ell^k} \frac{F_h^k(\beta^k(x))}{F_\ell^k(\beta^k(x))}}. \tag{13}$$

The limit of this equation is solved to obtain $\lim F_\omega^k$ as follows. Consider p such that $\lim F_\ell^k(p) = q \in (0,1)$ and recall that $x^k(q)$ is implicitly defined by $\left(G_\ell\left(x^k(q)\right)\right)^{n_\ell^k} = q$. Given the assumption on $\lim F_\omega^k$, by construction, $\beta^k(x^k(q)) \to p$, $x^k(q) \to \bar{x}$, and hence, $\frac{g_h(x^k(q))}{g_\ell(x^k(q))} \to \bar{g}$. Therefore, evaluating (13) at $x = x^k(q)$ and taking limits, we obtain

$$p = \frac{v_\ell + \rho \bar{g} r \frac{\lim F_h^k(p)}{q} v_h}{1 + \rho \bar{g} r \frac{\lim F_h^k(p)}{q}}. \tag{14}$$

Since $\lim \left(G_\ell\left(x^k(q)\right)\right)^{n_\ell^k} = q$, Lemma 1 implies

$$\lim F_h^k(p) = \lim \left(G_h\left(x^k(q)\right)\right)^{n_h^k} = q^{\bar{g} \lim(n_h^k/n_\ell^k)} = q^{\bar{g}r}.$$

Substituting this into (14) and solving for q, we get $q = \Phi_\ell(p|r)$ for every p. Thus, if it is strictly increasing and continuous, $\lim F_\omega^k = \Phi_\omega(\cdot|r)$. The proof in the appendix verifies that $\lim F_\omega^k$ must indeed be continuous and strictly increasing.

[13] The MLRP implies $\frac{G_h(x_-)}{G_\ell(x_-)} \leq \frac{G_h(x_+)}{G_\ell(x_+)}$, and hence $\frac{G_h(x_-)}{G_h(x_+)} \leq \frac{G_\ell(x_-)}{G_\ell(x_+)}$. Therefore, the right-hand side of (11) is weakly higher for $\omega = h$. The strict inequality below in (12) follows from $n_h > n_\ell$. If $n_h = n_\ell$, a strict inequality also follows if $\frac{g_h(x_-)}{g_\ell(x_-)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$, because then $\frac{G_h(x_-)}{G_\ell(x_-)} < \frac{G_h(x_+)}{G_\ell(x_+)}$.

3.4 Expected Revenue and Information Aggregation in Large Auctions

Theorem 1 has straightforward implications for the expected equilibrium revenue and information aggregation. In the "pooling" case of $\bar{g}r < 1$, the auction ends almost certainly with the same winning bid b^k in both states. Bidders' individual rationality then implies $\lim b^k \leq \mathbb{E}[v]$, for otherwise bidders' expected ex-ante payoff would be negative. This inequality may be strict, as we show by an example in Lauermann and Wolinsky (2017).

In the "partially separating" case of $\bar{g}r > 1$, the auction ends almost certainly with no tie at the winning bid and, from (9), it follows that $\beta^k(x) \approx \mathbb{E}^k[v|x,\text{win at } \beta^k(x)]$, for large k and for x with a significant winning probability.

These observations have immediate implications for the seller's expected revenue, $\mathbb{E}^k[p|\omega] \triangleq \mathbb{E}\left[p|\omega;\beta^k,n_\omega^k\right]$ for large k. In the "pooling" case, the revenue is the same across the states and may be strictly below the ex-ante expected value, $\mathbb{E}[v]$. In the "separating" case, the seller's ex-ante revenue is approximately equal to the ex-ante expected value, and the interim expected revenue is higher in state h.

Corollary 1 *Consider a sequence of bidding games* $\Gamma_0\left(\mathbf{n}^k, N^k\right)$ *with* $\min\{n_\ell^k, n_h^k\} \to \infty$, $\lim \frac{n_h^k}{n_\ell^k} = r$, *and a corresponding sequence of bidding equilibria* β^k.

1. *If* $\bar{g}r < 1$, *then*

$$\frac{v_\ell + \rho r \overline{g} v_h}{1 + \rho r \overline{g}} \leq \lim \mathbb{E}^k[p|\ell] = \lim \mathbb{E}^k[p|h] \leq \mathbb{E}[v].$$

2. *If* $\lim \bar{g}r > 1$, *then*

$$\lim \mathbb{E}^k[p|\ell] < \mathbb{E}[v] < \lim \mathbb{E}^k[p|h], \tag{15}$$

and

$$\rho_\ell \lim \mathbb{E}^k[p|\ell] + \rho_h \lim \mathbb{E}^k[p|h] = \mathbb{E}[v]. \tag{16}$$

Proof: The equality in Part 1 of the result is immediately implied by Part 1 of Theorem 1.

For Part 2, (16) follows from direct calculation using the explicit form of the winning bid distribution Φ_ω given by (5).[14] Then, (15) follows from (16) and the fact that Φ_h first-order stochastically dominates Φ_ℓ. ∎

The Extent of Information Aggregation is determined by $\bar{g}r (\equiv \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \lim \frac{n_h^k}{n_\ell^k})$. When $\bar{g}r < 1$, the price fails to aggregate the information since exactly the same price $\bar{b}^k$ prevails in both states with high probability. In the partially separating case of $\bar{g}r > 1$,

[14] The calculation is simplified by changing the integration variable to $y = \frac{1}{\rho \bar{g} r} \frac{p - v_\ell}{v_h - p}$ in the integral $\rho_\ell \lim \mathbb{E}^k[p|\ell] + \rho_h \lim \mathbb{E}^k[p|h] = \rho_\ell \int p d\Phi_\ell(p) + \rho_h \int p d\Phi_\ell(p)$. Alternatively, it follows from (9) and the law of iterated expectations.

the extent of aggregation can be evaluated by comparing the limit distributions of the winning bid in the two states. Inspection of (5)-(6) reveals that, when $\bar{g}r$ is near 1, then Φ_h and Φ_ℓ are nearly identical; and when $\bar{g}r$ is large, then Φ_ω is concentrated near v_ω in both states and actually approaches a mass point on v_ω as $\bar{g}r \to \infty$. Since the price is always in $[v_\ell, v_h]$, the latter is equivalent to $\mathbb{E}^k[p|\omega]$ approaching v_ω as $\bar{g}r \to \infty$.

Thus, a price observation is not a very informative signal of the state if $\bar{g}r$ is near 1, but it is so if $\bar{g}r$ is very large. More formally, the price is a signal of the state generated by the information structure (Φ_ℓ, Φ_h). Observe that, if (Φ_ℓ, Φ_h) corresponds to $\bar{g}r = \alpha > 1$ and $(\widetilde{\Phi}_\ell, \widetilde{\Phi}_h)$ corresponds to $\bar{g}r = \widetilde{\alpha} \in (1, \alpha)$, then the former is more informative in the sense of Lehman (1988)'s criterion. To see this, consider a probability q of a Type I error (rejecting the hypothesis that $\omega = h$ when it is true), and let $p(q)$ and $\widetilde{p}(q)$ be the thresholds that achieve it, $q = \Phi_h(p(q)) = \widetilde{\Phi}_h(\widetilde{p}(q))$. The corresponding Type II errors satisfy $1 - \Phi_\ell(p(q)) = 1 - q^{1/\alpha} < 1 - q^{1/\widetilde{\alpha}} = 1 - \Phi_\ell(\widetilde{p}(q))$, which implies Lehman's ranking. Since in this two state environment Lehman's ranking is equivalent to Blackwell's ranking (Jewitt, 2007), (Φ_ℓ, Φ_h) is more informative by that criterion as well.

Corollary 2 *Consider a sequence of bidding games $\Gamma_0\left(\mathbf{n}^k, N^k\right)$ such that $\min\{n_\ell^k, n_h^k\} \to \infty$, $\lim \frac{n_h^k}{n_\ell^k} = r$, and $\bar{g}r > 1$, and a corresponding sequence of bidding equilibria β^k. For any $\varepsilon > 0$, there are $\overline{\delta}$ and $\underline{\delta}$ such that $\overline{\delta} > \underline{\delta} > 1$ and*

$$\begin{aligned}
&\text{if} \quad \lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} > \overline{\delta}, \quad \text{then} \quad |\lim \mathbb{E}^k[p|\omega] - v_\omega| < \varepsilon, \qquad \omega = \ell, h; \text{ and} \\
&\text{if} \quad \lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \underline{\delta}, \quad \text{then} \quad |\lim \mathbb{E}^k[p|\omega] - \mathbb{E}[v]| < \varepsilon, \quad \omega = \ell, h.
\end{aligned}$$

Moreover, with higher $\bar{g}r$, the price is a more informative signal of the state in the sense of Blackwell's criterion.

3.5 A Comment on the Borderline Case

In the borderline case of $\bar{g}r = 1$, the distribution of the winning bid becomes degenerate on $\mathbb{E}[v]$, so that the expected price is independent of the state, with $\lim \mathbb{E}_h^k(p) = \lim \mathbb{E}_\ell^k(p) = \mathbb{E}[v]$. We do not know whether β^k is strictly increasing for large k but becoming increasingly flat at the top or β^k itself contains atoms, i.e., is flat for large k.

Proposition 3 *Consider a sequence of bidding games $\Gamma_0\left(\mathbf{n}^k, N^k\right)$ such that $\min\{n_\ell^k, n_h^k\} \to \infty$, $\lim \frac{n_h^k}{n_\ell^k} = r$, and $\bar{g}r = 1$. If β^k is a bidding equilibrium of $\Gamma_0\left(\mathbf{n}^k, N^k\right)$ for all k, then the limit of the winning bid distribution is a mass point on $\mathbb{E}[v]$, i.e., $\lim F_\omega^k(b) = 1$ for all $b > \mathbb{E}[v]$ and $= 0$ for all $b < \mathbb{E}[v]$.*

4 Strategic Solicitation—Model and Preliminaries

So far, the solicitation decision $\mathbf{n} = (n_\ell, n_h)$ has been taken as given, without inquiring about the seller's motives. In this section, we endogenize this decision in a specific way. The seller incurs a cost s per invited bidder and decides on the number optimally, given the anticipated bidding behavior. So, if n bidders are solicited and the winning bid is p, then the seller's payoff is $p - ns$. We can now inquire about the nature of participation patterns (n_ℓ, n_h) may emerge in equilibrium.

4.1 Strategic Solicitation and Equilibrium

Let $\Gamma(s)$ be the game that includes both, the strategic bidder solicitation by the seller and the strategic bidding by the buyers. A bidding strategy β is as before. A solicitation strategy $\mathbf{n} = (n_\ell, n_h)$, such that $1 \leq n_\omega \leq N_s$, prescribes the number of bidders solicited by the seller in each state. The potential number of bidders N_s is such that $N_s \geq \frac{v_h}{s}$, which guarantees that it is never profitable for the seller to solicit all potential bidders. The restriction $n_\omega \geq 1$ is imposed to avoid dealing with trivial equilibria. The expected winning bid in state ω is $\mathbb{E}[p|\omega; \beta, n]$.

A *pure equilibrium* of $\Gamma(s)$ consists of a strategy β and a solicitation strategy $\mathbf{n} = (n_\ell, n_h)$ such that (i) β is a bidding equilibrium of $\Gamma_0(\mathbf{n}, N_s)$, and (ii) the solicitation strategy is optimal for the seller, i.e.,

$$n_\omega \in \underset{n \in \{1,2,\ldots,N_s\}}{\arg\max} \left(\mathbb{E}[p|\omega; \beta, n] - ns\right).$$

Since a pure equilibrium might not exist, we allow for mixed solicitation strategies. Let $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$ denote a mixed solicitation strategy, where $\eta_\omega(n)$ is the probability with which $n = 1, \ldots, N_s$ bidders are invited in state ω.

The expected payoff $U(b|x; \beta, \boldsymbol{\eta})$ and the probability of winning $\pi_\omega(b|\beta, \boldsymbol{\eta})$ are now functions of the mixed strategy $\boldsymbol{\eta}$. Some explicit expressions of these magnitudes that are needed for the proofs are stated in Subsection B.1 of the appendix.

In a complete analogy to the definitions for pure strategies, $\Gamma_0(\boldsymbol{\eta}, N)$ is the bidding game given $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$, and $\Gamma(s)$ is the full game that includes the strategic solicitation. A bidding equilibrium of $\Gamma_0(\boldsymbol{\eta}, N)$ is a strategy β such that, for all x, $b = \beta(x)$ maximizes $U(b|x; \beta, \boldsymbol{\eta})$.

The strategy profile $(\beta, \boldsymbol{\eta})$ is an *equilibrium* of $\Gamma(s)$ if (i) β is a bidding equilibrium of $\Gamma_0(\boldsymbol{\eta}, N_s)$ and (ii) the solicitation strategy is optimal,

$$\eta_\omega(n) > 0 \Rightarrow n \in \underset{n \in \{1,2,\ldots,N_s\}}{\arg\max} \left(\mathbb{E}[p|\omega; \beta, n] - ns\right).$$

4.2 Optimal Solicitation Strategies are Essentially Pure

If η_ω is optimal, its support is either a single integer n or two adjacent integers $\{n, n+1\}$. This is because the seller's payoff, $\mathbb{E}[p|\omega;\beta,n] - ns$, is strictly concave in n—unless β is constant, in which case $n = 1$ is optimal.

Lemma 2 **(Optimal Solicitation)** *Given any bidding strategy β, there is an integer n^*_ω such that*

$$\{n^*_\omega, n^*_\omega + 1\} \supseteq \underset{n\in\{1,2,\cdots,N\}}{\arg\max}\ \mathbb{E}[p|\omega;\beta,n] - ns.$$

This result is proven in Lauermann and Wolinsky (2017). It is familiar from other contexts and is an immediate consequence of the concavity of the expectation of the first-order statistic in n.

Given the lemma, we restrict attention to mixed strategies η whose support contains at most two adjacent integers. Any such mixed strategy η_ω can be described by two numbers, namely, $n_\omega \in \{1, ..., N\}$ and $\gamma_\omega \in (0,1]$, where $\gamma_\omega = \eta_\omega(n_\omega) > 0$ and $1 - \gamma_\omega = \eta_\omega(n_\omega + 1) \geq 0$. A solicitation strategy is pure if $\gamma_\omega = 1$. Thus, from here on, when we talk about n_ω in the context of a strategy η_ω, we mean the bottom of the support of η_ω.

The introduction of mixed strategies does not alter the main qualitative features of the bidding equilibria. In Section B.3 of the Appendix, we state analogous versions of Proposition 1, Theorem 1, and Proposition 3 for the case of a mixed solicitation strategy η and call them Proposition 1', Theorem 1', and Proposition 3', respectively.

In particular, any bidding equilibrium is still monotone without loss of generality when the solicitation strategy is mixed. This is because the seller's mixing is of no consequence for the single crossing property of the buyers' preferences. Moreover, the characterization of the bidding equilibria of large auctions in Theorem 1 and Proposition 3 extends to mixed solicitation strategies η that are optimal. This is not surprising because, by Lemma 2, solicitation strategies that are optimal are almost pure when $\min\left\{n^k_\ell, n^k_h\right\} \to \infty$.

5 Equilibria with Small Sampling Costs

In Section 3, where the numbers of bidders are exogenous, the many-bidders case was considered by looking at a sequence of bidding games, $\Gamma_0\left(\mathbf{n}^k, N^k\right)$ in which $n^k_\omega \to \infty$, $\omega = \ell, h$, and a corresponding sequence of bidding equilibria β^k. Since the game now includes strategic solicitation, the primitive here is a sequence

$$\{s^k\}_{k=1}^{\infty}, \text{ with } s^k > 0 \text{ and } s^k \to 0. \tag{17}$$

This sequence induces a sequence of games $\Gamma\left(s^k\right)$, and we consider a corresponding sequence of equilibrium bidding and solicitation strategies $(\beta^k, \boldsymbol{\eta}^k)$, with $\boldsymbol{\eta}^k = (\eta^k_\ell, \eta^k_h)$. Here, too, we look at the limits of equilibrium magnitudes as $k \to \infty$. As it will turn out, with $s^k \to 0$, optimal solicitation usually results in $n_\omega^k \to \infty$. Therefore, here too $k \to \infty$ is associated with ever larger numbers of solicited bidders. We continue with the simplifications adopted above of omitting $(\beta^k, \boldsymbol{\eta}^k)$ from the arguments of equilibrium magnitudes and omitting the delimiter $k \to \infty$ from the expression lim.

5.1 Characterization of Equilibrium with Endogenous Solicitation

Recall the shorthand $F_\omega^k(p)$ for the c.d.f. of the winning bid. We study $\lim_{k\to\infty} F_\omega^k(p)$, thinking of it as an approximation for the winning bid distribution when s is small.[15] Following is our second main theorem.

Theorem 2 **(Equilibrium Characterization)** *There exists a number r^* such that for every sequence of games $\Gamma\left(s^k\right)$ with $s^k \to 0$ and every corresponding sequence of equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)$, one of the following two statements holds (but not both):*

1. *$\lim F_\omega^k$ is degenerate, with probability mass 1 on some number $C \leq \rho_\ell v_\ell + \rho_h v_h$ for $\omega = \ell, h$.*
2. *$\lim F_\omega^k = \Phi_\omega(\cdot|r^*)$, $\omega = \ell, h$, where $r^* = \lim \frac{n_h^k}{n_\ell^k}$ and $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$ (and where Φ_ω is described by (5)-(6)).*

Thus, in the limit, an equilibrium winning bid distribution, $\lim F_\omega^k$, takes one of two forms. Either it is a mass point below the ex-ante expected value, or it is the partially revealing function $\Phi_\omega(\cdot|r^*)$, with a uniquely determined limit ratio r^*. Equilibria of both types may coexist—and sometimes they do; we take up the question of equilibrium existence later.

Theorem 2 makes two new points relative to what we already know from Theorem 1'. First, it also covers the case of $\min\left\{n_\ell^k, n_h^k\right\} \nrightarrow \infty$ (not addressed by Theorem 1') to account for the possibility that optimal sampling results in a bounded number even as $s \to 0$.[16] This first point is fairly obvious, though it requires some work to establish.

Second, optimal sampling pins down the ratio $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k}$ for the partially revealing equilibrium at r^*. As we show below, given any $\bar{g}$ and ρ, the ratio r^* is the unique $r > \frac{1}{\bar{g}}$ that solves

$$\int_0^1 (x - \frac{1}{\bar{g}}) x^{\frac{1}{\bar{g}r-1}} \frac{\ln x}{(1 + x\rho\bar{g}r)^2} dx = 0. \tag{18}$$

Note that r^* depends only on $\bar{g}$ and ρ, and is independent of the other parameters (namely, the exact shape of the distribution G and the values v_ω). This second point is the more distinct contribution of the theorem.

[15] Also, recall that we represent a mixed equilibrium strategy η_w by n_w and γ_w, where $\gamma_w = \eta_w(n_w)$ and $1 - \gamma_w = \eta_w(n_w + 1)$, utilizing Lemma 2.

[16] This part also covers the trivial equilibrium with $\mathbf{n} = (1, 1)$ and $\beta(x) \equiv 0$ (subject to the constraint $n_\omega \geq 1$).

5.2 Proof of Theorem 2

We use a result of Lauermann and Wolinsky (2018) that establishes a relationship between the total solicitation cost and the distribution of the winning bid in first-price auctions with bidder solicitation. Specifically, for the common-value environment of this paper, it implies the following lemma.

Lemma 3 **(Total Solicitation Costs)** *Consider a sequence $s^k \to 0$ and a sequence of bidding strategies β^k. Suppose that η_ω^k is an optimal solicitation strategy given β^k in state ω, and the implied winning bid distribution F_ω^k converges pointwise. Then,*

$$\lim_{k\to\infty} n_\omega^k s^k = -\int_0^{v_h} \left(\lim_{k\to\infty} F_\omega^k(p)\right) \ln\left(\lim_{k\to\infty} F_\omega^k(p)\right) dp.$$

That is, the total cost of the optimal solicitation is proportional to a certain "dispersion" measure of the winning bid.

Proposition 4 addresses the main new insight of Theorem 2: the optimal sampling pins down $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k}$.

Proposition 4 *Consider a sequence of games $\Gamma(s^k)$ such that $s^k \to 0$, and a corresponding sequence of equilibria $(\beta^k, \boldsymbol{\eta}^k)$. Given ρ and $\bar{g}$, there exists a unique number $r^* \triangleq r^*(\rho, \bar{g}) > \frac{1}{\bar{g}}$ such that if $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\bar{g}\lim_{k\to\infty}\frac{n_h^k}{n_\ell^k} > 1$, then $\lim_{k\to\infty}\frac{n_h^k}{n_\ell^k} = r^*$. The number r^* is the unique solution to (18).*

Proof of Proposition 4: Let $r = \lim \frac{n_h^k}{n_\ell^k}$, and suppose that $r < \infty$ and $\bar{g}r > 1$. By Theorem 1', $\lim_{k\to\infty} F_\omega(p|\beta^k, \eta_\omega^k) = \Phi_\omega(\cdot|r)$. This and Lemma 3 together imply that

$$\lim_{k\to\infty} n_\omega^k s^k = -\int_{v_\ell}^{\frac{v_\ell+\rho\bar{g}rv_h}{1+\rho\bar{g}r}} (\Phi_\omega(p|r)) \ln(\Phi_\omega(p|r))\, dp.$$

Since $\lim(n_h^k s^k) = r\lim(n_\ell^k s^k)$, it follows that

$$\frac{1}{r}\int_{v_\ell}^{\frac{v_\ell+\rho\bar{g}rv_h}{1+\rho\bar{g}r}} (\Phi_h(p|r)) \ln(\Phi_h(p|r))\, dp = \int_{v_\ell}^{\frac{v_\ell+\rho\bar{g}rv_h}{1+\rho\bar{g}r}} (\Phi_\ell(p|r)) \ln(\Phi_\ell(p|r))\, dp. \tag{19}$$

In the appendix, we rewrite (19) using the explicit characterization of Φ_ω to prove the following lemma.

Lemma 4 *For any $\rho > 0$ and $\bar{g} > 1$, there is a unique number $r^* \in (\frac{1}{\bar{g}}, \infty)$ such that equation (19) holds. It is the unique $r > \frac{1}{\bar{g}}$ that solves (18).*

It follows from (19) and Lemma 4 that for any sequence of equilibria with $\bar{g}\lim\frac{n_h^k}{n_\ell^k} > 1$ and $\lim\frac{n_h^k}{n_\ell^k} < \infty$, it must be the case that $\lim\frac{n_h^k}{n_\ell^k} = r^*$.

Thus, Proposition 4 holds if $\lim \frac{n_h^k}{n_\ell^k} < \infty$ for any such sequence. Suppose to the contrary that $\lim \frac{n_h^k}{n_\ell^k} = \infty$. Then, Theorem 1' implies that $\lim F_\omega\left(\cdot|\beta^k, \eta_\omega^k\right)$ is a degenerate distribution with support v_ω. Lemma 3 implies that $\lim n_\omega^k s^k = 0$, so that seller type ω's equilibrium payoff converges to v_ω.

By reasoning analogously to Lemma 1, if $\lim F_h\left(p|\beta^k, \eta_h^k\right) = 0$, then $\lim F_\ell\left(p|\beta^k, \eta_h^k\right) = 0$. Therefore, if seller type ℓ solicits n_h^k bidders, $\lim \mathbb{E}\left[p|\ell; \beta^k, n_h^k\right] \geq v_h$. Since $\lim n_h^k s^k = 0$, for large k, seller type ℓ's payoff with this strategy is near v_h, which is larger than her equilibrium payoff near v_ℓ. Thus, $\lim \frac{n_h^k}{n_\ell^k} = \infty$ cannot hold. ∎

Theorem 1' dealt only with the case of $\min\left\{n_\ell^k, n_h^k\right\} = \infty$. We now state Proposition 5, which addresses the possibility of optimal solicitation resulting in a bounded number of participants.

Proposition 5 *Consider a sequence of games $\Gamma\left(s^k\right)$ such that $s^k \to 0$. Suppose that $(\beta^k, \boldsymbol{\eta}^k)$ is a corresponding sequence of equilibria such that $\min\left\{n_\ell^k, n_h^k\right\} \nrightarrow \infty$. Then $\lim_{k\to\infty} F_\omega^k$ has probability mass 1 on some $C \leq \rho_\ell v_\ell + \rho_h v_h$, for both $\omega = \ell$ and $\omega = h$.*

The previous results now combine to prove the theorem.

Proving Theorem 2.—First, suppose the limit distribution is not degenerate. Proposition 5 implies that then $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$. Given this and the hypothesis that the limit distribution is not degenerate, Theorem 1' and Proposition 3' together require that $\bar{g}\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} > 1$. Then, Proposition 4 implies that $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} = r^*\left(\rho, \bar{g}\right)$. Given that, Theorem 1' implies the exact form of the limit distribution.

Second, suppose the limit distribution is degenerate in state h. If, in addition, $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$, then it must be concentrated on $C \leq \rho_\ell v_\ell + \rho_h v_h$, either by Theorem 1' and Proposition 4 or by Proposition 3'. Otherwise, if $\min\left\{n_\ell^k, n_h^k\right\}$ remains bounded, Proposition 5 yields the same conclusion.

Thus, if the limit distribution is not degenerate, then the limit outcome is as in the first part of Theorem 2; otherwise, if the limit distribution is degenerate, it is as stated in its second part. This finishes its proof. ∎

6 Existence of Equilibria with a Grid

This section shows the existence of equilibria when bids must be chosen from a grid. In this case, a partially separating equilibrium of the form described in Part 2 of Theorem 1 always exists, while a pooling equilibrium of the form described in Part 1 exists at least under some conditions on the signal distribution. At the end of this section, we discuss the difficulty of establishing the existence of an equilibrium when bids can be chosen from the continuum $[0, v_h]$. All proofs from this section are in an online appendix.[17]

[17] Appended to the end of the present version.

6.1 Finite Grid: Setup

The set of feasible bids is now P_Δ, defined as

$$P_\Delta \triangleq [0, v_\ell] \cup \{v_\ell + \Delta, v_\ell + 2\Delta, \cdots, v_h - \Delta, v_h\},$$

where $\Delta > 0$ is the step size of the grid. Notice that we leave the continuum of prices on $[0, v_\ell]$. This avoids some irrelevant distinctions between the case in which the bottom equilibrium bid is v_ℓ and the case in which it is $v_\ell - \Delta$. The continuum set of possible bids considered so far is denoted by $P_0 = [0, v_h]$.

The description of the bidding game and of the full game will now include the grid, $\Gamma_0(\boldsymbol{\eta}, N, \Delta)$ and $\Gamma(s, \Delta)$ respectively. Other definitions, including that of an equilibrium, are naturally extended. With a finite grid, existence of bidding equilibria for any $\mathbf{n}$ is guaranteed.

Proposition 6 *Given $\Delta > 0$, every bidding game $\Gamma_0(\mathbf{n}, N, \Delta)$ has a bidding equilibrium β.*

Since it is a fairly immediate implication of standard existence results, Proposition 6 does not require a separate proof.

A full equilibrium exists for any $\Delta \geq 0$ because there is always a trivial equilibrium with $n_\ell = n_h = 1$ and $\beta \equiv 0$ (subject to the constraint $n_\omega \geq 1$). Thus, our existence results in this part concern nontrivial equilibria of the alternative forms discussed above.

As before, we will look at a sequence of games—either $\Gamma_0(\boldsymbol{\eta}^k, N^k, \Delta^k)$ or $\Gamma(s^k, \Delta^k)$—with the additional requirements that $\Delta^k > 0$, $\lim \Delta^k = 0$, and $P_{\Delta^k} \subset P_{\Delta^{k+1}}$, which will henceforth be assumed whenever we mention such a sequence of games.[18]

The introduction of the grid (like the introduction of mixed strategies earlier) does not alter the main qualitative features of the equilibrium. Corresponding versions of Theorems 1 and 2, named Theorem 1" and 2", also hold for the model with a finite grid that becomes finer along the sequence. Those theorems are stated and proven[19] in Section D.1 in the Online Appendix.

6.2 Existence of Partially Separating Equilibria

The full game always has an equilibrium whose winning bid distribution is $\Phi_\omega(\cdot|r^*)$ in the limit.[20]

Theorem 3 *For any sequence of games $\Gamma(s^k, \Delta^k)$, such that $\lim(s^k, \Delta^k) = (0, 0)$, there exists a sequence of equilibria that converges to the partially revealing outcome of Part 2 of Theorem 2.*

[18] Not all arguments require the strict $\Delta > 0$. We will point it out when it is of interest.

[19] The grid necessitates adjustment of the proofs when "slight undercutting" arguments are used.

[20] No condition is needed for the relative speed of convergence for Δ^k and s^k.

The proof introduces "constrained equilibria" $\left(\beta^k, \boldsymbol{\eta}^k\right)$ such that β^k is a bidding equilibrium given $\boldsymbol{\eta}^k$, as before, but $\boldsymbol{\eta}^k$ are optimal subject to the following constraints: (i) n_ω^k is forced to increase without a bound, i.e., $n_\omega^k \geq \underline{n}_\omega^k \to \infty$; and (ii) $\frac{n_h^k}{n_\ell^k} \geq \bar{r}$ for some $\bar{r} \in (\frac{1}{\bar{g}}, r^*)$. By Theorem 1, such a constrained equilibrium $\left(\beta^k, \boldsymbol{\eta}^k\right)$ must induce a non-degenerate winning bid distribution. The proof then uses two Lemmas. The first lemma establishes that, if a sequence of constrained equilibria $\left(\left(\beta^k, \boldsymbol{\eta}^k\right)\right)_{k=1}^{\infty}$ exists, then for large enough k, the constraints do not bind; therefore, $\left(\beta^k, \boldsymbol{\eta}^k\right)$ is an unconstrained equilibrium. This and Theorem 2 then imply that the limiting winning bid distribution is $\Phi_\omega\left(\cdot|r^*\right)$. The second lemma uses essentially standard existence arguments to establish that, for any sequence $\left(s^k, \Delta^k\right) \to (0,0)$ for which $\Delta^k > 0$ for all k, a constrained equilibrium exists. Together these lemmas establish the Theorem.

The key step in the proof is that, for k large enough, the constraint $\frac{n_h^k}{n_\ell^k} \geq \bar{r}$ does not bind. In particular, it cannot be the case that $\lim \frac{n_h^k}{n_\ell^k} = \bar{r}$ over a sequence of constrained equilibria. This is easier to see when $\bar{r}$ is close to $\frac{1}{\bar{g}}$. In this case, $\lim \frac{n_h^k}{n_\ell^k} = \bar{r}$ implies that $\bar{g} \lim \frac{n_h^k}{n_\ell^k}$ is close to 1, and hence $\Phi_\ell\left(\cdot|\bar{r}\right)$ is close to $\Phi_h\left(\cdot|\bar{r}\right)$. Therefore, by Lemma 3, the optimal n_ℓ^k and n_h^k would be similar as well, so that $\lim \frac{n_h^k}{n_\ell^k} \approx 1 > \frac{1}{\bar{g}} \approx \bar{r}$. Thus, $\lim \frac{n_h^k}{n_\ell^k} = \bar{r}$ cannot be a fixed point of the limit of constrained equilibria.

The grid is used only to show the existence of an equilibrium of the auxiliary game (in the first lemma of the proof). The second lemma of the proof characterizes the equilibria of the auxiliary game and holds for both, $\Delta^k > 0$ and $\Delta^k = 0$.

6.3 Existence of a Nontrivial Pooling Equilibrium

We establish here the existence of a non-trivial pooling equilibrium for a general class of discrete signals, that is, when the likelihood ratio $\frac{g_h}{g_\ell}$ is a step function with finite but arbitrary number of steps. This does not mean that such equilibria exist only under those circumstances, but rather that we know how to construct an equilibrium in these cases.

This extends a result in Lauermann and Wolinsky (2017) on the existence of a pooling equilibrium with a binary signal. The simplicity of that case allowed us to prove it without imposing a grid. Hence, the proof here is not an immediate extension.

Condition 1. Discrete Signals. The range of the signal values $[\underline{x}, \bar{x}]$ is divided into m subintervals, as follows:

$$[\underline{x}, \epsilon], (\epsilon, 2\epsilon], \cdots, (\bar{x} - \epsilon, \bar{x}]\,.$$

The density functions g_ω are step functions that are constant over each of these intervals.

Condition 2. Strengthening MLRP. The likelihood ratios satisfy

$$\frac{1}{G_\ell(\bar{x}-\epsilon)} < \frac{g_h(\bar{x})}{g_\ell(\bar{x})}, \tag{20}$$

and

$$\frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\frac{G_\ell(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)} \leq \frac{g_h(\bar{x})}{g_\ell(\bar{x})}. \tag{21}$$

Condition 1 means that, as far as the information is concerned, this is a discrete signal structure with m values. Consequently, the likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ is a step function as well, so there are at most m different likelihood ratios.[21] The continuum signal set is kept only for purification purposes.

Condition 2 can be thought of as a strengthening of the increasing likelihood-ratio requirement at the top.[22] The first part is naturally satisfied if ϵ is not too large, since then $G_\ell(\bar{x}-\epsilon)$ is near 1, while $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} > 1$. Both requirements of Condition 2 are satisfied, for example, when $G_\omega(x) = x^{z_\omega}$, with $z_h > z_\ell$.

Theorem 4 Existence of a Pooling Equilibrium *Consider a sequence of games* $\Gamma\left(s^k, \Delta^k\right)$ *such that* $\lim\left(s^k, \Delta^k\right) = (0,0)$*. Suppose that the signals satisfy Conditions 1 and 2 above. Then, there exist bids* $\underline{b} < \bar{b} < \rho_\ell v_\ell + \rho_h v_h$ *and a sequence of equilibria* $(\beta^k, \boldsymbol{\eta}^k)$ *such that* $\min\{n_\ell^k, n_h^k\} \to \infty$ *and*

$$\beta^k(x) \begin{cases} = \bar{b} & \text{if} \quad x > \bar{x}-\epsilon, \\ \leq \underline{b} & \text{if} \quad x \leq \bar{x}-\epsilon, \end{cases} \qquad \text{for sufficiently large } k.$$

Thus, for this sequence of equilibria, the winning bid converges to $\bar{b}$ almost surely. Since we look only at sequences such $P_{\Delta^k} \subset P_{\Delta^{k+1}}$, we are assured that $\bar{b} \in P_{\Delta^k}$, for all sufficiently large k. However, essentially the same result can be proved without this assumption on the grids by looking at a sequence of $\bar{b}^k$.

6.4 Existence without Grid

The difficulty in showing existence without a grid is the presence of atoms in equilibrium. Because there can be atoms, the buyers' *equilibrium* payoffs can be discontinuous in their bids. In particular, we cannot argue that the limit of a sequence of equilibria for a vanishingly small grid is an equilibrium of the continuum case. The reason is that there may be atoms in the limit that are absent in the sequence. To illustrate the problem, consider a sequence of games with grid P_{Δ^k} and suppose that along the sequence bidders bid either b or $b+\Delta^k$, depending on whether their signal is below or above some threshold

[21] The important assumption here is the finiteness of the set of values that the likelihood ratio takes on. Density functions that are also step functions are consistent with that assumption but are not necessary.

[22] In fact, for the existence proof we only need the implication that $\frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\frac{G_\ell(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)}\frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} \leq 1$. This inequality is implied by $\frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\frac{G_\ell(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)} \leq \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$ since $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = \frac{1-G_h(\bar{x}-\varepsilon)}{1-G_\ell(\bar{x}-\varepsilon)} < \frac{\ln G_h(\bar{x}-\varepsilon)}{\ln G_\ell(\bar{x}-\varepsilon)}$ because $\frac{1-z}{\ln z}$ is decreasing in z.

$\hat{x}$. The pointwise limit strategy as $\Delta^k \to 0$ would be that all bidders bid the constant b. However, the bidding strategy implies a strictly lower winning probability for buyers who bid $b + \Delta^k$ along the sequence, and a strictly higher winning probability for buyers bidding b (and the two winning probabilities are bounded away from each other). Thus, the limit strategy may not be an equilibrium of the game with a continuum of bids, even though the elements of the sequence may have been.[23]

A possible solution to the existence problem without a grid is to change the tie-breaking rule, as suggested by Jackson, Simon, Swinkels, and Zame (2002). Specifically, consider the following extension: Buyers submit two numbers, the first interpreted as a bid (just as before) and the second number interpreted as eagerness to trade. If there is a unique highest bid, the seller chooses to buy from that bidder. When several bids are tied, the seller may choose among the buyers based on their expressed eagerness. Extending our model in this way solves the existence problem, because the limit of a sequence of equilibrium strategies for a vanishingly small grid corresponds to an equilibrium of the extended game with a continuum of bids. For instance, in the example from the last paragraph, one may specify as the limit strategy of the extended game that buyers bid b for all signals. Buyers with signals above the threshold (i.e., those who bid $b + \Delta^k$ along the sequence) all express the same eagerness, say $\overline{e}$, and buyers with signals below the threshold (i.e., those who bid b along the sequence) express a different eagerness, say, $\underline{e}$. If multiple bidders are tied at b, then the seller picks first from among those bidders who express $\overline{e}$ and chooses randomly if there are multiple such bidders; if no bidder expressed $\overline{e}$, the seller chooses randomly among bidders expressing $\underline{e}$ (and, finally, chooses bidders who expressed anything else last). This limit strategy preserves the winning probabilities, and, hence, preserves the payoffs in a continuous way. Thus, if the elements of the described sequence of bidding strategies each constitute an equilibrium, so would the limit.[24]

7 Discussion and Extensions

7.1 Comparison with the Ordinary Common-Value Auction

The large ordinary common-value auction is the special case of $n_h = n_\ell = n$ and, by Theorem 1, the limit of the winning bid distribution for this case[25] is $\Phi_\omega(p|r = 1)$, $\omega = \ell, h$. The case of $n_h = n_\ell = n$ is of special interest since it fits the situation in which the seller

[23] There is no such problem for the seller's strategy because of the continuity of the seller's payoffs in β and η. If (β^k, η^k) converge pointwise to (β^*, η^*), and if η^k is an optimal solicitation strategy given β^k for all k, then η^* is an optimal solicitation strategy given β^*.

[24] In ongoing unpublished work, Lauermann and Speit, "Common value auctions with an uncertain number of bidders," show that equilibrium may indeed not exist in a related auction model with a state-independent, Poisson distributed number of bidders and verify the applicability of Jackson et al (2002).

[25] While this closed form solution does not appear in the literature, it is essentially implied by the analysis of Murto and Valimaki (2015b).

does not know the state.

We now compare the equilibrium outcome of this case to that arising in the case of an informed seller (as considered in this paper) in the partially revealing equilibrium with bidder solicitation. Consider a sequence of bidding games $\Gamma_0\left(\boldsymbol{\eta}^k, N^k\right)$ for which $\lim \frac{n_h^k}{n_\ell^k} = r$ and $\bar{g}r > 1$, and a corresponding sequence of bidding equilibria β^k. Since the distributions of the winning bid Φ_ω are functions of ρ, $\overline{g}$, and r, we can express the limit expected revenue as a function of these parameters:

$$\bar{E}_\omega\left[p|\rho,\overline{g},r\right] \triangleq \int pd\Phi_\omega\left(p|r;\rho,\overline{g}\right).$$

The limit revenue for the ordinary common-value auction is then $\bar{E}_\omega\left[p|\rho,\overline{g},r=1\right]$, while the limit revenue for the equilibrium with endogenous solicitation is $\bar{E}_\omega\left[p|\rho,\overline{g},r^*\right]$, where r^* is determined by Equation (18). Since $\Phi_h\left(p|r\right)$ is decreasing in r, it follows that $\bar{E}_h\left[p|\rho,\overline{g},1\right] > \bar{E}_h\left[p|\rho,\overline{g},r^*\right]$, if $r^* < 1$ (i.e., when there is a *solicitation curse*), while the inequality is reversed if $r^* > 1$ (i.e., when there is a *solicitation blessing*). So, when $r^* < 1$, there is less information revelation with bidder solicitation than in the ordinary auction; when $r^* > 1$, there is more information revelation with bidder solicitation. Figure 1 illustrates the shape of the ratio r^* as a function of $\overline{g}$ for a uniform prior, $\rho = 1$ (i.e., $\rho_h = \rho_\ell$), and compares the expected revenue of each type of seller with and without solicitation. As the figure shows, when $\bar{g}$ is small, $r^* < 1$ and when $\bar{g}$ is large, $r^* > 1$.[26]

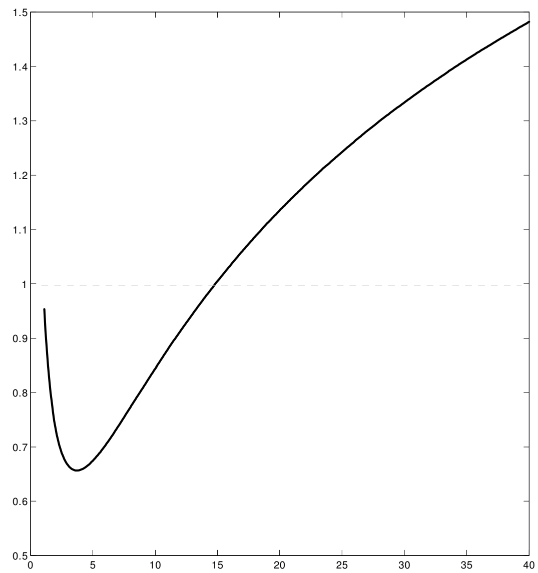
7.2 Affiliation of the First-Order Statistic and the State

The emergence of atoms can also be explained by the failure of the affiliation between the value and the highest signal. Let $y_{[\mathbf{n}]}$ denote the highest signal realization given participation $\mathbf{n} = (n_\ell, n_h)$. The likelihood ratio of $y_{[\mathbf{n}]} = x$ is

$$\frac{n_h}{n_\ell}\frac{g_h\left(x\right)}{g_\ell\left(x\right)}\frac{\left(G_h\left(x\right)\right)^{n_h-1}}{\left(G_\ell\left(x\right)\right)^{n_\ell-1}}. \tag{22}$$

In standard auctions with $n_h = n_\ell = n$, this likelihood ratio is increasing in x. Thus, the statistic $y_{[\mathbf{n}]}$, which in this case coincides with the first-order statistic $x_{(n)}$, is affiliated with the value. In contrast, with state-dependent participation, the likelihood ratio (22) need not be increasing—in fact, it is *decreasing* for x sufficiently close to $\bar{x}$ if $\frac{n_h}{n_\ell}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1$. Therefore, $y_{[\mathbf{n}]}$ might not be affiliated with the value.

[26]Figure 1 is generated by a numerical computation of the solution using maple. We conjecture that one can find a cutoff $\hat{g}\left(\rho\right)$ for all ρ such that $r^* \gtrless 1$ if $\bar{g} \gtrless \hat{g}\left(\rho\right)$, but we have not been able to verify this conjecture analytically.



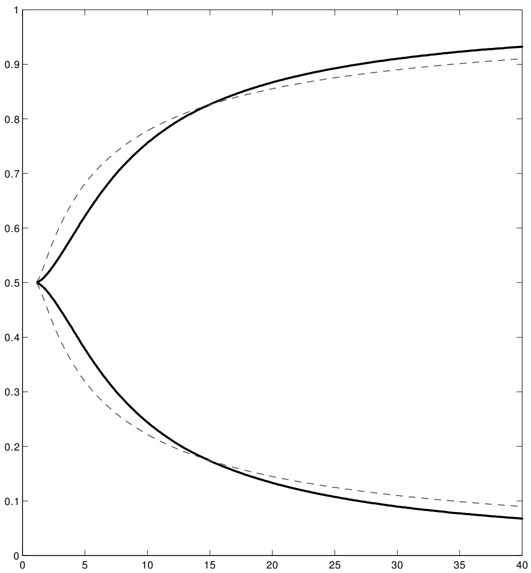


Figure 1: Left Panel: The ratio of the number of sampled bidders, r^*, as a function of $\bar{g}$, given $\rho = 1$. Right Panel: Expected revenue as functions of $\bar{g}$ given $\rho = 1$. Straight black lines are expected revenues with solicitation, $\bar{E}_h\left[p|\overline{g}, r^*\right]$ (top) and $\bar{E}_\ell\left[p|\overline{g}, r^*\right]$ (bottom); dashed grey lines are the expected revenues of the ordinary common-value auction, $\bar{E}_h\left[p|\overline{g}, r = 1\right]$ (top) and $\bar{E}_\ell\left[p|\overline{g}, r = 1\right]$ (bottom).

7.3 Simultaneous Search

Although our model is framed using the terminology of auctions, it can also be read as a simultaneous search model along the lines of Burdett and Judd (1983), with adverse selection as the added element. In that model, a buyer obtains a sample of prices from sellers of a homogeneous product. The seller in our model is the counterpart of the buyer in their model.[27] The important difference is that our variation on their model endows this buyer with private information that might affect the seller's cost. This could be relevant for markets of certain services, such as repair or the credit markets mentioned in the introduction. The private information implies both additional substantive insights and some additional analytical challenges. In particular, in Burdett and Judd's (1983) model, the more convincing equilibrium becomes competitive when the sampling cost becomes negligible, while this is not necessarily the case in our model.

7.4 Unboundedly informative signals

It has been assumed throughout that the signals are boundedly informative, $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \infty$. While this assumption was used in the analysis, some of the results extend to a setting

[27] The roles of the seller and the buyers in our model can be reversed to make the models exactly parallel.

with an unboundedly informative signal. Specifically, the online appendix[28] shows that, with an unbounded likelihood ratio, all equilibria of the full game are still either of the pooling form or of the separating form (which, in this case, is perfectly revealing in the limit when the numbers of bidders go to ∞). Moreover, for any unboundedly informative signal, a perfectly revealing equilibrium exists.

The more surprising result (also proved in the online appendix) is that a pooling equilibrium can actually exist for some signal distributions exhibiting an unbounded likelihood ratio.

Proposition 7 *There exists a signal distribution $\bar{G} = (G_\ell, G_h)$ with $\lim_{x \to \bar{x}} \frac{g_h(x)}{g_\ell(x)} = \infty$, a sequence $\left(s^k, \Delta^k\right) \to (0,0)$ and a sequence of equilibria of $\Gamma\left(s^k, \Delta^k\right)$ such that (i) $n_\omega^k \to \infty$, and (ii) for some $\bar{b} \leq E[v]$,*

$$\lim F\left(\bar{b} + \delta|\omega; \beta^k, \boldsymbol{\eta}^k\right) - F\left(\bar{b} - \delta|\omega; \beta^k, \boldsymbol{\eta}^k\right) = 1 \text{ for all } \delta > 0.$$

Notice that the proposition does *not* imply that, given $\bar{G}$, there exists such a sequence of equilibria for any sequence $\left(s^k, \Delta^k\right) \to (0,0)$. Yet, the sequences $\left(s^k, \Delta^k\right)$ for which the result holds are "generic" in a sense that is made clear in the proof.

7.5 Broader class of environments

We analyze large, first-price auctions in a binary state world and, strictly speaking, the results pertain to that environment. However, our main insights carry over to a broader class of environments. The previous subsection already presents another scenario to which this analysis applies directly.

Other auction formats. Although we have not performed the full analysis, it seems that the qualitative results continue to hold for a second price auction as well. In this case, the functional forms of the limit price distribution (5) and the magnitude of the auctioneer's solicitation cost in Lemma 3 will be different, but the main insights would not change.

Large auctions. The focus on large auctions is natural for discussing information aggregation. But the strategic effects of state-dependent participation are just as relevant for trading scenarios with few participants. Still we focus on large auctions because the analysis is simpler. For example, in the separating case, large numbers guarantee that bids are near expected values and thus simplify the argument. But such proximity may already hold for fairly low numbers and perhaps other arguments utilizing more directly the structure of the equilibrium might be used.

Two states. The qualitative insights of the strategic inference from the state-dependent participation do not seem to depend on the two state assumption. We use this assumption

[28] The online appendix is appended to the end of the present version.

to establish the monotonicity of the equilibrium bidding strategy. If monotonicity can be established for the multiple state case, perhaps by resorting to stronger assumptions, then the extension to a world of multiple states would probably be quite straightforward.

8 References

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A Characterization Results of Bidding Equilibrium with a Pure Solicitation Strategy

Auxiliary Result: Winning Probability at Atoms. The following lemma is restated from Lauermann and Wolinsky (2017). It derives an expression for the winning probability in the case of a tie. Recall from (10) that $x_-(b) \triangleq \inf\left\{x \in [\underline{x}, \bar{x}]\,|\beta(x) \geq \bar{b}\right\}$ and $x_+ \triangleq \sup\left\{x \in [\underline{x}, \bar{x}]\,|\beta(x) \leq \bar{b}\right\}$.

Lemma 5 [Lauermann and Wolinsky, 2017.] *Suppose β is nondecreasing and, for some $\bar{b}$, $x_- = x_-\left(\bar{b}\right) < x_+\left(\bar{b}\right) = x_+$. Then,*

$$\pi_\omega\left(\bar{b}\right) = \frac{G_\omega(x_+)^n - G_\omega(x_-)^n}{n\left(G_\omega(x_+) - G_\omega(x_-)\right)} = \int_{x_-}^{x_+} \frac{\left(G_\omega(x)\right)^{n-1} g_\omega(x)\, dx}{G_\omega(x_+) - G_\omega(x_-)}. \tag{23}$$

Observe that the last expression is the expected probability of a randomly drawn signal from $[x_+, x_-]$ to be the highest. Thus, $\pi_\omega\left(\bar{b}\right)$ "averages" what would be the winning probabilities of the types in $[x_+, x_-]$ if β were strictly increasing.

A.1 Proof of Theorem 1 (Large Bidding Equilibria)

Here and in the rest of the appendix, we often use the abbreviation

$$\lambda \triangleq \overline{g} \lim \frac{n_h^k}{n_\ell^k}.$$

A.1.1 Auxiliary Lemmas

The proof of Theorem 1 relies on the three lemmas stated and proven below. It also uses that the expected payoffs can be written as

$$U(b|x, \mathrm{sol}; \beta, \mathbf{n}) = \frac{\rho_\ell g_\ell(x)\, n_\ell \pi_\ell(b; \beta, n_\ell)(v_\ell - b) + \rho_h g_h(x)\, n_h \pi_h(b; \beta, n_h)(v_h - b)}{\rho_\ell g_\ell(x)\, n_\ell + \rho_h g_h(x)\, n_h}. \tag{24}$$

Lemma 6 ("Zero Profit") *For any $\varepsilon > 0$, there is an $M(\varepsilon)$ such that, if $n_\omega > M(\varepsilon)$, $\omega = \ell, h$, then $U(\beta(x)|x,$ sol$; \beta, \mathbf{n}) < \varepsilon$ for all x in every bidding equilibrium β.*

Remark: We do not suppress here $\beta, \mathbf{n}$ from the arguments of U, since the claim concerns a range of $\mathbf{n}$ and all corresponding equilibria β.

Proof of Lemma 6. By (24) and the right-continuity of $\frac{g_h}{g_\ell}$, $(U(b|\cdot, \mathrm{sol}; \beta, \mathbf{n}))_{b,\beta,\mathbf{n}}$ is a family of functions that is uniformly (right-)equi-continuous: For every $\varepsilon > 0$ and x, there is some $z_\varepsilon > 0$ such that

$$\left|U(b|x', \mathrm{sol}; \beta, \mathbf{n}) - U(b|x, \mathrm{sol}; \beta, \mathbf{n})\right| \leq \frac{\varepsilon}{2},$$

for all b, all $(\beta, \mathbf{n})$ and all x' such that $0 \leq x' - x \leq z_\varepsilon$; similarly at $\bar{x}$ for all x' s.t. $\bar{x} - x' \leq z_\varepsilon$.[29]

Suppose $U(\beta(x)|x, \text{sol}; \beta, \mathbf{n}) = \varepsilon > 0$ for some $x < \bar{x}$ (the case $x = \bar{x}$ is analogous and omitted). From β being a bidding equilibrium, for all $x' > x$ s.t. $x' - x \leq z_\varepsilon$,

$$\left|U(\beta(x)|x, \text{sol}; \beta, \mathbf{n}) - U(\beta(x')|x',\ \text{sol}; \beta, \mathbf{n})\right| \leq \frac{\varepsilon}{2}. \tag{25}$$

Therefore,

$$\begin{aligned} &U(\beta(x)|x, \text{sol}; \beta, \mathbf{n}) - \frac{\varepsilon}{2} \leq \inf_{x' \in [x, x+z_\varepsilon]} U(\beta(x')|x', \text{sol}; \beta, \mathbf{n}) \\ &\leq \sum_{\omega=\ell,h} \rho_\omega \frac{\int_x^{x+z_\varepsilon} [v_\omega - \beta(x')] \pi_\omega(\beta(x'); \beta, n_\omega) dG_\omega(x')}{G_\omega(x+z_\varepsilon) - G_\omega(x)} \leq \sum_{\omega=\ell,h} \rho_\omega \frac{v_\omega \int_{\underline{x}}^{\bar{x}} \pi_\omega(\beta(x'); \beta, n_\omega) dG_\omega(x')}{G_\omega(x+z_\varepsilon) - G_\omega(x)} \\ &\qquad = \sum_{\omega=\ell,h} \frac{\rho_\omega v_\omega}{n_\omega (G_\omega(x+z_\varepsilon) - G_\omega(x))} \leq \frac{\mathbb{E}[v]}{\min_{\omega \in \{\ell,h\}} (n_\omega (G_\omega(x+z_\varepsilon) - G_\omega(x)))}, \end{aligned}$$

where the first inequality follows from (25), the second follows from the definition of U, the third owes to increasing the term in the numerator, and the fourth from the fact that the expected probability of winning over all signals is $1/n_\omega$. Now, let $M(\varepsilon)$ be large enough so that, for $n_\omega \geq M(\varepsilon)$, the RHS is smaller than $\frac{\varepsilon}{2}$. Therefore, for any $\mathbf{n}$ such that $n_\omega \geq M(\varepsilon)$, $U(\beta(x)|x, \text{sol}; \beta, \mathbf{n}) < \varepsilon$. ∎

Corollary 3 *Let $(\mathbf{n}^k)_{k=1}^\infty$ be such that $\min\{n_\ell^k, n_h^k\} \to \infty$ and $(\beta^k)_{k=1}^\infty$ be a corresponding sequence of bidding equilibria.*
(i)

$$\lim \sup_{x \in [\underline{x}, \bar{x}]} U^k(\beta^k(x)|x,\ sol) = 0. \tag{26}$$

(ii) If, for some sequence $(b^k)_{k=1}^\infty$ of bids and some ω, $\lim \pi_\omega^k(b^k) > 0$, then for any sequence $(x^k)_{k=1}^\infty$,

$$\lim \mathbb{E}^k[v|x^k, sol, win\ at\ b^k] \leq \lim b^k. \tag{27}$$

(iii) If $\lim \pi_\omega^k\left(\beta^k(x^k)\right) > 0$ for some ω and sequence $(x^k)_{k=1}^\infty$, then

$$\lim \beta^k(x^k) = \lim \mathbb{E}^k[v|x^k, sol, win\ at\ \beta^k(x^k)]. \tag{28}$$

Proof of Corollary 3: From Lemma 1, $\lim \pi_h^k\left(\beta^k(x^k)\right) > 0 \Leftrightarrow \lim \pi_\ell^k\left(\beta^k(x^k)\right) > 0$. Therefore, $\lim \pi_\omega^k(b^k) > 0$ for some ω is sufficient for $\lim \pi_\omega^k(b^k) > 0$ for all ω.

Part (i) and (ii) follow immediately from Lemma 6 that would be contradicted if (26) or (27) did not hold. Part (iii) is immediate from (27) and the individual rationality condition,

$$\beta^k(x^k) \leq \mathbb{E}^k[v|x^k, \text{sol,win at } \beta^k(x^k)].$$

[29] The monotonicity of $U(\beta(x)|x, \text{sol}; \beta, \mathbf{n})$ in x, which is established in Lemma 10, implies that it would be sufficient to argue the result for $\bar{x}$.

∎

Recall that $\overline{g} \triangleq \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$.

Lemma 7 *Let $\mathbf{n}^k$ be such that $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\overline{g} \lim \frac{n_h^k}{n_\ell^k} < 1$, and let $(\beta^k)_{k=1}^\infty$ be a corresponding sequence of (nondecreasing) bidding strategies. If $(b^k)_{k=1}^\infty$ is a sequence of bids such that $b^k < \beta^k(\bar{x})$ for all k and $\lim \pi_\ell^k(b^k) \in (0,1)$, then,*

$$\lim \mathbb{E}^k[v|\bar{x}, sol, win\ at\ b^k] > \lim \mathbb{E}^k[v|\bar{x}, sol, win\ at\ \beta^k(\bar{x})].$$

Proof of Lemma 7. Divide through the numerator and denominator of (3) by $\rho_\ell g_\ell(x) n_\ell \pi_\ell(b)$ or $\rho_\ell g_\ell(x) n_\ell$ to express it in terms of the compound likelihood ratio $\frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{n_h}{n_\ell}\frac{\pi_h(b)}{\pi_\ell(b)}$ as

$$\mathbb{E}[v|x,\text{sol,win at } b] = \frac{v_\ell + \frac{\rho_h g_h(x) n_h \pi_h(b)}{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b)} v_h}{1 + \frac{\rho_h g_h(x) n_h \pi_h(b)}{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b)}}. \tag{29}$$

Hence, we have to show that

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} > \lim \frac{\pi_h^k(\beta^k(\bar{x}))}{\pi_\ell^k(\beta^k(\bar{x}))}. \tag{30}$$

Let

$$\begin{aligned} \hat{q} &\triangleq \lim \left(G_\ell(x_+^k\left(b^k\right))\right)^{n_\ell^k - 1}, \\ \hat{q}_- &\triangleq \lim \left(G_\ell(x_-^k\left(b^k\right))\right)^{n_\ell^k - 1}, \end{aligned}$$

with $1 \geq \hat{q} \geq \hat{q}_- > 0$ by $\lim \pi_\ell^k(b^k) \in (0,1)$. Recall $\lambda \triangleq \overline{g} \lim \frac{n_h^k}{n_\ell^k}$. We first show the following:

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \hat{q}^{\lambda - 1} > 1 \text{ if } \hat{q}_- = \hat{q} \tag{31}$$

and

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \frac{(\hat{q})^\lambda - (\hat{q}_-)^\lambda}{\lambda(\hat{q} - \hat{q}_-)} > \hat{q}^{\lambda - 1} \geq 1 \text{ if } \hat{q}_- < \hat{q}. \tag{32}$$

To derive (31), note that[30]

$$\left(G_\omega\left(x_-^k\right)\right)^{n_\omega^k - 1} \leq \pi_\omega\left(b^k|\beta^k, n_\omega^k\right) \leq \left(G_\omega\left(x_+^k\right)\right)^{n_\omega^k - 1}.$$

[30] This can be verified using Lemma 5. For example, expanding the formula for π_ω gives

$$\pi_\omega\left(b^k|\beta^k, n_\omega^k\right) = \frac{1}{n_\omega^k}[G_\omega\left(x_+^k\right)^{n_\omega^k - 1} + G_\omega\left(x_+^k\right)^{n_\omega^k - 2} G_\omega\left(x_-^k\right) + \cdots + G_\omega\left(x_-^k\right)^{n_\omega^k - 1}] \geq \frac{n_\omega^k G_\omega\left(x_+^k\right)^{n_\omega^k - 1}}{n_\omega^k}.$$

Hence, whenever $\lim \left(G_\ell\left(x_-^k\right)\right)^{n_\ell^k-1} = q_- = \hat{q} = \lim\left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k-1}$, Lemma 1 implies

$$\lim \frac{\pi_h^k\left(b^k\right)}{\pi_\ell^k\left(b^k\right)} = \frac{\hat{q}^\lambda}{\hat{q}} = \hat{q}^{\lambda-1}.$$

To derive (32), recall from Lemma 5 that

$$\pi_\omega^k\left(b^k\right) = \frac{\left(G_\omega\left(x_+^k\right)\right)^{n_\omega^k} - \left(G_\omega\left(x_-^k\right)\right)^{n_\omega^k}}{n_\omega^k[G_\omega\left(x_+^k\right) - G_\omega\left(x_-^k\right)]}, \tag{33}$$

and hence using Lemma 1,

$$\lim \frac{\pi_h^k\left(b^k\right)}{\pi_\ell^k\left(b^k\right)} = \lim \frac{n_\ell^k}{n_h^k} \frac{G_\ell\left(x_+^k\right) - G_\ell\left(x_-^k\right)}{G_h\left(x_+^k\right) - G_h\left(x_-^k\right)} \frac{G_h\left(x_+^k\right)^{n_h^k} - G_h\left(x_-^k\right)^{n_h^k}}{G_\ell\left(x_+^k\right)^{n_\ell^k} - G_\ell\left(x_-^k\right)^{n_\ell^k}} = \frac{(\hat{q})^\lambda - (\hat{q}_-)^\lambda}{\lambda\left(\hat{q} - \hat{q}_-\right)}.$$

To show the inequality $\frac{(\hat{q})^\lambda - (\hat{q}_-)^\lambda}{\lambda(\hat{q}-\hat{q}_-)} > \hat{q}^{\lambda-1}$, let $Q \triangleq \frac{\hat{q}_-}{\hat{q}} < 1$. Then, the inequality is equivalent to $Q^\lambda - \lambda Q + \lambda < 1$. Since $\lambda < 1$, the LHS is increasing in Q over $[0,1)$ and is equal to 1 at $Q = 1$, so the inequality holds.

Let

$$\bar{x}_-^k \triangleq x_-^k\left(\beta^k\left(\bar{x}\right)\right) \text{ and } q \triangleq \lim\left(G_\ell\left(\bar{x}_-^k\right)\right)^{n_\ell^k}.$$

Since, by the hypothesis, $b^k < \beta^k\left(\bar{x}\right)$ for all k, we have $q \geq \hat{q}$.

Case 1. Suppose that $q = 1$. Since

$$\pi_\omega^k\left(\beta^k(\bar{x})\right) \geq \left(G_\omega\left(\bar{x}_-^k\right)\right)^{n_\omega^k-1},$$

we have $\lim \pi_\ell^k\left(\beta^k(\bar{x})\right) = q(=1)$. By Lemma 1, $\lim\left(G_h\left(\bar{x}_-^k\right)\right)^{n_h^k} = q^\lambda = 1$ as well. So, $\lim \frac{\pi_h^k\left(\beta^k(\bar{x})\right)}{\pi_\ell^k\left(\beta^k(\bar{x})\right)} = 1$. This, (31), and (32) imply (30).

Case 2. Suppose that $q < 1$. So, there is an atom at $\beta^k(\bar{x})$. First, consider $\lambda \in (0,1)$. As before, using Lemmas 1 and 5, we have

$$\lim \frac{\pi_h^k(\beta^k\left(\bar{x}\right))}{\pi_\ell^k(\beta^k\left(\bar{x}\right))} = \frac{1 - q^\lambda}{\lambda\left(1 - q\right)} < q^{\lambda-1}, \tag{34}$$

where the last inequality follows from $\lambda \in (0,1)$, $q \in (0,1)$, and straightforward algebraic manipulation.[31]

Since $q \geq \hat{q} > 0$ and $\lambda < 1$, we have $\hat{q}^{\lambda-1} \geq q^{\lambda-1}$. Now, this together with (31), (32), and (34) imply (30).

If $\lambda = 0$, by Lemma 1, $\lim G_h(x_-^k\left(b^k\right))^{n_h^k} = 1$, and hence $\lim \pi_h^k\left(\beta^k\left(\bar{x}\right)\right) = 1$. Thus, (30) follows from $\lim \pi_\ell^k\left(b^k\right) < \lim \pi_\ell^k\left(\beta^k\left(\bar{x}\right)\right)$. ∎

[31] With $Q = \frac{1}{q}$, the inequality is equivalent to $(Q)^\lambda - \lambda Q + \lambda < 1$. The right-hand side equals 1 if $Q = 1$ and is increasing in Q on $(0,1)$ by $\lambda \in (0,1)$; hence, the inequality holds.

Lemma 8 *Let* $\mathbf{n}^k$ *be such that* $\min\{n_\ell^k, n_h^k\} \to \infty$ *and* $\bar{g} \lim \frac{n_h^k}{n_\ell^k} > 1$. *Let* $(\beta^k)_{k=1}^\infty$ *be a corresponding sequence of equilibrium bidding strategies. If* $(\beta^k)_{k=1}^\infty$ *contains a sequence of nonvanishing atoms* $(b^k)_{k=1}^\infty$, *i.e.,* $\lim \left(G_\ell\left(x_+^k\left(b^k\right)\right)\right)^{n_\ell^k} > \lim \left(G_\ell\left(x_-^k\left(b^k\right)\right)\right)^{n_\ell^k}$, *then*

$$\lim_{k\to\infty} b^k < \lim_{k\to\infty}\lim_{\varepsilon\to 0} \mathbb{E}^k[v|x_+^k, \textit{sol,win at } b^k + \varepsilon].$$

Proof of Lemma 8. By bidders' individual rationality, $\mathbb{E}^k[v|x_-^k,\text{sol,win at } b^k] \geq b^k$. Therefore, the claim will follow from $\lim \lim_{\varepsilon\to 0} \mathbb{E}^k[v|x_+^k,\text{sol,win at } b^k+\varepsilon] > \lim \mathbb{E}^k[v|x_-^k,\text{sol,win at } b^k]$, which in turn will follow from

$$\lim \frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)} \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} < \lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)} \frac{\left(G_h\left(x_+^k\left(b^k\right)\right)\right)^{n_h^k}}{\left(G_\ell\left(x_+^k\left(b^k\right)\right)\right)^{n_\ell^k}}. \tag{35}$$

Let $q_- = \lim G_\ell\left(x_-^k\right)^{n_\ell^k}$ and $q_+ = \lim G_\ell\left(x_+^k\right)^{n_\ell^k}$. Note that $G_\omega\left(x_+^k\right)^{n_\omega^k} \approx G_\omega\left(x_+^k\right)^{n_\omega^k - 1}$ for large k. By the hypothesis of the lemma, $q_+ > 0$. By Lemma 1, $\lim G_h\left(x_-^k\right)^{n_h^k} = (q_-)^\lambda$ and $\lim G_h\left(x_+^k\right)^{n_h^k} = (q_+)^\lambda$. Recall from Lemma 5 that

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \lim \frac{n_\ell^k}{n_h^k} \frac{G_\ell\left(x_+^k\right) - G_\ell\left(x_-^k\right)}{G_h\left(x_+^k\right) - G_h\left(x_-^k\right)} \frac{G_h\left(x_+^k\right)^{n_h^k} - G_h\left(x_-^k\right)^{n_h^k}}{G_\ell\left(x_+^k\right)^{n_\ell^k} - G_\ell\left(x_-^k\right)^{n_\ell^k}}. \tag{36}$$

Using this and the above observations,

$$\lim \frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)} \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \lim \left(\frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)} \frac{G_\ell\left(x_+^k\right) - G_\ell\left(x_-^k\right)}{G_h\left(x_+^k\right) - G_h\left(x_-^k\right)} \right) \frac{g_h\left(\bar{x}\right)}{g_\ell\left(\bar{x}\right)} \frac{(q_+)^\lambda - (q_-)^\lambda}{\lambda\left(q_+ - q_-\right)}. \tag{37}$$

Now, $\lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)} = \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$ and by MLRP

$$\frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)} \frac{G_\ell\left(x_+^k\right) - G_\ell\left(x_-^k\right)}{G_h\left(x_+^k\right) - G_h\left(x_-^k\right)} \leq 1.$$

Therefore, we may establish (35) by showing that,

$$\frac{(q_+)^\lambda - (q_-)^\lambda}{\lambda\left(q_+ - q_-\right)} < (q_+)^{\lambda - 1}. \tag{38}$$

Letting $Q = \frac{q_-}{q_+} < 1$, (38) is equivalent to $Q^\lambda - \lambda Q + \lambda > 1$. Since $\lambda > 1$, the LHS is decreasing in Q over $[0, 1)$ and is equal to 1 at $Q = 1$. Therefore, (38) holds and so does (35). ∎

A.1.2 Proof of Theorem 1 using the above Lemmas

By Proposition 1, we may assume that each bidding strategy β^k is monotone.

Part 1: Suppose $\bar{g}r < 1$.

Given any $\varepsilon \in (0,1)$, let (x^k) be such that $\lim \left(G_h\left(x^k\right)\right)^{n_h^k} = \varepsilon$ for all k. We show that

$$\lim \left(G_h\left(x_+^k\left(\beta^k\left(x^k\right)\right)\right)\right)^{n_h^k} = 1,$$

with $x_+^k(b) = \sup\{x|\beta^k(x) = b\}$. This implies

$$\lim \left(G_h\left(x_+^k\left(b^k\right)\right)\right)^{n_h^k} - \left(G_h\left(x_-^k\left(b^k\right)\right)\right)^{n_h^k} \geq 1 - \varepsilon.$$

Then by Lemma 1 and $\bar{g}r < 1$, this inequality holds for $\omega = \ell$ as well. Since we can choose ε arbitrarily small, this establishes the claim.

Let $y_+^k \equiv x_+^k(b)$, and suppose to the contrary that

$$\lim \left(G_h\left(y_+^k\right)\right)^{n_h^k} < 1. \tag{39}$$

Since $\beta^k\left(x^k\right) < \beta^k\left(\bar{x}\right)$, (39) implies that there exists b^k with $\beta^k\left(x^k\right) < b^k < \beta^k\left(\bar{x}\right)$ and

$$\lim \pi_\ell^k(b^k) \in (0,1)\,. \tag{40}$$

Hence, the zero-profit condition (27) from Corollary 3 requires that

$$\lim b^k \geq \lim \mathbb{E}^k[v|\bar{x},\text{sol,win at } b^k]. \tag{41}$$

Given (40) and (41), Lemma 7 implies that

$$\lim \mathbb{E}^k[v|\bar{x},\text{sol,win at } b^k] > \lim \mathbb{E}^k[v|\bar{x},\text{sol,win at } \beta^k\left(\bar{x}\right)]. \tag{42}$$

Individual rationality requires that

$$\lim \mathbb{E}^k[v|\bar{x},\text{sol,win at } \beta^k\left(\bar{x}\right)] \geq \lim \beta^k\left(\bar{x}\right). \tag{43}$$

Hence, (41)-(43) together imply a contradiction to $b^k < \beta^k\left(\bar{x}\right)$. Thus, (39) cannot hold, which proves the claim.

Part 2a: Suppose $\bar{g}r > 1$ and $r \neq \infty$.

Let us establish first that there are no atoms in the limit. Suppose to the contrary that $\beta^k\left(x\right) = b^k$ for all $x \in (x_-^k,\, x_+^k)$ and $\lim \left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k} > \lim \left(G_\ell\left(x_-^k\right)\right)^{n_\ell^k} \geq 0$. Thus,

$$\lim_{k\to\infty}\lim_{\varepsilon\to 0} \pi_\ell^k\left(b^k + \varepsilon\right) = \lim_{k\to\infty}\left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k} > 0. \tag{44}$$

This and Lemma 8 implies that

$$\lim_{k\to\infty}\lim_{\varepsilon\to 0} U^k(b^k+\varepsilon|x_+^k,\ \text{sol}) > 0. \tag{45}$$

contradicting the zero-profit condition (26). Thus, there can be no atom.

Next, let us derive the functional form. Take any $\alpha \in (0,1)$. Let $(x^k)_{k=1}^{\infty}$ be such that $\left(G_\ell\left(x^k\right)\right)^{n_\ell^k-1} = \alpha$ for all k. By the absence of atoms (just established above), $\lim \pi_\omega^k(\beta^k\left(x^k\right)) = \lim \left(G_\omega\left(x^k\right)\right)^{n_\omega^k-1} = \lim F_\omega^k(\beta^k\left(x^k\right))$. By Corollary (3)-(iii) ,

$$\lim \beta^k\left(x^k\right) = \lim \mathbb{E}^k[v|x^k, \text{sol, win at } \beta^k\left(x^k\right)]$$

Therefore, expressing $\mathbb{E}^k[v|x^k,\text{sol,win at } \beta^k\left(x^k\right)]$ in terms of the compound likelihood ratio as in (29) and using $\lim \pi_\omega^k(\beta^k\left(x^k\right)) = \lim \left(G_\omega\left(x^k\right)\right)^{n_\omega^k-1}$,

$$\lim \beta^k\left(x^k\right) = \lim \frac{v_\ell + \frac{\rho_h}{\rho_\ell}\frac{g_h\left(x^k\right)}{g_\ell\left(x^k\right)}\frac{n_h^k}{n_\ell^k}\frac{\pi_h^k(\beta^k\left(x^k\right))}{\pi_\ell^k(\beta^k\left(x^k\right))}v_h}{1+\frac{\rho_h}{\rho_\ell}\frac{g_h\left(x^k\right)}{g_\ell\left(x^k\right)}\frac{n_h^k}{n_\ell^k}\frac{\pi_h^k(\beta^k\left(x^k\right))}{\pi_\ell^k(\beta^k\left(x^k\right))}} = \lim \frac{v_\ell + \frac{\rho_h}{\rho_\ell}\frac{g_h\left(x^k\right)}{g_\ell\left(x^k\right)}\frac{n_h^k}{n_\ell^k}\frac{\left(G_h\left(x^k\right)\right)^{n_h^k-1}}{\left(G_\ell\left(x^k\right)\right)^{n_\ell^k-1}}v_h}{1+\frac{\rho_h}{\rho_\ell}\frac{g_h\left(x^k\right)}{g_\ell\left(x^k\right)}\frac{n_h^k}{n_\ell^k}\frac{\left(G_h\left(x^k\right)\right)^{n_h^k-1}}{\left(G_\ell\left(x^k\right)\right)^{n_\ell^k-1}}}. \tag{46}$$

From $\lim \left(G_\omega\left(x^k\right)\right)^{n_\omega^k-1} > 0$, we have $x^k \to \bar{x}$. This, Lemma 1, and $\lim \left(G_\omega\left(x^k\right)\right)^{n_\omega^k-1} = \lim F_\omega^k(\beta^k\left(x^k\right))$ imply

$$\lim \frac{\left(G_h\left(x^k\right)\right)^{n_h^k-1}}{\left(G_\ell\left(x^k\right)\right)^{n_\ell^k-1}} = \lim \frac{\left(G_h\left(x^k\right)\right)^{n_h^k}}{\left(G_\ell\left(x^k\right)\right)^{n_\ell^k}} = [\lim \left(G_\ell\left(x^k\right)\right)^{n_\ell^k}]^{\lambda-1} = \alpha^{\lambda-1},$$

where, as before, $\lambda = \overline{g}\lim \frac{n_h^k}{n_\ell^k}$. Using this observation and letting $\lim \beta^k\left(x^k\right) = p$, we can rewrite (46) as

$$p = \frac{v_\ell + \rho\lambda\alpha^{\lambda-1}v_h}{1+\rho\lambda\alpha^{\lambda-1}}. \tag{47}$$

Thus, for every $\alpha \in (0,1)$, we can find the unique p such that $\lim F_\ell^k(p) = \alpha$. This gives a function $\hat{p}(\alpha)$ that is continuous and strictly increasing on $(0,1)$. The limit distribution $\Phi_\ell(p)$ is simply the inverse of $\hat{p}$, meaning, the α solution of (47) for given p. Finally, from Lemma 1, $\lim F_h^k(p) = \Phi_h(p)$.

Part 2b: Suppose $r = \infty$.

In this case, $\Phi_\omega(\cdot|r)$ is degenerate with probability mass 1 on v_ω. Given bidders' individual rationality constraint, it is sufficient to show that $\Phi_h(\cdot|r)$ is degenerate with probability mass 1 on v_h. But this follows directly from the zero profit condition and the observation that, given $r = \infty$, if $\lim F_h^k(p) > 0$ for some $p < v_h$, then $\lim \mathbb{E}^k[v|x^k,\text{sol, win at } p] = v_h$. ∎

A.2 Proof of Proposition 2

Suppose that $n_h \geq n_\ell \geq 2$ and β is a nondecreasing bidding equilibrium given (n_ℓ, n_h).

Recall that $x_+(b) = \sup\{x|\beta(x) = b\}$ and $x_-(b) = \inf\{x|\beta(x) = b\}$, so that an atom at b means $x_-(b) < x_+(b)$. We omit the arguments and write x_- and x_+ when it is clear from the context.

Step 1: $\frac{G_h(x)^{n_h}}{G_\ell(x)^{n_\ell}}$ is weakly increasing on any interval (y,z) if $n_\ell \le n_h$. It is constant if and only if $n_\ell = n_h$ and $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for all $x < z$.

Proof of Step 1: For $z > y$,

$$\begin{aligned}\frac{G_h(z)^{n_h}}{G_\ell(z)^{n_\ell}} - \frac{G_h(y)^{n_h}}{G_\ell(y)^{n_\ell}} &= \left(\frac{G_h(z)}{G_\ell(z)}\right)^{n_\ell}(G_h(z))^{n_h-n_\ell} - \left(\frac{G_h(y)}{G_\ell(y)}\right)^{n_\ell}(G_h(y))^{n_h-n_\ell} \\ &\ge \left(\frac{G_h(z)}{G_\ell(z)}\right)^{n_\ell} - \left(\frac{G_h(y)}{G_\ell(y)}\right)^{n_\ell} \ge 0,\end{aligned}$$

where the first inequality is from $n_h \ge n_\ell$ and $G_h(z) > G_h(y)$, and the second inequality is from the MLRP. Hence, $\frac{G_h(x)^{n_h}}{G_\ell(x)^{n_\ell}}$ is weakly increasing on (y,z), and it is strictly increasing unless both $n_h = n_\ell$ and $\frac{G_h(x)}{G_\ell(x)}$ is constant on (y,z), which requires $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for all $x < z$.

To see the latter assertion, suppose that $\frac{G_h(x)}{G_\ell(x)} = C$ (a constant) on $[y,z]$. Thus, $\left(\frac{G_h(x)}{G_\ell(x)}\right)'_x = 0$, which implies $\frac{g_h(x)}{g_\ell(x)} = C$ on (y,z). Now, if there were some $\underline{x} < x' < y$ such that $\frac{g_h(x')}{g_\ell(x')} < C$, then by the monotonicity of $\frac{g_h}{g_\ell}$ and $\frac{g_h(x)}{g_\ell(x)} = C$ on (y,z),

$$\begin{aligned}C = \frac{G_h(z)}{G_\ell(z)} &= \frac{\int_{\underline{x}}^{y} \frac{g_h(x)}{g_\ell(x)} g_\ell(x)\,dx + \int_{x_-}^{z} \frac{g_h(x)}{g_\ell(x)} g_\ell(x)\,dx}{G_\ell(z)} \\ &< \frac{C\left[\int_{\underline{x}}^{y} g_\ell(x)\,dx + \int_{y}^{z} g_\ell(x)\,dx\right]}{G_\ell(z)} = C,\end{aligned}$$

a contradiction. Therefore, $\frac{g_h(x)}{g_\ell(x)} = C$ for all $\underline{x} < x' < z$. Finally, $\frac{g_h(\underline{x})}{g_\ell(\underline{x})} = C$ by right-continuity at $\underline{x}$. □

Step 2: If $x_-(b) < x_+(b)$ and $n_h \ge n_\ell$, then

$$\frac{G_h(x_-)^{n_h-1}}{G_\ell(x_-)^{n_\ell-1}} \le \frac{\pi_h(b;\beta,n_h)}{\pi_\ell(b;\beta,n_\ell)} \le \frac{G_h(x_+)^{n_h-1}}{G_\ell(x_+)^{n_\ell-1}},$$

with equalities holding if and only if $n_\ell = n_h$ and $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for all $x < x_+$. Otherwise, if either $n_\ell > n_h$ or $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for some $x < x_+$ or both, both inequalities are strict.

Proof of Step 2: Expanding the expression for $\pi_\omega(b;\beta,n_\omega)$ from (23) yields

$$\frac{\pi_h(b;\beta,n_h)}{\pi_\ell(b;\beta,n_\ell)} = \frac{n_\ell}{n_h}\frac{G_h(x_-)^{n_h-1} + G_h(x_-)^{n_h-2}G_h(x_+) + \cdots + G_h(x_+)^{n_h-1}}{G_\ell(x_-)^{n_\ell-1} + G_\ell(x_-)^{n_\ell-2}G_\ell(x_+) + \cdots + G_\ell(x_+)^{n_\ell-1}}.$$

Divide through by $\frac{G_h(x_+)^{n_h-1}}{G_\ell(x_+)^{n_\ell-1}}$ to obtain

$$\left(\frac{\pi_h(b;\beta,n_h)}{\pi_\ell(b;\beta,n_\ell)}\right)/\left(\frac{G_h(x_+)^{n_h-1}}{G_\ell(x_+)^{n_\ell-1}}\right) = \frac{\left[1+\frac{G_h(x_-)}{G_h(x_+)}+\left(\frac{G_h(x_-)}{G_h(x_+)}\right)^2+\cdots+\left(\frac{G_h(x_-)}{G_h(x_+)}\right)^{n_h-1}\right]/n_h}{\left[1+\frac{G_\ell(x_-)}{G_\ell(x_+)}+\left(\frac{G_\ell(x_-)}{G_\ell(x_+)}\right)^2+\cdots+\left(\frac{G_\ell(x_-)}{G_\ell(x_+)}\right)^{n_\ell-1}\right]/n_\ell}$$

$$\leq \frac{\left[1+\frac{G_h(x_-)}{G_h(x_+)}+\left(\frac{G_h(x_-)}{G_h(x_+)}\right)^2+\cdots+\left(\frac{G_h(x_-)}{G_h(x_+)}\right)^{n_\ell-1}\right]/n_\ell}{\left[1+\frac{G_\ell(x_-)}{G_\ell(x_+)}+\left(\frac{G_\ell(x_-)}{G_\ell(x_+)}\right)^2+\cdots+\left(\frac{G_\ell(x_-)}{G_\ell(x_+)}\right)^{n_\ell-1}\right]/n_\ell} \leq 1.$$

The first inequality follows from the fact that, since $\frac{G_h(x_-)}{G_h(x_+)}<1$, the numerator after the inequality is an average of the largest n_ℓ terms out of the n_h terms that are averaged on the numerator before the inequality sign. The second inequality follows from $\frac{G_h(x_-)}{G_h(x_+)} \leq \frac{G_\ell(x_-)}{G_\ell(x_+)}$, which in turn follows from $\frac{G_h(x_-)}{G_\ell(x_-)} \leq \frac{G_h(x_+)}{G_\ell(x_+)}$ which holds by MLRP.

Analogously, dividing through by $\frac{G_h(x_-)^{n_h-1}}{G_\ell(x_-)^{n_\ell-1}} > 1$ and exactly reversing the previous arguments,

$$\left(\frac{\pi_h(b;\beta,n_h)}{\pi_\ell(b;\beta,n_\ell)}\right)/\left(\frac{G_h(x_-)^{n_h-1}}{G_\ell(x_-)^{n_\ell-1}}\right) \geq 1,$$

where the inequalities are explained by noting that $\frac{G_h(x_+)}{G_h(x_-)} > 1$ and reversing the previous arguments.

In both cases, the two inequalities hold as equalities if and only if $n_h = n_\ell$ and $\frac{G_h(x_-)}{G_h(x_+)} = \frac{G_\ell(x_-)}{G_\ell(x_+)}$. The last equality is equivalent to $\frac{G_h(x_-)}{G_\ell(x_-)} = \frac{G_h(x_+)}{G_\ell(x_+)}$ which holds if and only if $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for all $x < x_+$, as argued before. □

Step 3: If for some p, $x_+(p) > x_-(p)$, then $v_\ell \leq p < v_h$.

Proof of Step 3: $p \geq v_h$ would imply strictly negative payoff. Slightly overbidding an atom at $p < v_\ell$ is necessarily profitable. □

Step 4: If for some p, $x_+(p) > x_-(p)$, then $n_h = n_\ell$, $U(p|x, \text{sol};\beta,\mathbf{n}) = 0$ for $x \in (x_-, x_+)$ and $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for all $x < x_+$.

Proof of Step 4: Recall $n_h \geq n_\ell \geq 2$. Use (1)—(2) to rewrite

$$U(p|x,\ \text{sol};\beta,\mathbf{n}) = \frac{\rho_\ell g_\ell(x)\, n_\ell \pi_\ell(p;\beta,n_\ell)}{\rho_\ell g_\ell(x)\, n_\ell + \rho_h g_h(x)\, n_h}\left[(v_\ell - p) + \frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{n_h}{n_\ell}\frac{\pi_h(p;\beta,n_h)}{\pi_\ell(p;\beta,n_\ell)}(v_h - p)\right]. \quad (48)$$

The expected payoff of a bidder with signal $x \in (x_-, x_+)$ who bids "just above" p is

$$\begin{aligned} & \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} U(p+\varepsilon|x,\ \text{sol};\beta,\mathbf{n}) && (49) \\ = & \frac{\rho_\ell g_\ell(x) n_\ell G_\ell(x_+)^{n_\ell-1}}{\rho_\ell g_\ell(x) n_\ell + \rho_h g_h(x) n_h}\left[(v_\ell - p) + \frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{n_h}{n_\ell}\frac{G_h(x_+)^{n_h-1}}{G_\ell(x_+)^{n_\ell-1}}(v_h - p)\right]. \end{aligned}$$

By optimality, $U(p|x,\ \text{sol};\beta,\mathbf{n}) \geq 0$ for all $x \in (x_-,\ x_+)$ and, by Lemma 5, $G_\ell(x_+)^{n_\ell-1} > \pi_\ell(p;\beta,n_\ell)$. Therefore, it follows from (48), (49), the MLRP, Step 2 and Step 3 that $\lim_{\varepsilon>0,\varepsilon\to 0} U(p+\varepsilon|x,\ \text{sol};\beta,\mathbf{n}) \geq U(p|x,\ \text{sol};\beta,\mathbf{n})$ for all $x \in (x_-,\ x_+)$. The inequality is strict if $U(p|x,\ \text{sol};\beta,\mathbf{n}) > 0$ or $n_h > n_\ell$ or $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for some $x < x_+$ (or any combination). Therefore, optimality of β implies the claim. □

Step 5: If $n_h = n_\ell \geq 2$ then $\beta(x'') = \beta(x')$ for all $x'', x' \in (\underline{x}, \hat{x})$, where $\hat{x} = \sup\left\{x | \frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}\right\}$.

Proof of Step 5: Suppose to the contrary that $\beta(x'') < \beta(x')$ for some $x'', x' \in (\underline{x}, \hat{x})$. Note that $U(\beta(\underline{x})|\underline{x},\ \text{sol};\beta,\mathbf{n}) = 0$ in every bidding equilibrium. (If there is no atom at $\underline{x}$, this is immediate from the monotonicity of β; if there is an atom at $\underline{x}$, this follows from the Step 4.) By the choice of $\hat{x}$, $\frac{g_h(x')}{g_\ell(x')} = \frac{g_h(x'')}{g_\ell(x'')} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$. This and $n_h = n_\ell$ imply via Steps 1 and 4 that $\frac{\pi_h(\beta(x'');\beta,n_h)}{\pi_\ell(\beta(x'');\beta,n_\ell)} = \frac{\pi_h(\beta(x');\beta,n_h)}{\pi_\ell(\beta(x');\beta,n_\ell)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$. To see this, consider the second equality. If there is no atom at $\beta(x')$, it follows from Step 1; if there is an atom, Step 4 implies $x_+(\beta(x')) \leq \hat{x}$ and it follows from Step 2. The argument for the equality between the first and third terms is completely analogous. In addition, $\frac{g_h(x'')}{g_\ell(x'')} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ and optimality implies $U(\beta(x'')|x'',\ \text{sol};\beta,\mathbf{n}) = 0$. Therefore, since $\pi_\omega(\beta(x);\beta,n_\omega) > 0$ for any $x > \underline{x}$ and $\omega \in \{\ell, h\}$ (by monotonicity of β), evaluating (48) yields

$$\begin{aligned} 0 &= (v_\ell - \beta(x'')) + \frac{\rho_h}{\rho_\ell}\frac{g_h(x'')}{g_\ell(x'')}\frac{n_h}{n_\ell}\frac{\pi_h(\beta(x'');\beta,n_h)}{\pi_\ell(\beta(x'');\beta,n_\ell)}(v_h - \beta(x'')) \\ &= (v_\ell - \beta(x'')) + \frac{\rho_h}{\rho_\ell}\frac{g_h(x')}{g_\ell(x')}\frac{n_h}{n_\ell}\frac{\pi_h(\beta(x');\beta,n_h)}{\pi_\ell(\beta(x');\beta,n_\ell)}(v_h - \beta(x'')) \\ &> (v_\ell - \beta(x')) + \frac{\rho_h}{\rho_\ell}\frac{g_h(x')}{g_\ell(x')}\frac{n_h}{n_\ell}\frac{\pi_h(\beta(x');\beta,n_h)}{\pi_\ell(\beta(x');\beta,n_\ell)}(v_h - \beta(x')) \\ &= U(\beta(x')|x';\beta,\mathbf{n}), \end{aligned}$$

where the second equality follows from the argument in the previous paragraph, and the inequality follows from the hypothesis $\beta(x'') < \beta(x')$. But $U(\beta(x')|x',\ \text{sol};\beta,\mathbf{n}) < 0$ contradicts the optimality of $\beta(x')$. Therefore, $\beta(x'') = \beta(x')$. □

Steps 4 establishes the first part and the second item of the second part of Proposition 2; Step 5 establishes the first item of the second part. ■

A.3 Proof of Proposition 3 (Borderline Case)

Proof of Proposition 3: From bidders' individual rationality,

$$\rho_\ell \lim \mathbb{E}^k[p|\ell] + \rho_h \lim \mathbb{E}^k[p|h] \leq \mathbb{E}[v]. \tag{50}$$

We show that, for any $p < \mathbb{E}[v]$, $\lim F_\omega^k(p) = 0$. This together with (50) implies the proposition, since if $\lim F_\omega^k(p) < 1$ for some $p > \mathbb{E}[v]$, (50) would be violated.

Suppose to the contrary that, for some $p < \mathbb{E}[v]$, $\lim F_\omega^k(p) > 0$. Therefore, given that bids are from the continuum, there is $p' < \mathbb{E}[v]$, such that $q \triangleq \lim \pi_\ell^k(p') > 0$. Then, there is a sequence $\left(b^k\right)_{k=1}^\infty$ such that β^k has no atom at b^k for any k, $b^k \geq p'$, and $\lim b^k = p'$. Letting $\hat{q} \triangleq \lim \pi_\ell^k(b^k)$, Lemma 1 and $\lambda = \overline{g} \lim \frac{n_h^k}{n_\ell^k} = 1$ imply

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \frac{\hat{q}^\lambda}{\hat{q}} = 1.$$

Thus, from (3), $\lim \mathbb{E}^k[v|\bar{x}\text{,sol,win at } b^k] = \mathbb{E}[v] > \lim b^k$. Since also $\lim \pi_\omega^k(b^k) > 0$ from $b^k > p'$ and $\lim \pi_\ell^k(p') > 0$, we have

$$\lim U^k(b^k|\bar{x},\ \text{sol}) > 0,$$

contradicting the zero-profit condition (26). Thus, such $\left(b^k\right)_{k=1}^\infty$ cannot exist. Therefore, $\lim \pi_\omega^k(p) = 0$ for all $p < \mathbb{E}[v]$, as needed. ∎

B Characterization of Bidding Equilibrium with a Mixed Solicitation Strategy

B.1 Notation for Mixed Strategies

Given a mixed solicitation strategy $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$, let

$$\overline{n}_\omega(\eta_\omega) \triangleq \sum_{n=1}^N n\eta_\omega(n), \text{ and } \overline{\pi}_\omega(b;\beta,\eta_\omega) \triangleq \sum_{n=1}^N \eta_\omega(n)\, n\pi_\omega(b;\beta,n)/\overline{n}_\omega. \tag{51}$$

These are the expected number of bidders and the weighted average probability of winning in state ω. To make the expressions less dense, we omit here and later the argument of $\overline{n}_\omega(\eta_\omega)$ and write just $\overline{n}_\omega$ instead. Also, as before, when there is no danger of confusion, we will continue to omit the argument β and $\boldsymbol{\eta}$ from U, π_ω, E, etc. The counterpart of (24)—the expected payoff to a bidder who bids b given a mixed solicitation strategy $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$—is

$$U(b|x,\ \text{sol}) = \frac{\rho_\ell g_\ell(x)\,\overline{n}_\ell \overline{\pi}_\ell(b)(v_\ell - b) + \rho_h g_h(x)\,\overline{n}_h \overline{\pi}_h(b)(v_h - b)}{\rho_\ell g_\ell(x)\,\overline{n}_\ell + \rho_h g_h(x)\,\overline{n}_h}. \tag{52}$$

Expressions (1)—(2) can also be adapted to mixed strategies, with $\overline{n}_\omega$ and $\overline{\pi}_\omega$ just taking everywhere the place of n_ω and π_ω.

B.2 Proof of Proposition 1' (Monotonicity with Mixing)

Proposition 1' *Suppose η is such that $\eta_\ell(1) = \eta_h(1) = 0$, and β is a bidding equilibrium.*

1. *If $x' > x$, then $U(\beta(x')\,|x',sol) \geq U(\beta(x)\,|x,sol)$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.*
2. *There exists an equivalent bidding equilibrium $\widetilde{\beta}$, such that $\widetilde{\beta}$ is nondecreasing on $[\underline{x}, \bar{x}]$ and coincides with β over intervals over which $\frac{g_h}{g_\ell}$ is strictly increasing.*

The proof of Proposition 1' relies on two lemmas.

Lemma 9 **(Single-Crossing)** *Given any bidding strategy β, any solicitation strategy $\boldsymbol{\eta}$ and any bids $b' > b \geq v_\ell$.*

1. *If $\overline{\pi}_\omega(b') > 0$ for some $\omega \in \{\ell, h\}$, then, for all $x' > x$,*
$$U(b'|x,\ sol) \geq U(b|x,\ sol) \Rightarrow U(b'|x',\ sol) \geq U(b|x',\ sol);$$
where the second inequality is strict if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.
2. *If $\overline{\pi}_\omega(b') = 0$ for some $\omega \in \{\ell, h\}$, then $\overline{\pi}_\omega(b) = 0$ for both ω, and $U(b'|x,\ sol) = U(b|x,\ sol) = 0$ for all x.*

The proof of Lemma 9 relies on the assumption that there are only two states. If bids are necessarily above v_ℓ (as is indeed implied by the next lemma), conditional on state ℓ, a higher bid is necessarily worse than a lower one. So, if two bids are optimal for some belief, the higher bid must be better if the state is h–implying that a higher belief must make the higher bid more attractive. This is the key role of that assumption.

The following lemma collects a number of additional properties of a bidding equilibrium β. One of them is a straightforward Bertrand property: when the seller solicits two or more bids in both states, then $\beta(x) \geq v_\ell$, for all x.

Lemma 10 **(Bertrand and Other Properties)** *Suppose $\eta_\ell(1) = \eta_h(1) = 0$ and β is a bidding equilibrium.*

1. *$\overline{\pi}_\omega(\beta(x)) > 0$ if $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$.*
2. *$\beta(x) \in [v_\ell, v_h)$ for almost all x.*
3. *$U(\beta(x')\,|x',\ sol) \geq U(\beta(x)\,|x,\ sol)$ if $x' > x$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.*

The proof of the lemma utilizes that the set of feasible bids is dense below v_ℓ. If the price grid is finite below v_ℓ as well, equilibrium may involve bids just below v_ℓ—just like in the usual Bertrand pricing game with price grid—but such equilibria would not add anything important.

Proof of Lemma 9: $b' > b \geq v_\ell$ implies $(v_\ell - b') < (v_\ell - b)$ and $\overline{\pi}_\ell(b') \geq \overline{\pi}_\ell(b)$. These together with the hypothesis $\overline{\pi}_\ell(b') > 0$ and $b' > b \geq v_\ell$ imply

$$\overline{\pi}_\ell(b')\left(v_\ell - b'\right) < \overline{\pi}_\ell(b)\left(v_\ell - b\right). \tag{53}$$

Hence, $U(b'|x, \text{sol}) \geq U(b|x, \text{sol})$ requires

$$\overline{\pi}_h(b')\left(v_h - b'\right) > \overline{\pi}_h(b)\left(v_h - b\right). \tag{54}$$

Rewriting $U(b'|x, \text{sol})$ yields

$$\frac{\rho_\ell g_\ell(x)\,\bar{n}_\ell}{\rho_\ell g_\ell(x)\,\bar{n}_\ell + \rho_h g_h(x)\,\bar{n}_h}\left[\overline{\pi}_\ell(b)(v_\ell - b) + \frac{\rho_h g_h(x)\,\bar{n}_h}{\rho_\ell g_\ell(x)\,\bar{n}_\ell}\overline{\pi}_h(b)\left(v_h - b\right)\right]. \tag{55}$$

It follows from $U(b'|x, \text{sol}) \geq U(b|x, \text{sol})$ and (53) that

$$\begin{aligned}
&\frac{\rho_h g_h(x)\,\overline{n}_h}{\rho_\ell g_\ell(x)\,\overline{n}_\ell}\left[\overline{\pi}_h(b')\left(v_h - b'\right) - \overline{\pi}_h(b)\left(v_h - b\right)\right] \\
\geq\ & \overline{\pi}_\ell(b)(v_\ell - b) - \overline{\pi}_\ell(b')(v_\ell - b') > 0.
\end{aligned}$$

Since $x' > x$ and $\frac{g_h(x)}{g_\ell(x)}$ is nondecreasing,

$$\begin{aligned}
&\frac{\rho_h g_h(x')\,\overline{n}_h}{\rho_\ell g_\ell(x')\,\overline{n}_\ell}\left[\overline{\pi}_h(b')\left(v_h - b'\right) - \overline{\pi}_h(b)\left(v_h - b\right)\right] \\
\geq\ & \overline{\pi}_\ell(b)(v_\ell - b) - \overline{\pi}_\ell(b')(v_\ell - b') > 0.
\end{aligned} \tag{56}$$

which implies

$$\begin{aligned}
&U(b'|x',\ \text{sol}) \\
=\ & \frac{\rho_\ell g_\ell(x')\,\overline{n}_\ell}{\rho_\ell g_\ell(x')\,\overline{n}_\ell + \rho_h g_h(x')\,\overline{n}_h}\left[\overline{\pi}_\ell(b')(v_\ell - b') + \frac{\rho_h g_h(x')\,\overline{n}_h}{\rho_\ell g_\ell(x')\,\overline{n}_\ell}\overline{\pi}_h(b')\left(v_h - b'\right)\right] \\
\geq\ & \frac{\rho_\ell g_\ell(x')\,\overline{n}_\ell}{\rho_\ell g_\ell(x')\,\overline{n}_\ell + \rho_h g_h(x')\,\overline{n}_h}\left[\overline{\pi}_\ell(b)(v_\ell - b) + \frac{\rho_h g_h(x')\,\overline{n}_h}{\rho_\ell g_\ell(x')\,\overline{n}_\ell}\overline{\pi}_h(b)\left(v_h - b\right)\right] \\
=\ & U(b|x',\ \text{sol}).
\end{aligned} \tag{57}$$

If $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$, then (56) and (57) hold with strict inequalities.

The last part of the lemma is immediate because G_h and G_ℓ are mutually absolutely continuous, so that $G_h\left(\{x|\beta(x) \leq b\}\right) = 0 \Leftrightarrow G_\ell\left(\{x|\beta(x) \leq b\}\right) = 0$. ∎

Proof of Lemma 10:

Step 0: If $\pi_\omega(b) > 0$ for some $n \geq 2$ and $\omega = \ell$ or h, then $\overline{\pi}_\omega(b) > 0$ for both ω and any η_ω.

Proof of Step 0: $\pi_\omega(b; \beta, n) > 0$ for some n and ω implies that $G_\omega(\{x|\beta(x) \leq b\}) > 0$. Since G_h and G_ℓ are mutually absolutely continuous, it follows that $G_{\omega'}(\{x|\beta(x) \leq b\}) > 0$ also for $\omega' \neq \omega$. Therefore, $\overline{\pi}_\omega(b) > 0$ for both ω and any η_ω. □

Step 1. $\beta(x) \geq v_\ell$ for almost all x.

Proof of Step 1: Let $\underline{b} \equiv \inf\{b|\pi_\omega(b) > 0 \text{ for some } n \text{ and } \omega\}$. Suppose $\underline{b} < v_\ell$. It may not be that β has an atom at $\underline{b}$ (i.e., $\int_{\{x:\beta(x)=\underline{b}\}} g_\omega(x)dx > 0$) since by a standard Bertrand argument $U(\underline{b}+\varepsilon|x, \text{sol}) > U(\underline{b}|x, \text{sol})$ for sufficiently small $\varepsilon \in (0, v_\ell - \underline{b})$. Therefore, there exists a sequence of x^k such that $\beta(x^k) \to \underline{b}$ and $\overline{\pi}_\omega(\beta(x^k)) \to 0$ (owing to $\eta_\omega(1) = 0$). Hence, equilibrium payoffs $U(\beta(x^k)|x^k, \text{sol}) \to 0$. However, by the definition of $\underline{b}$ and monotonicity of $\overline{\pi}_\omega$, $\overline{\pi}_\omega(b)$ is strictly positive for all $b \in (\underline{b}, v_\ell)$. Thus, for all $b \in (\underline{b}, v_\ell)$, the payoff $U(b|x, \text{sol}) > 0$. This contradicts the optimality of $\beta(x^k)$ for sufficiently large k, a standard Bertrand argument. Thus, $\underline{b} \geq v_\ell$. Finally, $\pi_\omega(b) = 0$ for all $b < v_\ell$ implies that $G_\omega(\{x|\beta(x) \geq v_\ell\}) = 1$, proving the step. □

Step 2. $\beta(x) < v_h$ for all x. □

Proof of Step 2: It clearly cannot be that $G_\omega(\{x|\beta(x) > v_h\}) = 1$ for any ω, since this would imply that bidders have strictly negative payoffs in expectations. Suppose that $\beta(x') \geq v_h$ for some x'. From $G_\ell(\{x|\beta(x) > v_h\}) < 1$, $\beta(x') \geq v_h$ implies $\overline{\pi}(\beta(x')) > 0$ and $U(\beta(x')|x', \text{sol}) < 0$, a contradiction to the optimality of $\beta(x')$. □

Step 3. $\overline{\pi}_\omega(\beta(x)) > 0$ for almost all x for $\omega \in \{\ell, h\}$.

Proof of Step 3: Fix $\omega \in \{\ell, h\}$. Let $X = \{x|\overline{\pi}_\omega(\beta(x)) = 0\}$. The probability that in state ω all bidders are from that set is $\Sigma_n \eta_\omega(n)[G_\omega(X)]^n$. Since in that event some bidder has to win, we have $\Sigma_n \eta_\omega(n)[G_\omega(X)]^n \leq \Pr[\{\text{Winning bidder has signal } x \in X\}|\omega] \leq \overline{n}_\omega \int_{x \in X} \overline{\pi}_\omega(\beta(x))\, g(x)\, dx = 0$. Hence, $G_\omega(X) = 0$. □

Step 4. For any $x' > x$, $U(\beta(x')|x', \text{sol}) \geq U(\beta(x)|x, \text{sol})$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$. Thus, $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ implies that $U(\beta(x')|x', \text{sol})$ is strictly positive.

Proof of Step 4: From (52) it follows (after dividing the numerator and denominator by $g_\ell(x)$) that

$$U(b|x,\ \text{sol}) = \frac{\rho_\ell \overline{n}_\ell \overline{\pi}_\ell(b)(v_\ell - b) + \rho_h \frac{g_h(x)}{g_\ell(x)} \overline{n}_h \overline{\pi}_h(b)\,(v_h - b)}{\rho_\ell \overline{n}_\ell + \rho_h \frac{g_h(x)}{g_\ell(x)} \overline{n}_h}. \tag{58}$$

Therefore, for any $x' > x$,

$$U(\beta(x')|x',\ \text{sol}) \geq U(\beta(x)|x',\ \text{sol}) \geq U(\beta(x)|x,\ \text{sol}) \geq 0, \tag{59}$$

where the first and last inequalities are equilibrium conditions; the second inequality owes to $\frac{g_h(x')}{g_\ell(x')} \geq \frac{g_h(x)}{g_\ell(x)}$ and $\overline{\pi}_h(\beta(x))\,(v_h - \beta(x)) \geq 0 \geq \overline{\pi}_\ell(\beta(x))(v_\ell - \beta(x))$, which follows from Steps 1 and 2.

Suppose $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$. Now, either $\overline{\pi}_\omega\,(\beta(x)) > 0$, in which case $\overline{\pi}_h(\beta(x))\,(v_h - \beta(x)) > 0$, and it follows from (58) and $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$ that the second inequality in (59) is strict, or $\overline{\pi}_\omega\,(\beta(x)) = 0$ and hence $U(\beta(x)|x, \text{sol}) = 0$. In the latter case, by Step 3, there is some $y \in (\underline{x}, x')$ such that $\overline{\pi}_\omega\,(\beta(y)) > 0$. We can choose y such that $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(y)}{g_\ell(y)}$ (recall that $\frac{g_h(\underline{x})}{g_\ell(\underline{x})} = \lim_{x\to\underline{x}} \frac{g_h(x)}{g_\ell(x)}$). By Step 2, $\overline{\pi}_h(\beta(y))\,(v_h - \beta(y)) > 0$. Since $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(y)}{g_\ell(y)}$, it follows from (58) and the fact that β is a bidding equilibrium that

$$U(\beta(x')|x',\ \text{sol}) \geq U(\beta(y)|x',\ \text{sol}) > U(\beta(y)|y,\ \text{sol}) \geq 0 = U(\beta(x)|x,\ \text{sol}).$$

Conversely, $\frac{g_h(x')}{g_\ell(x')} = \frac{g_h(x)}{g_\ell(x)}$ implies

$$U(\beta(x')|x',\ \text{sol}) = U(\beta(x')|x,\ \text{sol}) \leq U(\beta(x)|x,\ \text{sol}) = U(\beta(x)|x',\ \text{sol}) \leq U(\beta(x')|x',\ \text{sol}),$$

where the inequalities are equilibrium conditions while the equalities owe to the fact that x and x' contain the same information. Therefore, $U(\beta(x')|x', \text{sol}) = U(\beta(x)|x, \text{sol})$. □

Step 5. The strict positivity of $U(\beta(x)|x, \text{sol})$ implies immediately that $\overline{\pi}_\omega\,(\beta(x)) > 0$ for any x for which $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$. (Step 3 established this only for almost all x). This proves Part 1 of the Lemma.

Step 6. $\beta\,(x) > v_\ell$ for any x for which $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ (as opposed to just $\geq$ established in Step 1).

Proof of Step 6: The same standard Bertrand argument used in the proof of Step 1 implies that there cannot be mass point at v_ℓ. Therefore, $U(v_\ell|x, \text{sol}) = 0$. Since Step 4 implies that $U(\beta(x)|x, \text{sol}) > 0$ for all $x >\underline{x}$ for which $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$, it must be that $\beta\,(x) > v_\ell$ for all such x. □

This completes the proof of the lemma. ■

Proof of Proposition 1':

Part 1: Proved by Lemma 10.

Part 2**:** Suppose that $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$ for some $x, x' \in (\underline{x}, \bar{x}]$, but $\beta\,(x') < \beta(x)$. Since β is a bidding equilibrium, $U(\beta(x)|x, \text{sol}) \geq U(\beta(x')|x, \text{sol})$. By Lemma 10, $\overline{\pi}_\omega(\beta\,(x')) > 0$ and $\beta\,(x') \geq v_\ell$. Therefore, by Lemma 9, $U(\beta(x)|x', \text{sol}) > U(\beta(x')|x', \text{sol})$, contradicting the optimality of $\beta(x')$ for x'. Thus, the supposition $\beta\,(x') < \beta(x)$ is false. Hence, $\beta\,(x') \geq \beta(x)$ whenever $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.

Next, suppose that $\frac{g_h(x')}{g_\ell(x')} = \frac{g_h(x)}{g_\ell(x)}$ for some $x, x' \in (\underline{x}, \bar{x}]$, but $\beta\,(x') < \beta(x)$. Then there is some interval containing x and x' over which $\frac{g_h(x)}{g_\ell(x)}$ is constant, say, C. Let $[x_-, x_+]$ be the closure of this interval. By the above argument, $\beta\,(x'') \leq \beta\,(x)$ whenever $x'' < x_- < x$

and $\beta(x) \leq \beta(x''')$ whenever $x < x_+ < x'''$. Define $\widetilde{\beta}_1(x)$ by

$$\widetilde{\beta}_1(x) = \inf\{b : G_h(x) \leq G_h(\{t|\beta(t) \leq b\})\} \quad \text{if} \quad x \in [x_-, x_+]$$

Thus, on $[x_-, x_+]$ the signals are essentially "reordered" to make $\widetilde{\beta}_1(x)$ monotone. Outside $[x_-, x_+]$, $\widetilde{\beta}_1(x)$ coincides with $\beta(x)$. Note that $\tilde{\beta}(x') \leq \tilde{\beta}(x) \leq \tilde{\beta}(x'')$ for all $x' < x_-$ and $x_+ < x''$. With this definition,

$$G_h(\{x|\widetilde{\beta}_1(x) \leq b\}) = G_h(\{x|\beta(x) \leq b\}),$$

for all b. That is, the distribution of bids induced by $\widetilde{\beta}_1$ is equal to the distribution of bids induced by β in state h. It is also the same in state ℓ because $\widetilde{\beta}_1 = \beta$ outside $[x_-, x_+]$ and because the distributions G_ℓ and G_h conditional on $x \in (x_-, x_+)$ are identical (owing to the constant $\frac{g_h(x)}{g_\ell(x)}$).

The equality of the distributions of bids under $\widetilde{\beta}_1$ and β implies that, for any $x \notin \{x_-, x_+\}$, $\tilde{\beta}_1(x)$ is optimal: for $x \notin [x_-, x_+]$ this follows immediately from $\widetilde{\beta}_1(x) = \beta(x)$; for $x \in (x_-, x_+)$ this follows from $\tilde{\beta}_1(x) = \beta(y)$ where y is some value of the signal such that $\frac{g_h(y)}{g_\ell(y)} = \frac{g_h(x)}{g_\ell(x)}$. For $x \in \{x_-, x_+\}$, note that we can represent the distribution of signals by an equivalent pair of densities that is equal to the original densities almost everywhere, so that the resulting equilibrium still corresponds to the same distributional strategy. Here, $\tilde{\beta}_1$ can be rationalized at $\{x_-, x_+\}$ by changing the densities at the points $x \in \{x_-, x_+\}$. At x_-, if $\tilde{\beta}_1(x_-) = \tilde{\beta}_1(x_- + \varepsilon)$ for some ε (an atom), $\tilde{\beta}_1(x_-)$ is rationalized by setting $g_\omega(x_-) = \lim_{\varepsilon \to 0} g_\omega(x_- + \varepsilon)$. Otherwise, $\tilde{\beta}_1(x_-)$ is rationalized by setting $g_\omega(x_-) = \lim_{\varepsilon \to 0} g_\omega(x_- - \varepsilon)$. Similarly for x_+. It follows that $\widetilde{\beta}_1$ is monotone on $[x_-, x_+]$ and that it is equivalent to β.

Repeating this construction for all intervals over which $\frac{g_h(x)}{g_\ell(x)}$ is constant, we get a sequence of bidding strategies (constructing the sequence by starting with the longest interval of signals on which $\frac{g_h(x)}{g_\ell(x)}$ is constant). Let $\tilde{\beta}$ be the pointwise limit of this sequence on $(\underline{x}, \bar{x}]$ and let $\tilde{\beta}(\underline{x}) = \lim_{\varepsilon \to 0} \beta(\underline{x} + \varepsilon)$. Then, $\tilde{\beta}$ is an equivalent bidding equilibrium that is monotone on $[\underline{x}, \bar{x}]$, as claimed. ∎

B.3 Restatement of Theorem 1 for a Mixed Solicitation Strategy

We now restate Theorem 1 and Proposition 3 for mixed solicitation strategies.

Theorem 1' *Consider a sequence of bidding games $\Gamma_0(\boldsymbol{\eta}^k, N^k)$ such that supp$(\eta^k_\omega) = \{n^k_\omega, n^k_\omega + 1\}$, $\min\{n^k_\ell, n^k_h\} \to \infty$ and $\lim \frac{n^k_h}{n^k_\ell} = r$, and a corresponding sequence of bidding equilibria β^k.*

1. *If $\bar{g}r < 1$, then, for large enough k, there is a bid $\overline{b}^k$ at which F^k_ω has an atom,*

$$\lim \Pr(\textit{winning bid } = \overline{b}^k|\omega) \triangleq \lim dF^k_\omega(\overline{b}^k) = 1 \qquad \text{for } \omega = \ell, h,$$

and

$$\lim \mathbb{E}[v|\bar{x}, \text{sol}] \equiv \frac{v_\ell + \rho \bar{g} r v_h}{1+\rho \bar{g} r} \leq \lim \overline{b}^k \leq \mathbb{E}[v].$$

2. *If* $\bar{g}r > 1$*, then,* F_ω^k *converges pointwise to* $\Phi_\omega(\cdot|r)$*,* $\omega = \ell, h$*, (where* Φ_ω is described by (5)-(6)*).*

Proposition 3' *Consider a sequence of bidding games* $\Gamma_0(\boldsymbol{\eta}^k, N^k)$ *such that supp*$(\eta_\omega^k) = \{n_\omega^k, n_\omega^k + 1\}$*,* $\min\{n_\ell^k, n_h^k\} \to \infty$ *and* $\lim \frac{n_h^k}{n_\ell^k} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} = 1$*. If* β^k *is a bidding equilibrium of* $\Gamma_0(\boldsymbol{\eta}^k, N^k)$ *for all* k*, then the limit of the winning bid distribution is a mass point on* $\mathbb{E}[v]$*, i.e.,* $\lim F_\omega^k(b) = 1$ *for all* $b > \mathbb{E}[v]$ *and* $= 0$ *for all* $b < \mathbb{E}[v]$*.*

The following lemma shows that, for the purposes of this proof, $\boldsymbol{\eta}^k$ may be replaced by $\mathbf{n}^k$ without loss of generality. Once this is established, the proofs of Theorem 1 and Proposition 3 apply and need not be repeated. Recall $\overline{n}_\omega(\eta_\omega)$ and $\overline{\pi}_\omega(b;\beta,\eta_\omega)$, $\omega = \ell, h$, from (51). Since we deal here explicitly with $\boldsymbol{\eta}$ and $\mathbf{n}$, we do not suppress them in the arguments of π and $\mathbb{E}[v|...]$.

Lemma 11 *Consider a sequence of bidding games* $\Gamma_0(N^k, \boldsymbol{\eta}^k)$ *such that supp*$(\eta_\omega^k) = \{n_\omega^k, n_\omega^k + 1\}$*,* $\min\{n_\ell^k, n_h^k\} \to \infty$ *and* $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} = r$*, and a corresponding sequence of bidding equilibria* β^k*.*
(i)

$$\lim \frac{\overline{n}_h^k}{\overline{n}_\ell^k} = \lim \frac{n_h^k}{n_\ell^k}, \tag{60}$$

(ii) For any (b^k) *with* $\lim \left(G_\omega\left(x_+\left(b^k\right)\right)\right)^{n_\omega^k} > 0$*,*

$$\lim \frac{\overline{\pi}_h(b^k;\beta^k,\eta_h^k)}{\overline{\pi}_\ell(b^k;\beta^k,\eta_\ell^k)} = \lim \frac{\pi_h(b^k;\beta^k,n_h^k)}{\pi_\ell(b^k;\beta^k,n_\ell^k)}.$$

(iii) For any (b^k) *with* $\lim \left(G_\omega\left(x_+\left(b^k\right)\right)\right)^{n_\omega^k} > 0$*,*

$$\lim \mathbb{E}[v|x^k, \textit{sol,win at } b^k; \beta^k, \boldsymbol{\eta}^k] = \lim \mathbb{E}[v|x^k, \textit{sol,win at } b^k; \beta^k, \mathbf{n}^k].$$

Remark: The condition $\lim \left(G_\omega\left(x_+\left(b^k\right)\right)\right)^{n_\omega^k} > 0$ is needed for part (ii). For any fixed $x < \bar{x}$, if β^k is strictly increasing, it follows from $\pi_\omega(\beta^k(x);\beta^k,n_\omega^k) = (G_\omega(x))^{n_\omega^k - 1}$ that

$$\frac{\pi_h(\beta^k(x);\beta^k,n_h^k+1)}{\pi_\ell(\beta^k(x);\beta^k,n_\ell^k)} = G_h(x)\frac{\pi_h(\beta^k(x);\beta^k,n_h^k)}{\pi_\ell(\beta^k(x);\beta^k,n_\ell^k)} < \frac{\pi_h(\beta^k(x);\beta^k,n_h^k)}{\pi_\ell(\beta^k(x);\beta^k,n_\ell^k)}.$$

Therefore, since $G_h(x) < 1$, the difference between these ratios is not vanishing as would be required for the result of the lemma to hold for this x. However, when $\lim \left(G_\omega\left(x_+\left(b^k\right)\right)\right)^{n_\omega^k} > 0$, then $x_+\left(b^k\right) \to \bar{x}$ and hence $G_\omega\left(x_+\left(b^k\right)\right) \to 1$.

Proof of Lemma 11. Since η_ω chooses n_ω and $n_\omega+1$ with probabilities γ_ω and $1-\gamma_\omega$ respectively,

$$\begin{aligned}\overline{n}_\omega &= \gamma_\omega n_\omega + (1-\gamma_\omega)(n_\omega+1)\\ \overline{\pi}_\omega[b;\beta,\eta_\omega] &= \frac{\gamma_\omega n_\omega \pi_\omega(b;\beta,n_\omega)+(1-\gamma_\omega)(n_\omega+1)\pi_\omega(b;\beta,n_\omega+1)}{\gamma_\omega n_\omega+(1-\gamma_\omega)(n_\omega+1)}.\end{aligned}$$

This and $n_\omega \to \infty$, $\omega = \ell, h$, imply Part (i). Part (iii) follows from Part (i) and (ii). So, we show Part (ii). For this, it is sufficient to show

$$\lim \frac{\pi_\omega(b^k;\beta^k,n_\omega^k)}{\pi_\omega(b^k;\beta^k,n_\omega^k+1)} = 1.$$

From Lemma 5,

$$\frac{\pi_\omega(b^k;\beta^k,n_\omega^k)}{\pi_\omega(b^k;\beta^k,n_\omega^k+1)} = \frac{\int_{x_-^k}^{x_+^k}(G_\omega(x))^{n_\omega^k-1}g_\omega(x)\,dx}{\int_{x_-^k}^{x_+^k}(G_\omega(x))^{n_\omega^k}g_\omega(x)\,dx}.$$

The claim is immediate if $x_-^k \to \bar{x}$ since

$$\frac{1}{G_\omega\left(x_+^k\right)} \le \frac{\int_{x_-^k}^{x_+^k}(G_\omega(x))^{n_\omega^k-1}g_\omega(x)\,dx}{\int_{x_-^k}^{x_+^k}(G_\omega(x))^{n_\omega^k}g_\omega(x)\,dx} \le \frac{1}{G_\omega\left(x_-^k\right)}, \tag{61}$$

and $G_\omega\left(x_+^k\right) \to 1$. Otherwise, we can choose some $\varepsilon > 0$ with $x_-^k < \bar{x}-\varepsilon$ for all k. Observe that

$$\lim \frac{\int_{\bar{x}-\varepsilon}^{x_+^k}(G_\omega(x))^{n_\omega^k}g_\omega(x)\,dx}{\int_{x_-^k}^{x_+^k}(G_\omega(x))^{n_\omega^k}g_\omega(x)\,dx} = 1.$$

The claim now follows using the previous bounds (61) because

$$\lim \frac{\pi_\omega(b^k;\beta^k,n_\omega^k)}{\pi_\omega(b^k;\beta^k,n_\omega^k+1)} = \lim \frac{\int_{\bar{x}-\varepsilon}^{x_+^k}(G_\omega(x))^{n_\omega^k-1}g_\omega(x)\,dx}{\int_{\bar{x}-\varepsilon}^{x_+^k}(G_\omega(x))^{n_\omega^k}g_\omega(x)\,dx},$$

and because we can choose ε arbitrarily small such that $G_\omega(\bar{x}-\varepsilon) \cong 1$. ∎

Using Lemma 11, the proof of Theorem 1' essentially repeats the proof of Theorem 1. Analogously for Proposition 3'.

C Characterization of Full Equilibrium (with Solicitation)

C.1 Proof of Lemma 4: Existence and Uniqueness of $r^*(\rho, \overline{g})$

The proof relies on two lemmas. Define

$$J(r;\rho,\overline{g}) \triangleq \int_0^1 (x - \frac{1}{\overline{g}}) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{(1+x\rho\overline{g}r)^2} dx. \tag{62}$$

Lemma 12 *For the function defined in (62), for any $\rho \in (0,1)$ and $\overline{g} > 1$:*

1. *There is a unique number $r^* = r^*(\rho, \overline{g}) \in (\frac{1}{\overline{g}}, \infty)$ s.t. $J(r^*) = 0$.*
2. *$J(r) < 0$ for $r \in (\frac{1}{\overline{g}}, r^*)$ and $J(r) > 0$ for $r \in (r^*, \infty)$.*

Proof of Lemma 12:
Claim 1: For each $\overline{g} > 1$, there exists an $r' \in (\frac{1}{\overline{g}}, \infty)$ (close to $\overline{g}^{-1}$) such that $J(r') < 0$.
Proof: Write

$$J(r) = \int_0^{\frac{1}{\overline{g}}} \left(\frac{1}{\overline{g}} - x\right) \left\{ (-\ln(x)) (x)^{\left(\frac{1}{\overline{g}r-1}\right)} (1+\rho\overline{g}rx)^{-2} \right\} dx$$
$$- \int_{\frac{1}{\overline{g}}}^1 \left(x - \frac{1}{\overline{g}}\right) \left\{ (-\ln(x)) (x)^{\left(\frac{1}{\overline{g}r-1}\right)} (1+\rho\overline{g}rx)^{-2} \right\} dx.$$

The term in the brackets $\{\cdots\}$ is always nonnegative and therefore both integrals are positive.

Let $\sigma \triangleq \frac{1}{\overline{g}r-1}$. The first integral is

$$\int_0^{\frac{1}{\overline{g}}} \left(\frac{1}{\overline{g}} - x\right) \left\{ (-\ln(x)) (x)^{\left(\frac{1}{\overline{g}r-1}\right)} \frac{1}{(1+\rho\overline{g}rx)^2} \right\} dx$$
$$\leq \frac{1}{\overline{g}} \int_0^{\frac{1}{\overline{g}}} (-\ln(x)) x^{\sigma} dx$$
$$= \left(\frac{1}{\overline{g}}\right)^{\sigma+2} \left(\frac{1}{\sigma+1}\right)^2 \left(1 - (\sigma+1)\ln\frac{1}{\overline{g}}\right),$$

where the equality is derived by integration-by-parts.

Thus, the first integral vanishes to zero at a rate of at least $\left(\overline{g}^{-1}\right)^{\sigma}$ as σ approaches ∞ (or equivalently, $r \to \overline{g}^{-1}$).

The second integral is

$$\int_{\frac{1}{\overline{g}}}^{1}\left(x-\frac{1}{\overline{g}}\right)\left\{-\ln\left(x\right)\left(x\right)^{\left(\frac{1}{\overline{g}r-1}\right)}\left(1+\rho\overline{g}rx\right)^{-2}\right\}dx$$

$$=\int_{\frac{1}{\overline{g}}}^{1}\left(x-\frac{1}{\overline{g}}\right)\left\{\left(-\ln\left(x\right)\right)x^{\sigma}\left(1+\rho\frac{\sigma+1}{\sigma}x\right)^{-2}\right\}dx$$

$$\geq\left(1+\rho\frac{\sigma+1}{\sigma}\right)^{-2}\int_{\frac{1}{\overline{g}}}^{1}\left(x-\frac{1}{\overline{g}}\right)\left(-\ln\left(x\right)\right)x^{\sigma}dx$$

$$=\left(\frac{\sigma}{\sigma+\rho(\sigma+1)}\right)^{2}\left(\frac{1}{\left(\sigma+2\right)^{2}}-\frac{\overline{g}^{-1}}{\left(\sigma+1\right)^{2}}+\frac{\left(-\ln\left(\overline{g}^{-1}\right)\right)\left(\overline{g}^{-1}\right)^{\sigma+2}}{\left(\sigma+2\right)\left(\sigma+1\right)}+\frac{1}{\overline{g}}\frac{\left(\overline{g}^{-1}\right)^{\sigma+1}}{\left(\sigma+1\right)^{2}}-\frac{\left(\overline{g}^{-1}\right)^{\sigma+2}}{\left(\sigma+2\right)^{2}}\right),$$

where the second equality follows by integration-by-parts.

Thus, either the second integral stays positive or it vanishes at a rate of at most σ^{-2} as σ approaches ∞ (or equivalently, $r\to\overline{g}^{-1}$).

To sum up, $J(r)<0$ for $r\to\overline{g}^{-1}$. ∎

Claim 2: For sufficiently large r, $J(r)>0$.

Proof: We show that $\lim_{r\to\infty}r^{2}J(r)=\infty$. Let $\xi\left(x,r\right)$ denote the integrand of $r^{2}J(r)$. That is,

$$\xi\left(x,r\right)\equiv\left(x-\frac{1}{\overline{g}}\right)\ln\left(x\right)x^{\frac{1}{\overline{g}r-1}}\left(\frac{r}{1+\rho\overline{g}rx}\right)^{2}.$$

Observe that $\xi\left(x,r\right)$ is nondecreasing in r on the domain $x\in\left(0,\overline{g}^{-1}\right)$, and is non-increasing in r on the domain $x\in\left(\overline{g}^{-1},1\right)$. Therefore, by the monotone convergence theorem,

$$\lim_{r\to\infty}r^{2}J(r)\equiv\lim_{r\to\infty}\int_{0}^{1}\xi\left(x,r\right)dx=\int_{0}^{1}\lim_{r\to\infty}\xi\left(x,r\right)dx=\frac{1}{\left(\rho\overline{g}\right)^{2}}\int_{0}^{1}\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln\left(x\right)dx$$

$$=\frac{1}{\left(\rho\overline{g}\right)^{2}}\left[\int_{0}^{\overline{g}^{-1}}\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln\left(x\right)dx+\int_{\overline{g}^{-1}}^{1}\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln\left(x\right)dx\right].$$

Now, letting $a\in\left(0,\overline{g}^{-1}\right)$,

$$\int_{0}^{\overline{g}^{-1}}\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln\left(x\right)dx\geq\lim_{a\to0}\int_{a}^{\overline{g}^{-1}}\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln\left(x\right)dx$$

$$=\lim_{a\to0}\left[\frac{1}{2}\left(\left(\ln\left(\overline{g}^{-1}\right)\right)^{2}-\left(\ln\left(a\right)\right)^{2}\right)+\frac{1}{\overline{g}}\left[\overline{g}\left(\ln\left(\overline{g}^{-1}\right)+1\right)-a^{-1}\left(1+\ln\left(a\right)\right)\right]\right]=\infty,$$

while $\int_{\overline{g}^{-1}}^{1}\left(x^{-1}-\frac{1}{\overline{g}}x^{-2}\right)\ln\left(x\right)dx$ is obviously bounded. Therefore, $\lim_{r\to\infty}r^{2}J(r)=\infty$ hence $J(r)>0$ for large enough r. ∎

Claims 1 and 2 together with the continuity of $J(r)$ in r establish the existence of $r>1/\overline{g}$ such that $J(r)=0$.

Claim 3: Fix a $\overline{g} > 1$. For $r > \overline{g}^{-1}$, if $J(r;\rho,\overline{g}) = 0$, then $J_r(r;\rho,\overline{g}) > 0$.
Proof: By hypothesis,

$$J(r;\rho,\overline{g}) \equiv \int_0^1 \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{\left(1+\rho\overline{g}rx\right)^2} dx = 0.$$

Since $x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{(1+\rho\overline{g}rx)^2} < 0$ for all $x \in (0,1)$, the integrand is positive for all $x \in (0, \frac{1}{\overline{g}})$ and is negative for all $x \in (\frac{1}{\overline{g}}, 1)$. Therefore, at any $r > \overline{g}^{-1}$ that satisfies $J(r) = 0$,

$$\int_0^{\frac{1}{\overline{g}}} \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{\left(1+\rho\overline{g}rx\right)^2} dx = -\int_{\frac{1}{\overline{g}}}^1 \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{\left(1+\rho\overline{g}rx\right)^2} dx > 0.$$

Consider the function $r^2 J(r)$ and observe that

$$\frac{dr^2 J(r)}{dr} = r \int_0^1 \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{\left(1+\rho\overline{g}rx\right)^2} \left[\frac{-\overline{g}r \ln x}{\left(\overline{g}r-1\right)^2} + \frac{2}{\left(1+\rho\overline{g}rx\right)}\right] dx.$$

The integrand is equal to the integrand of $J(r)$ times the term $\left[\frac{-\overline{g}r\ln x}{(\overline{g}r-1)^2} + \frac{2}{(1+\rho\overline{g}rx)}\right]$ which is nonnegative and decreasing in x. Therefore, at r such that $J(r) = 0$, the positive part over $(0, \frac{1}{\overline{g}})$ is weighted more heavily than the negative part over $(\frac{1}{\overline{g}}, 1)$ implying $\frac{dr^2 J(r)}{dr} > 0$. Now, at r such that $J(r) = 0$, $sgn\left(J_r(r)\right) = sgn\left(\frac{dr^2 J(r)}{dr}\right)$. Therefore, $J_r(r) > 0$ as required. □

Claim 3 concludes the proof of the Lemma, since $J_r(r) > 0$ at any r such that $J(r) = 0$, there can be only one such r. ■

Lemma 13 *If r satisfies equation (19), then $J\left(r;\rho,\overline{g}\right) = 0$.*

Proof of Lemma 13: Using (5) to spell out $\Phi_\omega\left(p|r\right)$ and rearranging, we have

$$\begin{aligned}
&-\int_{v_\ell}^{\overline{p}} \Phi_h\left(p|r\right) \ln \Phi_h\left(p|r\right) dp + r \int_{v_\ell}^{\overline{p}} \Phi_\ell\left(p|r\right) \ln \Phi_\ell\left(p|r\right) dp \\
= \quad & -\frac{\overline{g}r}{\overline{g}r-1} \int_{v_\ell}^{\overline{p}} \left(\frac{1}{\rho\overline{g}r}\frac{p-v_\ell}{v_h-p} - \frac{1}{\overline{g}}\right) \left(\frac{1}{\rho\overline{g}r}\frac{p-v_\ell}{v_h-p}\right)^{\frac{1}{\overline{g}r-1}} \ln\left(\frac{1}{\rho\overline{g}r}\frac{p-v_\ell}{v_h-p}\right) dp
\end{aligned}$$

Changing the integration variable by substituting for p the function $\psi(x) = \frac{v_\ell + x\rho\overline{g}rv_h}{1+x\rho\overline{g}r}$ we get

$$\begin{aligned}
&-\int_{v_\ell}^{\overline{p}} \Phi_h\left(p|r\right) \ln \Phi_h\left(p|r\right) dp + r \int_{v_\ell}^{\overline{p}} \Phi_\ell\left(p|r\right) \ln \Phi_\ell\left(p|r\right) dp \qquad (63) \\
= \quad & -\frac{\overline{g}r}{\overline{g}r-1} \int_{\psi^{-1}(v_\ell)}^{\psi^{-1}(\overline{p})} \left(\frac{1}{\rho\overline{g}r}\frac{\psi(x)-v_\ell}{v_h-\psi(x)} - \frac{1}{\overline{g}}\right) \left(\frac{1}{\rho\overline{g}r}\frac{\psi(x)-v_\ell}{v_h-\psi(x)}\right)^{\frac{1}{\overline{g}r-1}} \ln\left(\frac{1}{\rho\overline{g}r}\frac{\psi(x)-v_\ell}{v_h-\psi(x)}\right) \psi'(x) dx \\
= \quad & -\frac{\overline{g}r}{\overline{g}r-1} \int_0^1 \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r-1}} \ln(x) \frac{r\overline{g}\rho\left(v_h - v_\ell\right)}{\left(1+rx\overline{g}\rho\right)^2} dx = -\frac{\left(\overline{g}r\right)^2}{\overline{g}r-1} \rho\left(v_h - v_\ell\right) J(r).
\end{aligned}$$

Now, this and (19) imply $J(r) = 0$. ∎

C.2 Proof of Proposition 5 (Bounded Number of Bidders)

If n_ω^k is bounded, then $\lim n_\omega^k s^k = 0$. Hence, there is a C_ω such that $\lim F_\omega$ is concentrated on some C_ω, since otherwise Lemma 3 would imply $\lim n_\omega^k s^k > 0$.[32]

Note that Lemma 3 does not require β^k to be monotone. This is useful because it may be that $n_\omega^k = 1$ for some state, and, hence, we cannot invoke Proposition 1 to argue that β^k must be monotone non-decreasing.

Case 1. Suppose that $\lim n_h^k < \infty$ and $\lim n_\ell^k < \infty$. Then, as was just argued, in each state, $\lim F_\omega^k$ is concentrated on some C_ω. However, $\overline{g} \equiv \frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \infty$ and $n_\omega^k < \infty$ imply that $\lim F_h^k$ and $\lim F_\ell^k$ are mutually absolutely continuous. Hence, $C_\ell = C_h = C$. (This case includes the trivial equilibrium with $C = 0 \le v_\ell$ and $n_h^k = n_\ell^k = 1$.)

Case 2. Suppose $\lim n_\ell^k = \infty$ and $\lim n_h^k = m < \infty$ for some $m \ge 1$.[33] As observed, $\lim F_h^k$ is concentrated on C_h. To start, by $n_h^k \to m < \infty$ and the bounded likelihood ratio, whenever $\lim F_h\left(p|\beta^k, n_h^k\right) = 0$ then $\lim F_\ell\left(p|\beta^k, n_\ell^k\right) = 0$, i.e., the lower bound on the support of $\lim F_\ell\left(\cdot|\beta^k, n_\ell^k\right)$ is weakly above C_h. If $C_h \ge v_\ell$, then the bidders' individual rationality and the law of iterated expectations rule out that $\lim F_\ell\left(p|\beta^k, n_\ell^k\right) < 1$ for any $p > C_h \ge v_\ell$. Hence, if $C_h \ge v_\ell$, then the distribution of the winning bid is degenerate on C_h in the low state as well.

The case $C_h < v_\ell$ can be ruled out. From $\lim n_h^k/n_\ell^k = 0$, a solicited bidder is certain that the state is ℓ, and, hence, $\lim U^k(p + \varepsilon|x, \text{sol}) \ge \lim F_\ell^k(p)(v_\ell - p - \varepsilon)$ for all x, p, and $\varepsilon > 0$. Since feasibility requires $\lim U^k(\beta^k(x)|x, \text{sol}) \to 0$ for all x (as in Lemma 6), it follows that $\lim F_\ell^k(p) = 0$ for all $p < v_\ell$. Also, as before, bidders' individual rationality and the law of iterated expectations rule out that $\lim F_\ell\left(p|\beta^k, n_\ell^k\right) < 1$ for any $p > v_\ell$. Thus, the distribution of the winning bid in the low state becomes degenerate on v_ℓ. Because the distribution of the winning bid is degenerate in the low state, Lemma 3 implies that $n_\ell^k s^k \to 0$. Now, the boundedness of the likelihood ratio implies that when the high type samples n_ℓ^k bidders, the winning bid is close to v_ℓ as well, while total solicitation costs are close to zero. Thus, the payoff of the high type is at least v_ℓ. This contradicts the assumption that $C_h < v_\ell$.

Case 3. (This is the remaining case.) Suppose $\lim n_h^k = \infty$ and $\lim n_\ell^k = m < \infty$ for some $m \ge 1$. Since $\lim n_h^k/n_\ell^k = \infty$, a solicited bidder is certain that the state is h, and, hence, $\lim U^k(p + \varepsilon|x, \text{sol}) \ge \lim F_h^k(p)(v_h - p - \varepsilon)$ for all x, p, and $\varepsilon > 0$. Since feasibility requires $\lim U^k(\beta^k(x)|x, \text{sol}) \to 0$, it follows that $\lim F_h^k(p) = 0$ for all $p < v_h$. So, $\lim F_h^k$ puts mass 1 on v_h. Since $\lim F_h^k$ is degenerate, by Lemma 3, $n_h^k s^k \to 0$.

[32] For notational simplicity, the proof is for pure solicitation strategies. It extends immediately to mixed solicitation strategies.

[33] Recall that we cannot assume β^k to be non-decreasing.

The boundedness of the likelihood ratio implies that when the low type samples n_h^k bidders, then the winning bid is close to v_h as well, so that $\lim \left[\mathbb{E}^k(p|\ell;\beta^k, n_h^k) - n_h^k s^k\right] = v_h$. Thus, the seller's revenue is v_h in both states. This means that, in the limit, the sum of the bidders' payoffs is strictly negative. Therefore, for k large enough, some bidder must have strictly negative expected payoff. This contradicts individual rationality, ruling out this case.

Thus, in all cases that are not ruled out, $\lim F_\omega^k$ has mass 1 on the same C in both states. By individual rationality and the law of iterated expectations, $C \leq \rho_\ell v_\ell + \rho_h v_h$. This proves Proposition 5.

Online Appendix
—Not for Publication—

This online appendix contains all proofs regarding the existence of equilibrium. The first section covers the existence result for a boundedly informative signal. The second section covers the existence result for unboundedly informative signals and also extends the characterization results to this case. The online appendix is for now included in the pdf for the convenience of the reader and intended to be separated out of the paper later.

D Proofs for Existence Results with bounded Informativeness

D.1 Restatements of Results for Auctions with Grid

We now restate Proposition 1, Theorem 1, Proposition 3, and Theorem 2 for bids from a grid.

Proposition 1" *Suppose η is such that $\eta_\ell(1) = \eta_h(1) = 0$, and β is a bidding equilibrium of $\Gamma_0\left(\boldsymbol{\eta}^k, N^k, \Delta^k\right)$.*

1. *If $x' > x$, then $U(\beta(x')\,|x', sol) \geq U(\beta(x)\,|x, sol)$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.*
2. *There exists an equivalent bidding equilibrium $\widetilde{\beta}$, such that $\widetilde{\beta}$ is non-decreasing on $[\underline{x}, \bar{x}]$ and coincides with β over intervals over which $\frac{g_h}{g_\ell}$ is strictly increasing.*

The proof of the monotonicity result extends immediately to a grid since it only relies on the single crossing property of $U(b|x, sol)$ in b and x. Note that we need the assumption that the feasible bid set includes $[0, v_\ell]$ for Lemma 10 in the proof of Proposition 1.

Theorem 1" *Consider a sequence of bidding games $\Gamma_0\left(\boldsymbol{\eta}^k, N^k, \Delta^k\right)$ such that $supp(\eta^k_\omega) = \left\{n^k_\omega, n^k_\omega + 1\right\}$, $\min\{n^k_\ell, n^k_h\} \to \infty$, $\lim \Delta^k = 0$, and $\lim \frac{n^k_h}{n^k_\ell} = r$, and a corresponding sequence of bidding equilibria β^k.*

1. *If $\bar{g}r < 1$, then, for large enough k, there is a bid b^k such that F^k_ω has an atom at b^k and possibly at $b^k + \Delta^k$, with*

$$\lim \Pr(\textit{winning bid } \in \left\{b^k, b^k + \Delta^k\right\} |\omega) = 1 \qquad \text{for } \omega = \ell, h,$$

and

$$\lim \mathbb{E}[v|\bar{x}, \text{sol}] \equiv \frac{v_\ell + \rho\bar{g}rv_h}{1 + \rho\bar{g}r} \leq \lim \overline{b}^k \leq \mathbb{E}[v].$$

2. *If* $\bar{g}r > 1$*, then* F_ω^k *converges pointwise to* $\Phi_\omega(\cdot|r)$, $\omega = \ell, h$ *(where* Φ_ω *is described by* (5)-(6)*).*

Proof of Theorem 1": Since the proof of Theorem 1" is almost identical to the proof of Theorem 1, we will only present the arguments that have to be adjusted, rather than reproduce the entire proof. These are in the instances where a "slight undercutting" argument is used and the proof in the appendix makes sure that, for a sufficiently fine grid, it goes through.

Case 1, $\bar{g}r < 1$**.** Given any $\varepsilon \in (0,1)$, let x^k be such that $\left(G_h\left(x^k\right)\right)^{n_h^k} = \varepsilon$ for all k. Let $b^k = \beta^k\left(x^k\right)$. As before, the result holds if

$$\lim\left(G_h\left(x_+^k\left(b^k + \Delta^k\right)\right)\right)^{n_h^k} = 1. \tag{64}$$

Suppose to the contrary that (64) fails and $\lim\left(G_h\left(x_+^k\left(b^k + \Delta^k\right)\right)\right)^{n_h^k} < 1$. Then,

$$b^k + \Delta^k < \beta^k(\bar{x}).$$

Moreover,

$$\lim \pi_h^k(b^k + \Delta^k) \le \lim\left(G_h\left(x_+^k\left(b^k + \Delta^k\right)\right)\right)^{n_h^k} < 1$$

and

$$\lim \pi_h^k(b^k + \Delta^k) \ge \lim\left(G_h\left(x_-^k\left(b^k + \Delta^k\right)\right)\right)^{n_h^k} \ge \lim\left(G_h\left(x^k\right)\right)^{n_h^k} = \varepsilon > 0.$$

Hence, the zero-profit condition (26) requires that

$$\lim\left(b^k + \Delta^k\right) \ge \lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } b^k + \Delta^k].$$

Now, $\lim \pi_h^k(b^k + \Delta^k) \in (0,1)$ and $b^k + \Delta^k < \beta^k(\bar{x})$ for all k implies via Lemma 7 (β^ks that have support only on the grid are a special case considered in that lemma) that

$$\lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } b^k + \Delta^k] > \lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } \beta^k(\bar{x})].$$

The bidders' individual rationality requires that

$$\lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } \beta^k(\bar{x})] \ge \lim \beta^k(\bar{x}).$$

Together, the last three displayed inequalities contradict $b^k + \Delta^k < \beta^k(\bar{x})$. ■

Case 2, $\bar{g}r > 1$**.**

The critical lemma for this case was Lemma 8, which should be adapted as follows.

Lemma 8" (Price Grid). *Let* n^k *be such that* $\min\{n_\ell^k, n_h^k\} \to \infty$ *and* $\lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} > 1$*. Let* $(\beta^k)_{k=1}^\infty$ *be a corresponding sequence of bidding equilibria. If* $(\beta^k)_{k=1}^\infty$ *exhibits a se-*

quence of non-vanishing atoms $(b^k)_{k=1}^{\infty}$*, i.e.,* $\lim\left(G_\ell\left(x_+^k\left(b^k\right)\right)\right)^{n_\ell^k} > \lim\left(G_\ell\left(x_-^k\left(b^k\right)\right)\right)^{n_\ell^k}$*, then*

$$\lim b^k < \lim \mathbb{E}^k[v|x_+^k, \text{sol,win at } b^k + \Delta^k].$$

Proof of Lemma 8".

If in the limit there is no atom at $b^k + \Delta^k$, i.e., if $\lim\left(G_\ell\left(x_+^k\left(b^k+\Delta^k\right)\right)\right)^{n_\ell^k} = \lim\left(G_\ell\left(x_-^k\left(b^k+\Delta^k\right)\right)\right)^{n_\ell^k}$, then the original proof of the lemma works directly. If in the limit there is an atom at $b^k + \Delta^k$, then instead of (35) we have to establish

$$\lim \frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)}\frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} < \lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)}\frac{\pi_h^k(b^k+\Delta^k)}{\pi_\ell^k(b^k+\Delta^k)}. \tag{65}$$

Let $x_{++}^k = x_+^k\left(b^k+\Delta^k\right)$ and note that $x_-^k\left(b^k+\Delta^k\right) = x_+^k\left(b^k\right) = x_+^k$. Also, recall $q_+ = \lim G_\ell\left(x_+^k\right)^{n_\ell^k}$ and let $q_{++} = \lim G_\ell\left(x_{++}^k\right)^{n_\ell^k}$. We already know from (35) that

$$\lim \frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)}\frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} < \lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)}\frac{\left(G_h\left(x_+^k\left(b^k\right)\right)\right)^{n_h^k}}{\left(G_\ell\left(x_+^k\left(b^k\right)\right)\right)^{n_\ell^k}} = \lim \frac{g_h\left(\bar{x}\right)}{g_\ell\left(\bar{x}\right)}\left(q_+\right)^{\lambda-1}$$

Analogous calculation to that of (36)-(37) in the proof yields

$$\begin{aligned}\lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)}\frac{\pi_h^k(b^k+\Delta^k)}{\pi_\ell^k(b^k+\Delta^k)} &= \lim \frac{n_\ell^k}{n_h^k}\frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)}\frac{G_\ell\left(x_{++}^k\right)-G_\ell\left(x_+^k\right)}{G_h\left(x_{++}^k\right)-G_h\left(x_+^k\right)}\frac{G_h\left(x_{++}^k\right)^{n_h^k}-G_h\left(x_+^k\right)^{n_h^k}}{G_\ell\left(x_{++}^k\right)^{n_\ell^k}-G_\ell\left(x_+^k\right)^{n_\ell^k}} \\ &= \frac{g_h\left(\bar{x}\right)}{g_\ell\left(\bar{x}\right)}\frac{\left(q_{++}\right)^\lambda-\left(q_+\right)^\lambda}{\lambda\left(q_{++}-q_+\right)}.\end{aligned}$$

Finally,

$$\left(q_+\right)^{\lambda-1} < \frac{\left(q_{++}\right)^\lambda-\left(q_+\right)^\lambda}{\lambda\left(q_{++}-q_+\right)}, \tag{66}$$

since letting $Q = \frac{q_{++}}{q_+} > 1$, (66) is equivalent to $Q^\lambda - \lambda Q + \lambda > 1$. Since $\lambda > 1$, the LHS is increasing in Q over $[1,\infty)$ and is equal to 1 at $Q = 1$. Therefore, (66) and so does (65). This completes the adaptation of Lemma 8 for the case of finite price grid**.** ∎

We can now adapt the proof of part 2 of Theorem 1. The theorem uses a slight overbidding argument. The paragraph containing equations (44) and (45) should be modified as follows

$$\lim_{k\to\infty}\lim_{\varepsilon^k\to 0}\pi_\ell^k(b^k+\Delta^k) \geq \lim_{k\to\infty}\left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k} > 0, \tag{67}$$

where the first inequality is strict if in the limit there is an atom at $b^k + \Delta^k$, i.e., if $\lim\left(G_\ell\left(x_+^k\left(b^k+\Delta^k\right)\right)\right)^{n_\ell^k} < \lim\left(G_\ell\left(x_-^k\left(b^k+\Delta^k\right)\right)\right)^{n_\ell^k}$. This and Lemma 8" implies that

$$\lim U^k(b^k+\Delta^k|x_+^k, \text{sol}) > 0. \tag{68}$$

Beyond that point, the proof of Theorem 1 continues unchanged. ∎

Proposition 3" *Consider a sequence of bidding games* $\Gamma_0\left(\boldsymbol{\eta}^k, N^k, \Delta^k\right)$ *such that* $supp(\eta_\omega^k) = \left\{n_\omega^k, n_\omega^k+1\right\}$, $\min\{n_\ell^k, n_h^k\} \to \infty$, $\lim \frac{n_h^k}{n_\ell^k} = r$ *and* $\lim \Delta^k = 0$, *and a corresponding sequence of bidding equilibria* β^k. *If* $r = 1$ *and* β^k *is a bidding equilibrium of* $\Gamma_0\left(\boldsymbol{\eta}^k, N^k, \Delta^k\right)$ *for all* k, *then the limit of the winning bid distribution is a mass point on* $\mathbb{E}[v]$, *i.e.,* $\lim F_\omega^k(b) = 1$ *for all* $b > \mathbb{E}[v]$ *and* $= 0$ *for all* $b < \mathbb{E}[v]$.

Proof of Proposition 3". Suppose $\bar{g}r = 1$. We adapt the original proof. The only necessary change is with respect to the choice of the sequence b^k. Note that given $\Delta^k \to 0$, under the stated hypothesis, there must still be a sequence b^k such that $b^k \geq p'$, $\lim b^k = p'$, and the probability of a tie at b^k is vanishing,

$$\lim \frac{\pi_\omega^k(b^k)}{\left(G_\omega\left(x_+^k\left(b^k\right)\right)\right)^{n_\omega^k}} = 1.$$

The remainder of the proof applies as before. ∎

Theorem 2" *There exists a number* r^* *such that for every sequence of games* $\Gamma\left(s^k, \Delta^k\right)$ *with* $\lim\left(s^k, \Delta^k\right) = (0,0)$ *and every corresponding sequence of equilibria* $\left(\beta^k, \boldsymbol{\eta}^k\right)$, *one of the following two statements holds (but not both):*

1. $\lim F_\omega^k$ *is a degenerate distribution for* $\omega = \ell, h$ *with probability mass* 1 *on some number* $C \leq \rho_\ell v_\ell + \rho_h v_h$;
2. $\lim F_\omega^k = \Phi_\omega\left(\cdot|r^*\right)$, $\omega = \ell, h$, where $r^* = \lim \frac{n_h^k}{n_\ell^k}$ and $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$ (and where Φ_ω *is described by* (5)-(6)).

Theorem 2" builds on Theorem 1" in exactly the same way that Theorem 2 built on Theorem 1' and the needed adjustments to a finite grid are already made in the proof of Theorem 1". Therefore, Theorem 2" does not require a separate proof.

D.2 Proof of Theorem 3: Existence of Partially Separating Equilibrium

Outline. The proof introduces "constrained equilibria" $\left(\beta^k, \boldsymbol{\eta}^k\right)$ for $\left(s^k, \Delta^k\right) \to (0,0)$ in which β^k is a bidding equilibrium given $\boldsymbol{\eta}^k$, as before, but $\boldsymbol{\eta}^k$ are required to satisfy additional constraints: (i) there is a lower bound sequence $\underline{n}_\omega^k \to \infty$ such that $n_\omega^k \geq \underline{n}_\omega^k$; and (ii) $\frac{n_h^k}{n_\ell^k} \geq \bar{r}$ for some $\frac{1}{\bar{g}} < \bar{r} < r^*(\rho, \bar{g})$. By Theorem 1", any such a sequence of constrained equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^\infty$ must be of the partially separating variety. The proof then uses two Lemmas. The first establishes that, if a sequence of constrained equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^\infty$ exists, then the constraints do not bind for large enough k, so that $\left(\beta^k, \boldsymbol{\eta}^k\right)$ is also an unconstrained equilibrium. It then follows from Theorem 2" that $\lim \frac{n_h^k}{n_\ell^k} = r^*$ and the winning bid distribution converges to $\Phi_\omega\left(\cdot|r^*\right)$. The second lemma establishes (using fairly standard arguments) that, for any sequence $\left(s^k, \Delta^k\right) \to (0,0)$ with $\Delta^k > 0$ for all

k, a constrained equilibrium exists. Thus, together these lemmas imply the existence of a sequence of equilibria that converge to the partially separating outcome described in Theorem 2". Note that the grid, $\Delta^k > 0$, is only needed in the second lemma to establish the existence of a constrained equilibrium.

One key element is that it is possible to choose $\bar{r} \in (\frac{1}{\bar{g}}, r^*)$ such that the constraints do not bind for k large enough. To see that this is so, consider $\bar{r}$ that is very close to $\frac{1}{\bar{g}}$ and let us argue why, over a sequence of constrained equilibria, $\lim \frac{n_h^k}{n_\ell^k} > \bar{r}$. Suppose to the contrary that that the restriction $\frac{n_h^k}{n_\ell^k} \geq \bar{r}$ is binding so that $\lim \frac{n_h^k}{n_\ell^k} = \bar{r}$. Then, since in this case, $\bar{g}\bar{r}$ is near 1, by Lemma 1, the winning bid distributions in the two states would be very similar. Therefore, by Lemma 3 the optimal solicitation would also be very similar, implying $\frac{n_h^k}{n_\ell^k}$ close to 1, in contradiction to $\frac{n_h^k}{n_\ell^k}$ being close to $\frac{1}{\bar{g}}$.

Constrained Equilibrium. Given numbers $(\underline{n}_\ell, \underline{r}) \geq (0,0)$, we define below a *constrained equilibrium* $(\beta, \boldsymbol{\eta})$ of $\Gamma(s, P_\Delta)$ in which $\boldsymbol{\eta}$ is optimal subject to the constraints $n_\ell \geq \underline{n}_\ell$ and $\frac{n_h}{n_\ell} \geq \underline{r}$.

By Lemma 2 and its proof, for any bidding strategy $\beta : [\underline{x}, \overline{x}] \to P_\Delta$, $\mathbb{E}[p|\omega;\beta,n] - ns$ is either strictly decreasing (e.g., if β is constant) or a strictly concave function in n. In either case, $\underset{n\in\{1,..,N\}}{\arg\max}\, \mathbb{E}[p|\omega;\beta,n] - ns$ consist of one or (at most) two adjacent integers. Let $m_\omega(\beta,s)$ be the lower one.[34]

Let D denote the set of probability distributions η over $\{1,...,N\}$. Given the bounds $(\underline{n}_\ell, \underline{r}) \geq (0,0)$, let the correspondence Ψ_2 map the bidding strategy β to the set of optimal solicitation strategies in state ℓ subject to the constraint that at least $\underline{n}_\ell$ bidders are solicited:

$$\Psi_2(\beta, s; \underline{n}_\ell) \triangleq \begin{cases} \{\eta \in D : \mathit{Support}(\eta) \subseteq \underset{n'\in\{1,..,N\}}{\arg\max}\, \mathbb{E}[p|\ell;\beta,n'] - n's\} & \text{if} \quad m_\ell(\beta,s) \geq \underline{n}_\ell, \\ \{\eta \in D \mid \eta(\underline{n}_\ell) = 1\} & \text{o/w} \end{cases}$$

The second line of Ψ_2 deals with the case in which the lower bound $\underline{n}_\ell$ binds. Since $\mathbb{E}[p|\omega;\beta,n] - ns$ is either strictly decreasing in n or strictly concave, if the constraint $n \geq \underline{n}_\ell$ binds, then the optimal constrained solicitation is exactly $\underline{n}_\ell$.

Similarly, let $\underline{n}_h(\beta,s) = \underline{r}\max\{m_\ell(\beta,s), \underline{n}_\ell\}$ and define the correspondence $\Psi_3(\beta, s; \underline{n}_\ell, \underline{r})$

$$\Psi_3(\beta, s; \underline{n}_\ell, \underline{r}) \triangleq \begin{cases} \{\eta \in D : \mathit{Support}(\eta) \subseteq \underset{n'\in\{1,..,N\}}{\arg\max}\, \mathbb{E}[p|h;\beta,n'] - n's\} & \text{if} \quad m_h(\beta,s) \geq \underline{n}_h(\beta,s), \\ \{\eta \in D \mid \eta(\lceil \underline{n}_h \rceil) = 1\} & \text{o/w} \end{cases}$$

Now, $(\beta, \boldsymbol{\eta})$ is a *constrained equilibrium* if (i) β is a bidding equilibrium of $\Gamma_0(N, \boldsymbol{\eta}, \Delta)$

[34] We use $m_\omega(\beta,s)$ rather than n_ω which was introduced earlier to denote the lower of the integers in the support of a mixed strategy η_ω, since, in the constrained environment considered below, these could be different numbers.

and (ii) $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$ is the seller's optimal constrained solicitation, i.e., $\eta_\ell \in \Psi_2(\beta, s; \underline{n}_\ell)$ and $\eta_h \in \Psi_3(\beta, s; \underline{n}_\ell, \underline{r})$.

A constrained equilibrium $(\beta, \boldsymbol{\eta})$ is also an equilibrium of $\Gamma(s, P_\Delta)$ if $m_\ell(\beta, s) \geq \underline{n}_\ell$ and $m_h(\beta, s) \geq \underline{r} m_\ell(\beta, s)$.

Lemma 14 *If $\Delta > 0$, $\Gamma(s, P_\Delta)$ has a constrained equilibrium.*

A constrained equilibrium exists by standard arguments given the single crossing properties established in Lemma 9; see, for example, Athey (2001). We state the proof for completeness.

Proof of Lemma 14: Recall that $P_\Delta = [0, v_\ell) \cup \{v_\ell, v_\ell + \Delta, v_\ell + 2\Delta, ..., v_h - \Delta, v_h\}$. Let B_Δ denote the set of monotone bidding functions using only bids from $\{v_\ell, v_\ell + \Delta, v_\ell + 2\Delta, ..., v_h - \Delta, v_h\}$. Let $m = \| \{v_\ell, v_\ell + \Delta, v_\ell + 2\Delta, ..., v_h - \Delta, v_h\} \|$. Using Athey's idea, Σ_Δ is a set of vectors of dimension $m+1$ whose coordinates belong to $[\underline{x}, \overline{x}]$

$$\Sigma_\Delta = \{\sigma = (\sigma_0, \sigma_1, ..., \sigma_m) \in [\underline{x}, \bar{x}]^{m+1} \mid \underline{x} \triangleq \sigma_0 \leq \sigma_1 \leq ... \leq \sigma_m \triangleq \bar{x}\},$$

where σ determines a bidding strategy β_σ by $\beta_\sigma(x) = v_\ell + i\Delta$ if $x \in [\sigma_i, \sigma_{i+1})$, $i = 0, ..., m-1$.

Define the correspondence $\Psi = \Psi_1(\sigma', \boldsymbol{\eta}') \times \hat{\Psi}_2(\sigma', \boldsymbol{\eta}') \times \hat{\Psi}_3(\sigma', \boldsymbol{\eta}')$ from $\Sigma_\Delta \times D \times D$ into itself: For any $\sigma' \in \Sigma_\Delta$ and $\boldsymbol{\eta}' \in D \times D$,

$$\begin{aligned}
\Psi_1(\sigma', \boldsymbol{\eta}') &= \left\{\sigma \in \Sigma_\Delta \mid \beta_\sigma \in \arg\max U(b|x, \text{sol}; \beta_{\sigma'}, \boldsymbol{\eta}')\right\}, \\
\hat{\Psi}_2(\sigma', \boldsymbol{\eta}') &= \Psi_2(\beta_{\sigma'}, s; \underline{n}_\ell), \\
\hat{\Psi}_3(\sigma', \boldsymbol{\eta}') &= \Psi_3(\beta_{\sigma'}, s; \underline{n}_\ell, \underline{r}).
\end{aligned}$$

Both Σ_Δ and D are closed convex sets. $\hat{\Psi}_2$ and $\hat{\Psi}_3$ are non-empty, convex-valued, and continuous by virtue of being the set of constrained maximizers of a concave problem on a finite convex set (see Lemma 2). Ψ_1 is non-empty, convex-valued, and upper hemi-continuous by the same arguments as in Athey (2001). For given σ', $\beta_{\sigma'}(\underline{x}) \geq v_\ell$, so that by the arguments from Lemma 10, the unconstrained best response has support only on bids $\geq v_\ell$. Moreover, by the single crossing properties of Lemma 9, the unconstrained best response is without loss of generality weakly monotone. Therefore, there exists some $\sigma \in \Sigma_\Delta$ such that β_σ is a best response to σ'. Convex valuedness also follows from Lemma 9. The upper hemi-continuity follows from the theorem of the maximum, given that $U(\cdot|x,\text{sol};\beta_{\sigma'}, \boldsymbol{\eta}')$ is continuous in $\boldsymbol{\eta}'$ and in σ'. It follows that Ψ is convex valued and upper hemi-continuous. By Kakutani's Theorem, Ψ has a fixed point. ∎

Next, we show that, for certain choice of the bounds $(\underline{n}_\ell, \underline{r})$, all the constrained equilibria are, in fact, unconstrained, for sufficiently large k.

Lemma 15 *Consider a sequence of games $\Gamma^B\left(s^k, P_{\Delta^k}\right)$ and parameters $\left(\underline{n}_\ell^k, \underline{r}\right)$ such that $\left(s^k, \Delta^k\right) \to (0,0)$, $\underline{n}_\ell^k = \frac{1}{\sqrt{s^k}}$ and $\underline{r} \in (\frac{1}{\overline{g}}, r^*(\rho, \overline{g}))$. For any sequence of constrained equilibria $(\beta^k, \boldsymbol{\eta}^k)$ of $\Gamma^B(s, P_{\Delta^k})$ given $\left(\underline{n}_\ell^k, \underline{r}\right)$: $n_\ell^k > \underline{n}_\ell^k$ for large k, $\lim \frac{n_h^k}{n_\ell^k} = r^*(\rho, \overline{g})$ and $\lim F_\omega\left(\cdot|\beta^k, \eta_\omega^k\right) = \Phi_\omega(\cdot|r^*)$, with Φ_ω defined by (5).*

Proof: Given the sequence of constrained equilibria, let $r = \lim_{k\to\infty} \frac{n_h^k}{n_\ell^k}$, with n_ω^k the lower of the integers in the support of η_ω^k, as before.

Step 1. $\lim F_\omega\left(p|\beta^k, n_\omega^k\right) = \Phi_\omega(p|r)$.
Proof of Step 1: The choice of $\underline{n}_\ell^k$ and $\underline{r}$ implies $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$ and

$$\lim\left(n_h^k/n_\ell^k\right) = r \geq \underline{r}. \tag{69}$$

Given (69), Theorem 1" implies that $\lim F_\omega\left(p|\beta^k, n_\omega^k\right) = \Phi_\omega(p|r)$, for all p and $\omega = \ell, h$.
□

Step 2. For k sufficiently large, $n_\ell^k > \underline{n}_\ell^k$ and $n_h^k > \underline{r} n_\ell^k$
Proof of Step 2: By (69) and the choice of r, $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} > \frac{1}{\overline{g}}$. The argument from the proof of Proposition 4) implies that $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} < \infty$. Hence, Φ_ℓ is not degenerate. As before, let $m_\omega^k = m_\omega\left(\beta^k, s^k\right)$ denote the lowest optimal n for type ω given β^k and s^k. By Lemma 3, m_ℓ^k satisfies

$$\lim_{k\to\infty} m_\ell^k s^k = -\int_0^{v_h} \Phi_\ell(p|r) \ln\left(\Phi_\ell(p|r)\right) dp > 0. \tag{70}$$

Since $\underline{n}_\ell^k s^k = \sqrt{s^k} \to 0$, $\lim m_\ell^k s^k > 0$ implies $\lim \frac{\underline{n}_\ell^k}{m_\ell^k} = 0$, so that $m_\ell^k > \underline{n}_\ell^k$ for sufficiently large k. Thus, $n_\ell^k > \underline{n}_\ell^k$, as claimed.

Suppose to the contrary that $m_h^k \leq \underline{r} m_\ell^k$. Then, the strict concavity of the seller's optimization implies $n_h^k = \underline{r} m_\ell^k$ (ignoring integer constraints). By $\frac{1}{\overline{g}} < \underline{r} < r^*(\rho, \overline{g})$ and Lemma 12, $J(\underline{r}; \rho, \overline{g}) < 0$.

We have observed in the proof of Lemma 3 that (for any sequence n_h^k and β^k—not only for optimal n_h^k)

$$\lim_{k\to\infty} n_h^k\left(\mathbb{E}\left[p|h; \beta^k, n_h^k + 1\right] - \mathbb{E}\left[p|h; \beta^k, n_h^k\right]\right) = -\int_0^{v_h} \Phi_h(p|r) \ln\left(\Phi_h(p|r)\right) dp.$$

This, (70) and $n_h^k = \underline{r} m_\ell^k$ imply

$$\begin{aligned}
&\lim_{k\to\infty} n_h^k\left(\mathbb{E}\left[p|h; \beta^k, n_h^k + 1\right] - \mathbb{E}\left[p|h; \beta^k, n_h^k\right]\right) - \lim_{k\to\infty} n_h^k s^k \\
= \quad &-\int_0^{v_h} \Phi_h(p|\underline{r}) \ln\left(\Phi_h(p|\underline{r})\right) dp + \underline{r}\int_0^{v_h} \Phi_\ell(p|\underline{r}) \ln\left(\Phi_\ell(p|\underline{r})\right) dp.
\end{aligned}$$

From Equation (63) in the proof of Lemma 4 the sign of the last expression is the same as $sign\left(-J(\underline{r})\right)$. Thus, $J(\underline{r})<0$ implies

$$\lim_{k\to\infty} n_h^k\left(\mathbb{E}\left[p|h;\beta^k,n_h^k+1\right]-\mathbb{E}\left[p|h;\beta^k,n_h^k\right]\right)>\lim_{k\to\infty} n_h^k s^k.$$

Hence, for sufficiently large k,

$$\mathbb{E}\left[p|h;\beta^k,n_h^k+1\right]-\mathbb{E}\left[p|h;\beta^k,n_h^k\right]>s^k.$$

That is, at n_h^k sampling an additional bidder is strictly profitable for type h. Therefore, $m_h^k>\underline{r}m_\ell^k$, as claimed. □

Step 3. $\lim n_h^k/n_\ell^k=\lim n_h^k/n_\ell^k=r^*\left(\rho,\overline{g}\right)$.

Proof of Step 3: By Step 2, n_h^k and n_ℓ^k are both unconstrained optimal given β^k. Hence, $(\beta^k,\boldsymbol{\eta}^k)$ is an (unconstrained) equilibrium for k large enough. Therefore, Lemma 4 implies that $\lim\frac{n_h^k}{n_\ell^k}=r^*\left(\rho,\overline{g}\right)$. □

Steps 1 and 3 together establish the lemma. ■

Lemma 14 implies that $\Gamma\left(s^k,P_{\Delta^k}\right)$ has a constrained equilibrium whenever $\Delta^k>0$. Lemma 15 implies that, for suitably chosen $\left(\underline{n}_\ell^k,\underline{r}\right)$ and for sufficiently small s_k and Δ_k, all constrained equilibria $(\beta^k,\boldsymbol{\eta}^k)$ are also (unconstrained) equilibria of $\Gamma\left(s^k,P_{\Delta^k}\right)$ and over any such sequence $\lim F_\omega\left(p|\beta^k,n_\omega^k\right)=\Phi_\omega\left(p|r\right)$. Therefore, there exists a sequence of equilibria $(\beta^k,\boldsymbol{\eta}^k)$ for $\Gamma\left(s^k,P_{\Delta^k}\right)$ that converges to the partially revealing outcome of Theorem 2. ■

Comment: The proof of Lemma 15 does not require $\Delta>0$. Thus, if without the grid ($\Delta=0$), there exists a sequence of constrained equilibria, then there also exists a sequence of equilibria that converges to the partially revealing outcome of Theorem 2.

D.3 Proof of Theorem 4 (Pooling Equilibria)

Before turning to the formal proof, let us discuss some key steps of this construction. First, it is immediate from the form of β^k that $s^k\to 0$ implies $\min\left\{n_\ell^k,n_h^k\right\}\to\infty$. Next, given the strategies β^k, the probability $\pi_\omega[\bar{b}|\beta^k,n_w^k]$ of winning with bid $\bar{b}$ is approximately $1/n_w^k\left[1-G_w(\bar{x}-\epsilon)\right]$ when k is large. This implies that the compound likelihood ratio $\frac{\rho_h}{\rho_\ell}\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{\pi_h[\bar{b}|\beta^k,n_h^k]}{\pi_\ell[\bar{b}|\beta^k,n_\ell^k]}$ approaches $\frac{\rho_h}{\rho_\ell}$. That is, in the limit, the sampling likelihood ratio $\frac{n_h^k}{n_\ell^k}$ and signal likelihood ratio $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$ exactly offset the winning likelihood ratio $\frac{\pi_h[\bar{b}|\beta^k,n_h^k]}{\pi_\ell[\bar{b}|\beta^k,n_\ell^k]}$. Therefore,

$$\lim_{k\to\infty}\mathbb{E}\left[v|\bar{x},\text{ win at }\bar{b};\,\beta^k,\,\mathbf{n}^k\right]=\rho_\ell v_\ell+\rho_h v_h,$$

independently of $\underline{b}$ and $\bar{b}$.

Thus, bidding $\bar{b}$ yields positive payoff to a bidder with signal $x>\bar{x}-\epsilon$. To verify that bidding $\bar{b}$ is indeed optimal for such bidder, we have to consider all possible deviations.

The deviation that requires a relatively more subtle argument is overbidding $\bar{b}$ by a bidder with signal $x > \bar{x} - \epsilon$. The payoff of such a bidder at $\bar{b}$ approaches 0 when k is large and slight overbidding assures a win. It turns out that optimality of the equilibrium sampling strategy assures that $\mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\, \beta^k,\, \mathbf{n}^k\right] < \bar{b}$. To see the essence of this argument, suppose that the bidding strategy is simply

$$\beta^k(x) = \begin{cases} \bar{b} & \text{if} \quad x > \bar{x} - \epsilon, \\ \underline{b} & \text{if} \quad x \leq \bar{x} - \epsilon, \end{cases}$$

and that the optimal solicitation strategy is pure. Ignoring integer constraints, optimal solicitation implies the equality of the marginal benefit of an additional bidder to its cost in each state,

$$\begin{aligned} (G_\ell(\bar{x} - \epsilon))^{n_\ell^k} (1 - G_\ell(\bar{x} - \epsilon)) \left(\bar{b} - \underline{b}\right) &= s^k, \\ (G_h(\bar{x} - \epsilon))^{n_h^k} (1 - G_h(\bar{x} - \epsilon)) \left(\bar{b} - \underline{b}\right) &= s^k. \end{aligned}$$

Substituting out s^k, making a logarithmic transformation, rearranging and then taking limits we get

$$\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} = \frac{\ln G_\ell(\bar{x} - \epsilon)}{\ln G_h(\bar{x} - \epsilon)}.$$

This ratio is smaller than one, so that being solicited is bad news. Moreover, $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} \frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} < 1$, which follows from $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = \frac{1-G_h(\bar{x}-\epsilon)}{1-G_\ell(\bar{x}-\epsilon)}$ and $\frac{1-z}{\ln z}$ being decreasing in z. Hence, if solicitation is optimal given β^k then

$$\frac{g_h(\bar{x})}{g_\ell(\bar{x})} \lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} < 1. \tag{71}$$

Note that the limiting ratio of the number of solicited bidders is independent of the choice of $\bar{b}$ and $\underline{b}$.

Since a bid $b > \bar{b}$ wins with certainty,

$$\mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\, \beta^k,\, \mathbf{n}^k\right] = \frac{v_\ell + \frac{\rho_h}{\rho_\ell} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \frac{n_h^k}{n_\ell^k} v_h}{1 + \frac{\rho_h}{\rho_\ell} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \frac{n_h^k}{n_\ell^k} v_h}.$$

Therefore, by (71) for large enough k,

$$\mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\, \beta^k,\, \mathbf{n}^k\right] < \rho_\ell v_\ell + \rho_h v_h.$$

Choosing $\bar{b}$ sufficiently close to $\rho_\ell v_\ell + \rho_h v_h$ assures that this upward deviation is unprofitable.

The formal proof deals with the above deviation without the special simplifying assumptions, addresses the other potential deviations and shows how to choose $\underline{b}$ and $\bar{b}$ to assure immunity against all the deviations simultaneously. However, the above discussion points out the more special argument that is tied to the endogenous sampling.

Proof of Theorem 4:

Auxiliary Game A: Let $\Gamma^A\left(s, P_\Delta|\underline{b}, \bar{b}\right)$ be an auxiliary game in which $\underline{b} < \bar{b}$ and the bidding strategies are constrained to satisfy

$$\beta(x)\begin{cases} = \bar{b} & \text{if} \quad x > \bar{x} - \epsilon, \\ \leq \underline{b} & \text{if} \quad x \leq \bar{x} - \epsilon. \end{cases} \tag{72}$$

A strategy profile $(\beta, \boldsymbol{\eta})$ is an equilibrium of $\Gamma^A\left(s, P_\Delta|\underline{b}, \bar{b}\right)$ if $\boldsymbol{\eta}$ is an optimal solicitation strategy for the seller given β, and, given $\boldsymbol{\eta}$, the strategy $\beta(x)$ is a best response subject to (72).

The heart of the proof consists of three lemmas on the equilibrium of the auxiliary game that are proved in the subsection following the main body of the present proof. The first establishes existence when $\Delta > 0$. In this case, the auxiliary game is a finite Bayesian game.

Lemma 16 *If $\Delta > 0$, $\Gamma^A\left(s, P_\Delta|\underline{b}, \bar{b}\right)$ has an equilibrium.*

The second lemma collects implications of the optimal sampling

Lemma 17 *Consider a sequence of auxiliary games $\Gamma^A\left(s^k, P_{\Delta^k}|\underline{b}, \bar{b}\right)$ such that $s^k \to 0$. Let β^k satisfy (72) and $\boldsymbol{\eta}^k$ be an optimal solicitation strategy given β^k, then:*

1. $\lim_{k\to\infty} n_w^k = \infty, \ w \in \{\ell, h\}$.

2. $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} = \frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} < 1$.

3. $\lim_{k\to\infty} \frac{G_h(\bar{x}-\epsilon)^{n_h^k-1}}{G_\ell(\bar{x}-\epsilon)^{n_\ell^k-1}} \leq \frac{(1-G_\ell(\bar{x}-\epsilon))}{(1-G_h(\bar{x}-\epsilon))}\frac{1}{G_h(\bar{x}-\epsilon)}$.

The third lemma utilizes the previous lemma to calculate limiting expected values conditional on winning.

Lemma 18 *Consider a sequence of auxiliary games $\Gamma^A\left(s^k, P_{\Delta^k}|\underline{b}, \bar{b}\right)$ such that $s^k \to 0$. Let β^k satisfy (72) and $\boldsymbol{\eta}^k$ be an optimal solicitation strategy given β^k. Then*

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x},\ \textit{win at}\ \bar{b};\ \beta^k,\ \boldsymbol{\eta}^k\right] = \rho_\ell v_\ell + \rho_h v_h,$$

and there are numbers v_1^*, v_2^*, v_3^* independent of $\underline{b}$, $\bar{b}$ such that

$$\begin{aligned}
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}-\epsilon,\ \textit{win at}\ \bar{b};\ \beta^k,\ \boldsymbol{\eta}^k\right] &\leq v_1^* < \rho_\ell v_\ell + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x},\ \textit{win at}\ b > \bar{b};\ \beta^k,\ \boldsymbol{\eta}^k\right] &\leq v_2^* < \rho_\ell v_\ell + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}-\epsilon,\ \textit{win at}\ b \in \left(\underline{b}, \bar{b}\right);\ \beta^k, \boldsymbol{\eta}^k\right] &\leq v_3^* < \rho_\ell v_\ell + \rho_h v_h.
\end{aligned}$$

Select any $\underline{b}$ and $\bar{b}$ that satisfy

$$\max\{v_1^*, v_2^*, v_3^*\} < \underline{b} < \bar{b} < \rho_\ell v_\ell + \rho_h v_h. \tag{73}$$

By Lemma 16, the auxiliary game $\Gamma^A\left(s^k, P_{\Delta^k}|\underline{b}, \bar{b}\right)$, $\Delta^k > 0$, has an equilibrium $(\beta^k, \boldsymbol{\eta}^k)$. We show next that $(\beta^k, \boldsymbol{\eta}^k)$ is an equilibrium of the original game for s^k sufficiently small by proving that the constraints (72) do not bind if (73) holds.

From Lemma 17, $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$.

Step 1. Bidding $\beta^k = \bar{b}$ is optimal if $x > \bar{x} - \epsilon$.

(i) Bidding $b > \bar{b}$ is unprofitable. By the choice of $\bar{b} > v_2^*$ and Lemma (18),

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\, \beta^k,\, \boldsymbol{\eta}^k\right] < \bar{b}. \tag{74}$$

Thus, there is some K_1 such that bidding $b > \bar{b}$ is strictly unprofitable for all $k \geq K_1$.

(ii) Bidding $b < \bar{b}$ is unprofitable. First, by Lemma 18 and the choice of $\bar{b}$,

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x},\ \text{win at } \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right] = \rho_\ell v_\ell + \rho_h v_h > \bar{b}. \tag{75}$$

For any $b < \bar{b}$, Lemma 5 implies

$$\lim_{k\to\infty} \frac{\bar{\pi}_w\left[\bar{b}|\beta^k, \eta_w^k\right]}{\bar{\pi}_w\left[b|\beta^k, \eta_w^k\right]} \geq \lim_{k\to\infty} \frac{\frac{1}{n_w^k}\frac{1}{1-G_w(\bar{x}-\epsilon)}\left(1-\left(G_w\left(\bar{x}-\epsilon\right)\right)^{n_w^k}\right)}{\left(G_w\left(\bar{x}-\epsilon\right)\right)^{n_w^k-1}} = \infty. \tag{76}$$

where the last equality follows from $n_w^k \to \infty$. By (75), the payoff conditional on winning at $\bar{b}$ is bounded away from 0. It now follows from (76) that there is some K_2 such that for all $k \geq K_2$, the payoff from bidding $b < \bar{b}$ is an arbitrarily small fraction of the payoff of bidding $\bar{b}$, so that undercutting $\bar{b}$ is unprofitable for $x > \bar{x} - \epsilon$. $\square$

Step 2. Bidding $b > \underline{b}$ is unprofitable for $x \leq \bar{x} - \epsilon$.

By Lemma 18, the choice of $\bar{b} > \max\{v_1^*, v_2^*, v_3^*\}$ and MLRP, for all $x \leq \bar{x} - \epsilon$:

(i) Bidding $b > \bar{b}$ is unprofitable. For $x \leq \bar{x} - \epsilon$,

$$\lim_{k\to\infty} \mathbb{E}\left[v|x, \text{ win at } b > \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right] \leq \lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\, \beta^k,\, \boldsymbol{\eta}^k\right].$$

Hence, (74) implies that bidding $b > \bar{b}$ is strictly unprofitable for $x \leq \bar{x} - \epsilon$ and all $k \geq K_1$.

(ii) Bidding $\bar{b}$ is unprofitable, since for all $x \leq \bar{x} - \epsilon$,

$$\lim_{k\to\infty} \mathbb{E}\left[v|x, \text{ win at } \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right] \leq \lim_{k\to\infty} \mathbb{E}\left[v|\bar{x} - \epsilon, \text{ win at } \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right] \leq v_1^* < \bar{b}.$$

Thus, there is some K_3, such that bidding $\bar{b}$ is unprofitable for $x \leq \bar{x} - \epsilon$ when $k \geq K_3$.

(iii) Bidding $b \in \left(\underline{b}, \bar{b}\right)$ is unprofitable, since for all $x \leq \bar{x} - \epsilon$,

$$\lim_{k\to\infty} \mathbb{E}\left[v|x, \text{ win at } b \in \left(\underline{b}, \bar{b}\right);\, \beta^k, \boldsymbol{\eta}^k\right] \leq \lim_{k\to\infty} \mathbb{E}\left[v|\bar{x} - \epsilon,\, b \in \left(\underline{b}, \bar{b}\right);\, \beta^k, \boldsymbol{\eta}^k\right] \leq v_3^* < \underline{b}.$$

Thus, there is some K_4 such that for all $k \geq K_4$ bidding any $b \in (\underline{b}, \bar{b})$ is unprofitable for all $x \leq \bar{x} - \epsilon$. □

Let $K = \max\{K_1, \cdots, K_4\}$. Step 1 and Step 2 imply that the additional constraints of the auxiliary game do not bind when $k \geq K$ and (73) holds. Thus, $(\beta^k, \boldsymbol{\eta}^k)$ is an equilibrium of the original game for $k \geq K$. For $k < K$, we can pick any equilibrium. By construction, $\beta^k(x) = \bar{b}$ for all $x > \bar{x} - \epsilon$ and $k \geq K$. ■

Proposition 4 establishes Part 1 of Theorem 4.

D.3.1 Proofs of three lemmas from proof of Theorem 4

Proof of Lemma 16: The proof relies closely on Athey (2001). The existence of a bidding equilibrium (given the constraints on β) for a given $\boldsymbol{\eta}$ is an immediate corollary of Athey's Theorem 1 and our Proposition 1. We have to establish that an equilibrium exists also when $\boldsymbol{\eta}$ is part of the equilibrium.

Recall that $P_\Delta = [0, v_\ell] \cup \{v_\ell + \Delta, v_\ell + 2\Delta, ..., v_h - \Delta, v_h\}$. Let B_Δ denote the set of monotone bidding functions using bids from $\{v_\ell, v_\ell + \Delta, v_\ell + 2\Delta, ..., \underline{b}, \bar{b}\}$ and let $m = \| \{v_\ell, v_\ell + \Delta, v_\ell + 2\Delta, ..., \underline{b}\} \|$. Using Athey's idea, Σ_Δ is a set of vectors of dimension $m+1$ whose coordinates belong to $[\underline{x}, \bar{x}]$

$$\Sigma_\Delta = \{\sigma = (\sigma_0, \sigma_1, ...\sigma_m) \in [\underline{x}, \bar{x} - \epsilon]^{m+1} \mid \underline{x} \equiv \sigma_0 \leq \sigma_1 \leq ... \leq \sigma_m \equiv \bar{x} - \epsilon\},$$

where σ determines a bidding strategy β_σ by $\beta_\sigma(x) = v_\ell + i\Delta$ if $x \in [\sigma_i, \sigma_{i+1})$, $i = 0, ..., m-1$. We set

$$\beta_\sigma(x) = \bar{b} \qquad \forall\ x > \bar{x} - \epsilon.$$

We say a bidding strategy β is a best response against $(\sigma, \boldsymbol{\eta})$ if for all $x \leq \bar{x} - \epsilon$,

$$\beta(x) \in \underset{b \in \{v_\ell, v_\ell + \Delta, v_\ell + 2\Delta, ..., \underline{b}, \bar{b}\}}{\arg\max} U(b | x, \beta_\sigma, \boldsymbol{\eta}).$$

Let D denote the set of probability distributions η over $\{1, ..., N\}$.

Define the correspondence Ψ from $\Sigma_\Delta \times D \times D$ into itself. For any $\sigma' \in \Sigma_\Delta$ and $\boldsymbol{\eta}' \in D \times D$,

$$\begin{aligned}
\Psi_1(\sigma', \boldsymbol{\eta}') &= \{\sigma \in \Sigma_\Delta \mid \beta_\sigma \text{ is best response against } (\sigma', \boldsymbol{\eta}') \text{ for } x \leq \bar{x} - \epsilon\}, \\
\Psi_2(\sigma', \boldsymbol{\eta}') &= \left\{\boldsymbol{\eta} = (\eta_\ell, \eta_h) \mid \text{if } \eta_w(n) > 0, \text{ then } n \in \underset{n \in \{1, .., N\}}{\arg\max} \mathbb{E}_w[p | \beta_{\sigma'}, n] - ns\right\}, \\
\Psi(\sigma', \boldsymbol{\eta}') &= \Psi_1(\sigma', \boldsymbol{\eta}') \times \Psi_2(\sigma', \boldsymbol{\eta}').
\end{aligned}$$

Both Σ_Δ and D are closed convex sets. Ψ_2 is convex valued and continuous by virtue of being the set of maximizers of a concave problem on a convex set (see Lemma 2). That

Ψ_1 is convex valued and upper hemi-continuous is established by Athey (2001). To be precise, the convex valuedness is established directly by Lemma 2 of that paper, while Lemma 3 establishes the upper hemi-continuity of $\Psi_1(\sigma', \boldsymbol{\eta}')$ only with respect to σ' (since $\boldsymbol{\eta}$ is exogenously fixed in Athey's model). Nevertheless, since the bidder's payoff function in our model, $U(\cdot|x, \beta, \boldsymbol{\eta})$, is continuous in $\boldsymbol{\eta}$, Athey's original argument establishes that $\Psi_1(\sigma', \boldsymbol{\eta}')$ is continuous with respect to $(\sigma', \boldsymbol{\eta}')$ as well.

It follows that $\Psi = \Psi_1 \times \Psi_2$ is convex valued and upper hemi-continuous. By Kakutani's Theorem, Ψ has a fixed point. Since the strategies in Σ_Δ are constrained to use only prices from $\{v_\ell, v_\ell + \Delta, v_\ell + 2\Delta, ..., \underline{b}, \bar{b}\}$, the bidding strategy determined by the fixed point satisfies

$$\beta_\sigma(x) \begin{cases} = \bar{b} & \text{if} \quad x > \bar{x} - \epsilon, \\ \leq \underline{b} & \text{if} \quad x \leq \bar{x} - \epsilon. \end{cases}$$

In order to claim that a fixed point of Ψ is indeed an equilibrium of the auxiliary game A, it only remains to argue that there is no profitable deviation to a price in $[0, v_\ell)$, which is also in P_Δ and which is allowed by the constraint $\beta(x) \leq \underline{b}$—but this obviously true and was also argued in Lemma **10**. ∎

Proof of Lemma 17. We want to show that whenever $s^k \to 0$, β^k satisfies (72) and $\boldsymbol{\eta}^k$ is an optimal solicitation strategy given β^k, then:

1. $\lim_{k\to\infty} n_w^k = \infty, \ w \in \{\ell, h\}$.
2. $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} = \frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} < 1$.
3. $\frac{(1-G_\ell(\bar{x}-\epsilon))}{(1-G_h(\bar{x}-\epsilon))} \frac{1}{G_h(\bar{x}-\epsilon)} \geq \lim_{k\to\infty} \frac{G_h(\bar{x}-\epsilon)^{n_h^k - 1}}{G_\ell(\bar{x}-\epsilon)^{n_\ell^k - 1}}$.

Given convexity of the seller's objective function, optimality requires that

$$\mathbb{E}_w\left[p|\ \beta^k, n_w^k\right] - \mathbb{E}_w\left[p|\beta^k, n_w^k - 1\right] \geq s^k \geq \mathbb{E}_w\left[p|\beta^k, n_w^k + 1\right] - \mathbb{E}_w\left[p|\beta^k, n_w^k\right]. \quad (77)$$

In particular, this implies

$$\mathbb{E}_h\left[p|\beta^k, n_h^k\right] - \mathbb{E}_h\left[p|\beta^k, n_h^k - 1\right] \ \geq \ \mathbb{E}_\ell\left[p|\beta^k, n_\ell^k + 1\right] - \mathbb{E}_\ell\left[p|\beta^k, n_\ell^k\right], \quad (78)$$
$$\mathbb{E}_\ell\left[p|\beta^k, n_\ell^k\right] - \mathbb{E}_\ell\left[p|\beta^k, n_\ell^k - 1\right] \ \geq \ \mathbb{E}_h\left[p|\beta^k, n_h^k + 1\right] - \mathbb{E}_h\left[p|\beta^k, n_h^k\right]. \quad (79)$$

Observe that

$$\mathbb{E}_w\left[p|\beta^k, n\right] = (1 - G_w(\bar{x} - \epsilon)^n)\, \bar{b} + G_w(\bar{x} - \epsilon)^n\, \mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n).$$

where $\mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n)$ is the expected winning bid conditional on $p \leq \underline{b}$. It follows that

$$\begin{aligned} &\mathbb{E}_w\left[p|\beta^k, n+1\right] - \mathbb{E}_w\left[p|\beta^k, n\right] = \\ &G_w(\bar{x}-\epsilon)^{n-1}(1-G_w(\bar{x}-\epsilon))[\bar{b} - \mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n-1)] \\ &+G_w(\bar{x}-\epsilon)^n[\mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n) - \mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n-1)]. \end{aligned} \tag{80}$$

Hence,

$$\mathbb{E}_w\left[p|\beta^k, n+1\right] - \mathbb{E}_w\left[p|\beta^k, n\right] \geq G_w(\bar{x}-\epsilon)^n(1-G_w(\bar{x}-\epsilon))[\bar{b} - \mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n)]. \tag{81}$$

Step 1: $n_w^k \to \infty$ for $w \in \{\ell, h\}$.
Proof of Step 1: By (77) and (81),[35]

$$s^k \geq \mathbb{E}_w\left[p|\beta^k, n_w^k+1\right] - \mathbb{E}_w\left[p|\beta^k, n_w^k\right] \geq G_w(\bar{x}-\epsilon)^{n_w^k}(1-G_w(\bar{x}-\epsilon))[\bar{b} - \mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n_w^k)].$$

It follows from $s^k \to 0$ that $G_w(\bar{x}-\epsilon)^{n_w^k} \to 0$ and, hence, $n_w^k \to \infty$. This establishes Part 1 of the Lemma. □

Step 2:

$$\begin{aligned} &G_\ell(\bar{x}-\epsilon)\frac{1-G_\ell(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)}\lim_{k\to\infty}\frac{[\bar{b} - \mathbb{E}_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k)]}{[\bar{b} - \mathbb{E}_h(p|p \leq \underline{b}, \beta^k, n_h^k)]} \\ &\leq \lim_{k\to\infty}\frac{G_h(\bar{x}-\epsilon)^{n_\ell^k-1}}{G_\ell(\bar{x}-\epsilon)^{n_h^k-1}} \leq \\ &\frac{1}{G_h(\bar{x}-\epsilon)}\frac{1-G_\ell(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)}\lim_{k\to\infty}\frac{[\bar{b} - \mathbb{E}_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k)]}{[\bar{b} - \mathbb{E}_h(p|p \leq \underline{b}, \beta^k, n_h^k)]}. \end{aligned} \tag{82}$$

Proof of Step 2: Using (80) for $w = h$ and (81) for $w = \ell$ to rewrite (78) and rearranging we get,

$$\begin{aligned} &\frac{G_h(\bar{x}-\epsilon)^{n_h^k-1}}{G_\ell(\bar{x}-\epsilon)^{n_\ell^k}}\left\{\frac{(1-G_h(\bar{x}-\epsilon))[\bar{b} - \mathbb{E}_h(p|p \leq \underline{b}, \beta^k, n_h^k-1)]}{(1-G_\ell(\bar{x}-\epsilon))[\bar{b} - \mathbb{E}_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k)]}+\right. \\ &\left.\frac{\mathbb{E}_h(p|p \leq \underline{b}, \beta^k, n_h^k) - \mathbb{E}_h(p|p \leq \underline{b}, \beta^k, n_h^k-1)}{(1-G_\ell(\bar{x}-\epsilon))[\bar{b} - \mathbb{E}_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k)]}\right\} \geq 1. \end{aligned} \tag{83}$$

Similarly,

$$\begin{aligned} &\frac{G_\ell(\bar{x}-\epsilon)^{n_\ell^k-1}}{G_h(\bar{x}-\epsilon)^{n_h^k}}\left\{\frac{(1-G_\ell(\bar{x}-\epsilon))[\bar{b} - \mathbb{E}_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k-1)]}{(1-G_h(\bar{x}-\epsilon))[\bar{b} - \mathbb{E}_h(p|p \leq \underline{b}, \beta^k, n_h^k)]}+\right. \\ &\left.\frac{\mathbb{E}_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k) - \mathbb{E}_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k-1)}{(1-G_h(\bar{x}-\epsilon))[\bar{b} - \mathbb{E}_h(p|p \leq \underline{b}, \beta^k, n_h^k)]}\right\} \geq 1. \end{aligned} \tag{84}$$

[35] The derivation of (81) does not assume monotonicity of β^k for $x \leq \bar{x} - \epsilon$.

Now, $n_w^k \to \infty$ implies

$$\lim_{k\to\infty} \mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n_w^k + 1) = \lim_{k\to\infty} \mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n_w^k) = \lim_{k\to\infty} \mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n_w^k - 1).$$

Hence,

$$\lim_{k\to\infty} \frac{\mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n_w^k) - \mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n_w^k - 1)}{(1 - G_w(\bar{x} - \epsilon))\,[\bar{b} - \mathbb{E}_w(p|p \leq \underline{b}, \beta^k, n_w^k)]} = 0.$$

Therefore, taking limits in (83) and (84) and combining them to a single chain of inequalities we get (82). □

Step 3: $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} = \frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} < 1$.

Proof of Step 3: From (82), $\lim \frac{G_h(\bar{x}-\epsilon)^{n_h^k}}{G_\ell(\bar{x}-\epsilon)^{n_\ell^k}} \in (0,\infty)$. Therefore,

$$\lim_{k\to\infty} \left(\frac{G_h(\bar{x}-\epsilon)^{\frac{n_h^k}{n_\ell^k}}}{G_\ell(\bar{x}-\epsilon)} \right)^{n_\ell^k} \in (0,\infty),$$

Since $n_\ell^k \to \infty$, this requires that

$$\lim_{k\to\infty} \frac{G_h(\bar{x}-\epsilon)^{\frac{n_h^k}{n_\ell^k}}}{G_\ell(\bar{x}-\epsilon)} = 1.$$

Therefore, $\ln G_h(\bar{x}-\epsilon) \lim \frac{n_h^k}{n_\ell^k} = \ln G_\ell(\bar{x}-\epsilon)$, hence,

$$\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} = \frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} < 1,$$

where the last inequality is a consequence of $G_\ell(\bar{x}-\epsilon) > G_h(\bar{x}-\epsilon)$. This establishes Part 2 of the lemma. □

Step 4: $\lim_{k\to\infty} \mathbb{E}_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k) \geq \lim_{k\to\infty} \mathbb{E}_h(p|p \leq \underline{b}, \beta^k, n_h^k)$

Proof of Step 4: Let $F_w(p|p \leq \underline{b}, \beta^k, n_w^k)$ denote the distribution of the winning bid conditional on being $p \leq \underline{b}$. Recall $(x_-^k(p) = \inf\{x \mid \beta^k(x) \geq p\}$. Consider a subsequence over which the following limits exist and let $q_w = \lim F_w(p|p \leq \underline{b}, \beta^k, n_w^k)$, $w = \ell, h$.

$$\begin{aligned} q_w \quad &\equiv \quad \lim_{k\to\infty} F_w(p|p \leq \underline{b}, \beta^k, n_w^k) = \lim_{k\to\infty} \left(\frac{G_w(x_-^k(p))}{G_w(\bar{x}-\epsilon)} \right)^{n_w^k} = \\ &\lim_{k\to\infty} \left(\frac{G_w(\bar{x}-\epsilon) - [G_w(\bar{x}-\epsilon) - G_w(x_-^k(p))]}{G_w(\bar{x}-\epsilon)} \right)^{n_w^k} \\ &\lim_{k\to\infty} \left(1 - \frac{\frac{G_w(\bar{x}-\epsilon) - G_w(x_-^k(p))}{G_w(\bar{x}-\epsilon)} n_w^k}{n_w^k} \right)^{n_w^k}. \end{aligned}$$

Observe that, for p such that $\limsup x_-^k(p) < \bar{x} - \epsilon$, we have $\liminf [G_w(\bar{x} - \epsilon) - G_w(x_-^k(p)] > 0$ and hence $q_w = 0$; for p such that from some point in the sequence $(\beta^k)^{-1}(p) = \bar{x} - \epsilon$, we have $q_w = 1$; for p such that $x_-^k(p) < \bar{x} - \epsilon$ and $\lim x_-^k(p) = \bar{x} - \epsilon$, we have

$$\lim_{k\to\infty} \frac{G_w(\bar{x} - \epsilon) - G_w(x_-^k(p))}{G_w(\bar{x} - \epsilon)} n_w^k = -\ln q_w.$$

Therefore,

$$\frac{-\ln q_h}{-\ln q_\ell} = \lim_{k\to\infty} \frac{[G_h(\bar{x} - \epsilon) - G_h(x_-^k(p))]}{[G_\ell(\bar{x} - \epsilon) - G_\ell(x_-^k(p))]} \frac{G_\ell(\bar{x} - \epsilon)}{G_h(\bar{x} - \epsilon)} \frac{n_h^k}{n_\ell^k}.$$

Now, since $\lim x_-^k(p) = \bar{x} - \epsilon$ and since g_w are step functions, it follows that, for k large enough, $G_w(\bar{x} - \epsilon) - G_w(x_-^k(p)) = g_w(\bar{x} - \epsilon)[\bar{x} - \epsilon - x_-^k(p)]$. Using this observation and $\lim \frac{n_h^k}{n_\ell^k} = \frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}$

$$\frac{-\ln q_h}{-\ln q_\ell} = \lim \frac{g_h(\bar{x} - \epsilon)}{g_\ell(\bar{x} - \epsilon)} \frac{G_\ell(\bar{x} - \epsilon)}{G_h(\bar{x} - \epsilon)} \frac{\ln G_\ell(\bar{x} - \epsilon)}{\ln G_h(\bar{x} - \epsilon)}.$$

By condition (21), the right side is smaller than one, and so $\frac{\ln q_h}{\ln q_\ell} \leq 1$ (see the footnote). Therefore,

$$\lim_{k\to\infty} F_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k) = q_\ell \leq q_h = \lim_{k\to\infty} F_h(p|p \leq \underline{b}, \beta^k, n_h^k).$$

Thus, $\lim F_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k)$ stochastically dominates $\lim F_h(p|p \leq \underline{b}, \beta^k, n_h^k)$. Hence, $\lim \mathbb{E}_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k) \geq \lim \mathbb{E}_h(p|p \leq \underline{b}, \beta^k, n_h^k)$. □

Step 5:

$$\lim_{k\to\infty} \frac{G_h(\bar{x} - \epsilon)^{n_h^k - 1}}{G_\ell(\bar{x} - \epsilon)^{n_\ell^k - 1}} \leq \frac{1}{G_h(\bar{x} - \epsilon)} \frac{1 - G_\ell(\bar{x} - \epsilon)}{1 - G_h(\bar{x} - \epsilon)}.$$

Proof of Step 5: Step 4 implies that $\lim \frac{[\bar{b} - \mathbb{E}_\ell(p|p \leq \underline{b}, \beta^k, n_\ell^k)]}{[\bar{b} - \mathbb{E}_h(p|p \leq \underline{b}, \beta^k, n_h^k)]} \leq 1$, which together with (82) implies the desired inequality. □

This proves Part (3) and concludes the proof of Lemma 17. ∎

Proof of Lemma 18. We want to show that there are numbers v_1^*, v_2^*,v_3^* independent of $\underline{b}, \bar{b}$ such that

$$\begin{aligned}
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } \bar{b};\, \beta^k,\, \boldsymbol{\eta}^k\right] &= \rho_\ell v_\ell + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x} - \epsilon, \text{ win at } \bar{b};\, \beta^k,\, \boldsymbol{\eta}^k\right] &\leq v_1^* < \rho_\ell v_\ell + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\, \beta^k,\, \boldsymbol{\eta}^k\right] &\leq v_2^* < \rho_\ell v_\ell + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x} - \epsilon, \text{ win at } b \in (\underline{b}, \bar{b})\,;\, \beta^k, \boldsymbol{\eta}^k\right] &\leq v_3^* < \rho_\ell v_\ell + \rho_h v_h.
\end{aligned}$$

Observe that

$$\begin{aligned}
\Pr\left[h|\bar{x},\text{ win at }\bar{b},\,\beta^k,\,\eta^k\right] &= \frac{\Pr\left[h,\,\bar{x},\,\beta^k,\text{ win at }\bar{b}\right]}{\Pr\left[\bar{x},\,\beta^k,\text{ win at }\bar{b}\right]} \qquad (85)\\
&= \frac{\rho_h g_h\left(\bar{x}\right)\frac{n_h}{N}\frac{1}{n_h}\frac{1-\left(G_h(\bar{x}-\epsilon)\right)^{n_h}}{1-G_h(\bar{x}-\epsilon)}}{\rho_h g_h\left(\bar{x}\right)\frac{n_h}{N}\frac{1}{n_h}\frac{1-\left(G_h(\bar{x}-\epsilon)\right)^{n_h}}{1-G_h(\bar{x}-\epsilon)}+\rho_\ell g_\ell\left(\bar{x}\right)\frac{n_\ell}{N}\frac{1}{n_\ell}\frac{1-\left(G_\ell(\bar{x}-\epsilon)\right)^{n_\ell}}{1-G_\ell(\bar{x}-\epsilon)}}\\
&= \frac{\rho_h\left(1-\left(G_h\left(\bar{x}-\epsilon\right)\right)^{n_h}\right)}{\rho_h\left(1-\left(G_h\left(\bar{x}-\epsilon\right)\right)^{n_h}\right)+\rho_\ell\left(1-\left(G_\ell\left(\bar{x}-\epsilon\right)\right)^{n_\ell}\right)}\\
&\to_{k\to\infty}\rho_h.
\end{aligned}$$

For the second equality, note that $\rho_w g_w\left(\bar{x}\right)\frac{n_w}{N} = \Pr\left[w,\,\bar{x},\,\eta^k\right]$ and that by Lemma 5, $\Pr\left[\text{win at }\bar{b}|w,\,\beta^k,\,\eta^k\right] = \pi_w\left[\bar{b}|\beta^k,n_w^k\right] = \frac{1}{n_w}\frac{1-\left(G_w(\bar{x}-\epsilon)\right)^{n_w}}{1-G_w(\bar{x}-\epsilon)}$. The third equality follows from $1-G_w\left(\bar{x}-\epsilon\right) = g_w\left(\bar{x}\right)\epsilon$ and cancellation of terms. Finally, the convergence to ρ_h follows from $n_w^k\to\infty$ by Lemma 17. Therefore

$$\lim_{k\to\infty}\mathbb{E}\left[v|\bar{x},\text{ win at }\bar{b},\,\beta^k,\,\eta^k\right] = \rho_\ell v_\ell+\rho_h v_h,$$

that is, the first equation holds.

Replacing $g_w\left(\bar{x}\right)$ by $g_w\left(\bar{x}-\epsilon\right)$ in (85), we obtain the corresponding expressions for $\Pr\left[h|\bar{x}-\epsilon,\text{ win at }\bar{b},\,\beta^k,\,\eta^k\right]$. Then using $g_h\left(\underline{x}\right)\epsilon = 1-G_h\left(\bar{x}-\epsilon\right)$ to substitute out $1-G_h\left(\bar{x}-\epsilon\right)$ and again canceling terms we get

$$\begin{aligned}
\Pr\left[h|\bar{x}-\epsilon,\text{ win at }\bar{b},\,\beta^k,\,\eta^k\right] &= \frac{\rho_h\frac{g_h(\bar{x}-\epsilon)}{g_h(\bar{x})}\left(1-G_h\left(\bar{x}-\epsilon\right)^{n_h}\right)}{\rho_h\frac{g_h(\bar{x}-\epsilon)}{g_h(\bar{x})}\left(1-G_h\left(\bar{x}-\epsilon\right)^{n_h}\right)+\frac{g_\ell(\bar{x}-\epsilon)}{g_\ell(\bar{x})}\rho_\ell\left(1-G_\ell\left(\bar{x}-\epsilon\right)^{n_\ell}\right)}\\
&\to_{k\to\infty}\frac{\rho_h\left(\frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\right)/\left(\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\right)}{\rho_h\left(\frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\right)/\left(\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\right)+\rho_\ell} < \rho_h.
\end{aligned}$$

The expression following the convergence sign is obtained by dividing through by $\frac{g_\ell(\bar{x}-\epsilon)}{g_\ell(\bar{x})}$ and noting that $G_w\left(\bar{x}-\epsilon\right)^{n_w}\to 0$ since $n_w^k\to\infty$ by Lemma 17. The last inequality owes to the increasing likelihood ratio $\frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)} < \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$. Let

$$v_1^* = \frac{\rho_\ell}{\rho_h\left(\frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\right)/\left(\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\right)+\rho_\ell}v_\ell+\frac{\rho_h\left(\frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\right)/\left(\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\right)}{\rho_h\left(\frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\right)/\left(\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\right)+\rho_\ell}v_h.$$

The second equation holds with

$$\lim_{k\to\infty}\mathbb{E}\left[v|\bar{x}-\epsilon,\text{ win at }\bar{b};\,\beta^k,\,\boldsymbol{\eta}^k\right] = v_1^* < \rho_\ell v_\ell+\rho_h v_h.$$

The winner at $b > \bar{b}$ does not learn anything from winning. Hence,

$$\begin{aligned}
\Pr\left[h|\bar{x},\text{ win at } b>\bar{b};\,\beta^k,\,\eta^k\right] &= \frac{\Pr\left[h,\,\bar{x},\text{ win at } b>\bar{b};\,\beta^k,\,\eta^k\right]}{\Pr\left[\bar{x},\text{ win at } b>\bar{b};\,\beta^k,\,\eta^k\right]} \\
&= \frac{\rho_h g_h\left(\bar{x}\right)\frac{n_h}{N}}{\rho_h g_h\left(\bar{x}\right)\frac{n_h}{N}+\rho_\ell g_\ell\left(\bar{x}\right)\frac{n_\ell}{N}} \\
&= \frac{\rho_h \overline{g}\frac{n_h^k}{n_\ell^k}}{\rho_h \overline{g}\frac{n_h^k}{n_\ell^k}+\rho_\ell} \\
&\to_{k\to\infty} \frac{\rho_h \overline{g}\frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}}{\rho_h \overline{g}\frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}+\rho_\ell},
\end{aligned}$$

where the limit follows from $\frac{n_h^k}{n_\ell^k} \to \frac{\ln G_\ell(\hat{x})}{\ln G_h(\hat{x})}$ established in Lemma 17. Let

$$v_2^* = \frac{\rho_\ell}{\rho_h \overline{g}\frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}+\rho_\ell} v_\ell + \frac{\rho_h \overline{g}\frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}}{\rho_h \overline{g}\frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}+\rho_\ell} v_h.$$

From $\frac{g_h(\bar{x})\epsilon}{g_\ell(\bar{x})\epsilon} = \frac{1-G_h(\bar{x}-\epsilon)}{1-G_\ell(\bar{x}-\epsilon)}$, $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)} = \frac{1-G_h(\bar{x}-\epsilon)}{1-G_\ell(\bar{x}-\epsilon)}\frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}$. Because the function $\frac{1-z}{\ln z}$ is strictly increasing in $z \in (0,1)$ and because $G_\ell\left(\bar{x}-\epsilon\right) > G_h\left(\bar{x}-\epsilon\right)$,

$$\frac{1-G_h\left(\bar{x}-\epsilon\right)}{1-G_\ell\left(\bar{x}-\epsilon\right)}\frac{\ln G_\ell\left(\bar{x}-\epsilon\right)}{\ln G_h\left(\bar{x}-\epsilon\right)} < 1.$$

Therefore, $\Pr\left[h|\bar{x},\text{ win at } b>\bar{b},\,\beta^k,\,\eta^k\right] < \rho_h$ and

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x},\text{ win at } b>\bar{b};\,\beta^k,\,\eta^k\right] = v_2^* < \rho_\ell v_\ell + \rho_h v_h.$$

Define v_3^* by

$$v_3^* \equiv \frac{\rho_h \frac{1}{G_\ell(\bar{x}-\epsilon)}\frac{1-G_\ell(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)}}{\rho_h \frac{1}{G_\ell(\bar{x}-\epsilon)}\frac{1-G_\ell(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)}+\rho_\ell} v_h + \frac{\rho_\ell}{\rho_h \frac{1}{G_\ell(\bar{x}-\epsilon)}\frac{1-G_\ell(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)}+\rho_\ell} v_\ell. \tag{86}$$

Since by condition (20), $\frac{1}{G_\ell(\bar{x}-\epsilon)}\frac{1-G_\ell(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)} < 1$, it follows that $v_3^* < \rho_\ell v_\ell + \rho_h v_h$. Now,

$$\begin{aligned}
& \Pr\left[v_h|\bar{x}-\epsilon, \text{ win at } b \in \left(\underline{b}^k, \bar{b}\right); \beta^k, \eta^k\right] \qquad (87) \\
= \quad & \frac{\rho_h \Pr[\bar{x}-\epsilon, \text{ win at } b \in \left(\underline{b}^k, \bar{b}\right)|h; \beta^k, \eta^k]}{\sum_{w=h,\ell}\rho_w \Pr[\bar{x}-\epsilon, \, b \in \left(\underline{b}^k, \bar{b}\right)|w; \beta^k, \eta^k]} \\
= \quad & \frac{\rho_h g_h(\bar{x}-\epsilon)\frac{n_h^k}{N^k}G_h\left(\bar{x}-\epsilon\right)^{n_h^k-1}}{\rho_h g_h(\bar{x}-\epsilon)\frac{n_h^k}{N^k}G_h\left(\bar{x}-\epsilon\right)^{n_h^k-1}+\rho_\ell g_\ell(\bar{x}-\epsilon)\frac{n_\ell^k}{N^k}G_\ell\left(\bar{x}-\epsilon\right)^{n_\ell^k-1}} \\
= \quad & \lim_{k\to\infty}\frac{\rho_h \frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\frac{G_\ell(\bar{x}-\epsilon)\ln G_\ell(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)\ln G_h(\bar{x}-\epsilon)}G_h\left(\bar{x}-\epsilon\right)^{n_h^k}}{\rho_h \frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\frac{G_\ell(\bar{x}-\epsilon)\ln G_\ell(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)\ln G_h(\bar{x}-\epsilon)}G_h\left(\bar{x}-\epsilon\right)^{n_h^k}+\rho_\ell G_\ell\left(\bar{x}-\epsilon\right)^{n_\ell^k}} \\
\leq \quad & \lim_{k\to\infty}\frac{\rho_h\frac{G_h(\bar{x}-\epsilon)^{n_h^k}}{G_\ell(\bar{x}-\epsilon)^{n_\ell^k}}}{\rho_h\frac{G_h(\bar{x}-\epsilon)^{n_h^k}}{G_\ell(\bar{x}-\epsilon)^{n_\ell^k}}+\rho_\ell} \leq \frac{\rho_h\frac{1}{G_\ell(\bar{x}-\epsilon)}\frac{1-G_\ell(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)}}{\rho_h\frac{1}{G_\ell(\bar{x}-\epsilon)}\frac{1-G_\ell(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)}+\rho_\ell} < \rho_h.
\end{aligned}$$

The expression after the 2nd equality sign is explained by $\Pr[\text{win at } b \in \left(\underline{b}^k, \bar{b}\right)|w; \beta^k, \eta^k] = G_h\left(\hat{x}\right)^{n_h^k-1}$. The expression following the convergence sign is obtained by dividing through by $\frac{g_h(\bar{x}-\epsilon)n_h^k}{G_\ell(\bar{x}-\epsilon)}$ and noting that $\lim\frac{n_h^k}{n_\ell^k} = \frac{\ln G_\ell(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}$. The following inequality follows from the assumption $\frac{g_h(\bar{x}-\epsilon)}{g_\ell(\bar{x}-\epsilon)}\frac{G_\ell(\bar{x}-\epsilon)\ln G_\ell(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)\ln G_h(\bar{x}-\epsilon)} \leq 1$. The next inequality follows from Lemma 17. The final inequality follows from $\frac{1}{G_\ell(\hat{x})}\frac{1-G_\ell(\hat{x})}{1-G_h(\hat{x})} < 1$ which is implied by condition (20). The definition of v_3^* together with (87) imply

$$\lim_{k\to\infty}\Pr\left[v_h|\bar{x}-\epsilon, \text{ win at } b \in \left(\underline{b}, \bar{b}\right); \beta^k, \eta^k\right] \leq v_3^* < \rho_\ell v_\ell + \rho_h v_h.$$

∎

E Proofs with Unboundedly Informative Signals

For this section, we include the state as an argument in the distributions of the signal and write $G\left(\cdot|w\right)$ and $g\left(\cdot|w\right)$ instead of $G_w\left(\cdot\right)$ and $g_w\left(\cdot\right)$.

We continue to assume that the signal distributions have identical supports and strictly positive densities. The likelihood ratio $\frac{g(x|h)}{g(x|\ell)}$ is non-decreasing and $\frac{g(\underline{x}|h)}{g(\underline{x}\ell)} = \lim_{x\to\underline{x}}\frac{g(x|h)}{g(x|\ell)}$. The signal is said to be *unboundedly informative* if the upper bound of its support is infinite,

$$G\left(x|h\right) < 1 \text{ for all } x \in \mathbb{R},$$

and the likelihood ratio becomes arbitrarily large,

$$\lim_{x\to\infty}\frac{g\left(x|h\right)}{g\left(x|\ell\right)} = \infty.$$

It will be useful to define the quantile $x_\lambda(\mathbf{n})$ of the first order statistic given a solicitation strategy $\mathbf{n}$,

$$x_\lambda(\mathbf{n}) : (G(x_\lambda(\mathbf{n})|h))^{n_h} = \lambda;$$

in the following, we often drop the argument $\mathbf{n}$ from x_λ.

E.1 Characterization with Unboundedly Informative Signals

The characterization result from Theorem 2 extends to unboundedly informative signals for all equilibrium in which at least two bidders are solicited.

Proposition 8 *Suppose the signal is unboundedly informative. For every δ, there are $(s', \Delta') >> 0$ such that for every $(s, \Delta) \leq (s', \Delta')$ and for every equilibrium (β, η) of $\Gamma(s, \Delta)$ for which $\eta(1|w) = 0$ for $w \in \{\ell, h\}$ either of the following is true.*

1. *Separation.*

$$F(v_w + \delta|w; \beta, \eta) - F(v_w - \delta|w; \beta, \eta) \geq 1 - \delta \text{ for } w \in \{\ell, h\}.$$

2. *Pooling. For some $\bar{b} \leq E[v]$,*

$$F(\bar{b} + \delta|w; \beta, \eta) - F(\bar{b} - \delta|w; \beta, \eta) \geq 1 - \delta \text{ for } w \in \{\ell, h\}.$$

That is, for sufficiently small solicitation costs and a sufficiently fine grid, in every equilibrium, the winning bid is either close to the true values v_w with probability close to one, or the winning bid is equal to some value $\bar{b}$ with probability close to one in either state (atom). A remark at the end of this section discusses equilibria in which only one bidder may be solicited.

We use the following auxiliary result.

Claim 5 *Suppose the signal is unboundedly informative. Consider any sequence (β^k, η^k) where η^k has support on at most two adjacent numbers, $n_h^k \to \infty$, and β^k is non-decreasing. Let $x_\lambda^k = x_\lambda(\mathbf{n}^k)$.*

1. $\lim_{\lambda \to 0} \lim_{k \to \infty} n_h^k (1 - G(x_\lambda^k|h)) = \infty$.
2. *If $\lim_{k \to \infty} F(b|h; \beta^k, \eta^k) = 0$ for some b and $\lim_{k \to \infty} F(b - \varepsilon|\ell; \beta^k, \eta^k) > 0$ for some ε, then,*

$$\lim \left(G\left(x_\lambda^k|\ell\right)\right)^{n_\ell^k} = 1 \text{ for all } \lambda \in (0, 1).$$

The first part states that the limit of the expected number of bidders with signals $x \geq x_\lambda^k$ is arbitrarily large for $k \to \infty$ when λ is small enough. The second part states that for all $\lambda > 0$ the probability of $x \geq x_\lambda^k$ must vanish in the low state if there is some b such that $\lim_{k \to \infty} F(b|h; \beta^k, \eta^k) = 0$ but $\lim_{k \to \infty} F(b - \varepsilon|\ell; \beta^k, \eta^k) > 0$.

Proof of the Claim: By the Poisson approximation from Lemma 1, $\left(G\left(x^k|h\right)\right)^{n_h^k} = \lambda$ implies

$$\lim_{k\to\infty} n_h^k \left(1 - G\left(x_\lambda^k|h\right)\right) = -\ln\lambda.$$

The first claim follows.

For the second claim, note that the hypothesis $\lim_{k\to\infty} F\left(b|h;\beta^k,\eta^k\right) = 0$ implies $\lim \beta^k\left(x_\lambda^k\right) \geq b$ for all $\lambda > 0$.

By definition,

$$\lim\left(G\left(x_\lambda^k|\ell\right)\right)^{n_\ell^k} \geq F\left(b-\varepsilon|\ell;\beta^k,\eta^k\right) \text{ for all } \lambda > 0. \tag{88}$$

Fix some $\tilde{\lambda}$. Let $\gamma = \lim\left(G\left(x_{\tilde{\lambda}}^k|\ell\right)\right)^{n_\ell}$. As before, $\lim_{k\to\infty} n_\ell^k\left(1-G\left(x_{\tilde{\lambda}}^k|\ell\right)\right) = -\ln\gamma$. Moreover, from the MLRP, for $m > 1$, $\tilde{\lambda}^m < \tilde{\lambda}$ implies

$$\frac{1-G\left(x_{\tilde{\lambda}^m}^k|\ell\right)}{1-G\left(x_{\tilde{\lambda}^m}^k|h\right)} \geq \frac{1-G\left(x_{\tilde{\lambda}}^k|\ell\right)}{1-G\left(x_{\tilde{\lambda}}^k|h\right)}.$$

Hence,

$$\frac{\lim_{k\to\infty} n_\ell^k\left(1-G\left(x_{\tilde{\lambda}^m}^k|\ell\right)\right)}{\lim_{k\to\infty} n_h^k\left(1-G\left(x_{\tilde{\lambda}^m}^k|h\right)\right)} \geq \frac{\ln\gamma}{\ln\tilde{\lambda}}.$$

Hence, from $\lim_{k\to\infty} n_h^k\left(1-G\left(x_{\tilde{\lambda}^m}^k|h\right)\right) = -m\ln\tilde{\lambda}$,

$$\lim_{k\to\infty} n_\ell^k\left(1-G\left(x_{\tilde{\lambda}^m}^k|\ell\right)\right) \geq -m\ln\gamma$$

from where for $\lambda' = \tilde{\lambda}^m$,

$$\gamma^m \geq \lim\left(G\left(x_{\lambda'}^k|\ell\right)\right)^{n_\ell^k}. \tag{89}$$

Thus, from (88) and the hypothesis $\lim_{k\to\infty} F\left(b-\varepsilon|\ell;\beta^k,\eta^k\right) > 0$, this implies that

$$\gamma^m \geq \lim_{k\to\infty} F\left(b-\varepsilon|\ell;\beta^k,\eta^k\right) > 0 \text{ for all } m > 1,$$

where the strict inequality is from the hypothesis. This requires $\gamma = 1$, that is, $\lim\left(G\left(x_\lambda^k|\ell\right)\right)^{n_\ell} = 1$ for all λ, as claimed. **QED.**

The main observation is that the distribution of the winning bid must be degenerate in the high state.

Lemma 19 *Suppose the signal is unboundedly informative. For every δ, there are s' and Δ' such that for every $s \leq s'$ and $\Delta \leq \Delta'$ and for every equilibrium (β,η) of $\Gamma(s,\Delta)$ there is some $\bar{b}_h$ such that $\eta(1|w) = 0$ for $w \in \{\ell, h\}$ implies*

$$F\left(\bar{b}_h + \delta|h;\beta,\eta\right) - F\left(\bar{b}_h - \delta|h;\beta,\eta\right) \geq 1-\delta.$$

Proof of Lemma: By contradiction. Let $\bar{F}(\cdot|w)$ be the limit of a sequence of distributions for $(s^k, \Delta^k) \to (0,0)$ and suppose $\bar{F}(\cdot|h)$ is not degenerate. It is clear that in this case, $n_h^k \to \infty$. In fact, from Lemma 3, if the limit distribution is not degenerate given optimal solicitation, then the limiting search cost is not zero,

$$\lim_{x\to\infty} \bar{n}_h^k s^k > 0.$$

Hence,

$$\lim_{x\to\infty} \frac{n_h^k}{n_\ell^k} > 0,$$

for otherwise $\lim n_\ell^k s^k = \lim \frac{n_\ell^k}{n_h^k} n_h^k (s^k) = \infty$. As a consequence of $\frac{g(x_\lambda^k|h)}{g(x_\lambda^k|\ell)} \to \infty$,

$$\lim_{x\to\infty} \left(G\left(x_\lambda^k|\ell\right)\right)^{n_\ell^k} = 1 \text{ for all } \lambda > 0, \tag{90}$$

and the interim belief with a signal $x = x_\lambda^k$ converges to one,

$$\Pr\left[w = h|x_\lambda^k, \text{ sampled}; \beta^k, \eta^k\right] \to 1 \text{ for all } \lambda > 0, \tag{91}$$

since the interim likelihood ratio $\lim_{x\to\infty} \frac{\rho(h)}{\rho(\ell)} \frac{g(x_\lambda^k|h)}{g(x_\lambda^k|\ell)} \frac{n_h^k}{n_\ell^k} = \infty$.

Since $\bar{F}(\cdot|h)$ is not degenerate, there are some $b_1 < b_2 < v_h$ such that

$$0 < \bar{F}(b_1|h) \leq \bar{F}(b_2|h) < 1.$$

From (90),

$$\bar{F}(b_1|\ell) = 1.$$

From (91) and since the probability of winning at b_2 is strictly positive in the limit if $w = h$ and one if $w = \ell$,

$$\Pr\left[w = h|x_\lambda^k, \text{ win at } b_2, \text{ sampled}; \beta^k, \eta^k\right] \to 1 \text{ for all } \lambda > 0.$$

Thus, for all λ, the interim expected payoffs conditional on $x \geq x_\lambda^k$ are bounded from below in the limit by

$$\bar{F}(b_1|h)(v_h - b_2).$$

For λ close to zero, the number of bidders with $x \geq x_\lambda^k$ is arbitrarily large. However, this means that the sum of the interim expected payoffs of the bidders conditional on $w = h$ is unbounded, a contradiction. This proves the Lemma. **QED.**

Proof of the Proposition: Again, suppose $\bar{F}(\cdot|w)$ is the limit of a sequence of distributions for $(s^k, \Delta^k) \to (0,0)$. Let b_1 be the point of the atom of $\bar{F}(\cdot|h)$, that is,

$$\bar{F}(b_1 + \delta|h) - \bar{F}(b_1 - \delta|h) = 1 \text{ for all } \delta. \tag{92}$$

From $\eta(1|w) = 0$ and Lemma 10, $b_1 \geq v_\ell$.

Case 1. Suppose that

$$\bar{F}(b_1 - \varepsilon|\ell) = 0 \text{ for all } \varepsilon > 0.$$

Because $b_1 \geq v_\ell$, (92), the law of iterated expectations, and individual rationality of the bidders requires that $\bar{F}(b_1 + \varepsilon|\ell) = 1$ for all $\varepsilon > 0$. Thus, in both states, the winning bid is close to b_1 with probability converging to one in both states. Hence, $b_1 \leq E[v]$, again by the law of iterated expectations and individual rationality of the bidders. Thus, the proposition holds in this case.

Case 2. Suppose that

$$\bar{F}(b_1 - \varepsilon|\ell) > 0 \text{ for some } \varepsilon.$$

We show first that this implies that

$$b_1 = v_h.$$

From Lemma 5,

$$\lim_{k\to\infty} \left(G\left(x_\lambda^k|\ell\right)\right)^{n_\ell^k} = 1 \text{ for all } \lambda > 0.$$

As a consequence,

$$\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} \frac{G\left(x_\lambda^k|h\right) - G\left(x_{\lambda'}^k|h\right)}{G\left(x_\lambda^k|\ell\right) - G\left(x_{\lambda'}^k|\ell\right)} = \infty \text{ for all } 0 < \lambda' < \lambda.$$

Therefore, the interim probability

$$\lim_{k\to\infty} \Pr\left[w = h|x_\lambda^k, \text{sampled}; \beta^k, \eta^k\right] = 1 \text{ for all } \lambda > 0.$$

Hence, because for all $b' > b_1$, $\bar{F}(b'|h) = 1$, the posterior probability

$$\lim_{k\to\infty} \Pr\left[w = h|x_\lambda^k, \text{win at } b', \text{sampled}; \beta^k, \eta^k\right] = 1 \text{ for all } \lambda > 0.$$

Thus, the interim expected payoff of all $x \geq x_\lambda^k$ are bounded from below by

$$v_h - b_1.$$

Since this is true for all $\lambda > 0$ and since there are arbitrarily many bidders with such signals for λ small by Lemma 5, feasibility requires that $v_h - b_1 \leq 0$. From individual rationality, $b_1 \leq v_h$. Therefore, $v_h = b_1$ follows.

Finally, from $b_1 = v_h$, it follows from the law of iterated expectations and individual rationality of the bidders that $\bar{F}(v_\ell|\ell) = 1$. From $\eta(1|w) = 0$ and Lemma 10, $\bar{F}(v_\ell - \varepsilon|\ell) = 0$ for all ε. Hence, the distribution becomes degenerate on v_ℓ in the low state. Thus, the proposition holds. **QED.**

Remark on $\eta(1|w) > 0$: When signals are unboundedly informative, there may be non-trivial equilibria in which $\eta(1|w) > 0$ in some state. Specifically, we cannot rule out asymmetrically revealing equilibria in which either $n_\ell^k \to 1$ and $n_h^k \to \infty$ or $n_h^k \to 1$ and $n_\ell^k \to \infty$. In both cases, the distribution of the winning bid becomes degenerate in all states. In the first case, this distribution becomes degenerate on v_h and c, respectively, and in the second case, it becomes degenerate on c and v_ℓ, respectively.

E.2 Existence of Pooling Equilibrium (Proposition 7)

For the discussion of existence, it will be useful to include the signal as a parameter of the game, that is, to write

$$\Gamma\left(s, P_\Delta, \bar{G}\right).$$

The Proposition 7 states that one can find a distribution of signals such that for every δ, for some arbitrarily small solicitation costs and arbitrarily fine price grid, there exists an equilibrium that is close to the pooling equilibrium with an atom of size at least $1-\delta$ at the top and with arbitrarily many solicited bidders. Equivalently, given $\bar{G}$, there exists a sequence $\left(s^k, \Delta^k\right) \to (0,0)$ and a sequence of equilibria of $\Gamma\left(s^k, P_{\Delta^k}, \bar{G}\right)$ such that $n_w^k \to \infty$ and for some $\bar{b} \le E[v]$

$$\lim F\left(\bar{b}+\delta|w;\beta^k,\eta^k\right) - F\left(\bar{b}-\delta|w;\beta^k,\eta^k\right) = 1 \text{ for all } \delta.$$

The distribution $\bar{G}$ is explicitly constructed using recursion. It will have countably many realizations of the likelihood ratio, extending the original construction of a pooling equilibrium sequence. In some sense, the distribution $\bar{G}$ is a "generic" signal in the class of signals with such a structure.

The Construction. We start with any boundedly informative signal distribution G that has constant densities (and, hence, likelihood ratios) on intervals $[0,\epsilon)$, $[\epsilon, 2\epsilon)$, ... $[\bar{x}-\epsilon, \bar{x}]$ and that strictly satisfies the assumptions for the existence of a pooling equilibrium sequence:

$$\frac{1}{G(\bar{x}-\epsilon|\ell)} < \frac{g(\bar{x}|h)}{g(\bar{x}|\ell)}, \tag{93}$$

and

$$\frac{\frac{g(\bar{x}-\epsilon|h)}{G(\bar{x}-\epsilon|h)}}{\frac{g(\bar{x}-\epsilon|\ell)}{G(\bar{x}-\epsilon|\ell)}} < \frac{g(\bar{x}|h)}{g(\bar{x}|\ell)}. \tag{94}$$

For given G and ε, let $B(G,\varepsilon)$ be the set of discrete signals $\hat{G}$ with support contained in $[0,\infty)$ (not necessarily full support) that satisfy the following conditions: (i)

$$\int_0^{+\infty} \left(\hat{g}_w(x) - g_w(x)\right)^2 dx \le \varepsilon,$$

(ii) $\hat{G}$ has constant densities on intervals (and, hence, likelihood ratios) on $[0,\epsilon)$, $[\epsilon, 2\epsilon)$, ... $[(n-1), n\epsilon)$,.... and (iii) $\hat{G}$ satisfies MLRP. The support of $\hat{G}$ and the likelihood ratio $\frac{\hat{g}_H(x)}{\hat{g}_L(x)}$ may be unbounded.

Remark: The construction ensures that whenever $\hat{G} \in B(G, \varepsilon)$, the densities $\hat{g}$ and g are uniformly close for $x \leq \bar{x}$ by the requirement that the densities are constant on the intervals. As a consequence, the likelihood ratios are uniformly close for $x \leq \bar{x}$ and small ε, which will be important for our proof.

The support of $\hat{G}$ may extend beyond $\bar{x}$, the upper bound of the support of G. To make the construction easier, we now consider symmetric and weakly increasing bidding strategies β that have the whole $\mathbb{R}_+$ as domain, that is, we consider bidding strategies that extend beyond the support of G. In addition, we extend the auxiliary game $\Gamma^A\left(s, P_\Delta, G|\underline{b}, \bar{b}\right)$ by allowing bidders with signals $x > \bar{x}$ to submit any bid, whereas bidders with signals $x \leq \bar{x}$ are restricted as before, that is

$$\begin{aligned} \beta(x) &= \bar{b} \text{ for } x \in [\bar{x} - \epsilon, \bar{x}] \\ \beta(x) &\leq \underline{b} \text{ for } x \leq \bar{x} - \epsilon. \end{aligned}$$

We also include G as a parameter of the auxiliary game Γ^A.

Lemma 20 *Suppose G satisfies conditions (93) and (94). For all $\delta > 0$, s', Δ', and ε', there are $0 < s \leq s'$, $0 < \Delta \leq \Delta'$, and $0 < \varepsilon \leq \varepsilon'$ such that for all $\hat{G} \in B(G, \varepsilon)$ there is an equilibrium of the original game $\Gamma\left(s, P_\Delta, \hat{G}\right)$ such that*

$$\beta(x) = \bar{b} \textit{ for all } x \in [\bar{x} - \epsilon, \bar{x}]$$

and

$$dF\left(\beta(\bar{x})|w; (\beta, \eta), \hat{G}\right) \geq 1 - \delta.$$

Proof:

Step 1: Given G, there exist $s \leq s'$, $\Delta \leq \Delta'$, $\underline{b}$, and $\bar{b}$ such that *all* equilibria of the auxiliary game $\Gamma^A\left(s, P_\Delta, G|\underline{b}, \bar{b}\right)$ are also equilibria of the original game (for that s and P_Δ) and the incentives not to violate the constraints are strict. Specifically, for some κ_1, for all equilibria (β, η) of $\Gamma^A\left(s, P_\Delta, G|\underline{b}, \bar{b}\right)$,

$$\begin{aligned} U\left(\bar{b}|\bar{x}, \beta, \eta; G\right) &> U(b|\bar{x}, \beta, \eta; G) + \kappa_1 \text{ for all } b \neq \bar{b} \\ U(b|x, \beta, \eta; G) &< -\kappa_1 \text{ for all } b > \underline{b} \text{ and } x \leq \bar{x} - \epsilon. \end{aligned}$$

and

$$dF(\beta(\bar{x})|w; (\beta, \eta), G) \geq 1 - \frac{1}{2}\delta.$$

The step follows from Lemma 18, which shows that there are numbers v_1^*—v_3^* that uniformly bound the various conditional expected values in the level of the solicitation costs.

We show that expected payoffs of the seller and the buyers (with signals $x \leq \bar{x}$) are 'uniformly continuous' in $\left(\beta, \eta, \hat{G}\right)$. A natural distance on strategies given G is

$$
\begin{aligned}
d\left(\beta_1, \beta_2\right) &= \int_0^{\infty} \left|\beta_1(x) - \beta_2(x)\right| dG(x|h), \\
d\left(\eta_1, \eta_2\right) &= \sum_{i=1}^{\infty} \left|\eta_1(i) - \eta_2(i)\right|.
\end{aligned}
$$

Restricting attention to $\beta(x) \in [0, v_h]$, $d(\beta_1, \beta_2) \leq v_h$ and $d(\eta_1, \eta_2) \leq 1$. Note that $d(\beta_1, \beta_2) = 0$ if they differ only on $x > \bar{x}$. For the step, recall that the number of potential bidders is bounded.

Step 2: Given G: For every $\kappa_2 > 0$ there is some $\varepsilon > 0$ such that $d(\beta_1, \beta_2) \leq \varepsilon$, $d(\eta_1, \eta_2) \leq \varepsilon$, and $\hat{G}_1, \hat{G}_2 \in B_\varepsilon(G, \varepsilon)$ imply that

$$
\begin{aligned}
\left|U\left(b|x; \beta_1, \eta_1, \hat{G}_1\right) - U\left(b|x, \beta_2, \eta_2, G_2\right)\right| &\leq \kappa_2 \text{ for all } b \text{ and } x \leq \bar{x}, \\
\left|E\left[p|w, n; \beta_1, \hat{G}_1\right] - E\left[p|w, n; \beta_2, \hat{G}_2\right]\right| &\leq \kappa_2 \text{ for all } n \text{ and } p.
\end{aligned}
$$

The step is immediate from the definitions, the fact that the posteriors of the buyers are uniformly continuous in ε for $x \leq \bar{x}$ and the fact that the total number of bidders is bounded by N (so that the probability $\left(1 - \hat{G}(\bar{x})\right)^N$ is arbitrarily close to one for ε small enough). The claim of the step does not need to hold for $x > \bar{x}$ because for these signals, interim beliefs are not necessarily close for ε small.

Step 3: Choose $s \leq s'$, $\Delta \leq \Delta'$, $\underline{b}$, and $\bar{b}$ such that for some $\kappa > 0$ the inequalities from Step 1 hold. For these parameter values, there exists an ε such that for every $\hat{G} \in B(G, \varepsilon)$ every equilibrium of the extended auxiliary game $\Gamma^A\left(s, P_\Delta, \hat{G}|\underline{b}, \bar{b}\right)$ is also an equilibrium of the original game $\Gamma\left(s, P_\Delta, \hat{G}\right)$.

By contradiction. Suppose not, then there is a sequence $\varepsilon^k \to 0$, a sequence $\hat{G}^k \in B\left(G, \varepsilon^k\right)$, and a sequence of equilibria $\left(\beta^k, \eta^k\right)$ of the auxiliary game $\Gamma^A\left(s, P_\Delta, \hat{G}^k|\underline{b}, \bar{b}\right)$ such that no $\left(\beta^k, \eta^k\right)$ is an equilibrium of the original game, that is, one of the constraints of Γ^A must be binding. Let $\left(\beta^0, \eta^0\right)$ be a limit point of the sequence $\left(\beta^k, \eta^k\right)$. From Step 2, $\left(\beta^0, \eta^0\right)$ is an equilibrium of $\Gamma^A\left(s, P_\Delta, \hat{G}|\underline{b}, \bar{b}\right)$. Given the κ_1 from Step 1, Step 2 and $\left(\beta^k, \eta^k\right) \to \left(\beta^0, \eta^0\right)$ imply that for $\kappa_2 \leq \frac{1}{3}\kappa_1$, there is some k' large enough such that

$$
\left|U\left(b|x; \beta^{k'}, \eta^{k'}, \hat{G}^{k'}\right) - U\left(b|x, \beta^0, \eta^0, G\right)\right| \leq \kappa_2 \text{ for all } b \text{ and } x \leq \bar{x}.
$$

Hence,

$$\begin{aligned}
U\left(\bar{b}|\bar{x},\beta^{k'},\eta^{k'},\hat{G}^{k'}\right) &> U\left(\bar{b}|\bar{x},\beta,\eta,\hat{G}\right)-\kappa_2 \\
&> U\left(b|\bar{x},\beta,\eta;G\right)+\kappa_1-\kappa_2 \\
&> U\left(b|\bar{x},\beta^{k'},\eta^{k'},\hat{G}^{k'}\right)+\kappa_1-2\kappa_2 \text{ for all } b\neq\bar{b}
\end{aligned}$$

and similarly

$$U\left(b|x,\beta^{k'},\eta^{k'},\hat{G}^{k'}\right) < -\kappa_1+2\kappa_2 \text{ for all } b>\underline{b} \text{ and } x\leq\bar{x}-\epsilon.$$

Since $\kappa_1-2\kappa_2>0$, this implies that $\left(\beta^{k'},\eta^{k'}\right)$ is an equilibrium of the original game $\Gamma\left(s,P_\Delta,G^{k'}\right)$, in contradiction to the hypothesis.

The lemma now follows from the fact that every auxiliary game $\Gamma^A\left(s,P_\Delta,\hat{G}|\underline{b},\bar{b}\right)$ has an equilibrium and that for ε small enough, $dF\left(\beta\left(\bar{x}\right)|w;\left(\beta,\eta\right),\hat{G}\right)$ can be made arbitrarily close to one. **QED.**

Inductive Construction of $\bar{G}$.

We now construct a signal distribution with an unbounded likelihood ratio as a limit of a sequence of signal distributions.

Fix some sequence $\delta_i\to0$, $\tilde{s}_i\to0$ and $\tilde{\Delta}_i\to0$. Fix some finite distribution G that satisfies conditions (93) and (94) (strictly) as before. We construct a sequence $\{G_i,s_i,\varepsilon_i,\Delta_i\}_{i=0}^{\infty}$

Let $G_0=G$ and take some $\varepsilon_0>0$.

Suppose G_{i-1} satisfies the conditions (93) and (94). Given any G_{i-1}, s_{i-1}, ε_{i-1}, Δ_{i-1} we construct G_i.

Pick $s_i\leq\min\{s_{i-1},\tilde{s}_i\}\,(=s')$, $\Delta_i\leq\Delta_{i-1}$, and $\varepsilon_i\leq\frac{1}{2}\varepsilon_{i-1}(=\varepsilon')$ such that Lemma 20 holds relative to G_{i-1}. Let $\bar{x}_{i-1}$ be the most favorable signal in the support of G_{i-1}. Choose $G_i\in B\left(G_{i-1},\varepsilon_i\right)$ such that $G_i\left(x|w\right)=G_{i-1}\left(x|w\right)$ for all $x\leq\bar{x}_{i-1}-\epsilon$, $\frac{\left(G_i(\bar{x}_{i-1}+\epsilon|H)-G_i(\bar{x}_{i-1}|H)\right)}{G_i(\bar{x}_{i-1}+\epsilon|L)-G_i(\bar{x}_{i-1}|L)}=2\frac{g_{i-1}(\bar{x}_{i-1}|H)}{g_{i-1}(\bar{x}_{i-1}|L)}$, and for $x\geq\bar{x}_{i-1}+\epsilon$, $G_i\left(x|w\right)=1$. [The Idea: G_i is equal to G_{i-1} for $x\leq\bar{x}_{i-1}-\epsilon$ and shifts some mass from $[x-\epsilon,\bar{x}]$ to the next interval $[\bar{x},\bar{x}+\epsilon]$ so that the likelihood ratio on the next interval is twice as large as the likelihood ratio according to G_{i-1} on the interval $[x-\epsilon,\bar{x}]$.]

Thus, we constructed have a sequence of finite distributions and parameters, $\{G_i,s_i,\varepsilon_i,\Delta_i\}_{i=0}^{\infty}$.

If ε_0 is chosen sufficiently small relative to G, all elements G_i of the constructed sequence of distributions satisfy the conditions (93) and (94) (because G satisfies the condition strictly).

Let $\bar{G}$ be the pointwise limit of the constructed sequence, $\bar{G}\left(x|w\right)=\lim_{i\to\infty}G_i\left(x|w\right)$. The limit is a distribution function because $\varepsilon_i\to0$. Moreover, the support of $\bar{G}$ and the likelihood ratio are both unbounded.

Claim: Given $\bar{G}$ and $\{\delta_i\}$, for the sequence $\{(s_i, \Delta_i)\}$ with $(s_i, \Delta_i) \to (0, 0)$ there is a sequence of equilibria $\{(\beta_i, \eta_i)\}$ and bids $b_i \le E[v]$ such that $dF\left(\bar{b}_i | w; \beta_i, \eta_i\right) \ge 1 - \delta_i$ for all $i \ge 1$.

Proof of the Claim: For each $i \ge 1$, $\bar{G}$ is within ε_i of G_{i-1} by construction, where ε_i, Δ_i, and s_i satisfy the conditions of Lemma 20 relative to G_{i-1} (recall that $\varepsilon_{i+1} \le \frac{1}{2}\varepsilon_i \le \frac{1}{4}\varepsilon_{i-1} \le ...$). Hence, there is an equilibrium (β_i, η_i) of $\Gamma\left(s_i, P_{\Delta_i}, \bar{G}\right)$ such that $dF\left(\bar{b}_i | w; \beta_i, \eta_i\right) \ge 1 - \delta_i$ for $b_i = \beta_i(\bar{x}_i)$. From $\delta_i \to 0$, the claim follows. **QED**

The Proposition 7 is implied by the claim.

E.3 Existence of Revealing Equilibrium Sequence

Proposition 9 *For every unboundedly informative signal with distribution $\bar{G}$ and for every $\delta > 0$, there are $(s', \Delta') >> 0$ such that for every $(s', \Delta') \ge (s, \Delta) > 0$ there exists an equilibrium (β, η) of $\Gamma\left(s, P_\Delta, \bar{G}\right)$ such that*

$$F(v_w + \delta | w; \beta, \eta) - F(v_w - \delta | w; \beta, \eta) \ge 1 - \delta \text{ for } w \in \{\ell, h\}.$$

Define the auxiliary game $\Gamma^C\left(s, P_\Delta, \bar{G} | b, \lambda\right)$ as follows. Given s and $b > E[v]$, let

$$\hat{n}(b, s) = \frac{1}{s}\frac{1}{4}(b - E[v]),$$

and

$$x_\lambda(b, s) : (G(x|h))^{\hat{n}(b,s)} = \lambda.$$

Then, in the auxiliary game, the bidding strategy is restricted to be

$$\beta(x) \ge b \text{ if } x \ge x_\lambda(b),$$

and otherwise the game is identical to the original game (there are no restrictions for bidders with $x < x_\lambda(b)$ and no restrictions on sellers relative to the original game).

Lemma 21 *For every sequence $\left\{\left(s^k, \Delta^k\right)\right\}$ with $\left(s^k, \Delta^k\right) >> 0$ but $\lim\left(s^k, \Delta^k\right) = 0$, there exists a sequence of equilibria $\left(\beta^k, \eta^k\right)$ of $\Gamma^C\left(s, P_\Delta, \bar{G} | b, \lambda\right)$ for all b, λ. If the distribution of the winning bid converges, then there exists some $\bar{b}$ such that*

$$\lim_{k \to \infty} F\left(\bar{b} + \delta | h; \left(\beta^k, \eta^k\right)\right) - F\left(\bar{b} - \delta | h; \left(\beta^k, \eta^k\right)\right) = 0 \text{ for all } \delta.$$

Proof: Existence of equilibrium is immediate given $\Delta > 0$. For the second part, note that the proof of Lemma 19 applies verbatim. **QED.**

Proof of the Proposition: Fix $\lambda = \frac{1}{2}$, any $b > E[v]$, and sequence $\left\{\left(s^k, \Delta^k\right)\right\}$ with $\left(s^k, \Delta^k\right) >> 0$ but $\lim\left(s^k, \Delta^k\right) = 0$. Take some sequence of equilibria of $\left(\beta^k, \eta^k\right)$ such that the distribution of the winning bid converges.

From Lemma 21, the equilibrium bid distribution of $\Gamma^C\left(s, P_\Delta, \bar{G}|b, \lambda\right)$ must become degenerate in the high state with an atom at $\bar{b}$.

The characterization from Proposition 7 applies to the equilibria of the auxiliary game as well. Thus, either $\bar{b} = v_h$ or $\bar{b} = E[v]$.

It cannot be that $\bar{b} = E[v]$. If $\bar{b} = E[v]$, then the seller having type h could increase her expected payoffs by sampling $\hat{n}\left(b, s^k\right)$ bidders: With probability $1-\lambda$ at least one bidder has a signal $x \geq x_\lambda(b, s)$ (by definition of $x_\lambda(b, s)$) who bids above b (by the restriction of the auxiliary game). Since the distribution of the winning bid becomes degenerate on $\bar{b} < b$ when sampling n_h^k bidders, it must be that $n_h^k < \hat{n}\left(b, s^k\right)$ for k sufficiently large. Hence, when sampling $\hat{n}\left(b, s^k\right)$, the expected winning bid is at least $(1-\lambda)\bar{b} + \lambda b$. From the definition of $\hat{n}\left(b, s^k\right)$ and $b > E[v]$, the limit of the expected payoff when soliciting $\hat{n}\left(b, s^k\right)$ bidders is at least

$$\begin{aligned} & (1-\lambda)\bar{b} + \lambda b - \lim s^k \hat{n}\left(b, s^k\right) - c \\ = & \left(1-\frac{1}{2}\right)\bar{b} + \frac{1}{2}b - \frac{1}{4}\left(b - E[v]\right) - c \\ > & \bar{b} - c \end{aligned}$$

a contradiction.

Thus, $\bar{b} = v_h$. As a consequence, sufficiently deep into the sequence, the restriction $\beta(x) \geq b$ if $x \geq x_\lambda(b)$ does not bind. First, since the distribution of the winning bid becomes degenerate, Lemma 3 implies that the total solicitation costs vanish to zero, $n_h^k s^k \to 0$. Therefore, $\frac{n_h^k}{\hat{n}\left(b, s^k\right)} \to 0$. Since $n_h^k << \hat{n}\left(b, s^k\right)$ sufficiently deep into the sequence, monotonicity and the fact that the winning bid is close to v_h requires that $\lim \beta^k\left(x_\lambda\left(b, s^k\right)\right) = v_h$. (If $\lim \beta^k\left(x_\lambda\left(b, s^k\right)\right) = b'$, then $\lim\left(G\left(x_\lambda\left(b, s^k\right)|h\right)\right)^{n_h^k} > \lambda = \frac{1}{2}$ implies that the winning bid is below b' with probability at least $1/2$. Hence, $b' = v_h$.)

Therefore, sufficiently deep into the sequence, the equilibria $\left(\beta^k, \eta^k\right)$ of the auxiliary game are also equilibria of the original game. Thus, there exists a sequence of equilibria of the original game for which the distribution of the winning bid becomes degenerate on v_h and v_ℓ, respectively. This finishes the proof of the proposition. **QED.**