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Skewed Information Transmission

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Abstract

This paper analyzes strategic information transition between skewed agents. More concretely, we study Crawford and Sobel's (1982) setting in which agents are not biased, but they differ on the relative importance they put on avoiding "upward" or "downward" mistakes. We show that even though agents can fully communicate when the state of the world is perfectly observed by the sender, their communication is significantly imprecise when there is an arbitrary small noise in the observation. Hence, contrary to what was previously thought, a small objective misalignment is not a sufficient condition for the existence of equilibria with precise information transmission. We illustrate the results through some applications.

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JEL Classifications: C72, D82, D83

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1 Introduction

Since the seminal work of Crawford and Sobel (1982) (henceforth CS) the cheap-talk literature has studied strategic information transmission in different economic environments. Typically, the main focus is on analyzing how the difference in the objectives of a sender and a receiver (i.e., the bias) affects their equilibrium communication. An important insight is that the equilibrium information transmission is coarse when the bias is large, whereas when the bias is small communication can be very precise (see, e.g., Spector, 2000).

This paper analyzes how the skewness of the preferences of the sender and the receiver, instead of the relative bias, shapes strategic communication. In contrast to the existing cheap-talk literature, we assume that the relative bias between the sender and the receiver is zero, and so their ideal actions coincide for each state of the world. Nevertheless, the sender and the receiver are assumed to be differently skewed: they differ in terms of the relative importance they put on avoiding "upward" or "downward" deviations from the optimal action.

Our main result establishes that even though the ideal actions of the sender and the receiver coincide for each state of the world, the communication between the sender and the receiver in *any* equilibrium is significantly coarse when the state is not perfectly observed, *regardless* of the precision of the observation. Our result contrasts with the conventional finding that a small conflict of interest permits precise equilibrium communication.

The paper begins with an analysis of a simple auxiliary model (see Section 2; the general case is studied in Section 3). As in the CS model, in the auxiliary model nature chooses a one-dimensional state of the world from a uniform distribution, a sender observes the state perfectly and sends a cheap talk message to a receiver, and, finally, the receiver takes an action. To fix ideas, consider the case where the receiver ("he") is "unskewed" (i.e., his payoff depends only on the size of the "mistake," that is, the difference between the state and the action), whereas the sender ("she") is "downward skewed" (i.e., for each given size of the mistake, she sustains a greater payoff loss when the mistake is negative than when it is positive).

We first characterize the set of partition equilibria of our auxiliary model, that is, equilibria where each message signifies an interval of the state space. In a partition equilibrium, there is an endogenous bias: if the state is known to be in some partition element, the receiver prefers his action to match the middle point of the partition element, while the sender prefers a higher action in order to avoid downward mistakes. The size of such endogenous bias is proportional to the size of the partition element. This is shown to imply that the size of the partition elements grows exponentially—in contrast of their linear growth in the standard case with quadratic preferences. Consequently, even though there are partition equilibria with an arbitrarily large number of partition elements, there is a uniform lower bound on the size of the biggest (and rightmost) partition element *across all* partition equilibria. However, our main result does not apply in our auxiliary model: there are equilibria with perfect information transmission.

Section 4 extends our auxiliary model by adding a small noise to the sender's observation of the state of the world. The sender and the receiver still agree on the ideal action for each realization of the state. However, since the state is observed imperfectly, there is a small disagreement between them on the ideal action to be taken after each given signal. Standard arguments apply: in the model with noise, *any* equilibrium is equivalent to a partition equilibrium. This is shown to imply that, *regardless* of how small the noise is, the information transmission is significantly coarse in *all* equilibria. In Section 5.1 we show that this result not only applies to the case where the state is imprecisely observed by the sender, but also when there is an arbitrarily small (exogenous) bias between the sender and the receiver (in addition to their different skewness). We further argue that the presence of any arbitrarily small learning costs leads to the same result; that is, it implies that the information transmission is significantly coarse.

The paper concludes with a discussion of different applications where skewed preferences arise naturally (Section 5.2). These applications illustrate how, in some situations, small objective misalignments may lead to significant welfare losses. The first application considers communication between two managers of different divisions of a firm. Both aim at maximizing the profits but, depending on their division, have different preferences with respect to under- or over-producing. Our results establish that, in this case, their communication is likely to be coarse. Hence, it may be optimal for firms not only to focus on reducing bias between managers, but also to minimize their relative skewness. The second application examines communication between politicians in a government. They agree on the (state-dependent) target spending level (or some other policy), but disagree on the political cost of missing the target (because of their different political biases or the preferences of their electorate). We find that prime ministers may prefer, when designing a cabinet, to include politicians with different ideal policies but similar skewness, rather than politicians with the same ideal policies but different skewness.

2 An Illustrative Example

We begin considering a simple auxiliary model (generalized in Section 3). Even though the model does not lead to the main result—it has equilibria with full information transmission—its analysis illustrates in a clear manner the drivers behind coarse information transmission. Sections 4 and 5.1 show that small variations of our auxiliary model lead to the result that *all* equilibria exhibit significantly imprecise information transmission.

Our auxiliary model is a version of the CS model. There is a sender (the s-agent), and a receiver (the r-agent). First, nature draws a state of the world t using a uniform distribution on $[0,1]$. After observing the state, the sender chooses a message m from a set $M \supset [0,1]$ (we assume that the message set is big enough for full communication to be possible). The receiver, after observing m but not t, decides on an action $a \in [0,1]$. If the realized state is t, the message sent is m, and the action taken is a, then the payoff of the θ-agent, for $\theta \in \{s,r\}$, is

$$u^{\theta}(t,a) \equiv \begin{cases} -(t-a) & \text{if } a < t, \\ -k^{\theta}(a-t) & \text{if } a \geq t, \end{cases} \tag{2.1}$$

where $k^r, k^s \in \mathbb{R}_{++}$ and $k^s \neq k^r$. Note that, for each state of the world $t \in [0,1]$, both the sender's payoff and the receiver's payoff are maximized when $a = t$ (i.e., their relative bias is zero). If, however, there is a mistake (i.e., $a \neq t$), their respective payoff loss depends on its size and direction.

A strategy of the sender is a map $\mu : [0,1] \to \Delta(M)$, and a strategy of the receiver is a map $\alpha : M \to \Delta([0,1])$. As in CS, we are looking for Bayes–Nash equilibria in this game.

For each $\theta \in \{s,r\}$, the *skewness parameter* k^{θ} is a measure of the skewness of the θ-agent's payoff function. The skewness parameter measures the loss of making an upward mistake relative to the loss of making a downward mistake of the same size. (Note that we have normalized the downward slope to 1 without loss of generality.) For example, the receiver is not skewed if $k^r = 1$, since his payoff loss from making an upward mistake is the same as it would be from making a downward mistake of the same size. Similarly, if $k^s = \frac{1}{2}$ then the sender's payoff loss from an upward mistake is half of what it would be from a downward mistake of the same size. We use $\kappa \equiv \frac{k^s}{k^r}$ to denote the *relative skewness* of the sender with respect to the receiver. Also, without loss of generality, we assume that $\kappa < 1$; that is, the receiver is assumed to have a stronger preference than the sender for avoiding upward mistakes relative to downward mistakes (in the opposite case one can just reverse the state space).

Finite equilibria. We begin our analysis by focusing on equilibria where a finite number of messages are used. It is easy to see that in our setting, similar to the CS model, each equilibrium in which a finite number of messages are used is essentially equivalent to a partition equilibrium, i.e., an equilibrium in pure strategies where each message used with positive probability signifies an interval of the state space. (See Section 3 and, in particular, footnote 1 for the formal condition for the equivalence.)

Assume that there is a partition equilibrium with exactly N partition elements, for some fixed $N \in \mathbb{N}$. We use $\{t_n\}_{n=0}^{N}$ to denote the boundaries of the partition elements, where $0 = t_0 < t_1 < \ldots < t_N = 1$. For each $n = 1, \ldots, N$, we denote by a_n the equilibrium action when the state is between t_{n-1} and t_n. Such an action satisfies

$$a_n \in \arg\max_a \int_{t_{n-1}}^{t_n} u^r(t,a)\,\mathrm{d}t \quad \Rightarrow \quad a_n = \tfrac{k^r}{1+k^r}\, t_{n-1} + \tfrac{1}{1+k^r}\, t_n \,. \tag{2.2}$$

Note that if, for example, $k^r > 1$, the receiver takes an action lower than the middle of the partition element to avoid an upward mistake. If $k^r = 1$, the action coincides with the state in the middle of the partition element.

In equilibrium, when the realized state is t_n, for $0 < n < N$, the sender is indifferent between inducing actions a_n and a_{n+1}. This implies that

$$-(t_n - a_n) = -k^s\,(a_{n+1} - t_n)\,. \tag{2.3}$$

If, for example, $k^s < 1$, the sender has a stronger incentive to avoid downward mistakes than to avoid upward mistakes, so she is indifferent between the two messages only if the action a_n is closer to t_n than the action a_{n+1} is.

We can then use equations (2.2) and (2.3) to compare the size of two adjacent partition elements:

$$t_{n+1} - t_n = \frac{t_n - t_{n-1}}{\kappa}\,. \tag{2.4}$$

Since $\kappa < 1$, communication is coarser for higher states. The size of the partition elements grows exponentially: for each two consecutive partition elements, the one to the right is $\kappa^{-1} > 0$ times bigger than the one to the left. Letting $t_0 = 0$ and $t_N = 1$, and by equation (2.4), we get

$$t_n = \frac{\kappa^{N-n}\,(1-\kappa^n)}{1-\kappa^N} \quad \text{for all } n = 0, \ldots, N. \tag{2.5}$$

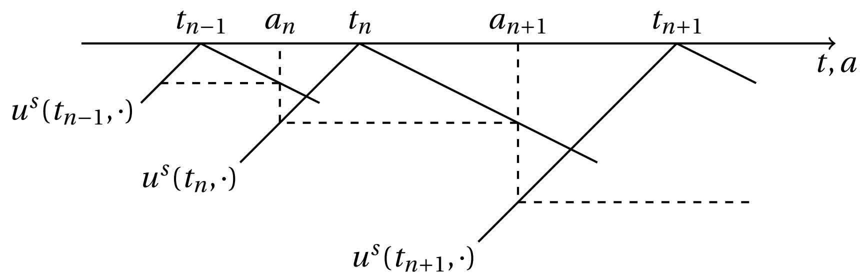

Figure 1: $u^s(t_{n-1}, \cdot)$, $u^s(t_n, \cdot)$ and $u^s(t_{n+1}, \cdot)$ for some equilibrium thresholds, in the case $k^s = \frac{1}{2}$ (sender's is negatively skewed) and $k^r = 1$ (receiver is not skewed), so $\kappa = \frac{1}{2}$. Notice that the $a_n = \frac{1}{2}(t_{n-1} + t_n)$ and $a_{n+1} = \frac{1}{2}(t_n + t_{n+1})$, and it is the case that $u^s(t_n, a_n) = u^s(t_n, a_{n+1})$.

Hence, the following claim holds:

Claim 2.1. *For each $N \in \mathbb{N}$ there is a partition equilibrium with N partitions.*

Figure 1 illustrates the previous arguments. In the picture, the receiver is not skewed, and so $k^r = 1$ and $a_n = \frac{1}{2}(t_{n-1} + t_n)$ for all n. The sender is downward skewed, $k^s = \frac{1}{2}$, and so her payoff loss from a downward mistake is twice as big as her payoff loss from an upward mistake of the same size. Hence, the sender is indifferent between two actions a_n and a_{n+1} only if t_n is at half the distance from a_n than from a_{n+1} (note that $\kappa = \frac{1}{2}$ in this example). As a result, the size of the higher interval is twice the size of the lower one.

Infinite equilibria. Assume now that there is a partition equilibrium with an infinite number of partition elements, each of them of positive measure (sometimes called an "infinite equilibrium"; see Gordon, 2010). Equation (2.4) applies for every pair of adjacent partition elements. In fact, it is not difficult to see that the boundaries of the partition elements of the unique infinite equilibrium, denoted with some abuse of notation $\{t_{\infty-n}\}_{n=0}^{\infty}$, satisfy

$$t_{\infty-n} = \lim_{N\to\infty} \frac{\kappa^{N-(N-n)}(1-\kappa^{N-n})}{1-\kappa^N} = \kappa^n \quad \text{for all } n \in \mathbb{Z}_+. \tag{2.6}$$

Additionally to partition equilibria with a discrete (finite or countably infinite) partition of the state space, referred to as "discrete equilibria," there exist equilibria with precise information transmission. In particular, there is a fully informative equilibrium, where $\mu(t) = t = \alpha(t)$ for all $t \in [0,1]$. In Section 5.1 we argue that non-discrete equilibria are not robust to adding a small noise to the observation of the state by the sender, as well as to the presence of a small bias between the sender and the receiver, and to the presence of small direct communication costs.

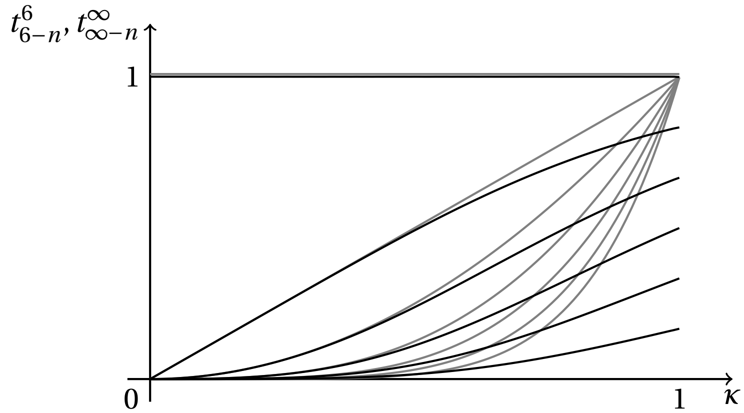

Figure 2: Black lines depict the values $\{t_{6-n}\}_{n=0}^{6}$ when $N = 6$, where high lines indicate low values of n ($t_{6-0} = 1$ and $t_{6-6} = 0$). Similarly, gray lines depict the values $\{t_{\infty-n}\}_{n=0}^{6}$ for the infinite equilibrium, where again high lines indicate low values of n.

2.1 Comparative Statics

We continue by providing some comparative statics results for finite equilibria. Let U_N^θ be the payoff of the θ-agent in the partition equilibrium with $N \in \mathbb{N}$ messages, for $\theta \in \{s, r\}$. Simple algebra shows that each agent's payoff increases when the other agent's skewness parameter becomes closer to hers/his. Also, as N increases, the payoff of both the sender and the receiver increases as well. Nevertheless, their payoffs are bounded above away from 0: $\lim_{N\to\infty} U_N^\theta = U_\infty^\theta < 0$ for both $\theta \in \{s, r\}$. Finally, for each $\kappa < 1$, the size of the rightmost partition element in any discrete equilibrium is significantly large (see Figure 2), regardless of how big N is. The following claim formalizes these results:

Claim 2.2. *U_N^θ is strictly increasing in N, for both $\theta \in \{s, r\}$. The size of the rightmost partition element is $\frac{1-\kappa}{1-\kappa^N} \geq 1-\kappa > 0$; that is, communication does not become arbitrarily precise as N increases.*

Claim 2.2 establishes that the information transmission is significantly coarse in discrete equilibria. Indeed, independently of the number of partition elements, the size of the rightmost interval is larger than $1-\kappa$. As we will see, this is going to set a limit on the information precision in *any* equilibrium when the state is imperfectly observed.

Pareto outcomes: We now compare the equilibrium outcomes of our model to constrained-efficient outcomes. To do so, we fix the number of available cheap-talk messages at some $N \in \mathbb{N}$. Then, we define a *constrained-Pareto outcome* (constrained to the usage of N

messages) as a pair of maps $\mu^N : [0,1] \to \{1,\dots N\}$ and $\alpha^N : \{1,\dots,N\} \to [0,1]$ that maximize

$$\mathbb{E}_t\big[\beta\, u^s(t,\alpha^N(\mu^N(t))) + (1-\beta)\, u^r(t,\alpha^N(\mu^N(t)))\big]$$

for some $\beta \in [0,1]$.

It is easy to see that any constrained-Pareto outcome divides the state space into N equally sized intervals (of size $\frac{1}{N}$) and assigns the same action to all states of each element of the partition. Hence, in contrast to standard cheap-talk models, the sender and the receiver in our setting "agree" on the optimal communication strategy, but they disagree on the action that should be taken after each message. In fact, the action that a constrained-Pareto outcome assigns to the states of the n-th interval coincides with the action that an agent with skewness parameter $\frac{k^\beta}{1+k^\beta} = \beta\,\frac{k^s}{1+k^s} + (1-\beta)\,\frac{k^r}{1+k^r}$ prefers in this interval. This action corresponds to the right-hand side of expression (2.2), except that k^β replaces k^r and $\frac{n-1}{N}$ and $\frac{n}{N}$ replace t_{n-1} and t_n, respectively. This gives to the θ-agent a payoff of

$$-\frac{\frac{k^\theta}{(1+k^\theta)^2} + \left(\frac{k^\beta}{1+k^\beta} - \frac{k^\theta}{1+k^\theta}\right)^2}{2N},$$

for both $\theta \in \{s,r\}$. As one might expect, the information transmission is precise in a constrained-Pareto outcome with a large number of messages N. Thus, for any sequence of constrained-Pareto outcomes with $N \to \infty$, the sender's and receiver's payoffs increase toward 0. This contrasts with our results that the equilibrium payoffs remain bounded away from zero as N increases.

3 General Case

We now generalize the previous result that the information transmission in equilibria where a discrete set of messages is used is significantly coarse when the agents are differently skewed. This result will illustrate how a slight relaxation of the standard assumption that payoff functions are twice-differentiable (in the CS model) changes the equilibrium predictions. To show this, we present first a general version of the auxiliary model introduced in Section 2.

As before, there are a sender and a receiver. First, nature draws a state of the world t, now using a distribution with a continuous density f in $[0,1]$ with full support. The sender chooses a message m from a set M containing $[0,1]$. The receiver, after observing m but not t, decides on an action $a \in [0,1]$. If the realized state is t, the message sent is m, and the action taken is a, then the payoff of the θ-agent, for each $\theta \in \{s,r\}$, takes the form

$$u^{\theta}(t,a) \equiv \begin{cases} u^{\theta-}(t,a) & \text{if } a<t, \\ -u^{\theta+}(t,a) & \text{if } a \geq t, \end{cases} \tag{3.1}$$

where $u^{\theta-}(t,\cdot)$ and $u^{\theta+}(t,\cdot)$ are strictly increasing and continuously differentiable functions for all $t \in [0,1]$, and $u^{\theta}(t,\cdot)$ is single-peaked at t. Assume also that, for all $a_1, a_2, t_1, t_2 \in [0,1]$ with $a_1 < a_2$ and $t_1 < t_2$, we have[1]

$$u^{s}(t_1,a_1) \leq u^{s}(t_1,a_2) \;\Rightarrow\; u^{s}(t_2,a_1) < u^{s}(t_2,a_2)\,,$$
$$u^{s}(t_2,a_1) \geq u^{s}(t_2,a_2) \;\Rightarrow\; u^{s}(t_1,a_1) > u^{s}(t_1,a_2)\,.$$

The following proposition establishes that, as long as the agents are differently skewed at some state of the world, there is a bound on how precise the equilibrium communication can be in any equilibrium with a finite number of used messages.

Proposition 3.1. *Assume that there exists some state* $\bar{t} \in (0,1)$ *such that* $u_2^{s-}(\bar{t},\bar{t}^-)\,u_2^{r+}(\bar{t},\bar{t}^+) \neq u_2^{s+}(\bar{t},\bar{t}^+)\,u_2^{r-}(\bar{t},\bar{t}^-)$. *Then, there exists some* $\Delta > 0$ *such that, for all discrete equilibria, the minimum interval length is greater than* Δ.

Proof. Assume that $\bar{t} \in (0,1)$ is such that $u_2^{s-}(\bar{t},\bar{t}^-)\,u_2^{r+}(\bar{t},\bar{t}^+) \neq u_2^{s+}(\bar{t},\bar{t}^+)\,u_2^{r-}(\bar{t},\bar{t}^-)$ and, without loss of generality (due to the full-support assumption), that $f(\bar{t}) > 0$. Assume also, for the sake of contradiction, that there is a strictly increasing sequence $(N_i)_{i=1}^{\infty}$ and a corresponding sequence of partition equilibria where the maximum length of a partition element, denoted by Δ_i, is such that $\Delta_i \to 0$. With some abuse of notation, let $[t^i_{n_i}, t^i_{n_i+1}]$ denote the partition element containing $\bar{t}$ in the i-th equilibrium.[2]

By continuity of the derivatives of the payoff functions there is some continuous, strictly increasing function $\delta(\cdot)$ with $\delta(0)=0$ such that, for each $\varepsilon > 0$,

$$\left|u_2^{\theta x}(t,a) - u_2^{\theta x}(\bar{t},\bar{t})\right| < \varepsilon \;\text{ and }\; |f(t) - f(\bar{t})| < \varepsilon$$

for all $x \in \{-,+\}$, $\theta \in \{s,r\}$, and $t,a \in [\bar{t}-\delta(\varepsilon), \bar{t}+\delta(\varepsilon)]$. Take a sequence $(\varepsilon_i)_i$ strictly decreasing towards 0 such that $\Delta_i < \delta(\varepsilon_i)$ for all i. Assume, without loss of generality, that

[1] This condition generalizes the standard condition $\frac{\partial^2}{\partial t \partial a} u^s(t,a) > 0$ to ensure that partition elements in partition equilibria are intervals.

[2] With some abuse of notation, we use closed intervals to denote the (interval) partition elements of the partition equilibrium (notice that partition elements are defined up to sets of measure zero).

$$k^s \equiv \frac{u_2^{s+}(\bar{t},\bar{t})}{u_2^{s-}(\bar{t},\bar{t})} < \frac{u_2^{r+}(\bar{t},\bar{t})}{u_2^{r-}(\bar{t},\bar{t})} \equiv k^r \ . \tag{3.2}$$

This assumption is equivalent to the assumption that $\kappa \equiv \frac{k^s}{k^r} < 1$ in our example in Section[3] 2. Hence, the incentive-compatibility condition for the receiver can be written as[4]

$$\begin{aligned} a^i_{n_i} &= \tfrac{k^r}{1+k^r}\, t^i_{n_i-1} + \tfrac{1}{1+k^r}\, t^i_{n_i} + o\big(\varepsilon_i\,(t^i_{n_i} - t^i_{n_i-1}) + (t^i_{n_i} - t^i_{n_i-1})^2\big) \text{ and} \\ a^i_{n_i+1} &= \tfrac{k^r}{1+k^r}\, t^i_{n_i} + \tfrac{1}{1+k^r}\, t^i_{n_i+1} + o\big(\varepsilon_i\,(t^i_{n_i+1} - t^i_{n_i}) + (t^i_{n_i+1} - t^i_{n_i})^2\big) \end{aligned}$$

as $i \to \infty$. Intuitively, the small size of the partition elements implies that the distribution of the states within them is approximately constant, and that the marginal payoff loss from upward and downward mistakes is approximately constant. Hence, equation (2.2) approximately holds. Similarly, the indifference condition of the receiver is now given by

$$-(t^i_{n_i} - a^i_{n_i}) = -k^s\,(a^i_{n_i+1} - t^i_{n_i}) + o\big(\varepsilon_i\,(t^i_{n_i+1} - t^i_{n_i}) + (t^i_{n_i+1} - t^i_{n_i})^2\big)$$

as $i \to \infty$. Hence, we have that

$$t^i_{n_i} - t^i_{n_i-1} = \kappa\left(t^i_{n_i+1} - t^i_{n_i} + o\big(\varepsilon_i\,(t^i_{n_i+1} - t^i_{n_i}) + (t^i_{n_i+1} - t^i_{n_i})^2\big)\right) < \tilde{\kappa}\,(t^i_{n_i+1} - t^i_{n_i}) \ ,$$

where $\tilde{\kappa} \equiv \frac{\kappa+1}{2} \in (\kappa, 1)$. Note that the size of the intervals decreases at least exponentially toward the left (with coefficient $\hat{\kappa}$); that is, lower intervals are smaller. Hence,

$$t^i_n \geq t^i_{n_i} - \frac{1}{1-\tilde{\kappa}}(t^i_{n_i} - t^i_{n_i-1}) \geq \bar{t} - \frac{1}{1-\tilde{\kappa}}\,\Delta_i$$

for all $n \in \{0,\ldots,n_i\}$. Since $\bar{t} > 0$, there exists a value of i such that $\bar{t} - \frac{1}{1-\tilde{\kappa}}\Delta_i > 0$. This is a contradiction since $t^i_0 = 0$ in any equilibrium.

Consider now the case where the sequence of partition equilibria contains discrete equilibria (not just finite equilibria). The previous argument implies again that, if i is high enough, the set of thresholds of the partition elements of the i-th equilibrium has a limit point $\bar{t}'_i$ in $(\bar{t} - \frac{1}{1-\tilde{\kappa}}\Delta_i, \bar{t})$ (i.e., there is a strictly decreasing sequence of thresholds converging to $\bar{t}'_i$). There are two possibilities. Assume first that there is some $\underline{t}'_i$ such that $[\underline{t}'_i, \bar{t}'_i]$ is

[3]Since $u_2^{s-}(\bar{t},\bar{t}^-)\,u_2^{r+}(\bar{t},\bar{t}^+) \neq u_2^{s+}(\bar{t},\bar{t}^+)\,u_2^{r-}(\bar{t},\bar{t}^-)$ and $u^{\theta-}(\bar{t},\cdot)$ and $u^{\theta+}(\bar{t},\cdot)$ are strictly increasing, we have that the denominators of equation (3.2) are strictly positive. The proof applies to the case $u_2^{s+}(\bar{t},\bar{t}) = 0$ or $u_2^{r-}(\bar{t},\bar{t}) = 0$ (or both), where $\kappa = 0$.

[4]The "error terms" appear because the derivatives u_2^{s-} and u_2^{s+}, as well as the density function, are not constant in the interval. The equation is then the standard Taylor approximation, where terms of order $\varepsilon_i\,(t^i_{n_i} - t^i_{n_i-1})$, order $(t^i_{n_i} - t^i_{n_i-1})^2$, and higher orders are omitted.

a partition element. Then, when the realized state is $\bar{t}'_i$, the sender can induce the receiver to take an action arbitrarily close to $\bar{t}'_i$ (by reporting a partition interval slightly to the right of $\bar{t}'_i$), while when she reports that the state is in $[\underline{t}'_i, \bar{t}'_i]$ the action of the receiver is bounded away from $\bar{t}'_i$. Since the sender strictly prefers the closer action, the equilibrium indifference condition of the sender is violated, a contradiction. Assume then that, instead, there is a strictly increasing sequence of thresholds converging to $\bar{t}'_i$ (so the size of the corresponding partition elements decreases toward the right). This is a contradiction since, if i is big enough, the inequality in equation (3.2) holds at $\bar{t}'_i$, and so the size of small partition elements increases, and not decreases, toward the right. □

Proposition 3.1 contrasts with the result in the standard CS model that a small bias permits precise communication when the number of messages is large (but finite). For instance, Ottaviani (2000), Dessein (2002), and Dilmé (2018b) consider the case where $u^s(t,a) = u^r(t, a + \varepsilon\, b(t))$ for some fixed function b. They obtain that for each $\Delta > 0$ there are equilibria with a finite number of messages where the size of the biggest partition element is lower than Δ if $\varepsilon > 0$ small enough.[5] The crucial assumption that drives this result is that, in these models, the limiting payoff functions are twice-differentiable, and so they are locally symmetric around the ideal action (Dessein allows for kinks, but considers only symmetric payoff functions). As a result, the growth of the size of small intervals is locally linear (globally linear in the uniform-quadratic case). Such a linear growth implies that even though all equilibria contain partition elements which are much bigger, in relative terms, than the size of the (possibly state-dependent) bias[6] $\varepsilon\, b$, there are equilibria in which the biggest partition element is smaller than any fixed $\Delta > 0$ if ε is small enough.

The same argument cannot be used when the sender and the receiver are differently skewed. In small intervals, where the distribution of states is approximately uniform and the payoff functions are approximately piecewise linear, equation (2.4) approximately holds. The difference in the skewness of the sender's and the receiver's preferences implies that, in relative terms, either the equilibrium action is not close to the middle of the partition

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[5] Spector (2000) obtains the same result assuming that $u^s(t,a) = u^r(t,a) + \varepsilon\, v(t,a)$, for some fixed function v.

[6] Ottaviani (2000), Dessein (2002) and Dilmé (2018b) have shown, with different levels of generality, that when the bias is small (i.e., when ε is small) the typical size of a partition element in an equilibrium with a maximal number of partition elements is much bigger than the bias in relative terms (of order $O(\varepsilon^{1/2})$). In particular, Dilmé allows for asymmetric payoff functions, and shows that the third derivative of the payoff functions is relevant for determining the coarseness of communication in different regions of the state space, but does not prevent the existence of equilibria with very precise information transmission.

element (by equation (2.2)) or the actions are not at a similar distance from the common boundary of neighboring partition elements (by equation (2.3)). This implies that the size of small partition elements changes approximately exponentially. The proof of Proposition 3.1 shows the thresholds of small partition elements have an accumulation point (in the direction where the size of the partition elements decreases). The indifference condition of the sender at such an accumulation point implies that it cannot be interior. It then follows that the size of the biggest partition element is uniformly bounded away from zero across all finite equilibria.

4 Noisy Observation of the State

This section illustrates the effect of adding a small noise onto the sender's observation of the state. As we will see, the noise generates a small conflict of interest between the sender and the receiver for each signal realization. Consequently, only equilibria that are essentially equivalent to partition equilibria will exist. We will conclude the section by showing that *all* equilibria feature a significantly coarse information transmission, *regardless* of how precise the observation of the state is.

Consider the following extension of the auxiliary model in Section 2, referred to as the "model with noise". As in the auxiliary model, nature initially chooses a state of the world $t \in [0,1]$ using a uniform distribution. But now, instead of observing the state of the world directly, the sender observes a noisy signal of it, $\sigma \in \mathbb{R}$. For simplicity, we assume that the signal is uniformly distributed in $[t-\varepsilon, t+\varepsilon]$ for some small $\varepsilon > 0$. Then, as in our auxiliary model, the sender sends a message m, and the receiver takes some action a.

For a given state $t \in [0,1]$, the preferences of the sender and the receiver are the same as in our auxiliary model defined in equation (2.1). As before, we assume without loss of generality that $k^s < k^r$, and therefore it is still the case that $\kappa < 1$. To make our arguments clear, we will focus on the case where the receiver is not skewed, i.e., $k^r = 1$. Now, for a signal σ, agents evaluate their expected payoffs using their beliefs about the state of the world. If, for example, the signal σ belongs to $[\varepsilon, 1-\varepsilon]$, the posterior belief about the state of the world is uniform in $[\sigma-\varepsilon, \sigma+\varepsilon]$. In this case, the expected payoff for the θ-agent when action a is taken, denoted by $\hat{u}^\theta_\varepsilon(\sigma, a) \equiv \mathbb{E}_t[u^\theta(t,a)|\sigma]$, is

$$\hat{u}^\theta_\varepsilon(\sigma,a) = \begin{cases} -\frac{1}{4\varepsilon}\left(\sigma + \frac{1-k^\theta}{1+k^\theta}\varepsilon - a\right)^2 - \frac{k^\theta}{(1+k^\theta)^2}\varepsilon & \text{if } a \in [\sigma-\varepsilon, \sigma+\varepsilon], \\ u^\theta(\sigma, a) & \text{if } a \notin [\sigma-\varepsilon, \sigma+\varepsilon], \end{cases} \tag{4.1}$$

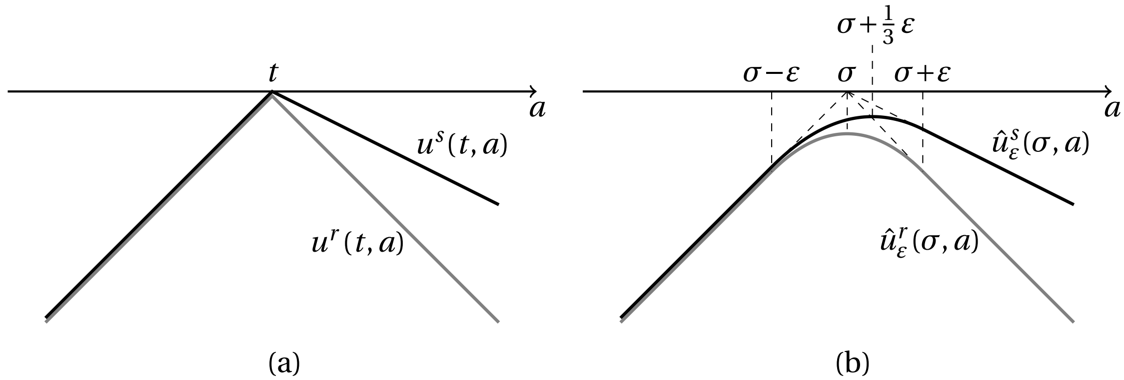

Figure 3: Payoff functions in (a) the state space and in (b) the signal space, for some $t \in [0,1]$ and $\sigma \in [\varepsilon, 1-\varepsilon]$. As in Figure 1, $k^s = \frac{1}{2}$ (the sender is downward skewed) and $k^r = 1$ (the receiver is not skewed), so $\kappa = \frac{1}{2}$. While the ideal actions of the agents coincide for each state t, the ideal action of the sender after signal σ is $\sigma + \frac{1}{3}\varepsilon$, while the ideal action of the receiver is σ.

for each $\theta \in \{s, r\}$. When $\sigma \in (-\varepsilon, \varepsilon) \cup (1-\varepsilon, 1+\varepsilon)$, the posterior belief distribution about the state is not uniform, but expressions similar to (4.1) can be obtained.

For each $\varepsilon > 0$ and $\theta \in \{s, r\}$, there are two important differences between the payoff function in the state space, u^θ, and the induced payoff function in the signal space, $\hat{u}^\theta_\varepsilon$. First, the presence of noise in the observation makes $\hat{u}^\theta_\varepsilon$ smooth (Figure 3). Hence, $\hat{u}^\theta_\varepsilon$ locally satisfies the smoothness conditions required in the CS model. Second, the ideal actions of the sender and the receiver after each signal are different.[7] If, for example, σ belongs to $[\varepsilon, 1-\varepsilon]$, since the receiver is not skewed (i.e., $k^r = 1$), his ideal action is $\sigma + \frac{1-k^r}{1+k^r}\varepsilon = \sigma$. The sender's ideal action is $\sigma + \frac{1-k^s}{1+k^s}\varepsilon > \sigma$; she prefers a higher action to avoid downward mistakes. (These expressions can also be obtained from equation (2.2) with $t_{n-1} = \sigma - \varepsilon$ and $t_n = \sigma + \varepsilon$.) Note that $\hat{u}^\theta_\varepsilon$ tends point-wise to u^θ as $\varepsilon \to 0$.

The difference between the ideal actions of the sender and the receiver for each given signal is $\left(\frac{1-k^s}{1+k^s} - \frac{1-k^r}{1+k^r}\right)\varepsilon > 0$ (when the signal is in $[\varepsilon, 1-\varepsilon]$; otherwise similar expressions can be obtained). Therefore, for small ε, the objectives of the sender and the receiver are different, but the difference between them is small. This implies, due to an argument analogous to that in Crawford and Sobel (1982), that there is no equilibrium and a set $[\sigma_-, \sigma_+]$, with $\sigma_- < \sigma_+$, such that signals in $[\sigma_-, \sigma_+]$ are communicated perfectly. In fact, as in the CS model, here each equilibrium is essentially equivalent to some discrete (partition) equilibrium.

[7]Formally, for each $\varepsilon > 0$ and $\sigma \in (-\varepsilon, 1+\varepsilon)$, the functions $\hat{u}^s_\varepsilon(\sigma, \cdot)$ and $\hat{u}^r_\varepsilon(\sigma, \cdot)$ have different maximizers.

To establish the "proximity" between the sets of partition equilibria in the model with noise (for small ε) and the auxiliary model (in Section 2), we first define the distance between the equilibria of the two models. We define the *distance* between a discrete equilibrium of the auxiliary model, $\{t_n\}_{n=0}^{N}$, and a discrete equilibrium of the model with noise, $\{\sigma_{\hat{n}}\}_{\hat{n}=0}^{\hat{N}}$, as

$$D\big(\{t_n\}_{n=0}^{N}, \{\sigma_{\hat{n}}\}_{\hat{n}=0}^{\hat{N}}\big) \equiv \sup_{n} \inf_{\hat{n}} \left| t_n - \sigma_{\hat{n}} \right| + \sup_{\hat{n}} \inf_{n} \left| t_n - \sigma_{\hat{n}} \right| .$$

The following proposition establishes that, if ε is small enough, the set of discrete equilibria in the auxiliary model and the set of equilibria in the model with noise are close:

Proposition 4.1. *Fix any $\Delta > 0$. There exists some $\bar{\varepsilon} > 0$ such that, if $\varepsilon \in (0, \bar{\varepsilon})$, the following holds:*

1. *For any equilibrium of the model with noise, there is a discrete equilibrium of the auxiliary model at a distance less than Δ from it.*
2. *For any discrete equilibrium of the auxiliary model, there is an equilibrium of the model with noise at a distance less than Δ from it.*

Proof. We prove the two parts of the statement separately:

1. In the model with noise, the induced bias (in the signal space) between the sender and the receiver is strictly positive in $(-\varepsilon, 1+\varepsilon)$. Gordon (2010) shows that, in such case, the set of thresholds of the partition elements in an infinite equilibrium (if it exists) has an accumulation point only at the lower end of the signal space, i.e., at $-\varepsilon$. It is then convenient, in this proof, to use $\{\sigma_n\}_{n=-N}^{0}$ to denote the thresholds of partition elements in a partition equilibrium. As before, $N \in \mathbb{N} \cup \{+\infty\}$ is the number of partition elements in the equilibrium (note that $\sigma_0 = 1$ and $\lim_{n \to -N} \sigma_n = 0$).

   We divide the proof of the first statement into 5 steps:

   Step 1: Some preliminary results. Fix a partition equilibrium of the model with noise, defined by the thresholds $\{\sigma_n\}_{n=-N}^{0}$. Fix also two consecutive partition elements denoted by $[\sigma_{n-1}, \sigma_n]$ and $[\sigma_n, \sigma_{n+1}]$. The sender's indifference condition can now be written as

$$\hat{u}_{\varepsilon}^{s}(\sigma_n, \alpha_{\varepsilon}(\sigma_{n-1}, \sigma_n)) = \hat{u}_{\varepsilon}^{s}(\sigma_n, \alpha_{\varepsilon}(\sigma_n, \sigma_{n+1})) , \tag{4.2}$$

   where $\alpha_{\varepsilon}(\sigma_n, \sigma_{n+1})$ maximizes the expected payoff of the receiver when the signal is in $[\sigma_n, \sigma_{n+1}]$, for a fixed value $\varepsilon > 0$. If, for example, $\sigma_{n-1}, \sigma_n \in [\varepsilon, 1-\varepsilon]$, the posterior of the receiver about the state is symmetric and has support $[\sigma_{n-1}-\varepsilon,\sigma_n+\varepsilon]$, and so $\alpha_\varepsilon(\sigma_{n-1},\sigma_n)=\frac{1}{2}(\sigma_{n-1}+\sigma_n)$. In general, it is easy to see that[8]

$$\alpha_\varepsilon(\sigma_{n-1},\sigma_n)\in\left(\tfrac{1}{2}(\sigma_{n-1}+\sigma_n)-\varepsilon,\tfrac{1}{2}(\sigma_{n-1}+\sigma_n)+\varepsilon\right)$$

and[9]

$$\left|\kappa(\sigma_{n+1}-\sigma_n)-(\sigma_n-\sigma_{n-1})\right|<4(1+\kappa)\varepsilon\,. \tag{4.3}$$

Equation (4.2) can be used to obtain the relationship between the sizes of consecutive intervals. If, for example, $\sigma_{n-1},\sigma_{n+1}\in[\varepsilon,1-\varepsilon]$, we can show that[10]

$$\sigma_{n+1}-\sigma_n\begin{cases}\geq\sigma_n-\sigma_{n-1}+4\,\frac{1-\kappa}{1+\kappa}\,\varepsilon & \text{if } \sigma_n-\sigma_{n-1}\leq 2\varepsilon,\\ =\kappa^{-1}(\sigma_n-\sigma_{n-1}) & \text{if } \sigma_n-\sigma_{n-1}>2\varepsilon.\end{cases} \tag{4.4}$$

Note that when the partition element is "big," i.e., $\sigma_n-\sigma_{n-1}>2\varepsilon$, then equation (4.4) coincides with equation (2.4). If, however, the partition element is "small," i.e., of size less than 2ε, the size of the interval grows at least linearly in n.

Step 2: Some definitions. In this step we define n^-, n^+, and n^b. Let $n^-\in\{-N-1,\ldots,0\}$ be such that $\varepsilon\in(\sigma_{n^--1},\sigma_{n^-}]$; that is, $[\sigma_{n^--1},\sigma_{n^-}]$ is the rightmost partition element containing signals below ε. Similarly, let $n^+\in\{-N,\ldots,-1\}$ be such that $1-\varepsilon\in[\sigma_{n^+},\sigma_{n^++1})$, that is, $[\sigma_{n^+},\sigma_{n^++1}]$ is the leftmost partition element containing signals

[8] The density of the posterior at state t conditional on the signal being in $[\sigma_{n-1}-\varepsilon,\sigma_n+\varepsilon]$ is proportional to the measure of $[t-\varepsilon,t-\varepsilon]\cap[\sigma_{n-1},\sigma_n]$. Hence, it is increasing from $\max\{0,\sigma_{n-1}-\varepsilon\}$ to $\min\{\sigma_{n-1}+\varepsilon,\sigma_{n+1}-\varepsilon\}$, then constant until $\max\{\sigma_{n-1}+\varepsilon,\sigma_{n+1}-\varepsilon\}$, and finally decreasing until $\min\{1,\sigma_n+\varepsilon\}$. Such a distribution is first-order dominated by one uniform on $[\sigma_{n-1}+\varepsilon,\sigma_n+\varepsilon]\cap[0,1]$, and so $\alpha_\varepsilon(\sigma_{n-1},\sigma_n)<\frac{1}{2}(\sigma_{n-1}+\sigma_n)+\varepsilon$.

[9] When the sender observes σ_n, she knows that the state is in $[\sigma_n-\varepsilon,\sigma_n,+\varepsilon]\cap[0,1]$. It can then be seen that the terms on each of the sides of the sender's indifference condition (4.2) can be bounded using the extreme states. For example,

$$\sigma_n-\varepsilon-\overbrace{\alpha_\varepsilon(\sigma_{n-1},\sigma_n)}^{<\frac{1}{2}(\sigma_{n-1}+\sigma_n)+\varepsilon}<\hat{u}^s_\varepsilon(\sigma_n,\alpha_\varepsilon(\sigma_{n-1},\sigma_n))=\hat{u}^s_\varepsilon(\sigma_n,\alpha_\varepsilon(\sigma_n,\sigma_{n+1}))<\kappa\,(\overbrace{\alpha_\varepsilon(\sigma_n,\sigma_{n+1})}^{<\frac{1}{2}(\sigma_n+\sigma_{n+1})+\varepsilon}-\sigma_n+\varepsilon)\,.$$

[10] If the middle points of the intervals $[\sigma_{n-1},\sigma_n]$ and $[\sigma_n,\sigma_{n+1}]$ lie in the quadratic part of $\hat{u}^s(\sigma_n,\cdot)$, then the inequality becomes an equality. To see this, recall that in the uniform-quadratic case of the CS model with bias b, we have $\sigma_{n+1}-\sigma_n=\sigma_n-\sigma_{n-1}+4b$. If both middle points lie in the linear part, then $\sigma_{n+1}-\sigma_n=\kappa^{-1}(\sigma_n-\sigma_{n-1})$. It is easy to prove that if one middle point lies in the linear part and the other in the quadratic one, the inequality holds.

above $1-\varepsilon$. When $n^- \le n^+$, we define $n^b \in \{n^-, ..., n^+\}$ as the unique index such that all partition elements contained in $[\sigma_{n^b}, \sigma_{n^+}]$ are big (i.e., greater than 2ε) and all contained in $[\sigma_{n^-}, \sigma_{n^b}]$ are small. When $n^- > n^+$ (which happens when $N = 1$ only), we let n^b be equal to 0. The existence and uniqueness of n^b follows from the fact that, by equation (4.4), the size of the partition elements contained in $[\sigma_{n^-}, \sigma_{n^+}]$ increases rightwards.

Step 3. Proof that σ_{n^+} is not close to 0 when ε is small. Take a sequence $(\varepsilon_i)_{i=1}^{\infty}$ strictly decreasing toward 0 and, for each i, a partition equilibrium $\{\sigma_n^i\}_{n=-N_i}^{0}$ of the model with noise with parameter ε_i. Let n_i^-, n_i^+ and n_i^b be, respectively, the values of n^-, n^+, and n^b corresponding to the i-th equilibrium. We assume, for the sake of contradiction, that $\sigma^i_{n_i^+} = 1-x_i$, where $(x_i)_i$ is a sequence converging to 0. Hence, if i is high enough, $n_i^+ > n_i^-$. Fix some i. Since $\sigma^i_{n_i^+ +1} > 1-\varepsilon_i$, we have that $\sigma^i_{n_i^+ +1} - \sigma^i_{n_i^+} < x_i + \varepsilon_i$. Using equation (4.3) we obtain

$$\sigma^i_{n_i^+} - \sigma^i_{n_i^+ -1} \le \kappa\,(\sigma^i_{n_i^+ +1} - \sigma^i_{n_i^+}) + 4\,(1+\kappa)\,\varepsilon_i < \kappa\, x_i - (4+6\kappa)\,\varepsilon_i \;.$$

It is therefore clear that, as i increases, $\sigma_{n_i^+ -2}$ converges to 1, and so $n_i^+ - 2 \ge n_i^-$ if i is high enough. Since all partition elements in $[\sigma^i_{n_i^b}, \sigma^i_{n_i^+}]$ are big, their size increases exponentially in this set (by equation (4.4)). We then have

$$\begin{aligned}\sigma^i_{n_i^b} &> \sigma^i_{n_i^+} - (1-\kappa)^{-1}\,(\sigma^i_{n_i^+} - \sigma^i_{n_i^+ -1}) > 1 - x_i - \tfrac{x_i\,\kappa + (4+6\kappa)\,\varepsilon_i}{1-\kappa} \\ &= 1 - \tfrac{x_i}{1-\kappa} - \tfrac{4+6\kappa}{1-\kappa}\,\varepsilon_i \;.\end{aligned}$$

There are two cases. Assume first that for all $\bar{i} \in \mathbb{N}$ there is some $i > \bar{i}$ such that $n_i^b = n_i^-$; that is, all intervals contained in $[\sigma_{n_i^-}, \sigma_{n_i^+}]$ are big. Using equation (4.3) and the fact that $\sigma^i_{n_i^- -1} < \varepsilon_i$ (by the definition of n_i^-), this implies that there are arbitrarily large values i such that[11]

$$\begin{aligned}\overbrace{4\,(1+\kappa)\,\varepsilon_i}^{\to 0} &> \left|\sigma^i_{n_i^-} - \sigma^i_{n_i^- -1} - \kappa\,(\sigma^i_{n_i^- +1} - \sigma^i_{n_i^-})\right| \\ &> \underbrace{\left|1 - \tfrac{x_i}{1-\kappa} - \tfrac{4+6\kappa}{1-\kappa}\,\varepsilon_i - 2\,\varepsilon_i - \kappa\left(1 + \varepsilon_i - (1 - \tfrac{x_i}{1-\kappa} - \tfrac{4+6\kappa}{1-\kappa}\,\varepsilon_i)\right)\right|}_{\to 1} \;.\end{aligned}$$

This is a contradiction. Assume then, alternatively, that $n_i^b > n_i^-$ for all $i > \bar{i}$, for some $\bar{i} \in \mathbb{N}$. Since all partition elements contained in $[\sigma^i_{n_i^-}, \sigma^i_{n_i^b}]$ are small, we can use

[11] To simplify notation, "$\to$" means that the left hand-side tends to the right-hand side as $i \to \infty$.

equation (4.4) to obtain

$$\begin{aligned}\sigma^i_{n^-_i+1} - \sigma^i_{n^-_i} &\le \sigma^i_{n^b_i} - \sigma^i_{n^b_i-1} - (n^b_i - n^-_i - 1)\,4\,\tfrac{1-\kappa}{1+\kappa}\,\varepsilon_i \\ &\le \big(2 - (n^b_i - n^-_i - 1)\,4\,\tfrac{1-\kappa}{1+\kappa}\big)\varepsilon_i\,. \end{aligned} \tag{4.5}$$

Thus, $n^b_i - n^-_i \le 1 + \frac{1}{2}\frac{1+\kappa}{1-\kappa}$. Furthermore, since the size of the partition elements contained in $[\sigma^i_{n^-_i}, \sigma^i_{n^+_i}]$ decreases leftwards we have that

$$\sigma^i_{n^-_i} > \sigma^i_{n^b_i} - \overbrace{(n^b_i - n^-_i)\,2\,\varepsilon_i}^{\to 0}\,. \tag{4.6}$$

Again, this is a contradiction by equation (4.3) and the fact that $\sigma^i_{n^-_i-1} < \varepsilon_i$ (by the definition of n^-_i), since we have that $\sigma^i_{n^-_i} - \sigma^i_{n^-_i-1} \to 1$ while $\sigma^i_{n^-_i+1} - \sigma^i_{n^-_i} \to 0$.

Step 4. Proof that $\sigma_{n^b_i-1} \to 0$. Take a sequence $(\varepsilon_i)^\infty_{i=1}$ and a corresponding sequence of equilibria as in Step 3. Assume, for the sake of contradiction, that $\liminf_{i\to\infty} \sigma^i_{n^b_i-1} > 0$. Again, we consider two cases. Assume first, taking a subsequence if necessary, that $n^b_i = n^-_i$ for all i. In this case, since $\sigma^i_{n^-_i-1} < \varepsilon_i$, we have $\lim_{i\to\infty}\sigma^i_{n^-_i-1} = 0$, a contradiction. Hence, there is no subsequence where $n^b_i = n^-_i$ for all i. There is then some $\bar{i}$ such that $n^b_i > n^-_i$ for all $i > \bar{i}$. In this case, for all $i > \bar{i}$, all partition elements contained in $[\sigma^i_{n^-_i}, \sigma^i_{n^b_i}]$ are small. Since equations (4.5) and (4.6) hold, we have $(\sigma^i_{n^b_i} - \sigma^i_{n^-_i}) \to 0$ and so $(\sigma^i_{n^-_i+1} - \sigma^i_{n^-_i}) \to 0$. This implies, due to equation (4.3), that $(\sigma^i_{n^-_i} - \sigma^i_{n^-_{i-1}}) \to 0$. Finally, since $\sigma^i_{n^-_i-1} \to 0$ (by the definition of n^-_i), we have $\sigma^i_{n^b_i-1} \to 0$, a contradiction.

Step 5. Proof of closeness. Fix $\Delta > 0$. Take a sequence $(\varepsilon_i)^\infty_{i=1}$ and a corresponding sequence of equilibria as in Step 3. Assume, for the sake of contradiction, that for all i, all equilibria in the auxiliary model are at a distance greater than Δ from the equilibrium $\{\sigma^i_n\}^0_{n=-N_i}$. Assume, taking a subsequence if necessary, that $(\sigma^i_{-1})_i$ and $(N_i)_i$ converge, respectively, to some $\bar{\sigma}$ and $N \in \mathbb{N} \cup \{+\infty\}$. Note that, by Step 3, we have $\bar{\sigma} < 1$. By equation (4.3) we have that, for all $n \ge -N$,

$$\sigma^i_n \to \bar{\sigma}_n \equiv 1 - (1-\bar{\sigma})\,\frac{1-\kappa^{-n}}{1-\kappa}\,.$$

Since, necessarily, $\lim_{n\to -N}\bar{\sigma}_n \to 0$ (since, by Step 4, $\sigma^i_{n^b_i-1} \to 0$), we have that $\bar{\sigma}_n = t_{N+n}$ as defined in equation (2.5) (in the finite case) and as defined in equation (2.6) (for the infinite case). This implies that, if i is big enough, the distance between the

i-th equilibrium and the discrete equilibrium with N signals in our auxiliary model is less than Δ, a contradiction.

2. Fix some $N \in \mathbb{N}$. We want to show that the partition equilibrium with N intervals in the model of Section 2 (with partition thresholds $\{t_n\}_{n=0}^N$ given in equation (2.5)), can be approximated by a sequence of equilibria as the noise becomes small (a similar argument applies for the infinite equilibrium). To do so, we construct a putative partition equilibrium of the model with noise as follows. Fix some small $\xi \in \mathbb{R}$, and define $\sigma_{0,\varepsilon}^{\xi} \equiv -\varepsilon$ and $\sigma_{1,\varepsilon}^{\xi} \equiv t_1 + \xi$. Then, we iteratively use the sender's indifference condition (4.2) to find $\sigma_{n+1,\varepsilon}^{\xi}$ for $n = 1, \ldots, N-2$. When ε is small, equation (4.2) approximates (2.4), and so each $\sigma_{n,\varepsilon}^{\xi}$ is strictly increasing and continuous in ξ for all $n = 1, \ldots, N-1$. Still, for each value of ξ, equation (4.2) does not necessarily hold for $n = N-1$ when $\sigma_{N,\varepsilon}^{\xi}$ is set to be the upper bound of the signal space, $1+\varepsilon$. In fact, for each $\xi > 0$, the right-hand side of equation (4.2) for $n = N-1$ is higher than its left-hand side if ε is small enough, and the reverse is true if $\xi < 0$. By continuity, there is some $\xi(\varepsilon)$ such that the equation holds. Thus, for any sequence $(\varepsilon_i)_i$ converging to 0 and for all $n = 1, \ldots, N-1$, we get

$$\hat{u}_{\varepsilon_i}^{s}\left(\sigma_{n,\varepsilon_i}^{\xi(\varepsilon_i)}, \alpha_{\varepsilon_i}\left(\sigma_{n-1,\varepsilon_i}^{\xi(\varepsilon_i)}, \sigma_{n,\varepsilon_i}^{\xi(\varepsilon_i)}\right)\right) = \hat{u}_{\varepsilon_i}^{s}\left(\sigma_{n,\varepsilon_i}^{\xi(\varepsilon_i)}, \alpha_{\varepsilon_i}\left(\sigma_{n,\varepsilon_i}^{\xi(\varepsilon_i)}, \sigma_{n+1,\varepsilon_i}^{\xi(\varepsilon_i)}\right)\right).$$

For each i, $\{\sigma_{n,\varepsilon_i}^{\xi(\varepsilon_i)}\}_{n=0}^{N}$ are the thresholds of a partition equilibrium with N partition elements of our model with noise with $\varepsilon = \varepsilon_i$. Furthermore, for each n, $\lim_{i\to\infty} \sigma_{n,\varepsilon_i}^{\xi(\varepsilon_i)} = t_n$.

□

The previous result can be generalized to a wider set of noise structures with the property that the distance between the state and the signal is, with an increasing probability, $O(\varepsilon)$ as $\varepsilon \to 0$. The crucial property that drives the result is that, as $\varepsilon \to 0$, the ratio between size of the partition elements and difference of ideal actions explodes (see footnote 6). Hence, for each $\theta \in \{s, r\}$, the induced payoff function in the signal space, $\hat{u}_{\varepsilon}^{\theta}$, can be approximated by the underlying payoff function u^{θ}, and the partition elements can be approximated by those described in Section 2 as a result. The induced payoff functions in the signal space are smooth for each given $\varepsilon > 0$, and also the difference between the ideal actions shrinks as $\varepsilon \to 0$. Hoverver, for each signal σ, $\hat{u}_{\varepsilon}^{\theta}(\sigma, \cdot)$ becomes increasingly more "kinky" as $\varepsilon \to 0$ (notice that the second derivative at their maximum is $-\frac{1}{2}\varepsilon^{-1}$) and the range where it is (approximately) symmetric becomes small. The simultaneous effect of reducing the noise on the endogenous bias and the shape of the payoff functions prevents the arguments used in the existing literature from holding. Thus, in contrast to the standard CS model, a small state-by-state objective misalignment is not sufficient to guarantee the existence of equilibria with precise communication in our setting.

Remark 4.1. Our result establishes that the usual (and natural) finding that a small objective misalignment is a necessary and sufficient condition for the existence of equilibria with precise information transmission does not hold in general. An alternative interpretation that reestablishes the above finding is the following. Our model illustrates that the bias may not be an adequate measure of objective misalignment in some settings. That is, the distance between the ideal actions of the sender and the receiver after each piece of information obtained by the sender is not enough to capture the relevant differences between their objectives. Our model illustrates that the relative skewness between the sender and the receiver plays an important role in determining the equilibrium communication even when the bias between the sender and the receiver is small.[12]

5 Discussion, Applications and Conclusions

5.1 Discussion

The models studied in Sections 2 and 3 contain equilibria other than discrete (partition) equilibria, e.g, equilibria with full information transmission. In Section 4 we showed that the existence of non-discrete equilibria crucially relies on the perfect observation of the state by the sender. In this section, we show than when a small bias or cognitive costs of using or learning a language are introduced to our auxiliary model, equilibrium communication becomes significantly coarse.

Bias

Partition equilibria appear naturally in settings where there is an "exogenous" bias between the sender and the receiver, that is, when they disagree on the ideal action to be taken after the realization of each state of the world. This section shows how the existence of a small bias may make communication significantly coarse when agents are differently skewed.

We illustrate the effect of a small bias on communication using a tractable example. Consider a setting analogous to the one presented in Section 2, the only difference being

[12] The author thanks Daniel Krähmer for suggesting this interpretation.

that, while the receiver still has the same payoff function (given by equation (2.1) with $\theta = r$), the sender's payoff is given by

$$\begin{cases} -((1+\varepsilon)\,t - a) & \text{if } a < (1+\varepsilon)\,t, \\ -k^s\,(a-(1+\varepsilon)\,t) & \text{if } a \geq (1+\varepsilon)\,t, \end{cases} \tag{5.1}$$

with[13] $\varepsilon > 0$. Note that now the ideal action of the sender when the state is $t \in (0,1]$ is $(1+\varepsilon)\,t > t$. It is not difficult to see that, for each $\varepsilon > 0$, only equilibria essentially equivalent to discrete equilibria exist. In fact, there is a unique infinite equilibrium, in which the thresholds of the partition elements are $\{\hat{\kappa}^n\}_{n=0}^{\infty}$ (similar to equation (2.6)), where $\hat{\kappa} \in (0,1)$ solves

$$\hat{\kappa} = \kappa - \varepsilon\,\frac{\hat{\kappa}}{1-\hat{\kappa}}\,\frac{(1+k^r)\,(1+k^s)}{k^r}\,.$$

The value $\hat{\kappa}$ is increasing in κ and decreasing in ε; that is, a lower relative skewness and a lower relative bias lead to better communication. As the bias disappears (i.e., as $\varepsilon \to 0$), the limit of the thresholds of the partition elements of the (unique) infinite equilibrium satisfy equation (2.6). It is then not difficult to show that only (and all) the discrete equilibria described in Section 2 can be approximated by equilibria in the model with bias as the bias becomes small.[14]

Costly talk

The assumption that talk is "cheap," i.e., all messages have the same cost (normalized to 0), seems plausible when the number of messages used in equilibrium is not too large. This is typically the case in partition equilibria of the CS model when the bias is not small. When the number of used messages is large, however, it seems more plausible that communication is costly *per se*. This may be due to the cognitive costs of learning or using complex

[13] This specification simplifies the analysis. Indeed, Melumad and Shibano (1991) show that, in the CS model with uniform distribution and payoff functions $\hat{u}^r(t,a) = -(t-a)^2$ and $\hat{u}^s(t,a) = -((1+\varepsilon)\,t-a)^2$, with $\varepsilon > 0$, there is an infinite equilibrium with thresholds satisfying an equation analogous to (2.6).

[14] Our example illustrates that, under some specifications, communication is significantly coarse regardless how small the bias is. Under some other specifications, communication may be precise in the presence of a small bias. Consider for example the case where the sender is both skewed and has a constant bias, so her payoff function is given by equation (5.1) except that $(1+\varepsilon)\,t$ is replaced by $t+\varepsilon$. In this case, communication is significantly coarse in any equilibrium if $\varepsilon > 0$, while there may be equilibria with precise communication if $\varepsilon < 0$. If, for example, $\varepsilon = -\frac{1-k}{2\,(1+k)\,N} < 0$ for some $N \in \mathbb{N}$, there is an equilibrium with N equally-sized intervals.

language structures (see Hertel and Smith, 2013). Direct communication costs are (implicitly or explicitly) assumed, for example, in communication models with no conflict of interest (e.g., Cŕemer et al., 2007, Jäger et al., 2011, Sobel, 2015 and Dilmé, 2018a), where the number of messages that can be used for communication is exogenously or endogenously limited. Our result highlights that the presence of small cognitive costs may have a significant impact on the existence of equilibria with precise communication.

5.2 Applications

There are many economic situations where agents have a common target, but they disagree on the relative cost of missing the target upwards or downwards. In what follows, we discuss two such situations. They exemplify how skewed preferences naturally arise in settings where communication between strategic agents takes place, and illustrate how seemingly small departures from standard assumptions can significantly affect predictions.

Intra-firm communication. In this example, the sender and the receiver jointly own (or are division managers in) a firm producing some good. Assume, for simplicity, that they equally share the profits, and that the price of the good is 1. The sender is an expert on estimating the demand t, distributed uniformly in $[0,1]$. She communicates her estimation to the receiver. The receiver decides the production $a \in [0,1]$. The marginal cost of the production materials is $c \in (0,1)$, and the receiver incurs an extra cost $c' \in (0, \frac{1-c}{2})$ per unit produced for the time and effort spent on production. Thus, the payoffs of the sender and the receiver when the realized demand is t and the production choice is a are given by

$$u^s(t,a) = \tfrac{1}{2}\left(\min\{a,t\} - c\,a\right) \text{ and } u^r(t,a) = \tfrac{1}{2}\left(\min\{a,t\} - c\,a\right) - c'\,a\,.$$

It is not difficult to see that both the sender and the receiver prefer $a = t$ above any other production level when the realized demand is equal to[15] t. Simple algebra shows that

$$\frac{k^s}{1+k^s} = c \text{ and } \frac{k^r}{1+k^r} = c + 2\,c'\,.$$

As in Section 2, $\kappa < 1$. As a result, if the demand is not observed perfectly, there will be a significant amount of imprecision in the information transmission in any equilibrium. While low demand realizations are precisely communicated, the communication of high

[15]In contrast to our example in Section 2, here the payoff functions of the sender and the receiver have a "level" that depends on the realized state. Since their incentives and the equilibrium construction do not depend on the levels, our analysis also applies in this case.

demand realizations is significantly coarse, even when the estimation of the demand is very accurate. Firms should therefore design contracts to reduce not only the difference between the objectives of their employees, but also the relative weight their employees put on avoiding mistakes of different types.

Politicians with different electorates. This example illustrates how communication between politicians may be affected by the fact they are chosen by different constituencies. The sender is a (finance or public works) minister with some private knowledge about the suitability of a policy (such as the optimal amount of taxes to be collected or the geographical suitability of a new public facility). The receiver is the prime minister in charge of making the final decision about the policy choice. They belong to the same party but are chosen by different constituencies. If the state of the world is t and the policy a is chosen, the payoff of the θ-agent is equal to

$$u^{\theta}(t,a) = |t-a| + \beta^{\theta}\,|a^{\theta}-a|\ ,$$

where $a^{\theta} \in [0,1]$ and $\beta^{\theta} \in [0,1)$, for all $\theta \in \{s,r\}$, and where either $a^{s} \neq a^{r}$ or $\beta^{s} \neq \beta^{r}$ (or both). The first term of the payoff function is interpreted as the preference for choosing the ideal policy for the country (or their party), normalized to be equal to the state. One can then interpret a^{θ} as the preferred action of the median voter of each of the politician's constituencies, or reflecting his or her own political bias, and β^{θ} as the relative weight of this term on each politician's preferences. The fact that $\beta^{\theta} < 1$ captures that both politicians care more about the country (or the party) than their own constituencies, and implies that if the state of the world t were known, they would agree on their preferred policy, $a = t$. It is easy to see that this setting generates differently skewed preferences (in some part of the state space), and therefore the equilibrium communication is significantly coarse regardless of the precision of the observation of the state of the world, or the amount of resources invested on making the observation more precise.

5.3 Conclusions

This paper shows that when agents are differently skewed, equilibrium communication between them tends to be coarse. Even though a sender and a receiver may agree on the ideal action for each state of the world, a slight noise in the observation of the state will prevent the transmission of precise information. This implies that a small conflict of interest between a sender and a receiver is not always sufficient to ensure the existence of equilibria with precise information transmission.

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